Mathematics-II
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Module 2
Lecture - 5
Solution of System Equation

Good morning, viewers. Today we will be discussing solution of system of equations. We will be starting with m equations in n unknowns: x 1, x 2 xn.

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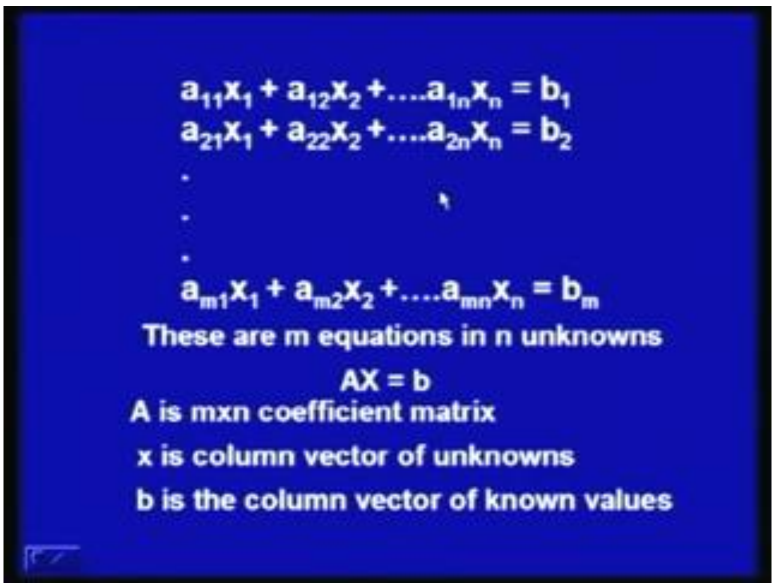

These are m equations which are linear. This can be represented by a matrix equation AX is equal to b where the matrix A is m by n coefficient matrix and the x is the column vector of unknowns consisting of x 1 x 2 x 3 xn, and b is the column vector of known values which appear on the right hand side of this system of equations.

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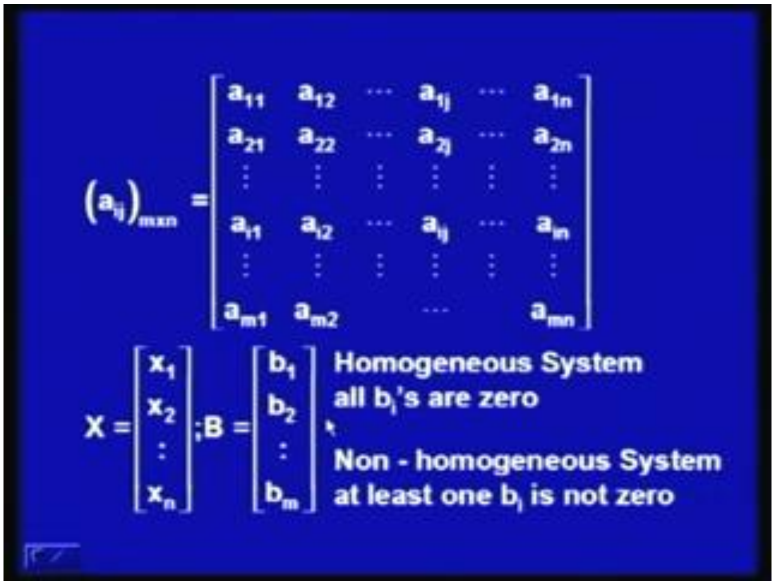

The coefficient matrix aij is m by n matrix having a 1 1, a 1 2 a 1 n in the first row and the last row is corresponding to the last equation am 1 am 2 and amn. As I told, x is the matrix of unknowns consisting of x 1 x 2 xn, while B is column vector of known values b 1 b 2 bm. This system of equations is homogeneous system if all bi's are 0; if all of these are 0 then we have a system of homogeneous equations or we call it a homogenous system. But, if any one of them is non 0 or at least 1 of them is non 0, then the system is a non homogeneous system.

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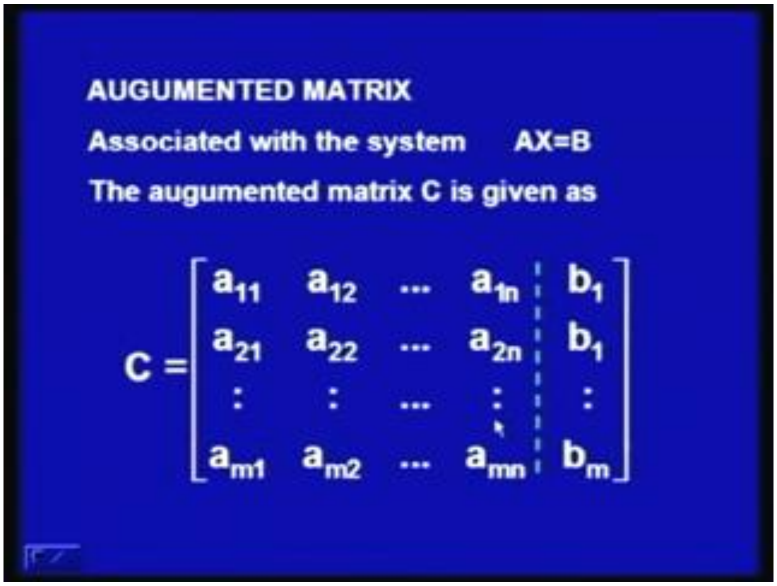

We will define an augmented matrix associated with the given system AX is equal to B and it is given as this C. You may notice that this equation has n plus n plus 1 columns. These n columns are the same as the coefficient matrix and this is the right hand side. Now, this is a convenient way of expressing the equation in a compact form; the system of equation is written in this compact form.

