Mathematics-II
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Module - 1
Lecture - 2
Contour Integration

Welcome to the lecture series on complex analysis for undergraduate students. Today's topic is contour integration. We have talked about the integral of complex functions defined on real domain and there we had found out that the complex function - we can break into 2 parts; both parts could be treated as real functions on the real domain. Then we have come up with the complex function on the complex domain, that is, the function argument is also complex variable. There we had seen that we would be talking about the integral between the 2 points, as the 2 points on the plane, and we can reach to the from 1 point to the other point in a thousand many manners. So we thought is that is it could be that is it is depending upon how we are moving along which path to reach to the other point, so that the function could be defined on that path and then we could do. So we have defined certain paths; we call them arcs, curves, simple curves and so on. So, what is this contour? Let us define the contour.

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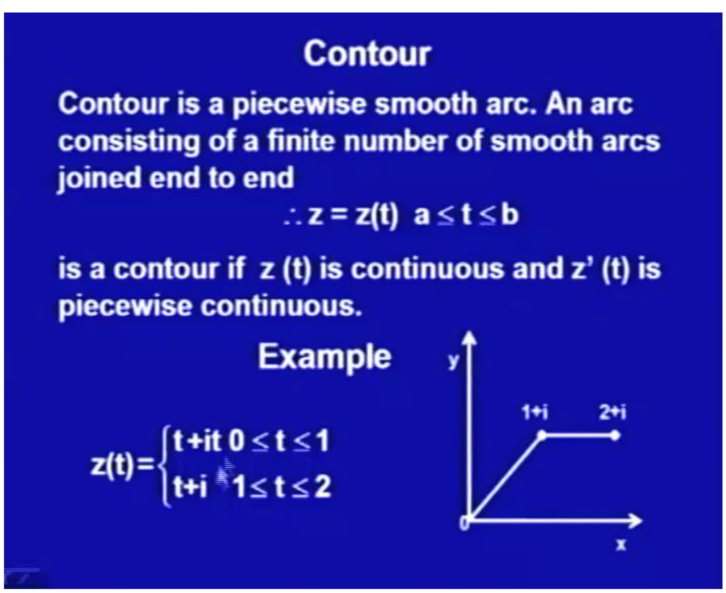

Contour is a piecewise smooth arc; that means, an arc consisting of finite numbers of a smooth arcs joined end to end. What it says is that, if z is equal to zt defined from on the interval a to b, is a contour if zt is continuous and z dash t is piecewise continuous. If do you remember, a smooth arc we mean meant is that the arc is differentiable and its derivative is not zero on the whole interval. So here, we say is that piecewise, a smooth means it could be zt is continuous, and z dash t is also piecewise continuous and its derivative that z dash t is not zero at all at any point on the interval a to b. Say For example, if I do take this function zt, which is defined as t plus i t between the range 0 to 1 and t plus i in the range 1 to 2. So, let's see that is what this would look like.

This is actually is t plus i t from the for the range zero to one; this is actually the straight line from 0 to 1 plus i and then we do have the straight line from 1 plus i to 2 plus i. So, we do have here 2 arcs which are joined from 1 end, that is, where this arc first arc is ending, the other arc is starting; from here so they are joined. Let's see: that is we say that it should be smooth curve, that says is, that I should have the differentiable arc - of course, we see from here, that these functions are continuous and differentiable. If I take the derivative of this zt by using this complex differentiation, I would get here, 1 plus i and here, I would get 1. So at 1, we see this function is continuous because at 1, the limit from the left would be 1 plus i and from the right, it would also be 1 plus i. But if I do take the derivative at 1, it would be here 1 plus i and here it would be 1 so it is discontinuous at this point - the derivative. So, it is it maybe that is at most piecewise continuous. While as at all other points, the derivative will also be continuous. So, this is 1 example of a smooth arc. As we have done this in the arcs and curves - that is, simple arcs and simple curves – similarly, we can define here also simple close contour.

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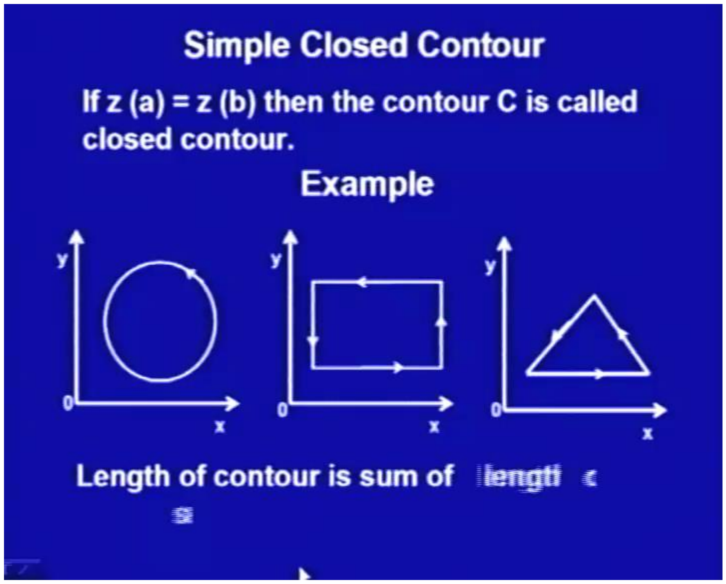

That is, if the initial point and the end point - that is, a and b- they are same, then the contour is called the close contour. Again, we do have the same kind of examples as any close arc - oriented one. So say For example, this is circle oriented towards anticlockwise; this is 1 contour. If the orientation is clockwise, it would be another contour. Arc This rectangle, if is if you say that this point we are starting; then we are orientated in this manner, that is, the we are starting from this point, moving over here then in this direction then in this direction and then in this direction. Now, this is a contour because all the arcs they are being joined end to end and they are all the straight lines and they have been joined end to end and this is the same; the orientation is in this manner. Now the same arc we could says that it would be contour if I take the orientation other way round. That is, if I take the orientation from this way around, then again also it is a contour; but that would be separate contour that what we are having. Similarly, if I do have this triangular kind of a shape, again if I start from here, we do have that orientation is from this side to here, then from here to here and then here to here; so again we are joining 3 straight lines in this orientation.

Now, if I change the orientation, that I, let's say that is if I am starting from this side and this way, or if we take any other point and then the orientation is being changing, all those would be different contours. From here, 1 thing is clear: because we are talking about contours - contour means is its z dash t is piecewise continuous and it is not zero at any point - that says is actually the z dash t is integrable in the whole range t is equal to a to b. What it says is that Length of contour can also be find it out since contour is nothing, but the sum of or that is joints of simple arcs or simple curves which are differentiable. So, we do say is that length of a contour, if it is single piece, that is, as in this first example it would be simply integral of z dash t along this line arc from the point a to b, or if it is joining of many smooth arcs then it would be the sum of length of those arcs. So, we do say is length of contour is sum of length of smooth arcs.

