Mathematics-II
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Module - 2
Lecture No - 7
Linear Algebra Part – 2

Welcome, viewers. This is second lecture on linear algebra. In my first lecture, we have given the definition of real vector space. To re - view it, I will first again give you the definition of a real vector space. A triplet V - a set - and two operations plus and star forms a vector space provided certain axioms are satisfied.

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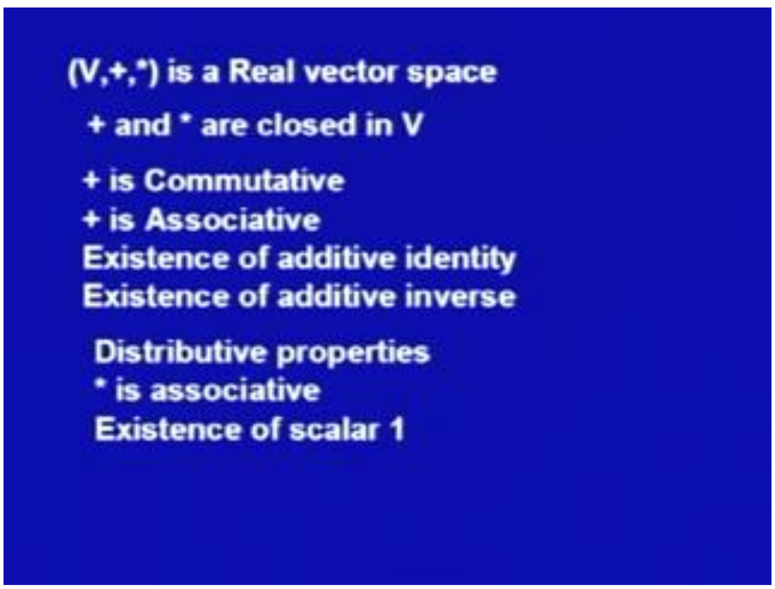

The first axiom is that the operations plus and star - we call it scalar multiplication - are closed in V. The addition is commutative and associative. Further the existence of additive identity is there in the set V and existence of additive inverse is also there in the set V. As far as the scalar multiplication star is concerned satisfies distributive properties with respect to vector addition and star is also associative and there exists a scalar 1 - if all these properties are satisfied - then V plus star is a real vector space. We have discussed a number of examples in my earlier lecture. I will continue with few more examples to fix up the ideas.

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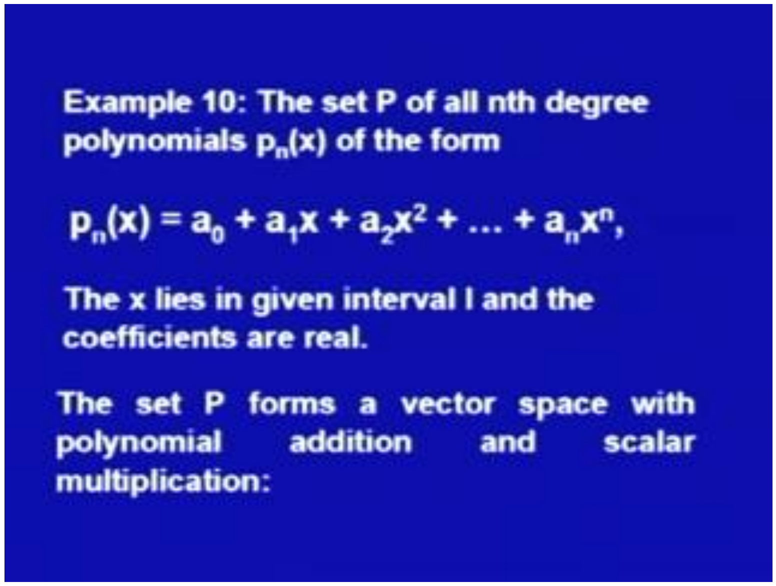

The example I take now is from the set of n th degree polynomials pnx of the form pnx is equal to a naught plus a 1 x plus a 2 x square plus an xn where the x lies in given interval i and all the coefficients a naught a 1 a 2 an etcetera are real. Then this set P forms a vector space with polynomial addition and scalar multiplication which is defined in this manner.

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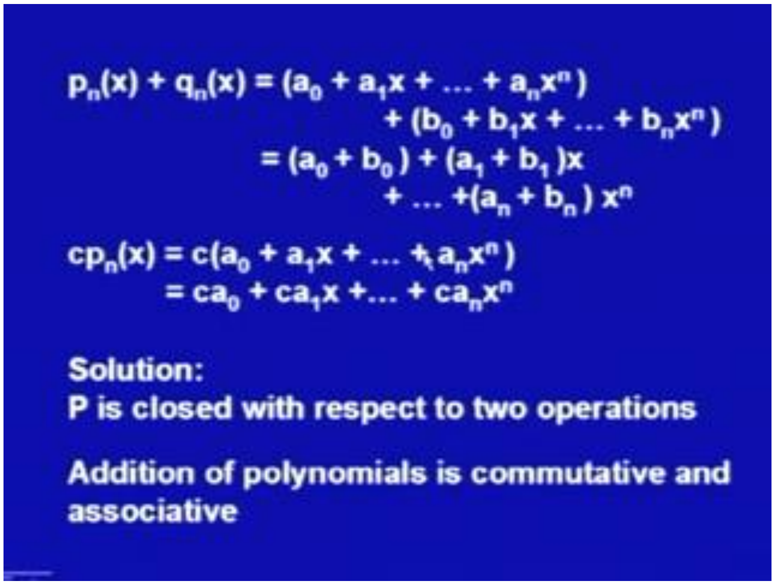

If I have 2 polynomials pnx and qnx where the polynomial pnx is a naught plus a 1 x plus an x raised to power n and another polynomial qnx of the form b naught plus b 1 x plus bn x raised to power n, then the addition of 2 polynomial is defined as a naught plus b naught plus a 1 plus b 1 x and so on and the last term is an plus bn x raised to power n; the idea is the coefficients are added. a naught b naught, a 1 plus b 1 is the coefficient of x, an plus bn is a coefficient of xn. Now this is called polynomial addition while the scalar multiplication by a scalar c is defined as c pnx is equal to c times the polynomial and if you simplify, it is c a naught plus c a 1 x plus c an xn.

Now with these two operations, we say that this set forms a vector space. For proving this, first thing we have to do is we have to say that p is closed with with respect to two operations - the addition and scalar multiplication. Now by this, I mean to say that if I have a polynomial pnx in p and a polynomial qnx in p, then their sum is also polynomial in p. So if you look at this definition, one can easily observe that when we add the two, the right hand side is also polynomial of degree n with real coefficients - a naught plus b naught, a 1 plus b 1 etcetera. So this operation is closed with respect to - the set p is closed with respect to – addition. Now as far as scalar multiplication is concerned, the same is true. Look at this. This cpnx is also polynomial of degree n; the only difference is all coefficients are multiplied by c, but this still remains real; that is why this cpnx is a polynomial of degree n or we say that p is closed with respect to the two operations.

The next property is that this addition is commutative as well as associative. Now this can be easily observed because pnx plus qnx - if this is so, then the coefficients of qnx plus pnx will be b naught a naught b 1 plus a 1 bn plus an and so on and this simply means these are real coefficients; so they are commutative. So pnx plus qnx will be the same as qnx plus pnx. Similarly one can easily establish that this operation is associative.

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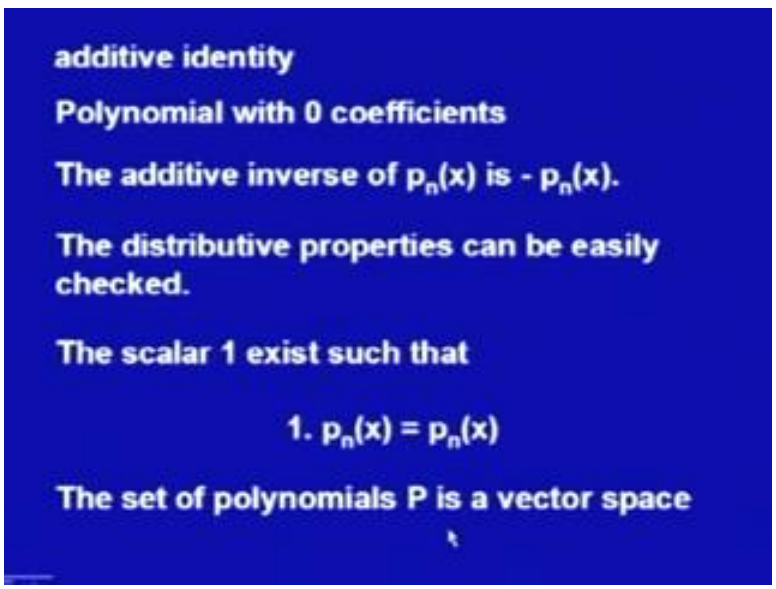

As far as additive identity in the set is concerned, one can easily see that polynomial with 0 coefficients is the additive identity and this belongs to the set of polynomials. Similarly the additive inverse of pnx is minus pnx and that means the additive inverse also exists in the set p. As far as the distributive properties they are concerned, they can also be easily checked because all the coefficients are real numbers and they satisfy distributive properties; so these properties are satisfied. Then comes the existence of 1 such that 1 multiplied by pnx is pnx. So the effect of this scalar multiplication with pnx is that all the coefficients are multiplied by 1 and multiplying a real number by 1 will not change the coefficient and that is why 1 times pnx is pnx and this means that the set of polynomials P is a vector space.