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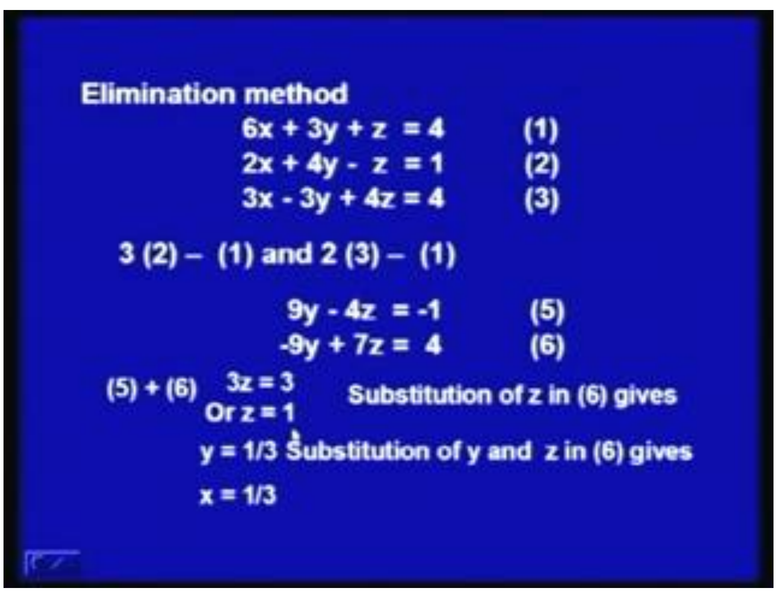

We know many methods to solve the system of equations especially when we have 3 or 4 equations. So if we apply this elimination method for a system of 3 equations in 3 unknowns, then one can multiply the second equation by 3 and subtract it from 1 and the third equation is multiplied by two and subtracted from; the idea is that we first eliminate x from the remaining 2 equations and what we have is a system of 2 equations in 2 unknowns. So If we do this simplification, then we can have 9 y minus 4 z is equal to minus 1, while this second operation gives us minus nine y plus 7 z is equal to four. Then one can add the equation number 5 and 6. This y will be eliminated and what we have is 3 z is equal to 3 which gives us z is equal to 1. Now if you substitute z in 6 we can get y as 1 by 3 and substitution of y and z in 6 gives us x is equal to 1 by 3. In fact, substitution of y and z in 6 will be satisfied automatically, but when we substitute this value of y and z in equation 1 we will get x is equal to 1 by 3. So this system of equation is solved.

Or We are applying this method from our school days; the idea here is that we want to extend this method so that it can be applicable to large system of equations. Like in engineering problems and the real life problems, the we have number of equations - number of situations - where we have number of equations which are linear but the number of equations are 10, sometimes 20, may be 50 or may be more than 100. So how to extend this method, so that we can conveniently get the solution of such equations, if at all the solution exists? So Now we see how this elimination method can be generalized and applied to higher order system of equations.

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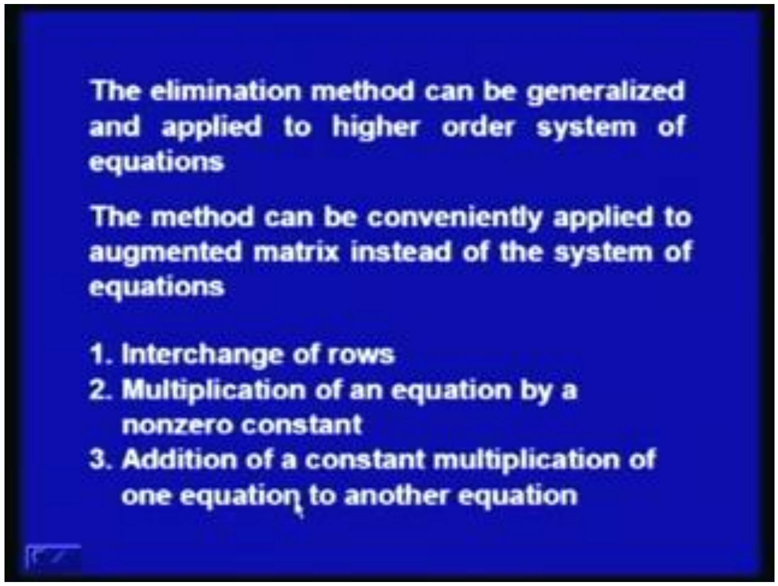

One may notice that the method which you have applied for system of equation - the same operations - can be applied to augmented matrix. You so do you don't have to write down xyz every time. You simply apply those operations on augmented matrix and the result will be the same. Now while applying the method, we have applied number of operations. Those operations can be categorized into 3 categories. One is we have made interchanges in the rows. We have multiplication of an equation by a non 0 constant or the third type of operation may be that addition of a constant multiplication of one equation to another equation. So these three types of equations may be used these 3 these 3 types of operations may be use to obtain the solution of equations.