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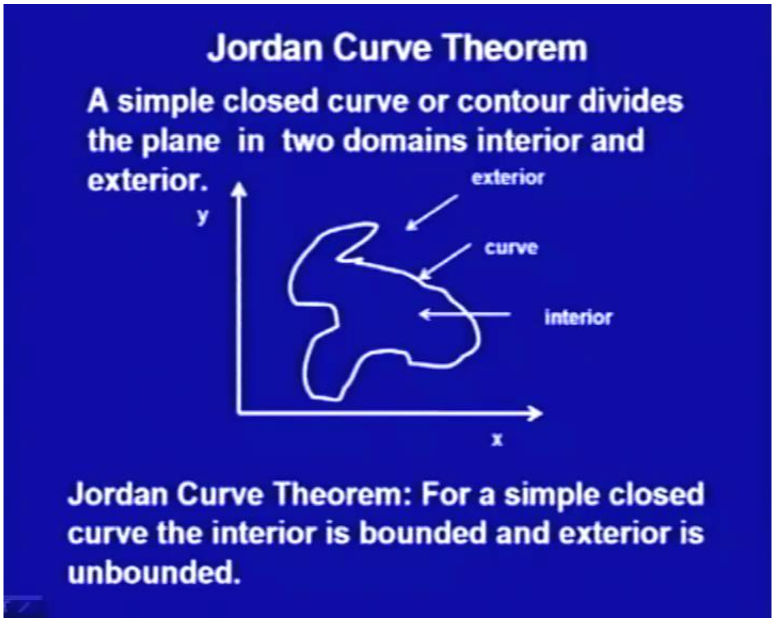

Let us come to 1 basic result - Jordan curve theorem. A simple closed curve or contour divides the whole plane into 2 domains, of which the only common points are the points which are on the curve c and that is called the boundary points. You see Suppose this is a simple Jordan curve, you see, we are not crossing it at any point; on this one, that the curve is not crossing itself at any point, this is closed one. So, this is a Jordan curve.

From here, if we see this whole space or whole plane is being divided into 2 parts. 1 part is this inside this close curve – that, we call interior domain; and the other is outside this curve, that is called the exterior domain. From here, we do see is that for these 2 domains - these 2 sets or these sets of plane - we do have the common points are only the points which are on the curve; that they are called the boundary points of interior as well as of the exterior. In this what it is being very clear is that all these points whatever they are being bounded by this curve and the points, which are in the exterior one, they could be anywhere. So we do give 1 basic result, that is called Jordan Curve Theorem: for a simple close curve, the interior is bounded and exterior is unbounded.

The proof of this theorem is a little bit involved, which requires a hard mathematics understanding; so we are not going to do this proof mathematically. But geometrically, we could understand that what we mean by bounded and unbounded interior and exterior. Now, let us move to this contour integration or the come to the integral of complex function of complex variable. We will define along the path and that we would call complex line integral.

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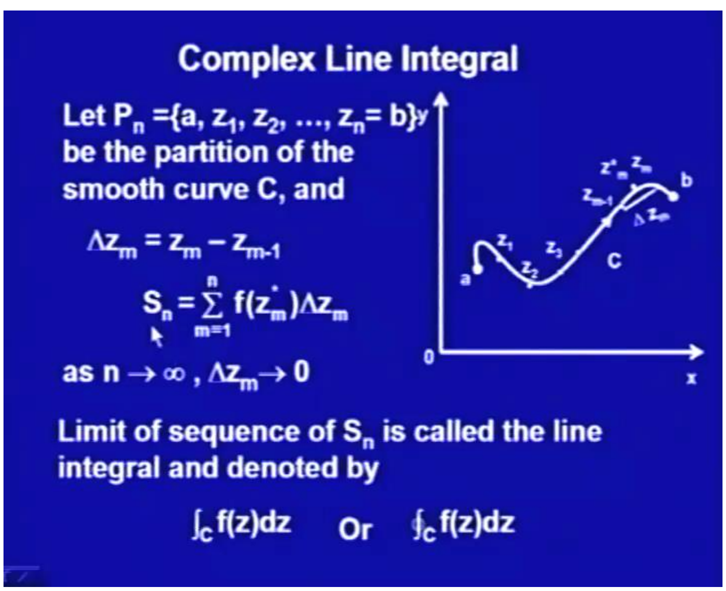

So, let us define complex line integral. Let us say we do have a curve from a to b along which a function fZ, that is, Z is complex variable. fZ, that is the complex function or the complex variable, is defined on this line or on this curve, we want the integral along this one. How do we proceed? We will proceed in the simple manner that is similar to that real integral, that is, what we done in the calculus. What we do is let us partition this smooth arc or this curve in n parts. That is, what we are saying is let us take z1, z2, z3,and so on zm minus 1, zm, zm and so on till zn, say b; n points on this one.

In what manner should I take these points? These points - because we are saying is that any arc or any curve, simple one, we are defining with the parameter presentation; that is, we say z is equal to Zt. So these points which we are finding it out, z1, z2 and so on - so a is equal to z naught we are seeing - we take in this manner such that I do have whatever the parameterization t I am having. z naught we say is z of t naught; z1 we would say z of t 1 and so on. z of m we would say z of tm and this t naught, t1, t2, tm, tn - they must be in order, that is, t naught must be smaller than t1; t1 must be smaller then t2. Iin this manner, this partition has to be done. So what we are doing is, now we are taking this complex domain; my function, my curve is from a to b. a and b are complex variables - are complex numbers -and all these points we are taking in the complex one. Now we are saying is that this arc we are defining by the a equation z is equal Zt; that says with the partition, we do we take in the increasing order of the parameterization parameter t. In this, let's say now if I do take a point between say z naught and z1, z1 and z2, say like this one - so here, I had just noted here that between Zm minus 1 and Zm, let's say take any arbitrary point Zm star. So between a and z1, I take one arbitrary point z1; between z1 and z2, lets take any arbitrary point z2 like that; and let us denote the difference between Zm and Zm minus 1.These 2 points - the distance between these 2 points - that is denoted by delta Zm.

We do know the distance between these two points would be given by this straight line. Then, if I just go with the real analysis, that in the calculus how we have defined the integral, we say is that we multiply this value of the function at this point with the distance between the two points and then we add it up. So let us define this sum sn – n, I have taken because I am taking this partition of n n values - as summation m is running from 1 to n f of Zm star multiplied with delta Zm. Let us define this sum. Now, this sum is depending upon two things: one is where I have taken this point Zm star and another is that is what is this partition, because accordingly I would change this distance between Zm minus 1 and Zm that would be changing. So as this n is changing or this partition P is changing, this my sum Sn will also change.

Now make a sequence of this Sn. In what manner? We try to take the partitions according to this n - that is why i have used here n - according to this n in such a manner such that the distance between these two points, that is, we are defining these points say z1 and z2 as z of t1 and z of t2. So, let us take this partition in such a manner that if n increases, that is, if I take the number of partition points more then the distance between tm and tm minus 1 - that is, the distance of the parameter t at this point and this point that approaches to 0. If that is approaching to zero what it says this that my delta Zm that would also approach to 0. Why? What we are saying is t is actually t t will move on this line. So if tm tm minus tm minus 1 would be this arc length and delta Zm is this one, so if this is approaching to zero, this can never be more than this arc length - so that would certainly has to approach to 0. So let us take the partitions in such a manner that if n is increasing, delta Zm approaches to 0. In this manner, I would get a sequence of sums Sn.

Now let us define the complex line integral in the similar manner as we have defined the line in line integral and the real ones. We say the limit of the sequence Sn - that we would call the complex line integral; so limit of the sequence of Sn is called the line integral and it will be denoted by either integral cfz dz or by this notation, where this circle is in between cfz and dz. This notation we are using when we do have any arc or any smooth curve; this notation we are using when we do have closed smooth curve. These are a this is the further notations, not necessarily that we have to use it; but when this notation is being used, it simply says it is on the close curve. So, we are talking about the smooth curves and we are defining this integral of complex function of complex variable.

So thus now, this integral would be existing, that says this integral would be defined if the limit of this sequence is coming up or is existing. This limit would exist if i just take this parameterization - parameterization is along this line. That is, along this line I can treat it as a real one. In the real one, if this function is such that that is existing, then we would say its sequence limit of the sequence is existing; we would say that our complex line integral is existing. So now this is the definition; the existence part we will see later on; if it we could see let us come definition we have done how to really find out this integral that is what is the method to evaluate this integral. So let us find out what is the method to evaluate this contour integral.