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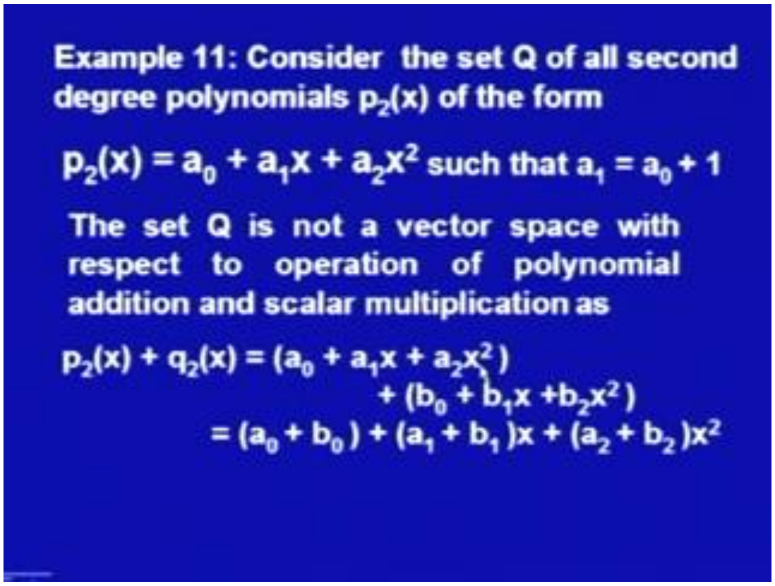

In the next example, we consider the set Q of all second degree polynomials of the form p 2 x is equal to a naught plus a 1 x plus a 2 x square such that the coefficients are related by the relation a 1 is equal to a naught plus 1. So it's not all polynomials but only those polynomials which satisfy this relation. Then this set Q is not a vector space with respect to operation of polynomials of addition and scalar multiplication defined as p 2 x plus q 2 x is equal to a naught plus a 1 x plus a 2 x square; this is p 2 x plus b naught plus b 1 plus b 2 x square is q 2 x. It is defined as a naught plus b naught plus a 1 plus b 1 x plus a 2 plus b 2 x square, that is, the coefficients are added together. The scalar multiplication c times p 2 x is defined as c a naught plus 1 x plus a 2 x square which is ca naught pus c 1 c times a 1 x plus ca 2 x square.

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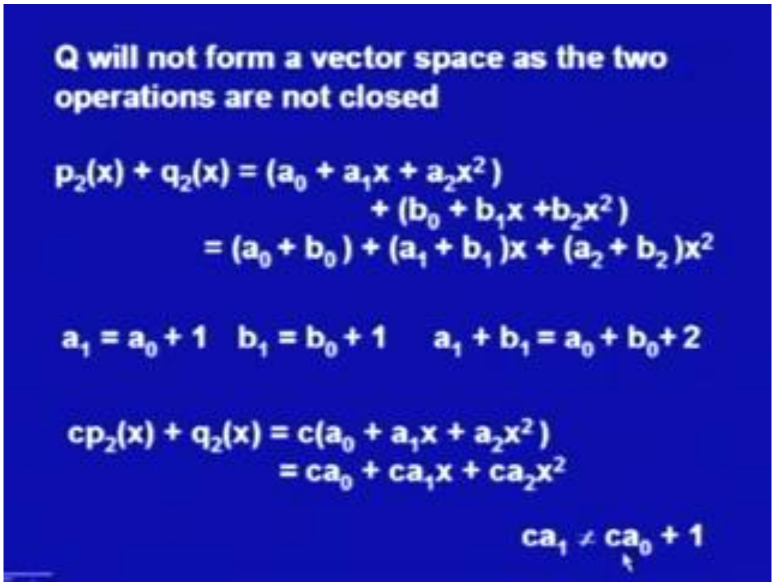

One may notice that this Q will not form a vector space as these two operations are not closed. By this, I mean to say that if i add the two polynomials - p 2 x and q 2 x - belonging to the set Q, then the resulting polynomial is a naught plus b naught plus a 1 plus b 1 x plus a 2 plus b 2 x square according to the definition which is given to us. a1 is equal to a naught plus 1 because p 2 x belongs to the set Q. Similarly b 1 is equal to b naught plus 1 - Q 2 belongs to Q - and from this, one can see that a 1 plus b 1 is equal to a naught plus b naught plus 2; but if you apply this here, then this property requires - this sum to be in the set Q requires - that a 1 plus b 1 is equal to a naught plus b naught plus 1 instead of 2. So this polynomial will not belong to the set Q or we say this operation is not close in Q. Similarly cp 2 x is equal to c a naught plus a 1 x plus a 2 x square; if we take q 2 x to be zero, then ca naught plus ca 1 x plus ca 2 x square - and this means ca 1 should be equal to ca naught plus 1; but this is not true because a 1 is equal to a naught plus 1 being p p 2 x belonging to q. So this property is also not satisfied; so this R is also not closed in Q and that makes V q not to be a vector space. Now So far, we were discussing real vector spaces.

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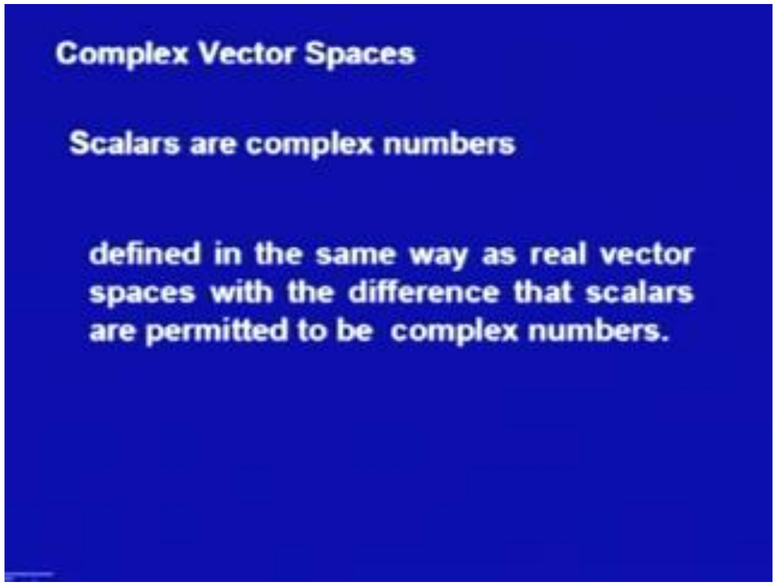

Now we will be coming to complex vector spaces. The basic difference between the two is in the real vector spaces, the scalars are taken from the set of real numbers but in case of complex vector spaces, scalars are taken from set of complex numbers. So we say that complex vector spaces are defined in the same way as real vector spaces, with the difference that scalars are permitted to be complex numbers; that means all the axioms need to be satisfied for this set of scalars.

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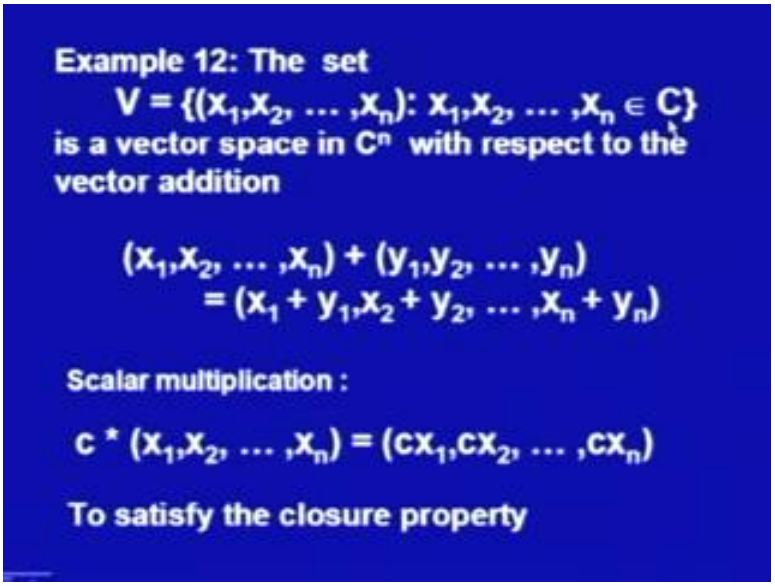

I will Let us take this example. The set of n-tuples V is equal to x 1 x 2 xn where x 1 x 2 xn belongs to C is a vector space in Cn with respect to vector addition which is defined as x 1 x 2 xn belonging to V, y 1 y 2 yn belonging to V - the sum of the two is again an n-tuple where they are components; y's addition is taking place. So this sum is equal to x 1 plus y 1 comma x 2 plus y 2 and the last is xn plus yn while this scalar multiplication is defined as c star x 1 x 2 xn is equal to component wise multiplication cx 1, cx 2 cxn - the last term. Now you may notice that here I am considering all these numbers belonging to c; this is required to satisfy the closure property because - if these are not - if c is a complex number then cx 1 will also be a complex number. So if these x 1 x 2 xn are real, then the closure property will not be satisfied. So this set forms a vector space in Cn. If you consider r and real vector space, then there will be a problem. So to satisfy the closure property, we consider set c this coefficient - this x 1 x 2 xn - belongs to c instead of r, and means that the set V is nothing but Cn. So Cn forms of vector space with respect to these operations.

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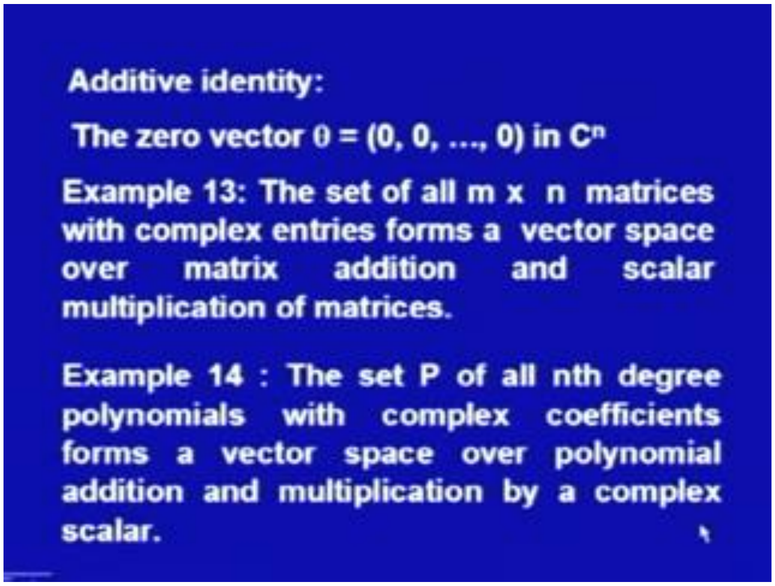

The additive identity can be checked easily because the zero vector theta is also in Cn - zero zero zero is the theta vector, it is in Cn. So this is a rest of the properties can easily be checked, so this forms a vector space. Now in the next example, I am considering m by n matrices with complex entries instead of real entries; this example, I have taken earlier but at that time, these were real entries and we have seen that they form a vector space and now we will see that with complete entries, they also form a vector space with respect to matrix addition and scalar multiplication. The next example is the set P of all n th degree polynomial with complex coefficients. They also form vector space with respect to polynomial addition and multiplication by complex scalars. These are some easy straightforward examples and one can easily but them out.