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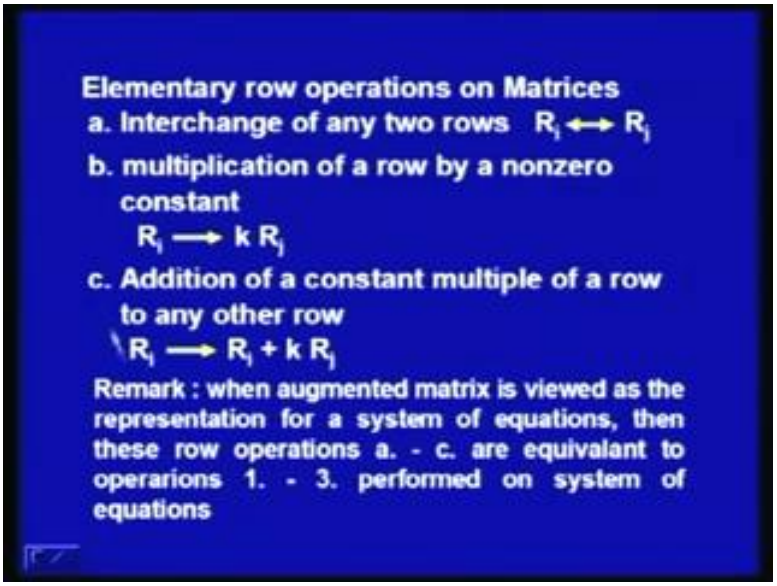

Now, after that, I will introduce the elementary row operations on matrices. The idea is we will not be using these operations on the system of equations; rather we will be using these operations on the augmented matrix and will get the solution. So correspondingly we have 3 operations: the first operation is interchange of any 2 rows gives - symbolically we write Ri interchange Ri Rj, that means the rows i and j are interchanged. The second operation is that multiplication of a row by a nonzero constant; so if k is nonzero then i th row is multiplied by k to give us modified i th row, And the third operation is that Ri is modified as k times Rj added into Ri. So these are 3 different types of operations which can be performed on matrices. We call these operations as elementary row operations. We call them row operations because they are applied to rows of a matrix. Now, we can make a remark here that when augmented matrix is viewed as a representation for a system of equations, then these row operations which we have discussed are equivalent to operations 1 to 3 performed on system of equations.

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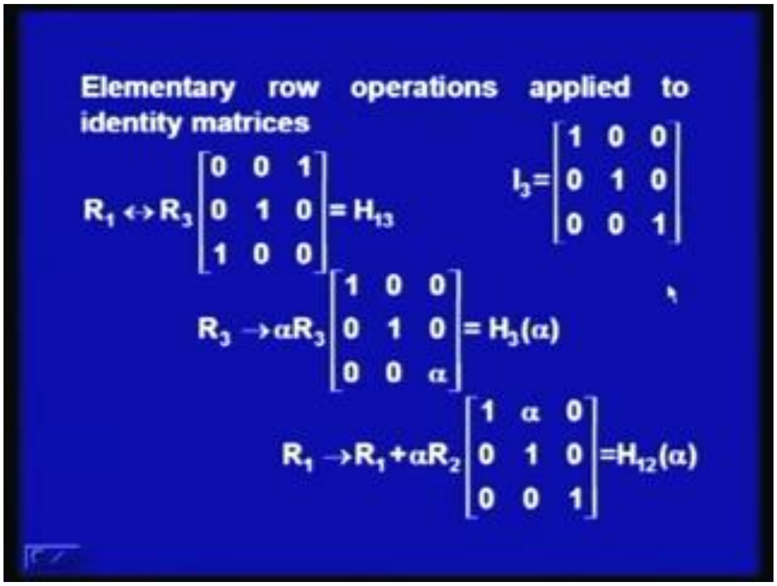

Now these elementary row operations - when they are applied to identity to matrix: Let us say i consider I three. That is a 3 by 3 identity matrix and I apply an operation R 1 interchange R 3. So when I interchanged the first row with the third, then this one will come here and this one will come here. So when I apply this operation, then what I have is a matrix H 1 3. So H 1 3 represent an elementary row matrix which is obtained by applying row elementary row operation of the type where R 1 is interchanged by R 3. That is the significance of this 1 3. Then other row operation is that a particular row is multiplied by a constant. So let us say we are multiplying that constant particular row, say R 3, by a constant alpha. So If I apply this operation on identity matrix, then third row which is 0 0 1 will become 0 0 alpha. I call this matrix as H 3 alpha because third row is multiplied by alpha and I am using only one index, because only one row is affected. Then the third type of operation may be written in the form R 1 goes to R 1 plus alpha times R 2, meaning thereby, that second row is multiplied by alpha and added into first row. So if I apply this operation on this, then second row will become 0 alpha 0; when added into this, it is 1 plus alpha and then 0. So This operation is called H 1 2 alpha, because alpha is the constant to be multiplied by the second row and added into 1. So we have 3 different operations: H 1 3, H 3 alpha and H 1 2 alpha. Now these matrices are called elementary matrices. So, elementary matrices are obtained from identity matrix by applying elementary row operations.