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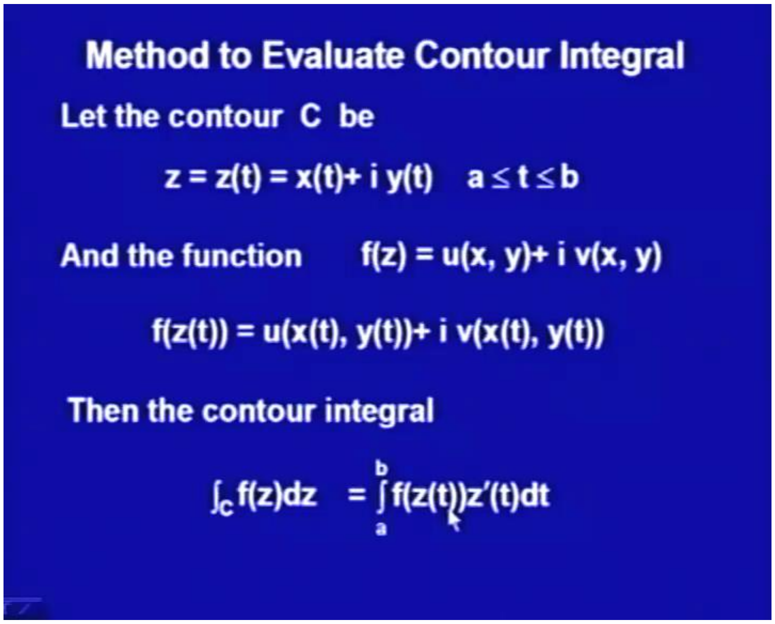

Let the contour c be given by this z equal to zt. Now this since this is complex variable, so lets say I am writing it as xt plus iyt for t between the interval a to b and the function fz is as u plus i v - as usual we as to define. Now the function of z : z itself depending on the two variables x and y; that says is that f which is depending on z, that I could say u, which is depending on z rather than z. Now, I will take the two real parts: that is, u is depending on x and y and v is depending on x and y. So you see this is the complex function on the complex variable we are again trying to break it into 2 parts; see where we are taking both the parts as the real one. So, this is a real function of 2 variables, this is also a real function of 2 variables and they have been joined with this square root of minus 1, that is, we are getting is this complex one.

While is this x and y are nothing but the real and imaginary parts of the z. So f of z of t, now I could write as now if I write this z z as a function of t, so certainly z is function of t, then x is function of t, y is function of t; so certainly I would get different parameterization - u as x of t, y of t and v as x of t and y of t. Moreover, we are talking about the contour c. Contour means is that my c is differentiable arc and its derivative is not zero - on the whole range - that says is z dash t would be x dash t plus iy dash t. So the contour integral then, we are defining as integral from a to b f of z of t times z dash t dt. Why? What I am doing is changing the variable. That is, instead of integrating in the z we try to integrate it on t; so z is a function of t - so z of t. If I am talking z of t, then dz - I have to write in the terms of dt. What will be dz? would be actually If z is zt, then dz would be z dash t dt. So that is just by the change of variable as usual.

We use to do in the real integrals in the similar manner we have done here; because why we could apply it here? We could see is that is we have get actually my z also into the two real integrals and in this your u and this f also in the two real integrals; so we could do it. So that is what we are getting. Lets see that is whether we could do it or not. So what we have defined now, our contour integral and contour c of a function fz with respect to z we are defining as integral from a to b f of z of t z dash t dt., where zt is actually the equation of this contour c in the parameter t; Now that says is this the integral along this contour. So this a and b and this zt that is actually defining that is what is the contour. Let's see the proof of this one.

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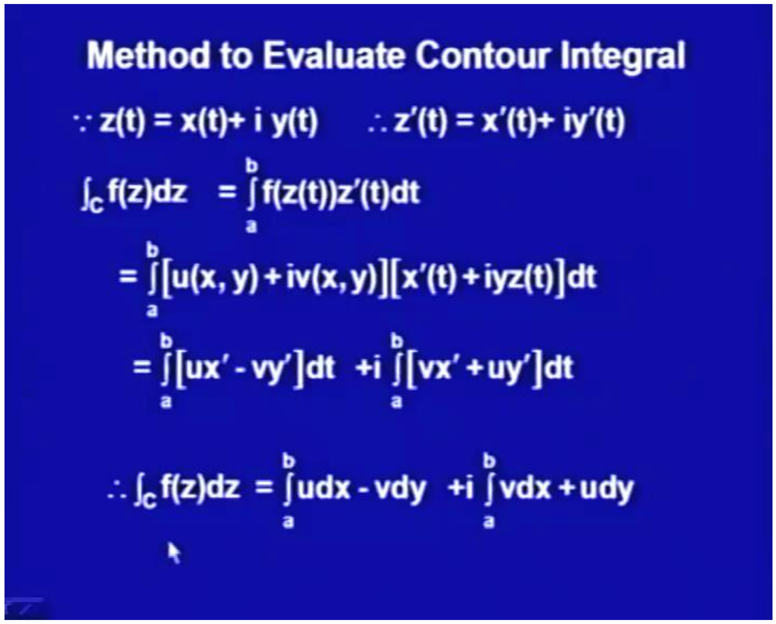

Since zt is xt plus i yt, so this says is z dash t would be x dash t plus iy dash t. So cfz dz - we could write which we are saying is that in this manner. What could be it? fz tb - we could write u of x y plus iv of xy into z dash t, is that is, x dash t plus iy dash t- it should be y dash t dt. Now, if I multiply these two, I get from here ux dash and i times - i will be multiplied - minus vy dash; so we do get minus v by dash and then the part which is containing i, I would get v and x dash plus u and y dash. So we do get plus i times vx dash plus uy dash. Now, since my u,v,x and y all are real, so we could treat it as a real integrals and using the properties of definite integrals we could just break it into two integrals. Now, since all these are real ones, you see is that is what we could write - u is a function of x and y and x is a function of t - so what we could write ux dash dt - that we could simply write as udx; so you see we could it as write udx minus vdy.

Similarly this one, I could write i times integral a to b vdx plus udy. So what we are having is a if i say is that i can i could write it as double the 2 integrals: integral a to b udx minus integral a to b vdy plus i times integral a to b vdx plus i times integral a to b udy. What we are doing is actually these a and b they are defining they are being defined; these limits are being defined according to the parameterization whatever we have taken - according to that. Since all this if the functions are continuous, then they are they are integrable; so we do get is that all these are integrals would be existing. So we do find it out this method of evaluation of the contour integral will give me the integral which is required and that would be again a complex number. This satisfies certain basic properties.