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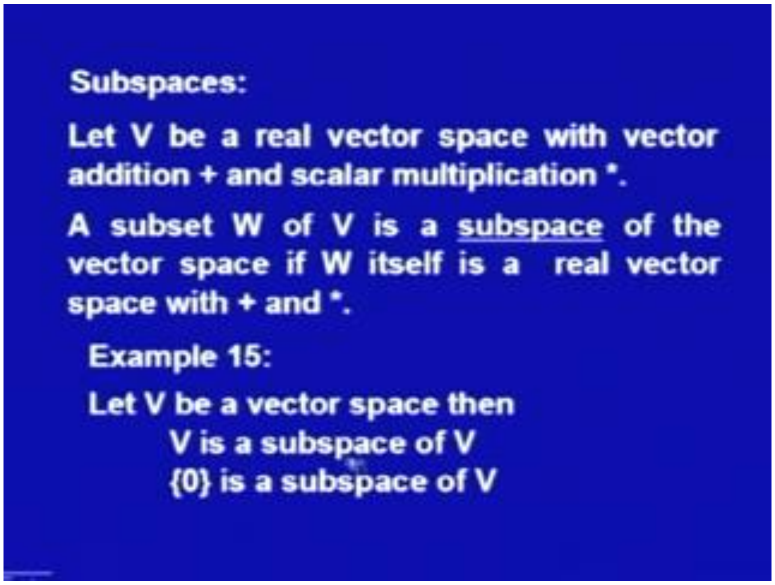

Now the next topic is subspaces. So first the definition of subspaces: you say that let V be a real vector space with vector addition plus and scalar multiplication star, then a subset W of V is a subspace of the vector space if W itself is a real vector space with respect to the 2 operations plus and star. So W is ah W itself is a vector space with respect to the two operations. Now this is an example: if V be a vector space then V is a subspace of V; we can say that V is a subset of V and since V is a vector space, so this is a trivial example that V is subspace of V. The other trivial example is the single term 0 is a subspace of V because this is additive identity; if it belongs to V then it will be a subspace. These are two examples.

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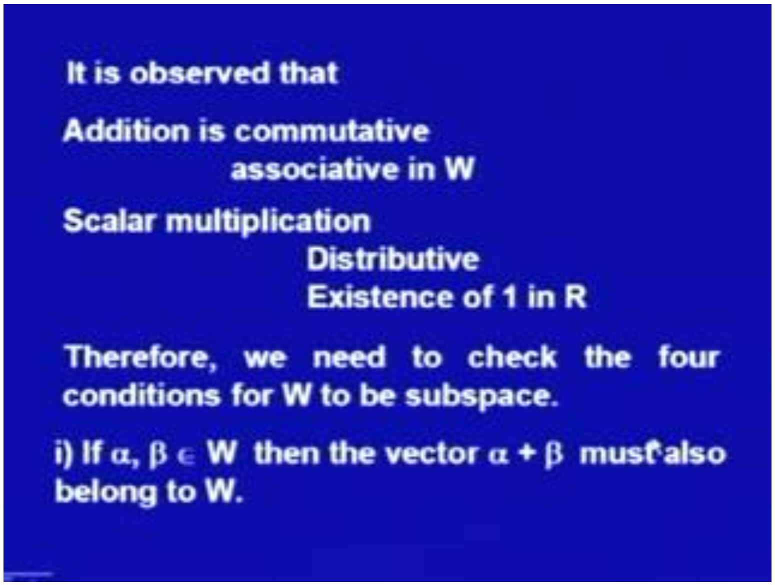

Further it is observed that addition is commutative as well as associative in W because all the members in W they belong to V; so if it is associative and commutative in V then it has to be in W. So that is straightforward. The next thing is scalar multiplication. Again the same logic is applied. It is distributive in V; so it has to be distributive in W also because W is a subset of V. The next thing is - existence of 1 in R - is also implied. So if you have to prove that W is a subspace of a given a vector space V, then we need to check the following 4 conditions:

The first condition is that if alpha beta belongs to W, then the vector alpha plus beta must also belong to W. The idea is alpha beta belongs to W but the sum alpha plus beta may be in V but may not be in W. So one has to check that for each pair alpha and beta in W, its sum also belong to W, that is, this addition is closed in W. So closure property is satisfied.

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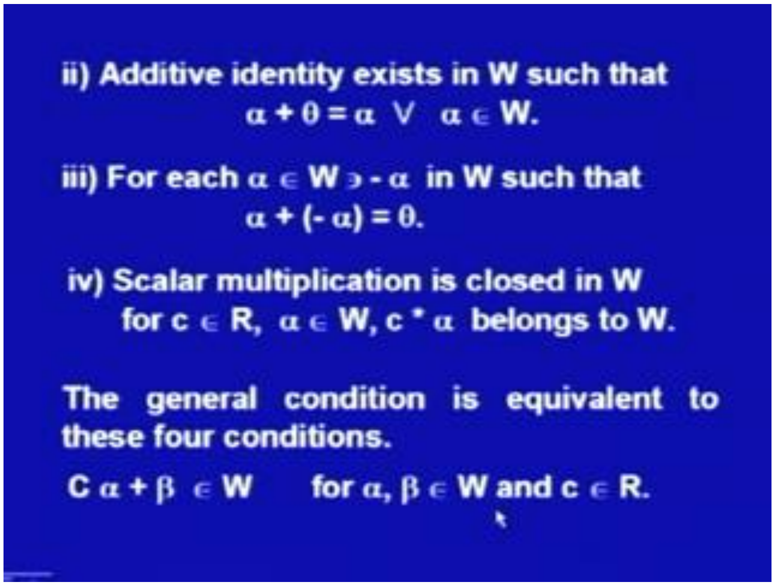

Second is additive identity exist in W such that alpha plus theta is equal to alpha for all alpha in W. So if the additive identity must also be in W, without that it will not be a vector space.

Next thing is that for each alpha belonging to W, there exists minus alpha in W such that alpha plus minus alpha is equal to the null vector theta. So if alpha belongs to W, its inverse must also belong to W; so this is also a requirement that is to be fulfilled by W to be a vector space.

and The last property is that scalar multiplication is closed in W. For example, if c belongs to R and alpha belongs to W then c star alpha must belong to W. So the closure property with respect to scalar multiplication must be closed in W.

So all these four properties are satisfied; rest of the properties are trivially satisfied because they are satisfied for V. So this W will form a vector space.

Now, one can notice that these four conditions can be translated into a single condition - like c alpha plus beta a belongs to W for alpha beta belongs to W and c belonging to R; so if this single condition is satisfied then all these four conditions are implied. So let us prove this; so the first thing is a closure property.

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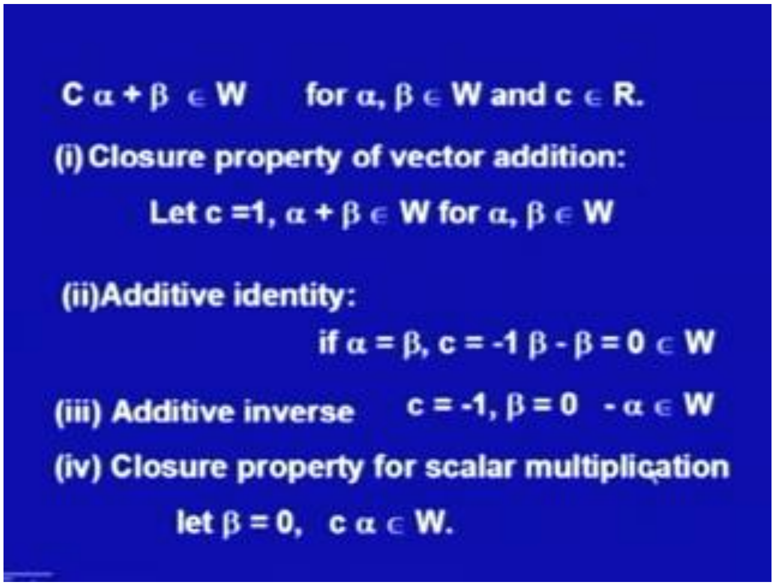

So If I take c is equal to 1 - because this property is satisfied for all c belonging to R, in particular if c is equal to 1 - then alpha plus beta belongs to W for alpha beta belongs to W. So if these properties are satisfied for all c then it will be satisfied for c is equal to 1 and this amounts to the closure property of vector addition in W. So alpha plus beta belongs to W for all alpha beta belonging to W. So If this property is satisfied so this property is implied.

Let us see the second property - additive identity. Here, if alpha is equal to beta and if c is equal to minus 1 - if you consider this - then in this property, because this alpha is equal to beta, so beta minus beta is nothing but null vector; so if c alpha plus beta belongs to W for all values of alpha beta and c in R then beta minus beta is equal to zero will also belong to W. So this shows that there exists additive identity in W. So existence of additive identity W is established in terms of this condition. Then the next is additive inverse. We have given this condition. So If i take c is equal to minus 1 and beta is equal to 0 - the existence of beta is equal to 0, i have already established; so if we take beta is equal to zero and c is equal to minus 1- substitute it here - it is nothing but minus alpha belonging to W. So if alpha is there, minus alpha is also in W and that establishes the additive inverse in W.

So three properties have been established and the fourth one is that closure property for scalar multiplication. So To satisfy to check this property - with respect to this given general condition - when beta is equal to zero, if we substitute then c alpha belongs with W; that means for c is belonging to R and alpha belonging to W, c alpha is also in W and that means closure property for scalar multiplication is also satisfied if this condition is satisfied.