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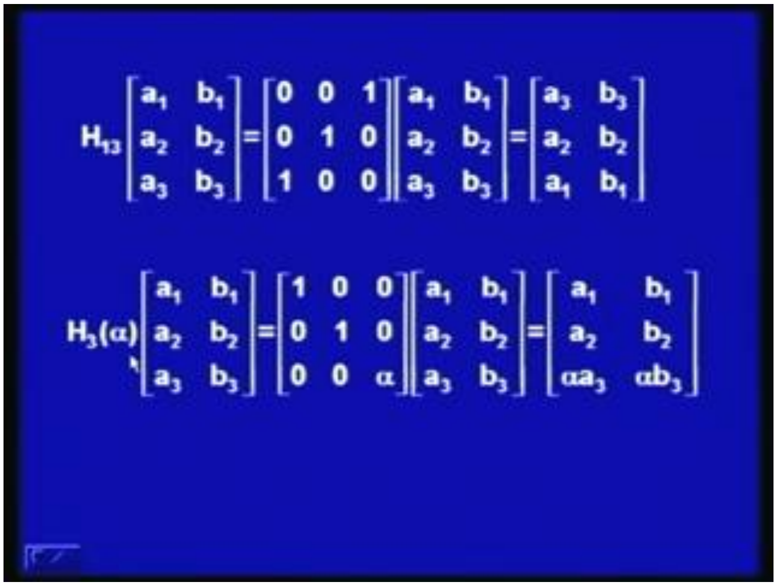

So If I have a 2 by 3 matrix and apply an operation H 1 3 on this matrix, then this actually is equivalent to this product where I am applying H by where i am multiplying H 1 3 by the given matrix. What should be the order of the matrix - this identity matrix? This the must be such that the this multiplication is confirmed - it is possible. So this On identity matrix, I apply this operation, that is, first row and third row are interchanged. So we got this matrix this multiplied by this; one can check if it has a same effect as if we are applying operation on this matrix. Let us see - 0 0 and 1; so this becomes a 3. Then if I multiply by this, I will get this element which is 0 0 and b 1. So we will have finally this matrix as if this matrix is interchanged - first row and third row. So that is the final matrix. Now this type of operation can be done for also for all type of operations, like H 3 alpha when multiplied by this - that means we want to change this matrix. we want to apply the row operation on this matrix, so that the third row is multiplied by alpha. So what I do is I first apply this operation on the identity matrix. So if third row is multiplied by alpha in the identity matrix, only this element will be affected. So we will be having 1

1 alpha multiplied by this and if you realize perform this multiplication, what you will be having is this matrix.

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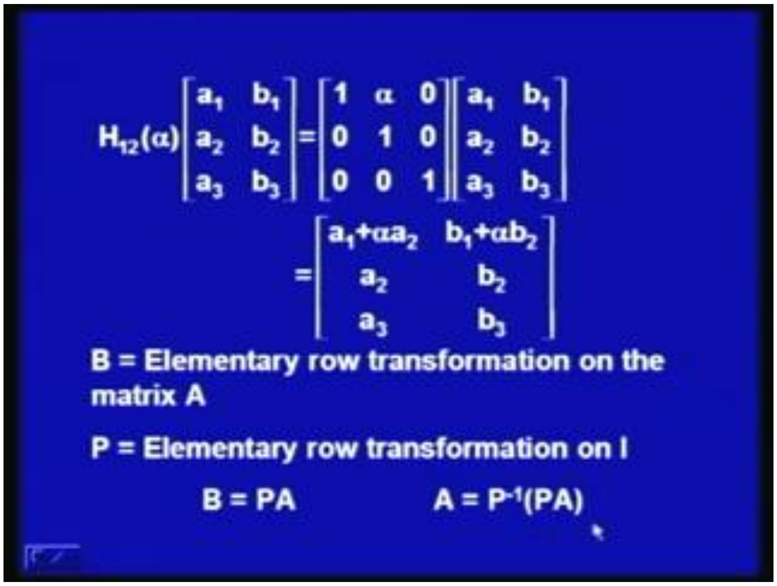

Similarly H 1 2 alpha multiplied by this means we will be having the first row plus second row multiplied by alpha. So what is H 1 2 alpha is this matrix multiplied by this matrix and if you perform this multiplication, we will be having this matrix. Now we say that B is the elementary row transformation of the matrix A. So let us say we have a matrix A; we perform elementary row of transformations on the matrix. Let us say after performing the transformation, what we get is B - the matrix B. Then, if P is the elementary row operation on I - the same elementary row operation which you have applied on A - so if P is the elementary row transformation on I then B is equal to P into A. So it is pre multiplied by the matrix and the result is the same. Now this actually gives us the result that A can be obtained as P inverse PA. So from after applying row transformation, A is transformed to B and you again apply a set of other transformations, and you will come back to the original matrix A - provided this P inverse exists. So if P is nonsingular, then P inverse will exist and in that case A can be obtained as P inverse PA. So one can obtain from B, the matrix A after applying elementary transformations. Now what is the matrix P? P is the elementary row transformations on I. I is nonsingular. After performing elementary row transformation, matrix remains nonsingular and that means P inverse exists. So it is possible that you obtain A from the matrix B.