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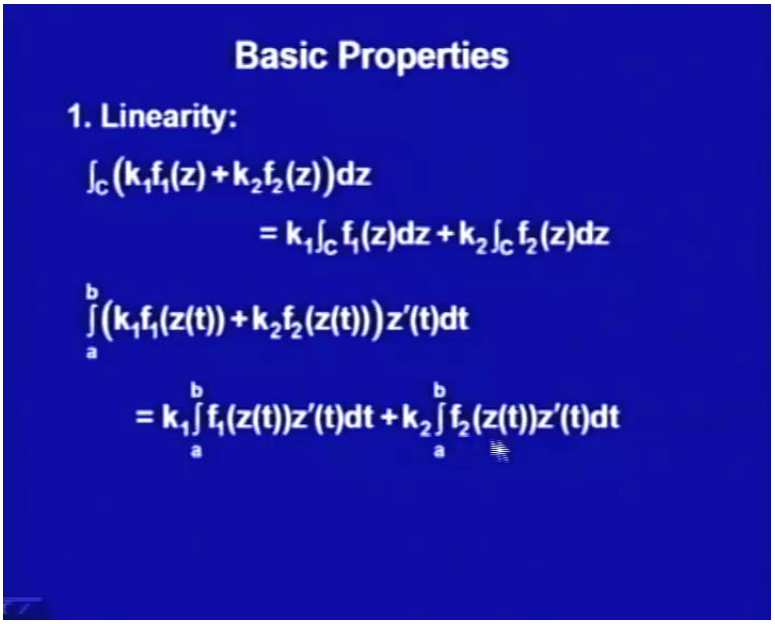

As in the first one we had seen that is those basic properties; they are also satisfying these properties. First is linearity which says is that if k 1 and k 2 are any two real numbers, here I am taking this k 1 and k 2 as real numbers and f 1 and f 2 are any 2 complex functions defined on complex variable, then the contour integral k 1 f 1 z plus k 2 f 2 z can be given as k 1 times the contour internal of f 1 z plus k 2 times the contour integral of f 2 z. We can show it very easily using the definition. If I just go by definition of the contour integral, we say is that with certain parameter zt - which says is that z is zt for t belonging t ranging between a and b - we could write it as k 1 f 1 zt plus k 2 f 2 zt z dash t dt. Now, once we have come it up, now we see is that is all these things we could use by the definition, we can make it that 2 integrals and then we would be getting is k 1 a to b f 1 zt z dash dt plus k 2 a to b integral f 2 zt z dash t dt. This is nothing but the contour integral of f 1 z and this is nothing but the contour integral of f 2 z. So thus the linearity is being kept, that is, which says that contour integral here integral is a liner operator.

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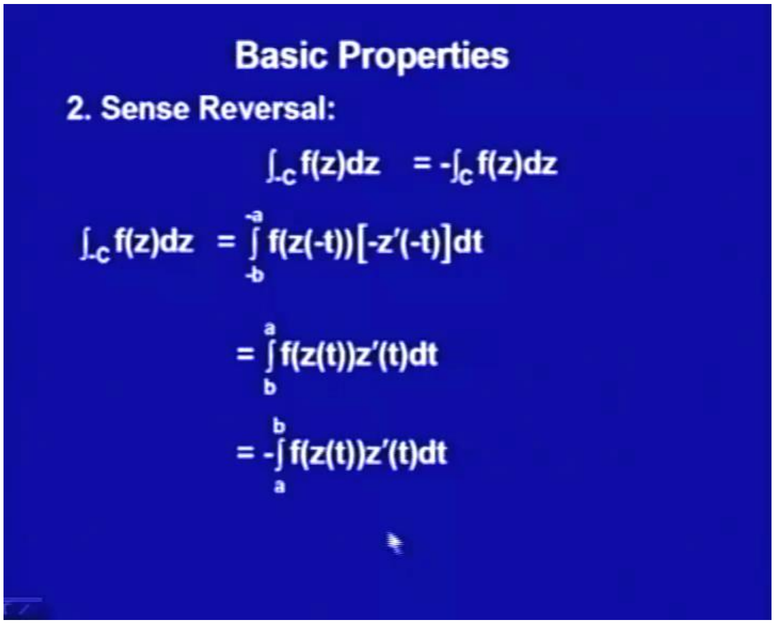

Now the other property: sense reversal. In the definite integral on the real line, you do have you have done it that is if we change the limits from lower to the upper we do get that the integral is negative. This property, we had seen in the complex integral of complex function on the real domain also; here also it is being done. That is what it says is Here, what do we mean by the sense reversal? There we do have is that because a is less than b kind of thing. Here what do we have if the orientation is changing? That is, if rather than having the contour c if I do have contour minus c, that is if the orientation has been changed. then it would be minus of contour integral on the contour c of fz tz. Again we can just prove it using thus our definition. We want contour integral and the contour minus cfz dz; that means the orientation has been changed. That is, if z is equal to zt is the equation of the contour c, what we do get is that t is increasing from a to b.

What we do take is that if I take instead of t minus t, we could define this my curve as minus c; so we just take z of minus t. So that is t that minus c, that is the contour orientation is being changed, that says is, instead of t, if I take the minus t on the same one- so if I am taking this minus t, then certainly if t is moving from a to b - we do have minus t would be moving from minus b to minus a. So it is minus b to minus a f of z of minus t and the derivative of z of minus t with respect to t that would be minus z dash at minus t and dt. Now, use the change the variable from minus t to plus t. That is, take the minus t is equal to some ,say v, and then again do so; I am doing is simply keeping it minus t is equal to t. What would be it? If minus t is say v, then as t is reaching to the minus b, I would take this minus t would be b ; as t would be reaching to the minus a, minus t would be reaching to the a. So integral b to a(these limits), f of z of t z dash at minus t - so that is minus t is t since I have change the variable from minus t to v. So what do we get? Minus dt would be dvr, that is, we are writing here dt. So we get integral b to a f zt z dash t dt.

Now here, what we have just only thing is that is the order of the limit has been changed. So using the simple property of the definite integral, we could say it is minus a to b f of z of t z dash t dt because t, is now we are treating as a real one so this is a to b. So we have got that this property - is being this is nothing but the contour integral of the contour c. Then third property: partitioning of path.

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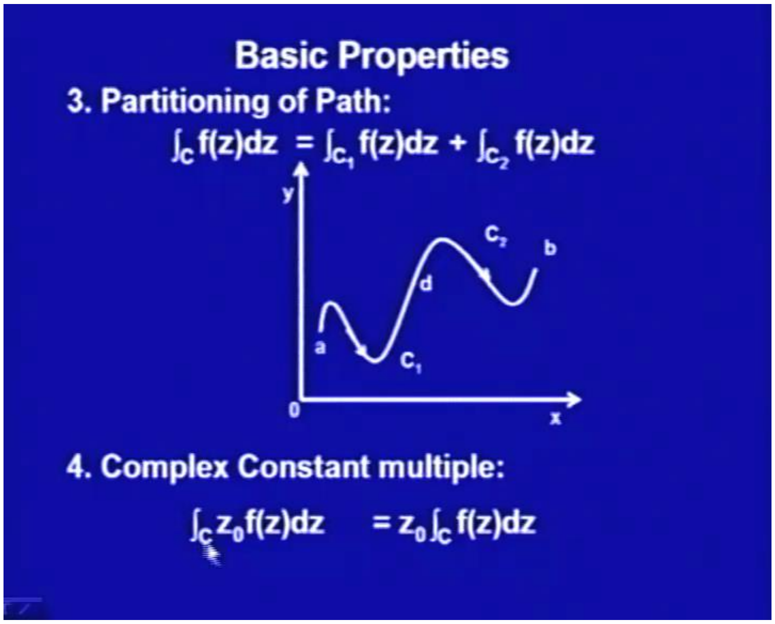

Say, I want this integral fz tz is on the contour c. This can be written as the sum of two contour integrals on the different contours c 1 and c 2, where c 1 and c 2 are nothing but the partitions of the contour c. Let's see here: suppose this is my contour c and suppose at this point d, this contour is being partitioned into 2 parts that is from a to d, this is the contour c 1 and from d to b, this is contour c 2. Then we could get this integral, that is, as the integral from a to d or rather, you could say is the on contour c 1 and then on contour c 2. Now here it as actually this in this example, what this contour I have it is not looking very impressive; that is why I should divide it in between, while I could go ahead directly in this one, because this is a same contour.