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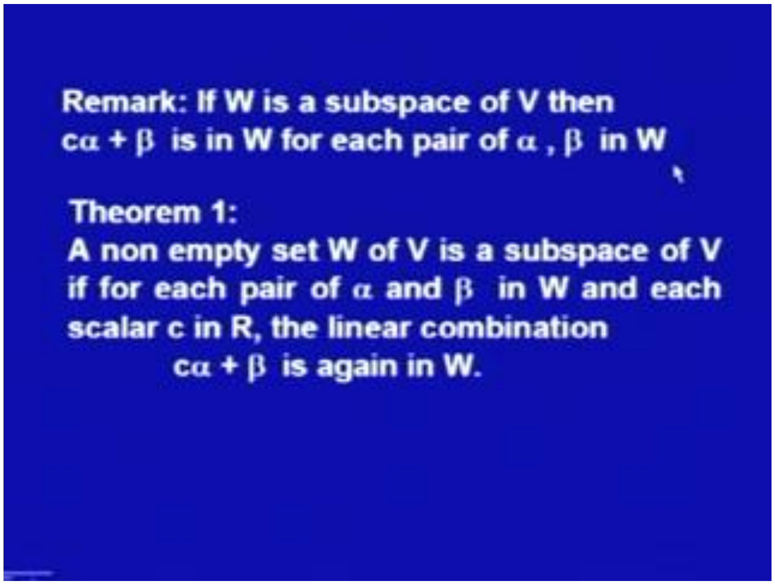

So we can put a remark here that if W is a subspace of V then c alpha plus beta is in W for each pair of alpha beta in W and if this is there, then we can say that W is a subspace. So this is given in the form of the theorem - it states that a non empty set W of V is a subspace of V if for each pair of alpha and beta in W and each scalar c in R this linear combination c alpha plus beta is again in W. So if this condition is satisfied, then W is a subspace or if W is a subspace then c alpha plus beta is again in W. So we have proved this theorem this property is also use as a definition for subspaces. So this is a very powerful property and many times we use it as a definition for subspaces.

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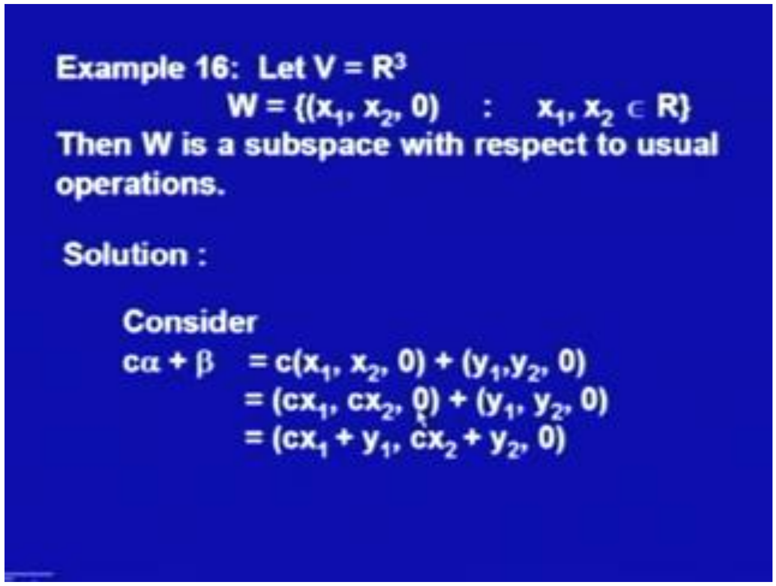

Now, I will explain the definition with the help of an example. So example is taken from R cube. So If we consider V is equal to R cube and W is the set of all triple x where x 1 and x 2 belongs to R, but the third component is 0, then W is a subspace with respect to usual operation. We know that R cube is a subspace with respect to usual addition and scalar multiplication. We have to establish that W is also a subspace in this. So to prove this, we consider c alpha plus beta; so I am using that very definition that if alpha and beta belonging to W, then c alpha plus beta belongs with W; then it becomes a subspace.

so Let us consider c alpha plus beta ; alpha happens to be x 1 x 2 zero belonging to the set W; y 1 y 2 0 is nothing but beta in W. Then if you add the 2, it is cx 1 comma cx 2 comma zero plus y 1 y 2 zero and finally we have cx 1 plus y 1 cx 2 plus y 2 and 0. Now cx 1 plus y 1 belongs to R because x 1 and y 1 and c all they are real. So this also belongs to R; cx 2 plus y 2 also belongs to R in the third component is 0. So this vector belongs to W; so c alpha plus beta belongs to W for given alpha beta in W and that means this W is a subspace - that is it.

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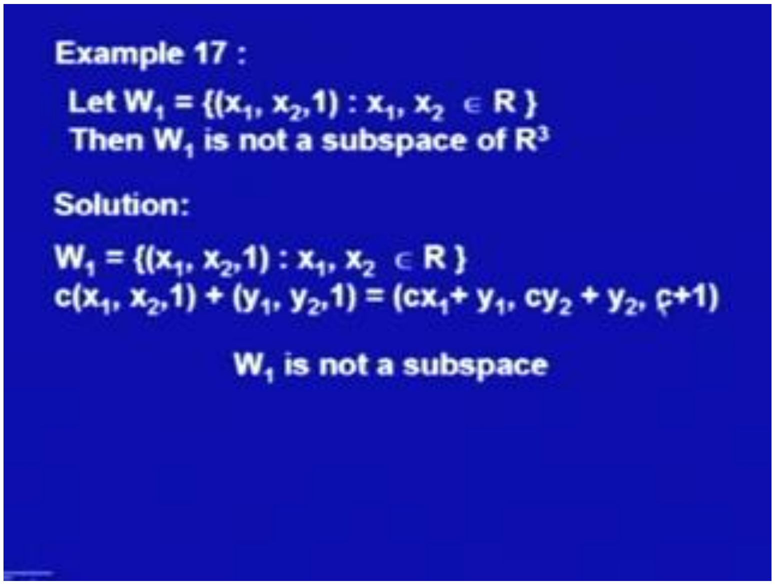

Now in the next example, I consider again another subspace of R cube. Here, W 1 is equal to x 1 x 2 1 such that x 1 x 2 belongs to R. So Instead of 0, now I am having 1 here but one can see that W 1 is not a subspace of R cube with respect to usual operations in R cube. So let us consider W 1 as being given here. Now, consider cx 1 x 2 comma 1 plus y 1 plus y 2 comma 1. So this like To check whether c alpha plus beta also belongs to W 1 or not, so we cancel this left hand side and the right hand side according to the definition of the 2 operations it becomes cx 1 plus y 1 cx 1 plus y 1; second component is cx 2 plus y 2 and the third component is 1 plus 1 is equal to c plus 1. So this is not a subspace because we want here to be 1. Now this will be 1 only when c is equal to 0. For general value of c, this will not be 1 and that is why this right hand side will not belong to W 1 or this operation is now this scalar this linear combination does not belong to W 1 and we say that W 1 is not a subspace.

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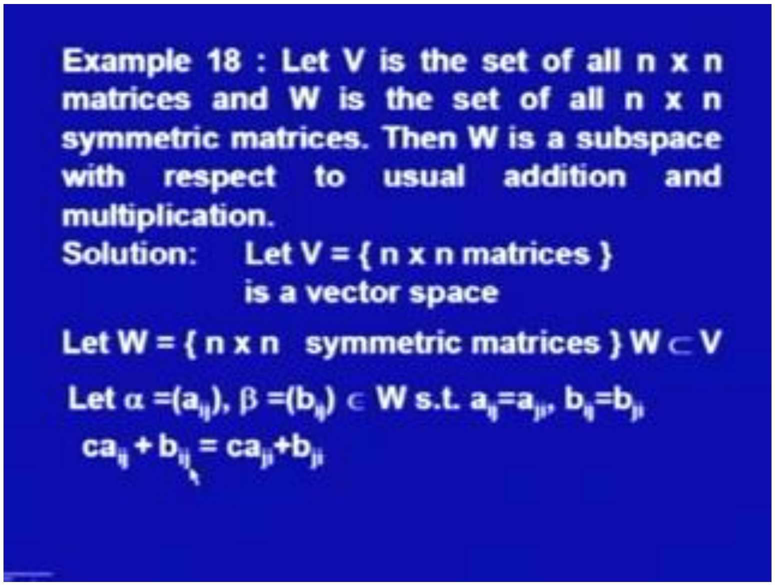

This is another example taken from the set of matrices. Let V be the set of all n by matrices and W is a set of all n by n symmetric matrices. Then W is a subspace with respect to usual addition and multiplication. We have already seen that V is this vector space when V is this set of square matrices and we have the usual addition and multiplication operations - usual addition and scalar multiplication operations in V; so it's a vector space. But we have to see that whether this set W of symmetric matrices is a subspace or not.

So let us start with V which is set n by n matrices; it is a vector space. Now consider set W set W n by n square matrices which are symmetric. We clearly know that W is a subset of V. This is the very first condition - W to be a subspace of V, that W should be a first subset of V; so this - we have established. Now let us consider 2 members in W alpha as aij a square matrix beta is bij another square matrix such that aij is equal to aji. Why? Because they belong - these 2 matrices - belong to W and they are asymmetric matrices; so aji j must be equal to aji. Similarly, bij is equal to bji. Now consider the linear combinations cij plus bij. So we consider typical element of this matrix cij plus bij. Now this is equal to - aij becomes aji - so this caiji plus bji. So we can say that as c alpha plus beta is also symmetric matrix. So the operation is closed and W is a subspace.