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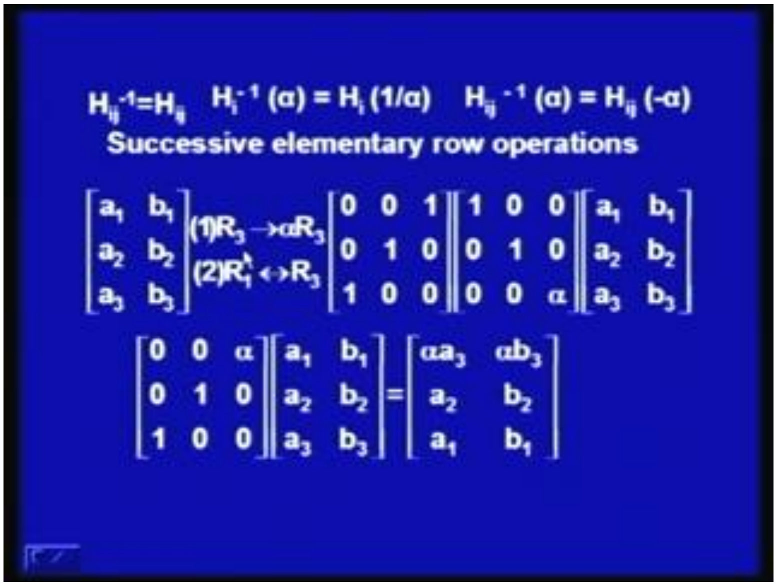

Now this means one has to see about the inverse transformations. Hij inverse will be Hij; the idea is Hij is the transformation which interchanges i th and j th rows. So if you want to come back to the original matrix, then you have to apply Hij inverse - that means, you again interchange i and j. So interchanging i and j twice - you will be getting this same matrix. So That is why Hij inverse is equal to Hij. Now Hi inverse alpha - that is, the inverse of Hi alpha is Hi inverse alpha - this can be obtained as Hi times 1 upon alpha. See If I apply if I multiply the i th row by alpha to get some matrix B, then if I want to come back to the original matrix, I have to multiply the same row but with the constant 1 upon alpha - provided alpha is not 0. That is why you say Hi inverse alpha is equal to Hi 1 upon alpha. Then the third operation - Hij inverse alpha is equal to Hij minus alpha; this operation simply means that j th row is multiplied by alpha added into i.

Now if you want want to come back to the original matrix, what I do is I multiply the j th row by minus alpha so that the result will be the same. So whatever in row transformation is being done here that will be nullified and we will get the original equation. So that is how Hij minus 1 alpha is equal to Hij minus alpha. The idea is that you can perform number of operations on a – elementary - on a matrix. So Let us say I have been given a matrix - a 3 by 2 matrix a 1 b 1 a 2 b 2 a 3 b 3 -and I apply 2 operations successively on this matrix. The first is R 3 multiplied by R 3 changes to alpha times R three and the second operation is that the R 1 and R 3 are interchanged. Now these 2 successive operations can be performed on this matrix; then we will be getting this matrix. The idea is What is this? This matrix is nothing but H 3 alpha and this matrix is nothing but H 1 3. So these are the elementary row operations. So it is it has the same effect; whether you apply these 2 operations on this matrix or you multiply these 2 matrices and then multiply with this the effect will be the same.

So let us see this. If you really perform these multiplications, then first this this product - 0 0 and 0; so this element is zero. 0 0 and 0 - this element is 0; 0 0 and alpha - so we will have first row as 0 0 alpha. Similarly one can perform the second row multiplied by this; we will be having 0. This multiplied by this is 1 because of this product; this multiplied by this is again 0. But this row multiplied by this row is 1; 1 multiplied by 1 this is 1, 0 multiplied by 0, 0 multiplied by zero - so the result is 1 here. Then this row multiplied by second row is 0 and third row multiplied by third column is again 0. So product of these 2 matrices - elementary matrices - comes out to be this and when you apply this product to this, what will be having is this matrix. So let us first check this element; it is 0, this and alpha times a 3 is the only nonzero element, rest of them will be 0 and this multiplied by this will give me alpha b 3 as this element. When you multiply second row by this, this will be 0, this will be a 2 and nothing will be contributed from this product. So we will have only a 2. Similarly when you multiply this by this, we will be having a 1; so what we are having is a 1. So the idea is when we apply these 2 operations successively, we will have this matrix. This can be obtained directly applying to this or you apply series of elementary matrices.