You could see if I do have the contours as in the close contours i was having is a square or a triangle, you do find it out that is suppose it is a rectangle one. So I do have in this line and then i do go in as upwards and then I do go over this one. So while i could be easy it it would be very difficult to find out um z is equal to zt parametric equation for the contour even if it is a simple line in this direction and in this direction and this direction. Rather if I break it into the parts, that is, I just go with this integral on this direction first and then on this direction and then on direction and then on this path it would be better it would be much easier one. So we do get there this property helps, very much helpful that is, I could integrate on the different contours which has been added up, and I could get the sum of those integrals integral values as the sum as the integral on that same contour.

Now as I have done this linearity with real constant, now let us take this if i do have a complex constant as multiple. This property we have done with the real domain actually - z naught fz dz; as we have done the in the real domain in the similar manner actually it would be same as z naught times contour integral on the contour c fz dz. That is, whatever be this contour, this z naught, if it is a constant complex constant then we can always take it outside the integral sign and the proof is very easy. You can again go ahead with the z is equal to zt and the definition and you could find it out that it would be easy. But from this property, what we have got is if I add up this property with our first property of the linearity, that says is, that those constants k 1 and k 2, now we can change to the complex constant as well; even then our property of the linearity would hold. So we come to the basic property about the absolute value of the integral.

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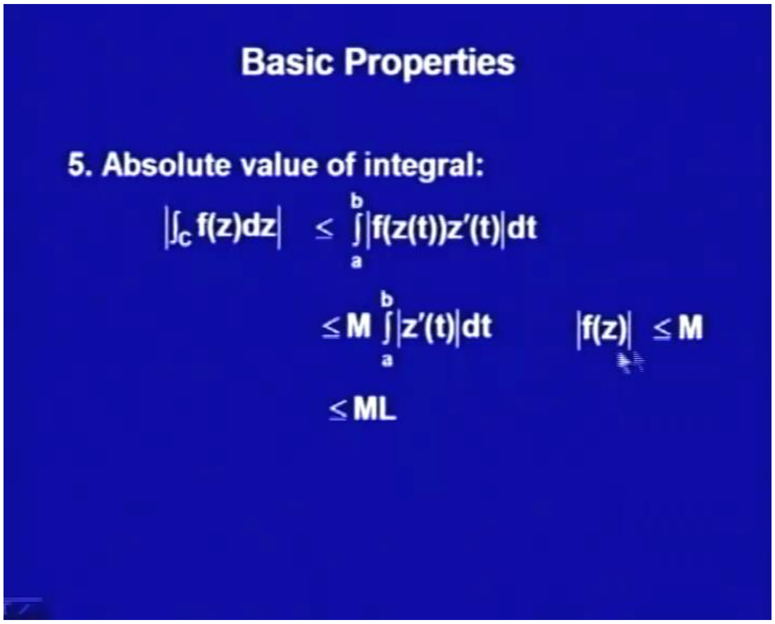

That is, what will be the absolute value of the contour integral fz tz on the contour c? We have defined by definition this is nothing but the integral a to b fz t z dash zt dt and then t is actually the real variable. So we could use the similar property of the real variable and from there, we could get that this should be less than or equal to integral of a to b fz tz dash t dt. Now from using this one this fz t into z dash, its modulus value/ absolute value - we do know that absolute value of the product of two complex numbers would be nothing but the product of absolute values. So we could write it as f absolute value of fz t into absolute value of z dash t.

Now, if this fzt - that is, it is absolute value of fz - if I could find it out at upper bound of this fz and the contour c - let say that upper bound is capital M - then that would be a constant; this M may be complex polynomial as well or the real one. Since it it is fz absolute value of fz, so it has to be real constant. So this real constant, I can take it out, and then what it would be the less than or equal to M times integral a to b absolute value of z dash t dt. So this this integral a to b z dash t dt, this if you do remember, this is nothing but the length of arc; so we could say this is ML. So what we do have got is that absolute value of this contour integral is less than or equal to the M into L, where M is nothing but the maximum or that upper bound of the function a absolute value of function fz on the contour. Let us do one example.

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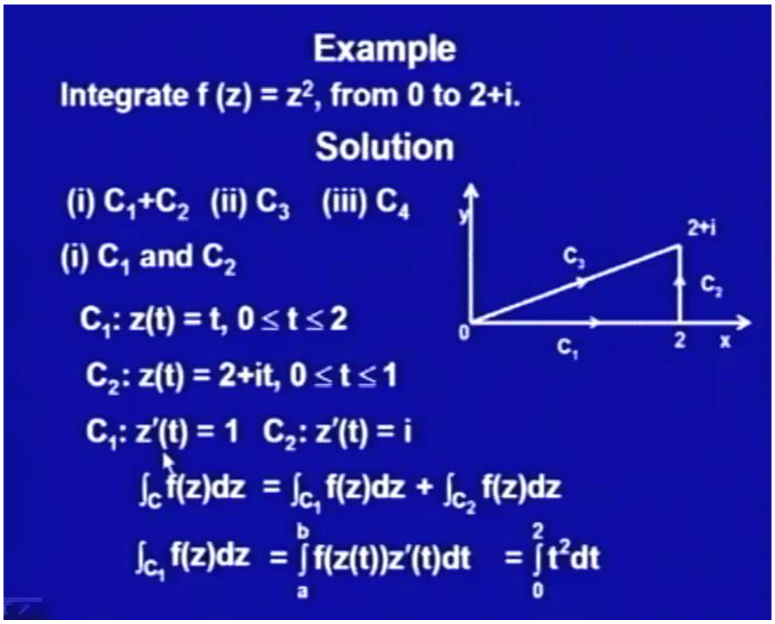

Integrate the function fz is equal to z square, from 2 to 2 plus i. Now let us this example, which we would be doing is, we would be actually trying to find it out; as we have seen, that is from a point a to b on the complex plane, we can reach in all infinite many ways. So we will try to reach, to that is, to we will try to find out this integral along different contours, that is different paths, and then we will try to see if this path or this contour is really making a difference in the evaluation of integral arc it is not making.

So let us take this first example where I am taking is that function fz is z square, and the points - two points we are taking - are zero and 2 plus i. So let us take the figure - this one. We do have this point 0 , this is 0, 0 in the complex plane, the point 2 plus i; that is this is a we would say is the complex plane, we can always define as the 2 dimensional plane where this one dimension - this is this is the x dimension and this is y. So z is x plus y, that is any point on this one x plus iy - we are defining as x comma y. So 2 plus i means is that is 2 and the multiple of i is 1; that is 2,1 so I have to reach from this point to this point. Now, I can reach to this point in many manners; either I can reach first from zero to 2 and then 2 to 1, that is along this contour c 1 and then contour c 2, or we can directly reach along this straight line from zero to zero zero to 2 comma 1, that is this contour c 3. Or I could treat a different parameterization. That is, here what I have when I have said is that c 1 plus c 2, I mean is that is I will take one contour c 1 and another contour c 2.or I can take this complete line, that is from zero to this 2 plus i in this path as single contour, and that c 4, that would be seeing you see, is that parameterization.