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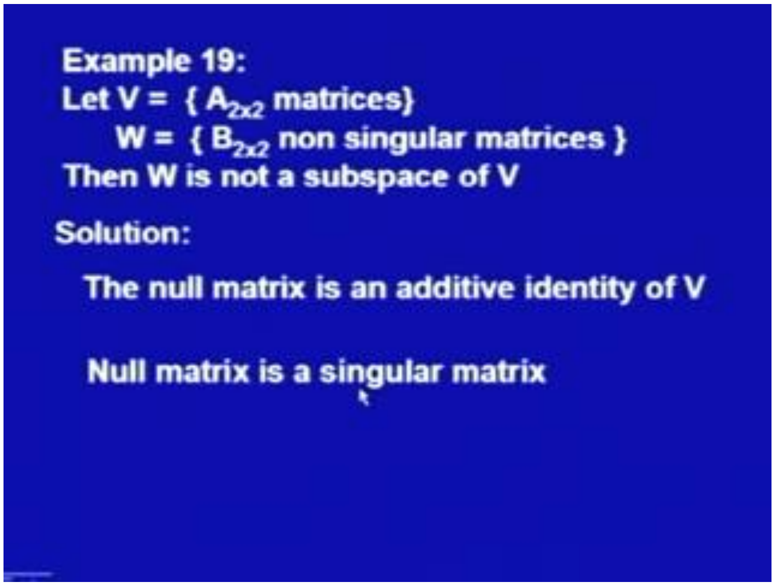

This is another example taken from the set of matrices but this time W is not a subspace. So let us consider V to be 2 by 2 matrices and W is again set of all 2 by 2 matrices. But the condition is that these are non - singular matrices. So we have bigger set considering all square matrices of order 2 while W is only those square matrices of order to which are non - singular. Then W is not a subspace of V, where V is a vector space with respect to usual vectors - matrix addition and scalar multiplication. So one can say that null matrix is an additive identity of V; this belongs to V. It is a 2 by 2 matrix; it is a vector space. So null matrix has to be there in V; but if we have null matrix also in W, then it is non - singular matrix. It is not a non - singular matrix. So if null matrix does not belong to W - because null matrix has determinant 0, null matrix is a singular matrix; so null matrix does not belong to W - but W has to be a vector space in its own right, then additive identity has to be there in W and because of this reason, W will not be a subspace of V. So W is not a subspace of V.

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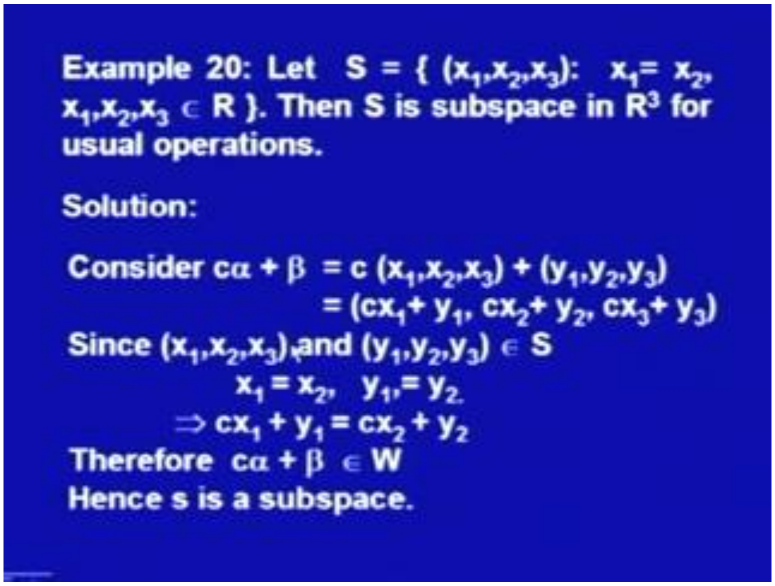

So In the next example, we are again considering a triplet - x 1 comma, x 2 comma, x 3; they all belong to R with the condition that x 1 is equal to x 2. Then S is a subspace in R cube for usual operations. I have different types of sets. Some form subspaces, some do not form subspaces. In this example, if this condition is satisfied, then they form a subspace. So let us see the solution. Again, consider the combination c alpha plus beta which is equal to cx 1 comma x 2 comma x 3; this triplet belonging to S plus y 1 y 2 y 3 belonging to S; then simplifying right hand side, we will have cx 1 plus y 1 comma cx 2 plus y 2 comma cx 3 plus y 3. Now since x 1 x 2 x 3 and y 1 y 2 y 3 belong to S, so they must satisfy the given constrain, that is, x 1 is equal to x 2 and y 1 is equal to y 2 and using this condition we can see that cx 1 plus y 1 will also be the same as cx 2 plus y 2. That means this constrain will be satisfied by this right hand side of c alpha plus beta and therefore c alpha plus beta belongs to W and hence it is a subspace.

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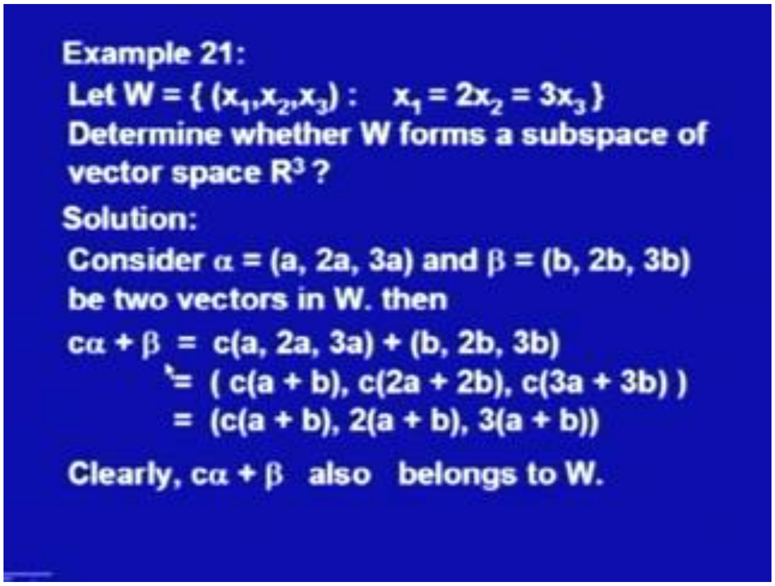

In the next example, W is equal to x 1 comma x 2 comma x 3 but the constraint is now different. x 1 is equal to 2 x 2 is equal to 3 x 3. We have to check whether it forms a subspace of R 3 or not. The solution is on the same lines. We consider an alpha in W which is a comma 2 a comma 3 a, for this condition has to be maintained and beta is equal to b comma 2 b comma 3 b with the two vectors in W. Then c alpha plus beta is c times the alpha vector plus the beta vector. Simplifying it, we have ca plus b comma 2 times a plus b comma 3 times a plus b and that means, the condition x 1 is equal to 2 x 2 is equal to 3 x 3 is also satisfied for c alpha plus beta and that means c alpha beta also belongs to W, and this proves that W is a subspace.

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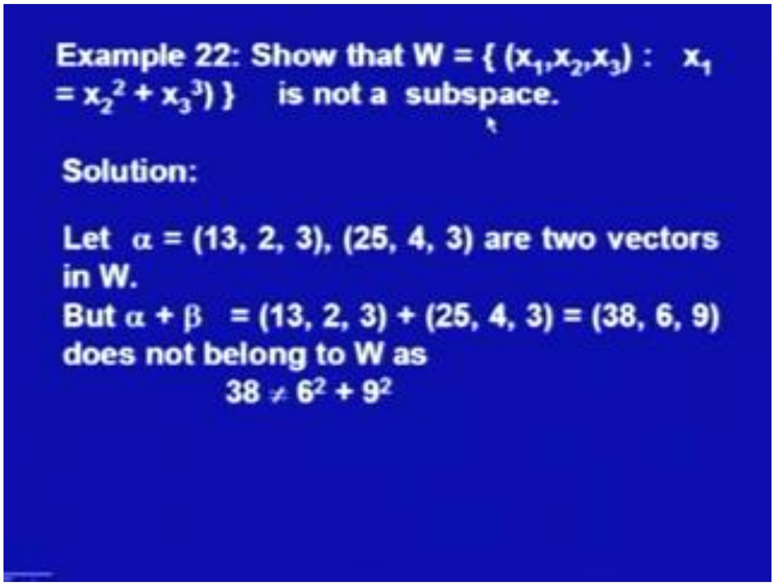

x 2 square plus x 3 square and we will see that it is not a subspace. Now the solution in the solution, we considered an example where alpha is 13 comma 2 comma 3 and beta is 25 comma 4 comma 3 are 2 vectors in W. One can easily see that they satisfy this constraint. 13 is equal to x 2 square plus x 3 square 4 plus 9 and in this case, 25 is equal to 4 square plus 3 square, that is, 16 plus 9 is 25; so these two satisfy the condition. But if you consider alpha plus beta, then the sum of these two vectors is 38 is equal to 6 38 comma 6 comma 9, but this vector does not belong to W. One can easily check that 38 is not equal to 6 square plus 9 square; that means if alpha and beta are these two vectors, then their sum alpha plus beta will not belong to W. Now this is a usual way to show that set is not a subspace.

So if you have to show that a set is a subspace, then you have to consider c alpha plus beta and try to show that c alpha plus beta also belongs to W. But if you know if you have to show that it is not a subspace, then we will take some typical examples which show that c alpha plus beta is belonging to W. So this one simple example is enough to show that W is not a subspace. Now this is slightly different example. Here, I am considering the system of equations represented in the matrix form x is equal to theta.

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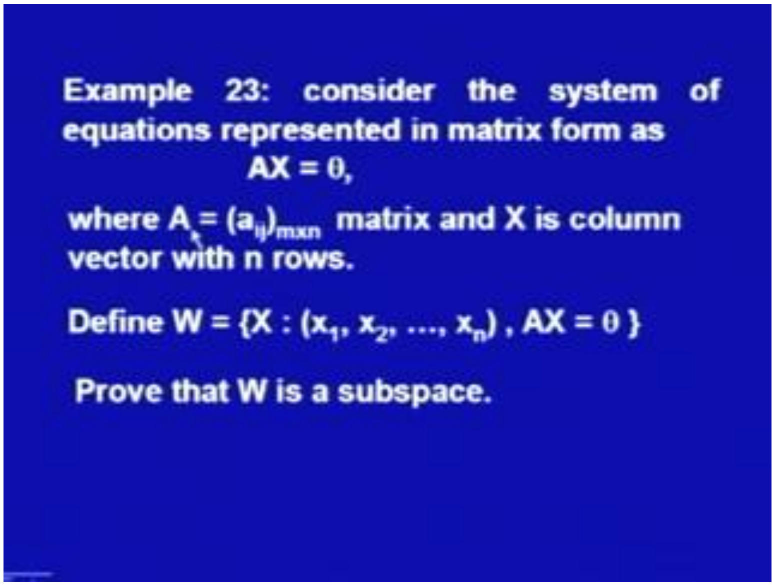

So x is a column vector and theta is null vector. So If i consider A as square A as a rectangular matrix of order m by n, and x is a column vector with n rows, then one can define subset W as all the solution vectors will satisfy x is equal to theta. Then we have to prove that W is a subspace. So we have A - the set of all rectangular matrices of order m by n with respect to usual operations of matrix addition and scalar multiplication; they form a subspace, they form a vector space. But as far as this W is concerned, W is a subset of A. But this is not this will not be a subspace.