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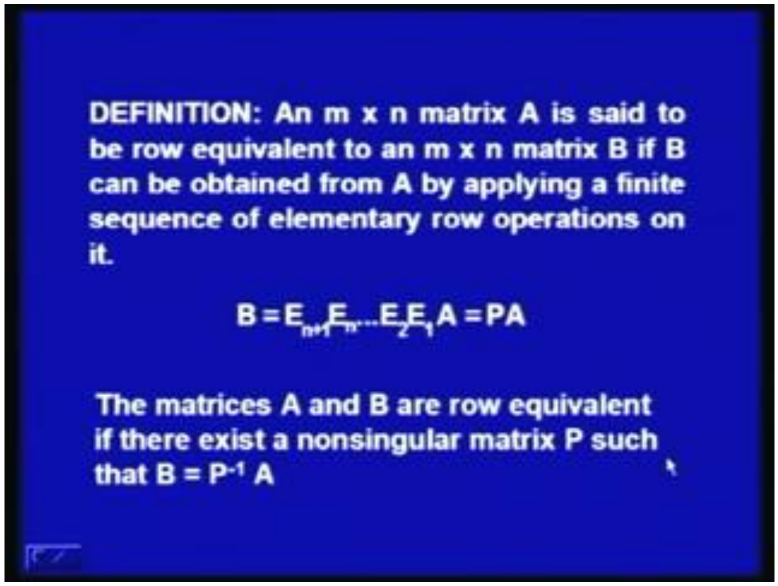

Now this leads to a definition which says that an m by n matrix A is said to be row equivalent to an m by n matrix B if B can be obtained from A by applying a finite sequence of elementary row operations on it. So two matrices are row equivalent, if one can be obtained from another matrix by applying finite sequence of elementary row operations on it. Now this means that we have a matrix A, on this matrix A, I apply sequence of elementary row operations. What are those operations? E 1 E 2 E 3 En En plus one. They are multiplied - pre multiplied - on the left hand side of A. What are E 1 E 2 En? These are elementary matrices, that is, the same row operations applied on identity matrices. So all these matrices when multiplied together - let us say this is P; so if B can be written as a matrix P times A, then we say B and A are row equivalent matrices or we say the matrix A and B are row equivalent if there exist a non singular matrix P such that B is equal to P inverse A since all E 1 E 2 En's they are identity matrices and on those elementary matrices we have applied row operations. So all these are nonsingular; when they are multiplied together they reaming nonsingular and that is why this P is nonsingular; and nonsingular means P inverse exist. So, if we can write down B is equal to P inverse A, then A and B are row equivalent.

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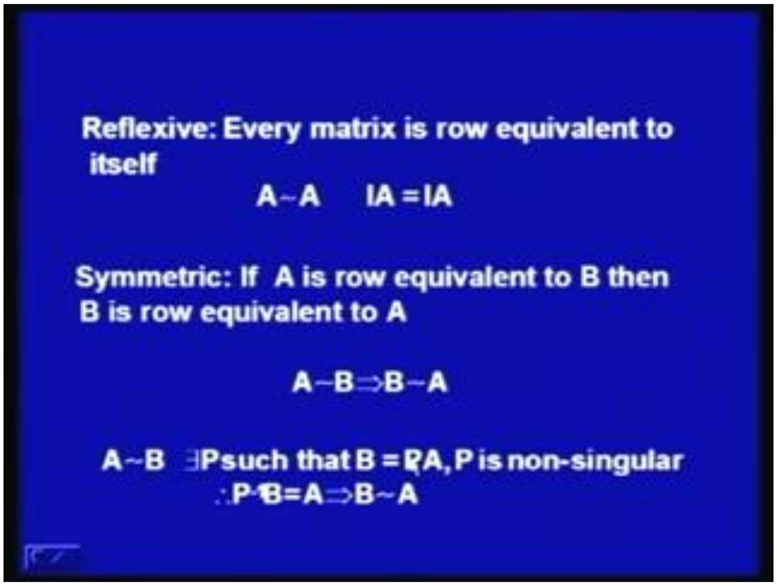

Now this row equivalence is actually an equivalence relation. For that we have to apply we have to prove that the relationship is reflexive. By this, I mean to say that every matrix is row equivalent to itself. The proof is simple. I write A is equivalent to B; this symbol is use for equivalence. So the matrix A is equivalent to A. That means you apply the you there is a matrix A, there this a matrix I which when applies to A on the left hand side; we will have AI the same matrix. So A is equivalent to A. Then the relationship is symmetric.

By this, I mean to say if A is row equivalent to B then B is row equivalent to A. A proof of this is simple; like if A is equivalent to B, then there exists some matrix P such that B is equal to PA. This is the basic definition of equivalence relationship where P is a nonsingular matrix. This is P is nonsingular, then P inverse B will be A. So that means if we are multiplied by P inverse on both sides of this, that is P inverse B is P inverse P multiplied by A is nothing but A. So P inverse B is A and if P inverse B is equal to A, that simply means that B is equivalent to A. That means you have started with A equivalence to B and what we have is B equivalent to A. That is, if A is equivalent to B then B is also equivalent to A and that proves the reflexive property of the operations. The third is the transitive property.

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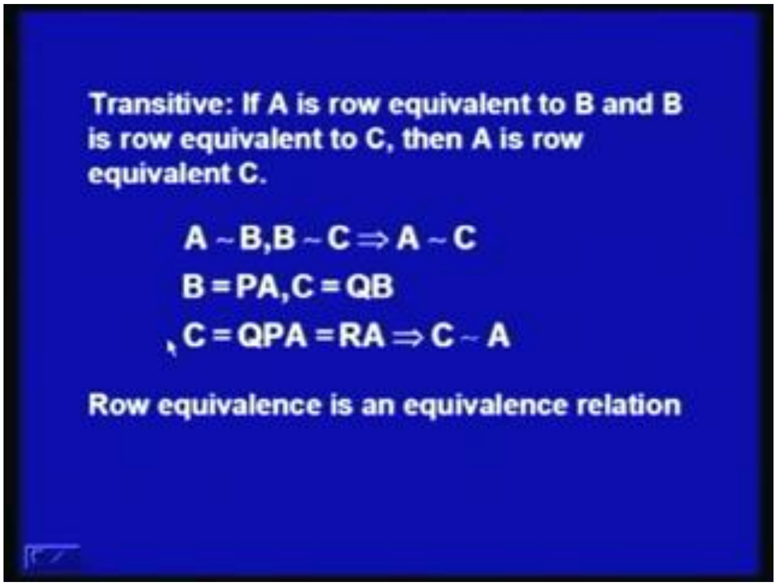