As I said, is that is why this property of a partitioning of path is important. That is, we do have is the parameterization would be a 1 this here is the our first simple example I have taken in which I could say is that from here to here I could either use single parameterization for whole contour, or I can have one contour, one parameterization for this, another parameterization for this. So let us start one by one. So first, I am taking integral along this path and this path I am breaking into two paths - one is c 1 and another is c 2. So c 1, if I see here I am having is I am moving along the x axis, that is along the single line - this one. And y is being kept as a constant, so and that constant y is actually 0 because it is on the x axis. So if I take x is equal to t - the parameterization, so I do get that y is zero so I would be getting this x plus iy, that is xt plus iyt. xt, I am taking as t; so t plus zero, that is t only and my t is varying from 0 to 2. And c 2, now I am taking the different parameterization; the c 2, if I do take if i treat simply this xy plane, this c 2 is the line which says is that your x is equal to 2 and y is varying from 0 to 1. So, I take the parameterization for y, that is, y is equal to t- where t is varying from zero to 1 - and x is constant, 2.

So I could use this parameterization zt as 2 plus it, for t between 0 and 1. So you see here what I have done is I have taken one contour c 1 for which I have taken 1 parameter - that parameter is actually x - and in this, I have taken another parameterization in this one - actually my parameter t is y. So these paths we have find it out; now we will go to contour integral along the path c 1 and along the path c 2. So first c 1: for the c 1 z dash t would be because zt is t so z dash t would be 1 for the range between 0 to 2. And for c 2, z dash t would be - this is constant - so it would be i only in the range 0 to 1. So now, using this property of the partitioning of path integral along this contour c, fz dz, I could write contour integral along the contour c 1 of fz dz and c 2 along the c 2 - fz is z square. So let us say evaluate first this c 1 integral. c 1 fz dz- I could write now; c 1, we see the parameterization is from zero to 2, so my a and b are zero to 2. So if I take this by definition, it should be f integral a to b fz tz dash t dt; I would have zero to t fz t. z is z square, so I would get is from here z of t is t, so z square would be t square and my z dash t is 1; so I would get simply dt. Its integral is simply eight by 3 because t cube by 3 evaluated from zero to 2 it is giving me eight by 3. Now, come to the second contour.

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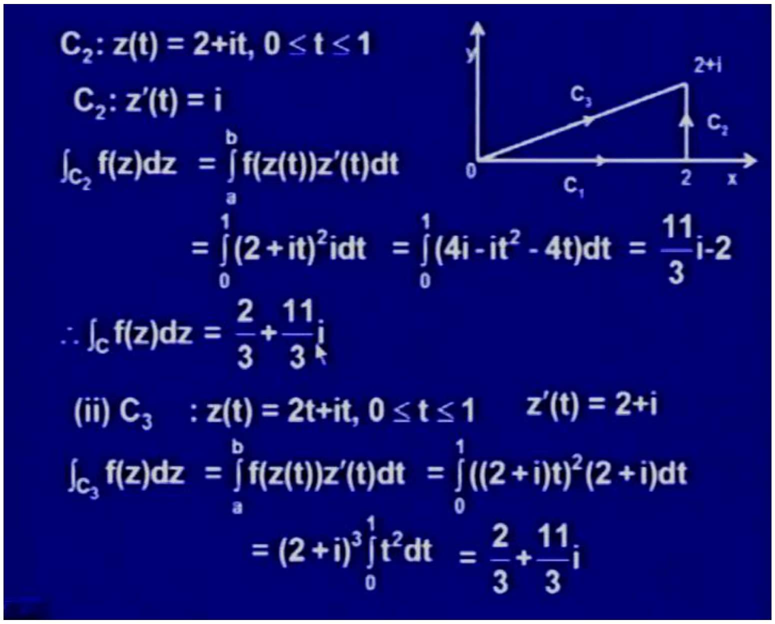

Second one, is my contour c 2 is 2 plus it for zero to 1 and z dash t is i. So again using the definition integral along the contour c 2, fz dz would be integral a to b fz tz dash t dt. Now substitute it, so my z fz is z square. So here f of z of t, if I do write i have to write 2 plus i t whole square; so 2 plus i t whole square z dash t is i, that dt. So this whole square will 4 plus i square t square, that is 4 minus t square plus 4 i t multiplied with I; we would get it integral zero to 1, 4 i minus i t square minus 4 t - because plus 4 i t into i that would give me i square, i square is minus 4, so minus 4 t. Now dt. So integrate it with respect to t, here I would get 4 i into t evaluated from zero to 1 - that would give me simply 4 i; minus i t square integral from zero to 1, that would give me t cube t cube by 3 evaluated from zero to 1, that is i by 3; and then minus 4 t - its integral would be 2 t square, evaluated from zero to 1 will give me only 2. So what we would be getting is 11 by 3 i minus 2.

First integral we have got as 8 by 3. So now the whole integral, that is along this path from point 0 to 2 plus i - along this path c 1 and c 2 - what we do have we do add up both the integrals : that is integral along this contour c 1 and the integral along this contour c 2. So along the contour c, we have got 2 plus 2 by 3 plus 11 by 3 i - because there we were getting is eight by 3 - so eight by 3 minus 2 would be giving me 2 by 3 plus eleven by 3 i. Now let us come to the second part c 3. c 3 means that is the straight line from zero to 2 plus i. What will be the equation of this arc? This arc we are having is that is this is a straight line with the orientation as 1 by 2. So what would we be getting this line if you do see y is equal to 2 x from 0 to 2. So lets again put my parameter as y as t then sorry x as t, let's say y is equal to half x, actually what we would be getting is y is equal to 1 by 2 x; so 2 y is equal to x. So let us take y as t, so y is ranging from 0 to 1 and then x would be 2 t; so now I am getting is x is 2 t and y is t. So I am getting this zt as 2 t plus i t and t, we have taken as y, so t is ranging from 0 to 1. So this is the parameterization of this straight line from 0 to 2 plus i - this one. So what we do get is that is along this path, if I find out what its derivative is, it would be - z dash t - would be 2 plus i.

Now, evaluate the integral along this path c 3, that is contour c 3 fz dz - we just go with the definition a to b fzt z dash t dt. Now a and b are here 0 and 1, fz is z square, so when I write zt, zt is my 2 t plus i t, so we could write it as 2 plus i times t actually ; so that whole square and z dash t is also 2 plus i. So, we are getting is 2 plus it whole square into 2 plus i dt. Now this 2 plus i - this is a complex constant; this is also complex constant, so what actually we are getting is 2 plus i whole cube into t square dt. Now t plus i cube i whole cube - that is a complex constant. And we do know with the property of that if it is a complex constant, that we can take outside the integral sign. So then, the integral left would be only t square with respect to t on the interval 0 to 1. So what i would get is this one - 2 plus i cube zero to 1 t square dt. This integral is nothing but 1 by

3 because t cube by 3 evaluated from 0 to 1 will give me only 1 by 3. So I would get 2 plus i whole cube. 2 plus i whole cube is what? eight 2 cube – 8, i cube would be minus i plus 2 into 3 - that is six i times 2 plus I; again you simplify it. Finally you will get it is 2 by 3 plus 11 by 3 i. You can evaluate by yourself; its very simple calculation. So we have got when I have gone with this path I have got 2 by 3 plus eleven by 3 i; when I have gone directly along this path, again I have got 2 by 3 plus eleven by 3 i. Now with the same path, if I use the different parameterization, that is what we have called c 4. So let us see is that is how we can have the different parameterization; once more we want along this path different parameterization c 4.