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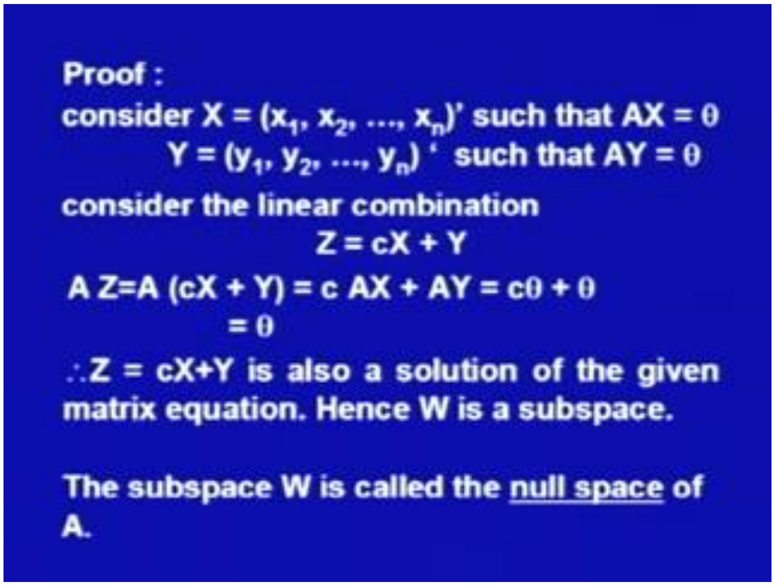

To prove this, we consider x as x 1 x 2 xn column vector such that x is equal to theta y is equal to y 1 y 2 yn such that ay is equal to theta. Now we consider linear combination Z is equal to cX plus Y and we will see that whether AZ it belongs to this given set or not; so we consider AZ is equal to A times column vector cX plus Y which is equal to c times AX plus AY. since cx is: x is theta, so this is c theta is 0 and AY is theta, so c theta plus theta is also theta. So AZ is equal to theta; so this Z is also solution vector; so it forms a subspace. So W is a subspace. In particular, the subspace W is called the null space of A.

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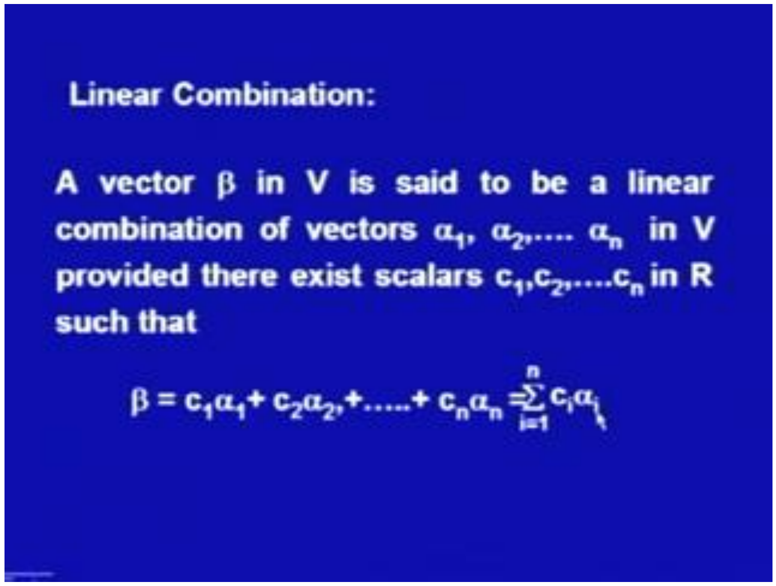

Now we define linear combination. A vector beta in V is said to be linear combination of vectors alpha 1 alpha 2 alpha n in V, provided there exists scalars c 1 c 2 cn in R such that beta is expressed as c 1 alpha 1 plus c 2 times alpha 2 plus c 3 times alpha 3 and cn alpha n. In short we write it as summation i is equal to 1 to n ci alpha i – it is a compressed form, representing this linear combination.

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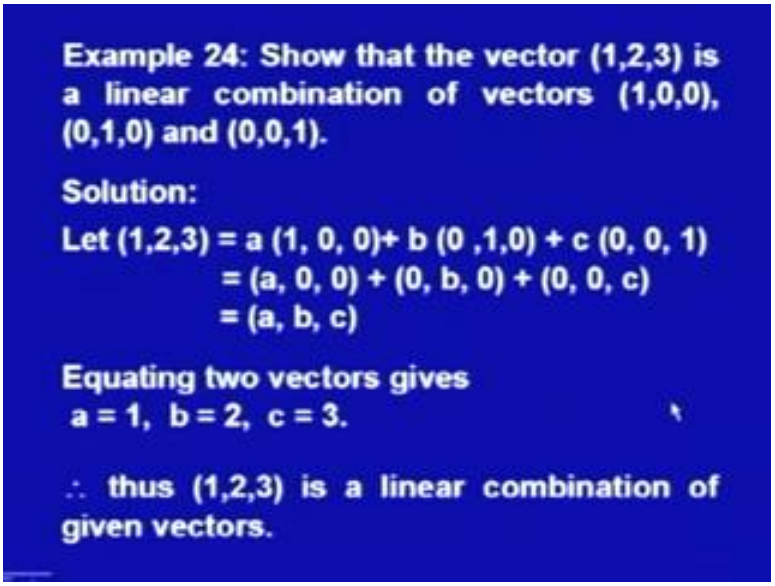

Let us illustrate this with example. In this example, we have to show that the vector 1 2 3 is a linear combination of vectors 1 0 0, 0 1 0 and 0 0 1. So what we are supposed to do is you have express the given vector 1 2 3 in the as linear combination of vectors 1 0 0, 0 1 0, 0 0 1. That means there exists scalars a b and c such that 1 2 3 is equal to a times the first vector plus b times the second vector plus c times the third vector. So if you add them up, then we will have a 0 0 plus - this is - 0 b 0 plus 0 0 c - and this gives me a b c. so if there are ah 2 vectors So if we equate the 2 vectors - left from left hand side and right hand side - they will be equal component wise and that means a will be 1, b will be 2 and c will be 3. So we can we have found a is equal to 1, b is equal to 2, c is equal to 3. If you substitute these values here, this is nothing but 1 2 3. In short, we have represented 1 2 3 as 1 times 1 0 0 plus 2 times 0 1 0 plus 3 times 0 0 1. So we can say that 1 2 3 has been represented as linear combination of the given 3 vectors - 1 0 0, 0 1 0 and 0 0 1 .

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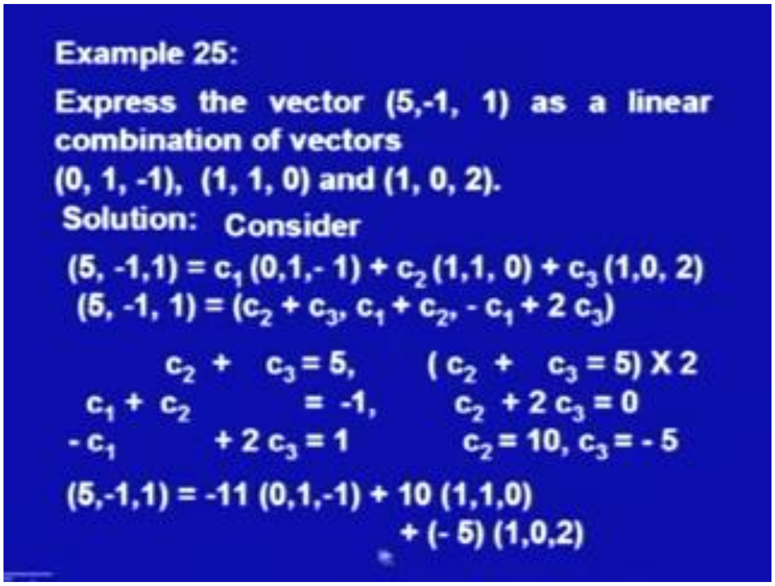

In the next example, again we are having vectors 5 minus 1 1 and you have to express it as a linear combination of vectors - 0 1 minus 1, 1 1 0 and 1 0 2. The procedure is the same. We consider left hand side - the given vector; right hand side - the linear combination of the three given vectors 0 1 minus 1, 1 1 0 and 1 0 2. This right hand side is simplified. It is c 1 plus if - the first component is - c 2 plus c 3 the second component is c 1 plus c 2 - no component is coming from this; then we will have minus 1 into c 1. So it is minus c 1 plus 2 times c 3. No component is coming from the second vector. If you equate left hand side and right hand side component wise, then c 2 plus c 3 is equal to 5 and c 1 plus the next is c 1 plus c 2 is equal to minus 1 and the last component is minus c 1 plus twice c 3 is 1. So we have to solve these three equations to get the values of c 1 c 2 and c 3. In the earlier example, the things were simple and we can straightaway get the constants, but in this case we are getting three equations to solve them simultaneously. Now to solve these three equations, one can observes that if you add these two equations, we will have first equation remains the same and the addition of these two equations gives rise to c 2 plus 2 c 3 is equal to 0; and that means we will have two equations in two unknowns - c 2 and c 3. So to simplify it, if I multiply this equation by 2 then and subtract - so this and this will get canceled and - what we have is c 2 is equal to 10 from this equation. This will get cancelled. c 2 is equal to 10 and once you have c 2 is equal to

10, you can substitute it here to get c 3 is equal to minus 5; and once we get c 2 and c 3, one can get c 1 from this equation and that gives me a 5 minus 1 one as minus 11 times zero 1 minus 1 - that is, the value of c 1 plus c 2 which is 10 1 1 0 plus c 3 minus 5 1 0 2. So this vector 5 minus 1 by 1 is expressed as linear combination of these three given vectors.