It says that if A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C. So I now I have 3 matrices: reflexive - I have one matrix which is equivalent to itself; symmetric - I have 2 matrices they are equivalent to each other; transitive - I have 3 matrices. If A is equivalent to B and B is equivalent to C then A is row equivalent to C. That means this is been given to us: A is equivalent B B is equivalent to C and this implies that A is equivalent to C. That is the meaning of the transitive property. To prove this, let us say A is equivalent to B means B is equal to PA; that means, there exist matrix P such that the B is equal to PA. Now C B equivalent to C means there exist an elementary matrix Q so such that C is equal to QB. Now if I write down B is equal to PA from here, then C is equal to QP into A. What is QP? QP is a product of elementary matrices. What is P? P is also product of elementary matrices. What is QP? Another elementary matrix - another matrix R. So C is equal to RA; that simply means that C is equivalent to A. So we have proved reflexive property; we have proved symmetric property and the transitive property and this means that row equivalence is an equivalence relation.

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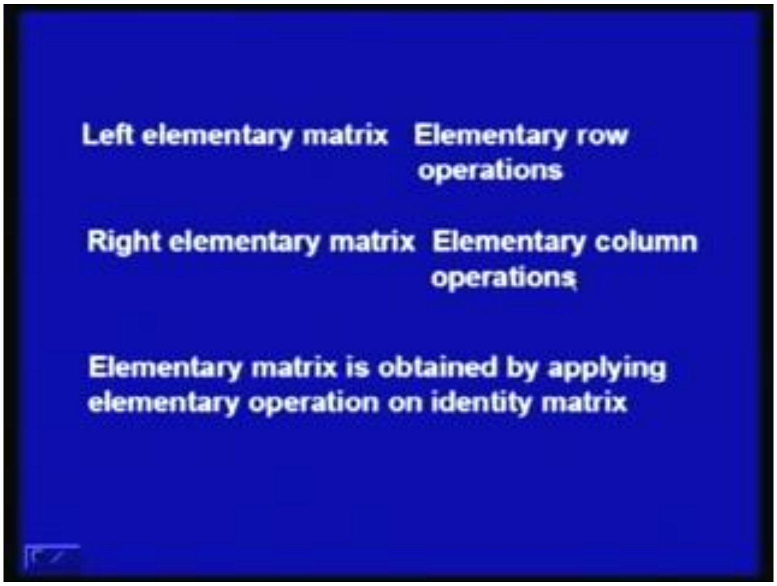

Now in all this discussion, we are to applying elementary row operations actually the same way one can define elementary column operations. Elementary column operations are same operations but they are applied to columns. The operations are: 2 columns are interchanged, one column is multiplied by a constant or the third type of elementary column operation is that one column is multiplied by constant added into the other column. So we will have 3 types of elementary column operations and these are again equivalent to the column operations applied on elementary identity matrix. What we have is again an elementary matrix; but this time we call it right elementary matrix because when we apply a row operation, then elementary matrix is pre multiplied and we get the final result. But when we apply column operations, then the this elementary matrix must be multiplied on the right hand side. That is how we differentiate the 2 matrices. So it is left elementary matrix as far as row operations are concerned and we have right elementary matrix when we apply the column operations.

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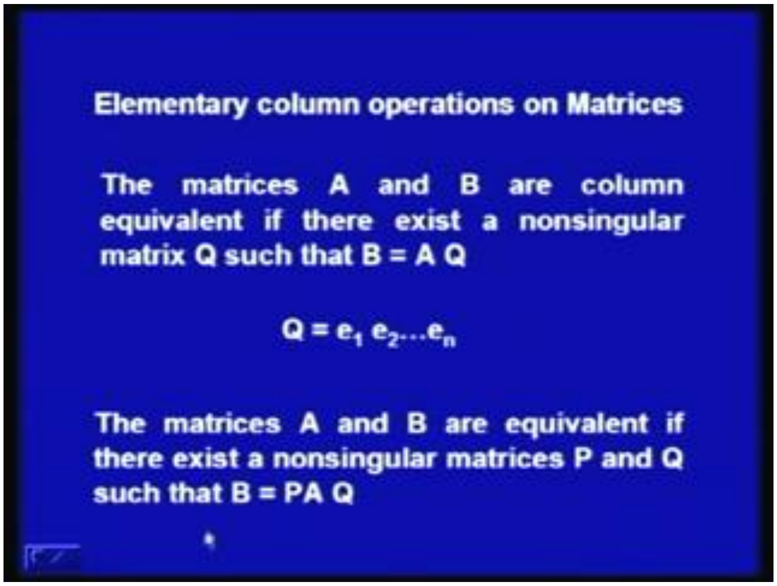