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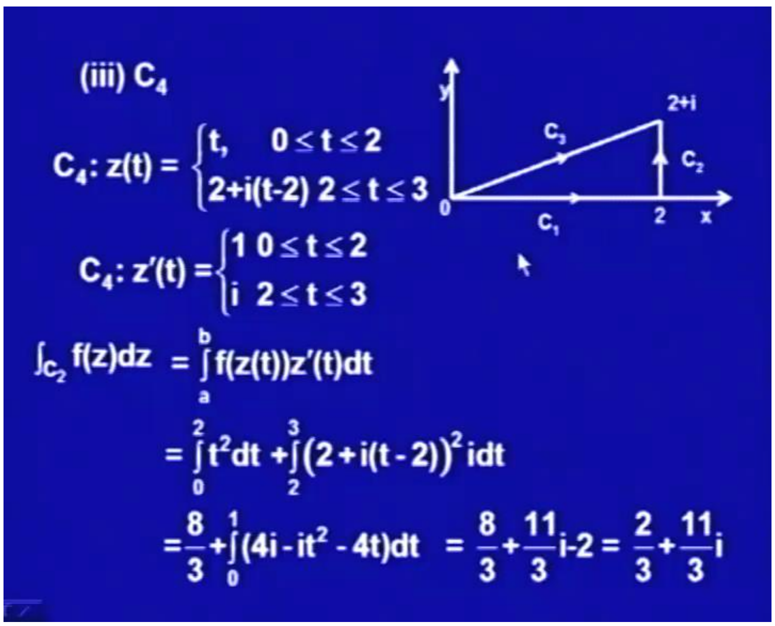

That is with the same parameter t, lets see you see that is here I am not going to explain that is how we have found it out. Let's see first this parameterization: zt is t from 0 to 2 and then 2 plus i t minus 2 for 2 to 3. You see this part is easy, because along this path it is just the x 1; so we do has is um it is a simply tx is equal to t and y is 0. Along this path when we are going, I just want the parameterization, the same continuation. That is, in the first one when I have broke into c 1 and c 2, I have taken as a different contour and the different parameterization.

But now here what do I want? I want the whole thing as the same contour. That means I want same equation. In the same equation, if you see you can check that is here what we are getting is that is we are getting is that x is being fixed at as 2; so that is all right. Now for this, y is ranging from 0 to 1. But if I do take this 2 plus y, then y i am not having this parameterization. Parameterization - we do had is for in the first part as x is equal to t. So let's say is that is rather then taking this x and y, let us try a different kind of parameterization; just the thinking about it its not that is in the simple manner we are finding it out. So we have got this as t, and from here to here I require so i have got that is it has to be this x part has to be constant and y part has to vary so that t has come into the y part. And this part we want from 0 to 1, so I have used t minus 2 so that now this onwards, that is I am taking a contour which is a smooth curve joint end to end. That means my t has to move along, that is my t has to be more than 2.

Remember it that is what how we are finding out this parameterization it. This t cannot go back as zero to 1 - in that I have to have the different contour. So what we are having is that is this one, we are moving from here and we are saying is that it has to move from 2 to 3.The partition, if you do remember, what i said is that we take t 1 is less than t 2 and so on. So this is 2 to 3, so for that because I require here only zero to 1, so we have taken t minus 2; so it is 2 to 3. I cannot move any other ahead also because this has to be same. So you could say this is as such it is doing a very simple one, but I do find it out that if I am making it a different contours and for the different contours if i am taking different parameterization it is not making any difference, since we have already proved that is in the length of arc that different parameterization does not change the length of arc.

Now here is that is the integral of this arc. So here when I have taken this one, then z dash t would be 1 in the range 0 to 2 and it would be i in the range 2 to 3. So now find out the contour integral along this contour c 4. fz tz by definition it has to be a to b fzt z dash t. Now this we have to write in the 2 parts - one is t ranging from 0 to 2 and then t is ranging from 2 to 3; the function is changing. So here z is square - would be coming to t square - and in the second case the 2 to 3 z square would be coming as 2 plus i times minus 2 whole square and z dash t is ,in the first range zero to 2 is 1, and the second range 2 to 3 is i. So we do get zero to 2 t square dt plus 2 to 3 2 plus i times t minus 2 whole square idt.

So this first integral is easy, that is, it is t square evaluated - so integral would be t cube by 3 evaluated from zero to 2; it will give me 8 by 3. The second one: now change this variable t minus 2 to, lets say, t again; so we would get the range is from - if t is from 2 to 3 - then t minus 2 would be from zero to 1. So it would be 2 plus i t whole square; that is again multiplied with I, we get 4 i minus i t square minus 4 t dt integrated from zero to 1; I would get it as 4 minus i by 2 i by 3 t square by 3. So it should be minus i by 3 minus 4 t 4 t - integral would be 2 t square; evaluated from zero to 1 would be minus 2. So what we would be getting is eight by 3 plus eleven by 3 i minus 2. Again we are getting it as 2 by 3 plus eleven by 3 i. Now what have we got if I have taken this way the path or I have taken this way the path, both times the integral of the function is coming same. That is the in which path we are moving that is not making any difference. You can say that is I have used only the straight lines. And for this is second this first path, that is c 1 plus c 2, I have used another parameterization; also even then, we have got it is going ahead and we are finding it out that is the same integral.

You can choose some other path - also you can use a part of parabola or part of circle - something anything like that like that kind of thing you can also choose, and think of the parameterization. And then you can calculate this integral and is this function, you will find it out that all the time you would this integral is same. Now what it says is that if this integral is same, which is not depending on the path that is on which path on which contour we are evaluating, its says is that if this kind of thing is happening - of course we are not knowing is that is when it will happen or we are not having here any condition that is under what for what functions fz this will happen or what contours it will happen.

The Actually contour is not mattering; so it says is that the property of function it has to be some important thing, that is which would tell us that is and which functions this can happen. So before reaching to those properties let us just find it out that if this is happening, that is if for some function we are finding it out that integral along the different contours is coming as the same. In that case we can use another formula or another method to evaluate this integral.

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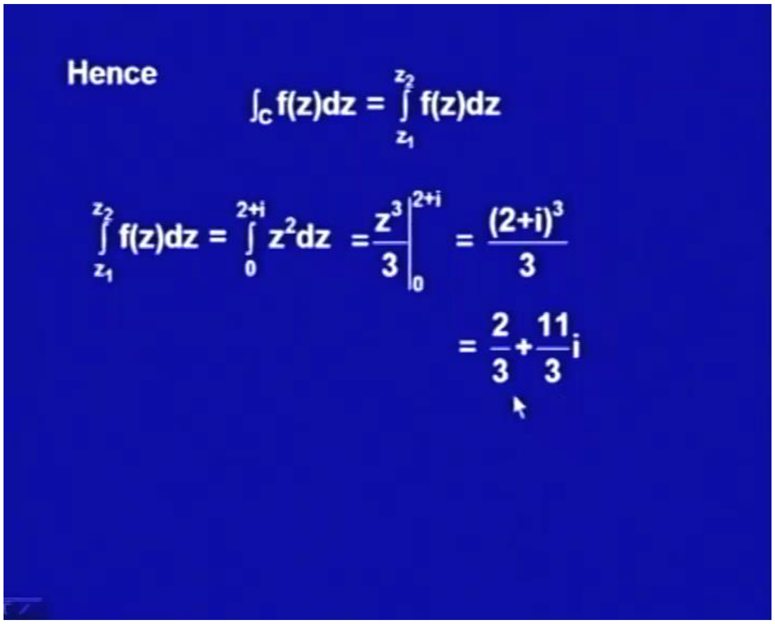

And that method we are getting is simply directly integrating the function from z 1 to z 2, where the z 1 is the point from where this contour is starting, that is t is equal to a and the z 2 is the point where the contour is ending, that is t is equal to b. So if it you to remember that is differentiation of complex functions, so if you just take this anti-derivative formula which we had used in the real domain as well also. So we will use the similar kind of result here. we can use So, lets see in this example itself z 1 to z 2 fz dz - we could say 0 to 2 plus i z square dz. Now z square - we are taking this variable itself, integrating with respect to z. So we do get is z cube by 3 evaluated from zero to 2 plus I, which is nothing but 2 plus i whole cube by 3, that you have to see is that this is how we have got in the second one, that is in c 3. So it is 2 by 3 plus eleven by 3 i. In that case it is 3 of the path of integration. Let us take one more example.