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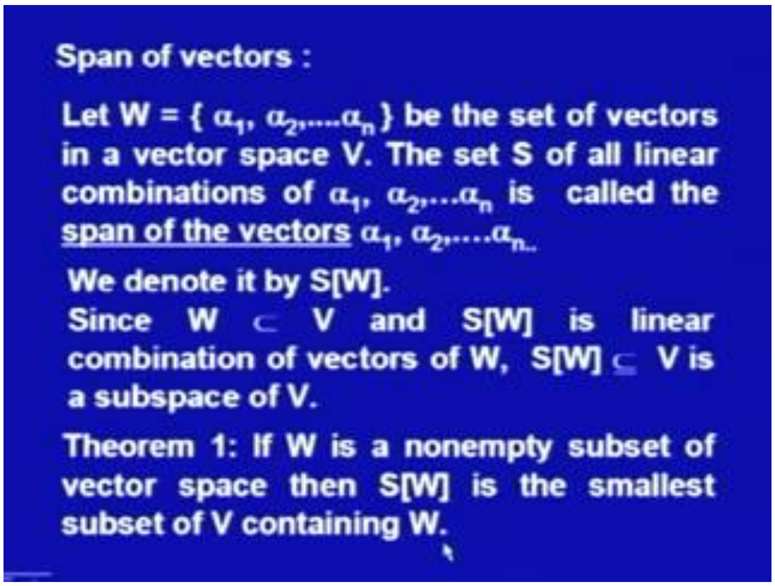

Now the next definition is span of vectors. Let us say we have been given a set W of vectors alpha 1 alpha 2 alpha n with the set of vectors in a vector space V. Then the set S of all linear combinations of alpha 1 alpha 2 alpha n is called the span of vectors alpha 1 alpha 2 alpha n. So all the linear combination of this vector will form the set S and that set will be call the span of vectors and we denote this span of vectors as SW - that means the span generated from W. Now since W is a subset of V and SW is linear combination of vectors of W - that means SW may be a subset of V - and since all the linear combinations are in SW, so SW is a subspace of V. Now we are in a position to state theorem. We say that if W is a nonempty subset of vector space, then SW is the smallest subset of V containing W is a smallest. It is not only the subset but is the subspace containing W.

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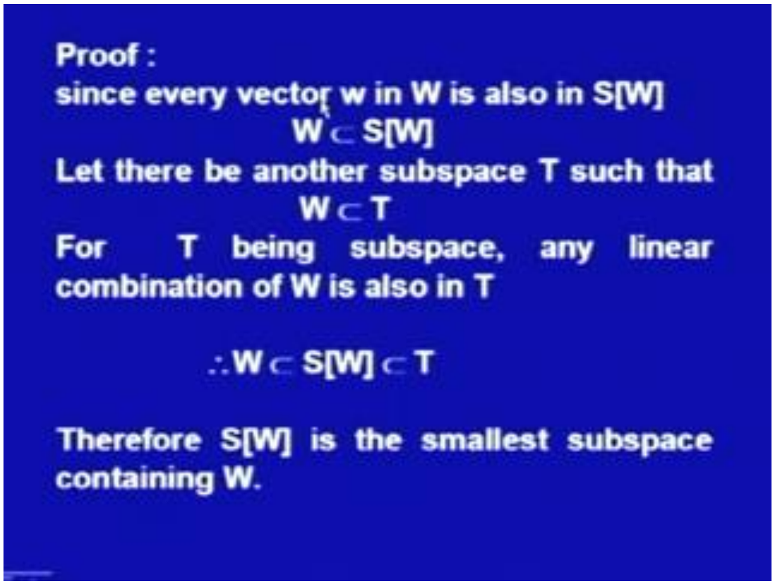

Now to prove this, since every vector in since every vector w in W is also in SW - because SW is a set of linear combinations of vectors of W, so this w in W is also will be in SW and that simply means that W will be a subset of SW - now we say that there will be another - this SW let us say is not the smallest subspace - let us see there has to be another subspace T such that W is a subset of T. Now we say that for such a subspace T, any linear combination of W is also in T because it is a subspace. So W is a subset of SW which is a subset of T because all the linear combinations are in SW. So they will be subset of T and that means that SW is the smallest subspace containing W.

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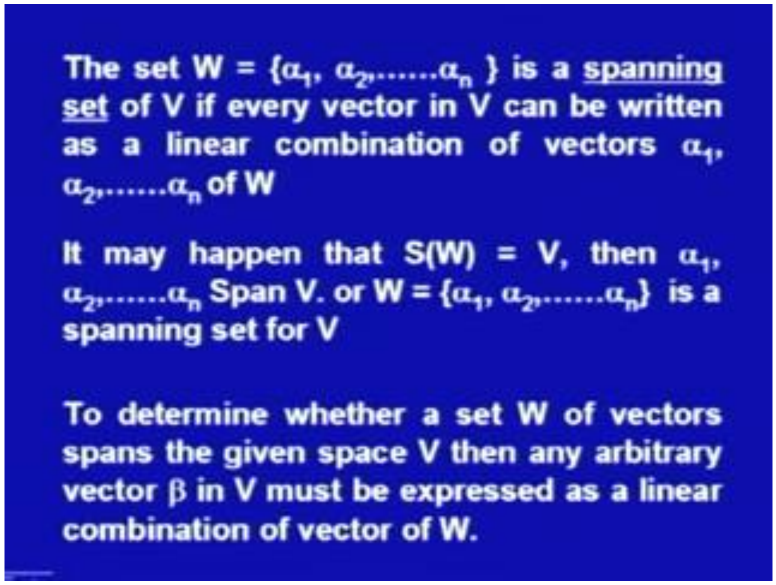

Now the next definition is that set W is a spanning set of V. If every vector in V can be written as the linear combination of vectors of alpha 1 alpha 2 alpha n of W, now it may happen that SW is equal to V the span of W is V. Then alpha 1 alpha 2 alpha n actually span V or in other words using this definition, one can say that W is equal to alpha 1 alpha 2 alpha n is a spanning set for the vector space V. Now to determine whether a set W of vector spans the given space, then any arbitrary vector beta in V must be expressed as linear combination of vectors of W.

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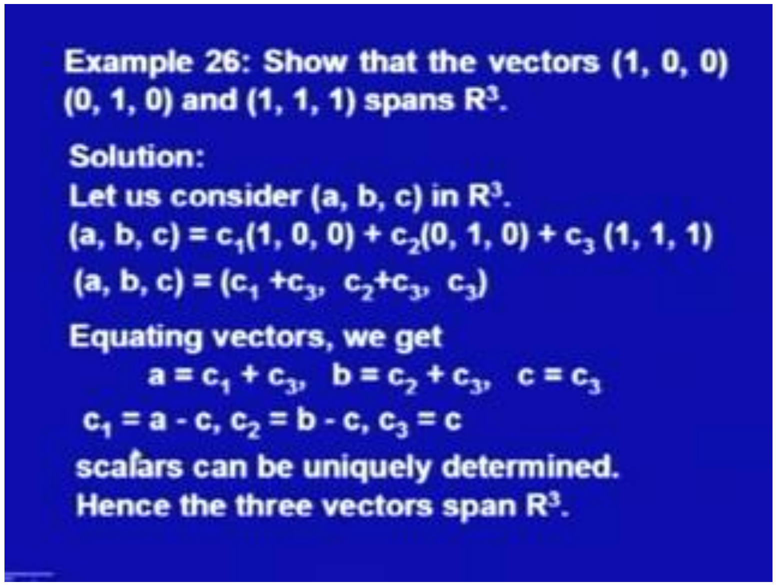

Now this We lets take with the example example is to show that the vectors 1 0 0 0 1 0 and 1 1 1 spans R cube. So let us consider any arbitrary vector a b c in R 3. Then it can be expressed as a linear combination of these three vectors - c1 1 1 0 plus c2 0 1 0 plus c3 1 1 1. So let us simplify; this comes out to be c 1 plus c 3 - the first component; the second component is c 2 plus c 3 and third component is simply c 3. Equating right hand side and the left hand side, we get c is equal to c 3. The last component and then from here, b is equal to c 2 plus c 3 and here, it is a is equal to c 1 plus 1 c 3. So this c 3 can easily be seen to be c and once you get the value of c 3 as c, c 2 can be calculated as b minus c and from this equation, if you put the value of c 3 as c, then c 1 comes to be a minus c. So for given values a b c, one can uniquely determine the scalars c 1 as a minus c, c 2 as b minus c, c 3 as c. Thus we can say that these three vectors 1 0 0, 0 1 0 and 1 1 1 spans the vector space R 3.

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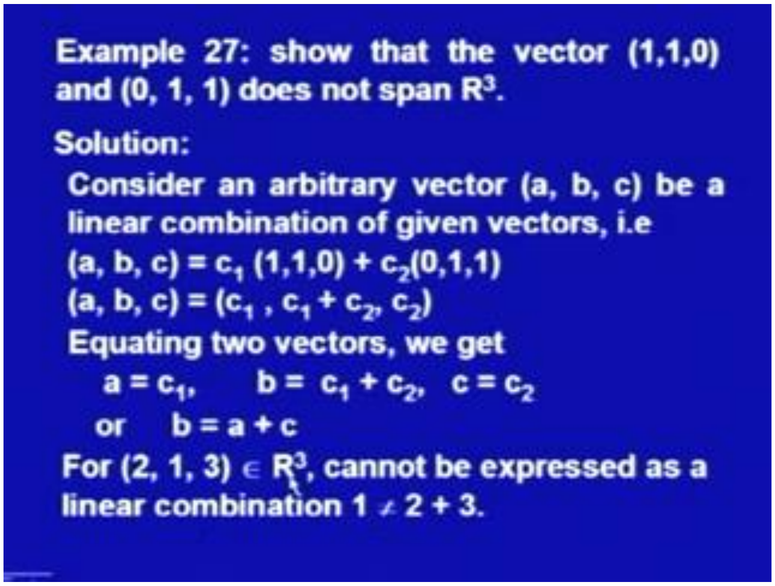

However in this example, we have to show that the vectors 1 1 0 and 0 1 1 do not span the vector space R 3. So let us see the solution. We again consider an arbitrary vector a b c and consider it to be linear combination of given vectors. So we write down abc is equal to c 1 times 1 1 0 plus c 2 times 0 1 1; simplify it. It is a b c is equal to c 1 comma c 1 plus c 2 and finally c 2 and this means a is equal to c 1 b is equal to c 1 plus c 2 and c is equal to c 2.and Simplifying this, we will get b is equal to a plus c. Now for the vector 2 1 3 belonging to R 3 cannot be expressed as a linear combination of these 3 vectors, because if it has to satisfy if it has if it has to be represented as this combination, then it must satisfy this constrain; but if you substitute the value of b as 1 a as 2 and c as 3, then one can notice that 1 is not equal to 2 plus 3. That simply means that this vector 2 1 3 cannot represented as a linear vector linear combination of these 2 vectors.

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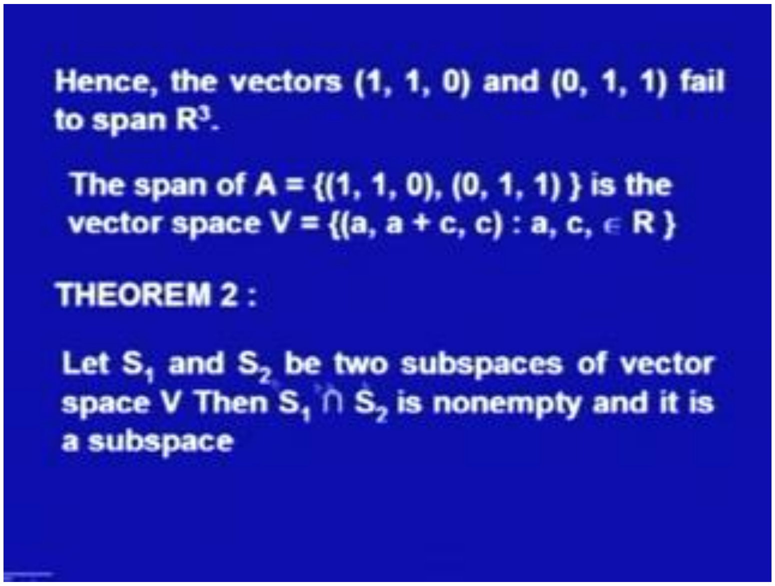

So the vectors 1 1 0 and 0 1 1 fail to span R 3 – that is the conclusion. However the span of these 2 vectors, that is 1 1 0 and 0 1 1, is the vector space V such that a comma a plus c comma c, where a and c belongs to R; so this is a vector space. So this is a span of this. But the vector which we have concern will not fall in this vector space V. Now we have another theorem and according to this, if we have two subspaces S 1 and S 2 of the vector space V, then the intersection of the two subspaces is nonempty and it is a subspace.

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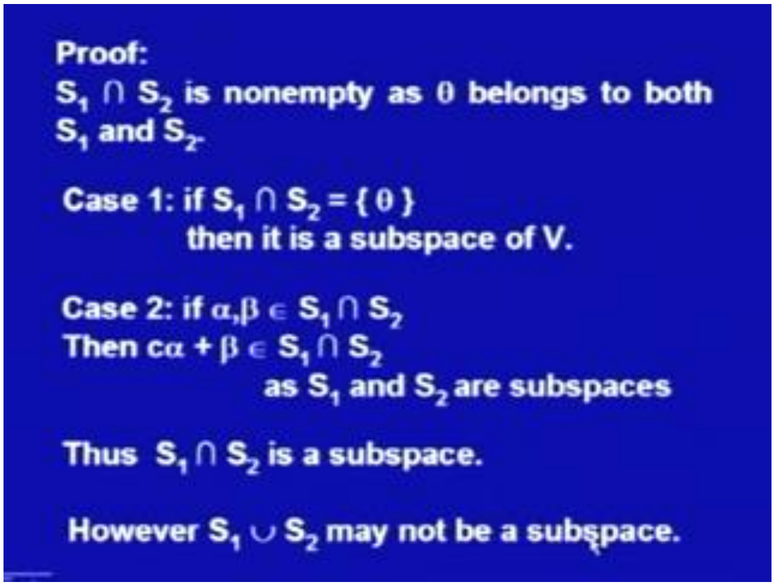

Now to prove this, it is observed that the intersection of S 1 and S 2 is nonempty. Why? because S 1 is a subspace, S 2 is a subspace; so theta has to belong to both S 1 and S 2. So theta must belong to S 1 intersection S 2 and that is why S 1 intersection S is nonempty. Now we consider two cases. The first case is if S 1 intersection S 2 is a single term theta; if that is the case then it is subspace of V, because this we have already established. But in the second case, if there is there are vectors other than theta in it - so let us say there are 2 vectors alpha and beta belonging to this intersection S 1 intersection S two - then let us consider c alpha plus beta; that means we are trying to apply the basic definition of subspaces. So for alpha beta belonging to a set, if c alpha plus beta also belongs to the set, then that set forms a subspace. That is what we are going to apply. So let us consider c alpha plus beta also belongs to S 1 intersection S 2. Now since S 1, S 2 are subspaces and that means S 1 intersection S 2 is also a subspace; c alpha beta belongs to S 1, c alpha plus beta belongs to S 2; so they belong to S 1 intersection S 2 also and that means S 1 intersection S 2 is subspace that proves the theorem. However, if you consider the union of the two sets, it may not be a subspace. Now to prove this, we have to establish by example.

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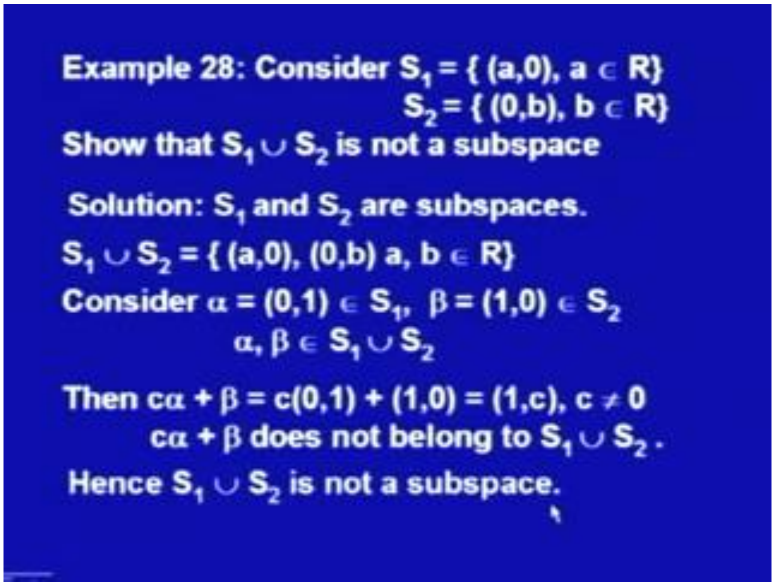

So this is an example which we consider in example. S 1 is equal to sets a comma zero a belongs to R; so it is a subset of R 2. S 2 is 0 comma b - b belongs to R. Now we have to show that S union S 2 is not a subspace. now The first thing is in the solution; we show that S 1 and S 2 are subspaces. One can easily see that a comma 0 - a belongs to R - is a vector space. S 2 is 0 comma b, is also a vector space. So S 1 and S 2 are subspaces of R 2 because this is a subset of R 2; this is also subset of R 2. So S 1 and S 2 are subspaces. Let us consider what S 1 union S 2 is. S 1 union S 2 is all sets - all vectors - which are either of the form a comma a zero or of the form zero comma b and a b belongs to R; so the all this type of sets belong to S 1 union S 2. Now for this, we consider alpha as 0 1 belonging to S 1. Here I am taking b is equal to 1. beta is equal to 1 comma 0 belonging to S 2 - 1 is a. Now alpha beta belongs to S 1 union S 2. Let us check this. c alpha plus beta is c 0 comma 1 plus 1 comma 0 is equal to - if you look at this it is - c 0 plus 1 it is 1 c plus 0 is 0 1 comma c where c is not 0. For c is equal to 0, things will be okay but for c not zero this property may not be satisfied. So if c is not 0, then 1 comma c does not belong to S 1 union S 2; so S 1 union S 2 is not a subspace. So while S 1 intersection S 2 will be subspaces for S 1 S 2 subspaces, but S 1 union S 2 need not be a subspace.

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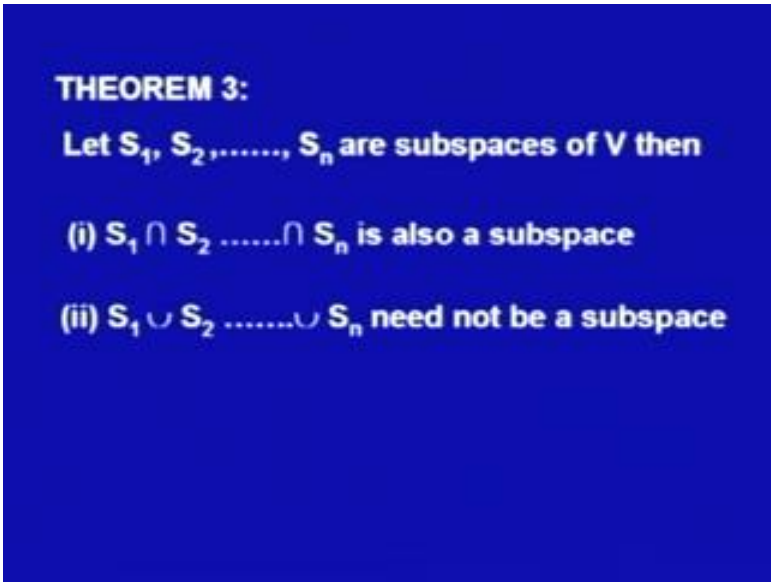

Now this result is generalized in the form of theorem 3 and what we say is that if S 1 S 2 Sn are subspaces of V, then the first result says that S 1 intersection S 2 intersection Sn is also a subspace although we have proved for n is equal to 2. Similarly S 1 union S 2 union Sn need not be a subspace - we have generalized it. The result - we have proved for S 1 union S 2; but it can be further generalized to finite n subspaces. The result has been established for n is equal to 2. The viewers can prove these results and the hint is they can use the mathematical induction to prove these results. Now towards the end of this lecture, let me summarize what we have done today. I have started with review for the first lecture in which I have discussed i have given the definition of vector spaces and then I had taken some examples. I have taken example of polynomials; some of the polynomials they form vector spaces. Some of the polynomials, when they when they are put under certain constrains, then they do not form vector spaces. I have discussed complex vector spaces. Then I have defined subspaces; after that, we have discussed linear combination, the span of vectors and in the next time, I will be discussing direct sum and independent and dependent vectors and basis; that is all. Thank you.