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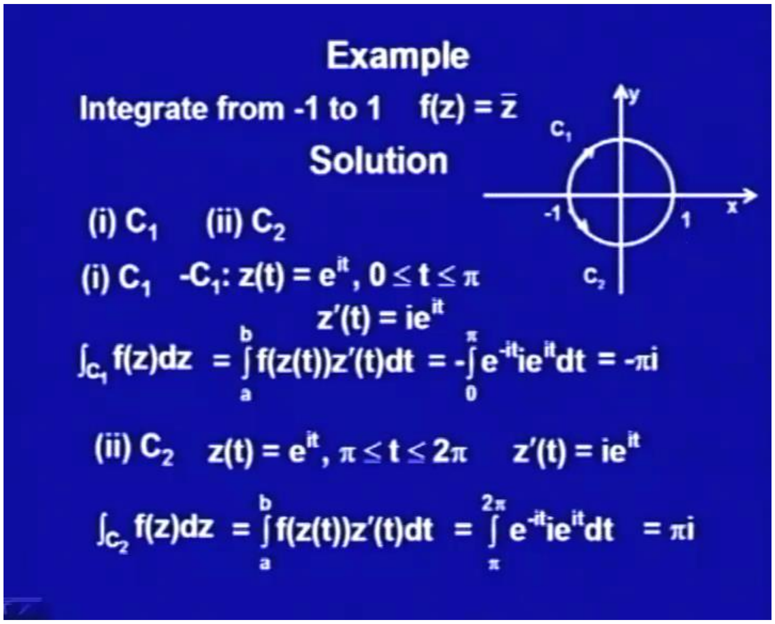

Integrate from minus 1 to plus 1, the integral z bar - that is complex conjugate of z, that is x minus iy. Let us take this path - we have minus 1 to plus 1; so let us take one path from minus 1 to plus 1 in this orientation. That is, this upper half circle and another path here taking is lower half circle in this orientation. So let us start, so these two paths - c 1 and c 2. For c 1 this is in this orientation; if you do remember we have done one example where we have introduced the circle and we said is that it should be actually there the circle was moving anticlockwise. So it would be actually minus of c 1 - that would be my circle anticlockwise; so it could be denoted it.

The parameterization would be e to the power i t for zero to pi because what if we do just get is the polar coordinates, kind of thing is that is your x is this one and i t - this t - is the angle between this one; so t is ranging from zero to pi. Then using the property of that contour minus c 1 - we could say minus of contour c 1 like that one - so z dash t along this would be i times e to the power i t. So if I just evaluate this integral around the contour c 1 fz dz, what we would be getting is integral a to b fz z dash t dt. is Using that property of sense reversal, it should be minus integral zero to pi e to the power minus i t - function is z bar, z bar means it is complex conjugate; so if the function is e to the power i t, then its complex conjugate would be e to the power minus i t i e to the power i t dt. So what we have got it is just i from here this e to the power minus i t into e to the power i t is 1; so minus i integral zero to pi i. That is, we would be getting minus pi i.

Now let us take the another path. This path, if I do take, this is moving anticlockwise. So the c 2 - my parameterization would be zt is e to the power i t where this t is moving from pi to 2 pi. Again my z dash t would be i times e to the power i t; so along this contour c 2, fz dz would be fz z dash t, that is pi to 2 pi e to the power minus i t i e to the power i t dt because my function is z bar. So it is e to the power minus i t. Again what we are getting is i integral of i dt - i is the constant, that we could take it out. 2 pi minus pi that is pi; so what we are getting is pi i. Now we see if I have taken this function - complex conjugate of the value z itself - then along these two paths - which paths we have taken on this circle - we have got that our integrals are not same. You can take actually this line - minus 1 to plus 1 - you can take along this x axis; also so that is another path you can take and find out what the integral is. So here we have got 1 example - 1 function - in which this path is mattering, that is according to the path, our the value of the integral is changing. The properties are which say is that this is happening - that we will see later on. Let us do one more, that is that absolute value of the integral; that property also I just want to have one example, that is estimation of integral.

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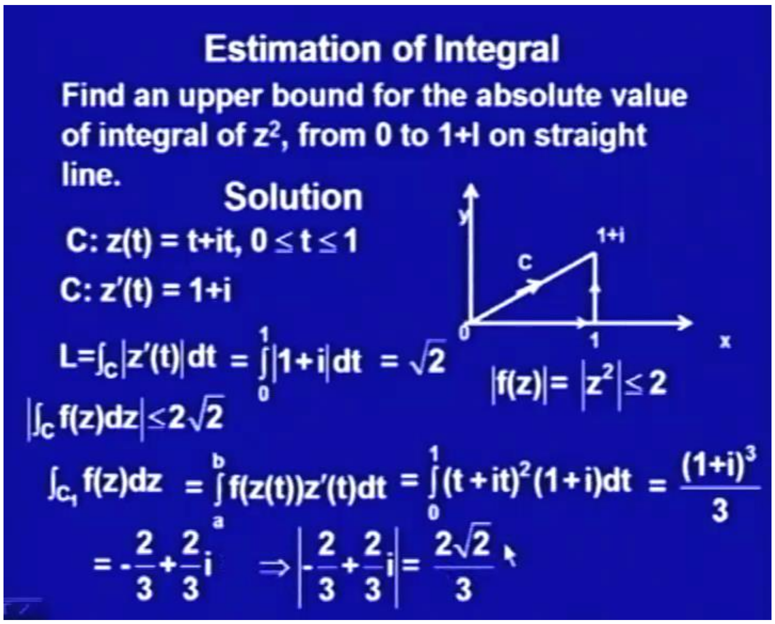

So find the upper bound for the absolute value of the integral of z square from zero to 1 plus i on the straight line. Let us take this is a straight line 0 to 1 plus i and this - I want the integral z square and its absolute value. So what the I will take this contour is - straight line - 0 to 1 plus i. So this would be t plus i t because this line is expressed x y is equal to xy, would have the t plus i t for t ranging between 0 and 1. z dash t would be 1 plus i.

Now if you do remember, that property says is that absolute values less than m times L, where L is the length of arc and M is the value highest value which is the upper bound of absolute value of fz. So first the length of arc: length of arc either I can get it, that is the length of arc this is the length of this is straight line - because it is zero 1 and this is 1 - so it should be square root 2 directly; or we can calculate using this formula z dash t is 1 plus i - absolute value of this one absolute value of 1 plus I, square root 2 - evaluated integral zero to 1 we do get it simply square root 2. And fz is z square - absolute value of z square - along this line if we see z is going to be along this line, the maximum value could be less than 2. So what we do get is that the absolute value - if I do use that inequality - ML inequality - we do get 2 root 2. What is actual value of this integral? We just know is that this integral we evaluated fz tz dash t form, I would get it 1 plus i cube time a 1 plus i cube into t square dt; that is 1 plus i cube by 3 which is minus 2 by 3 plus 2 by 3 i. Its absolute value is nothing but 2 root 2 by 3 which is already smaller than 2 root 2.

So is we have established this. So today we have learned actually the integral of a complex function on the complex variable along some path, that is we call contour integral. That is all for today. Thank you.