So we can apply elementary column operations on matrices and one can define same equivalence relationships like, the matrices A and B are column equivalent if there exists some they are exist a nonsingular matrix Q such that B is equal to AQ and here I am multiplying on the right hand side; for row operation it was on the left hand side. That is the basic difference between the row operations and column operations and any of this matrix Q is actually series of elementary matrices elementary column matrices. And these are e 1 e 2 en - I am using small letters e 1 e 2 en to denote the elementary column matrices. So Q is equal to e 1 e 2 en. Now, the matrices A and B are equivalent if there exist a nonsingular matrix P and Q such that B is equal to PAQ. See you you have a matrix A and a matrix B; one can obtain matrix B by applying simply row transformations or one can obtain a matrix B by applying column transformation but some times we apply a row as well as column operations from to the matrix A to get the matrix B. If there is a such a situation, then we say B is P times A into Q. What are what is P? P are the elementary row operations and Q are the column operations. So the 2 matrices are row of they are equivalent if B is equal to PAQ; so we have row equivalence, we have column equivalence and simply the 2 matrices are equivalent.

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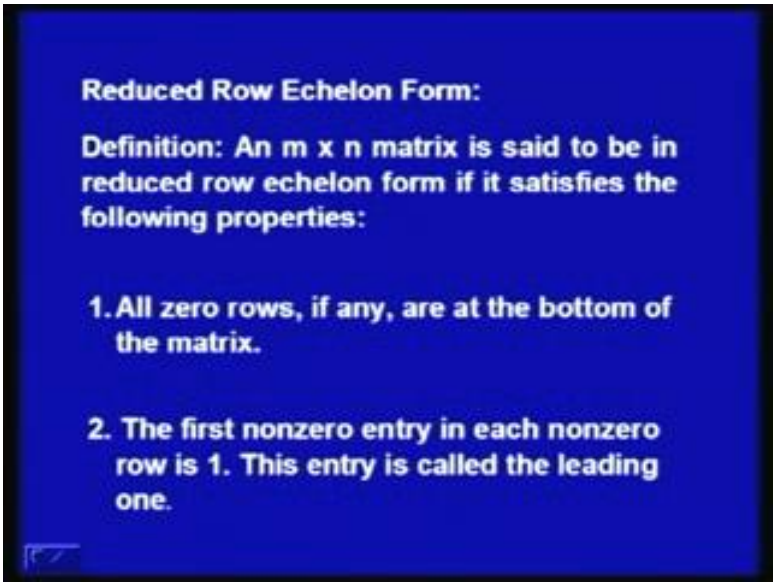

Now all these concepts will be used to obtain the solution of equations. Now Before I really apply these concepts to solve system of equations, I like to introduce the concept of reduced row echelon form. Now we define a matrix m by n to be in reduced row echelon form if it satisfies certain properties. So what are those properties? The first property is that all zero rows, if there are any, are at the bottom of the matrix. So matrix is in the reduced row echelon form if there are some zero rows then they will be at the bottom of the matrix and the second property is that the first nonzero entry in each nonzero row is 1; and this entry is called the leading one.

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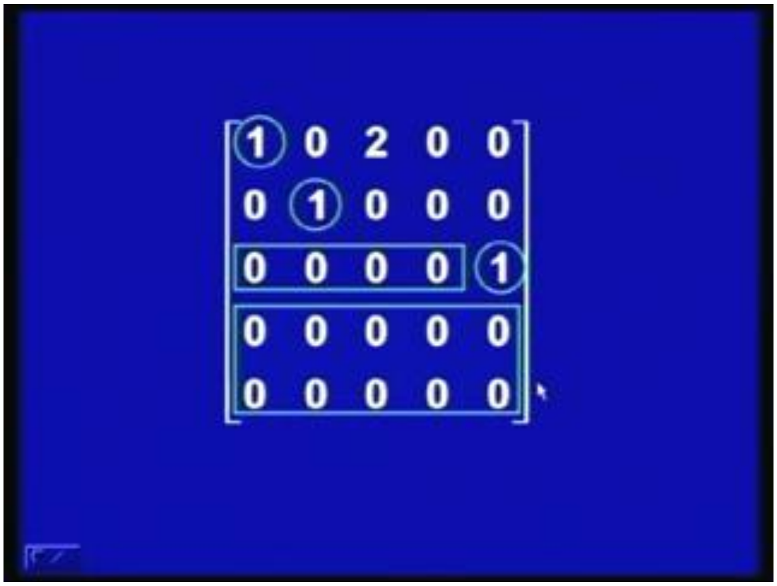

For example, I have this matrix consisting of 5 rows and 5 columns - 5 rows and 5 columns. There are two 0 rows - a fourth row and fifth row - they are 0 rows. They are at the bottom of the matrix. So this line is a candidate for row reduced form. This is got to be the first property. In the second property, we say that if there is a row which is having nonzero elements; like the first row it has 2 nonzero elements, then the first element in this row has to be 1. So this is 1 here - this element can be any value - but this has to be 1. This also has to be 1, and even the third column has nonzero element and this also has to be one. So the this may be a candidate for row reduced form.

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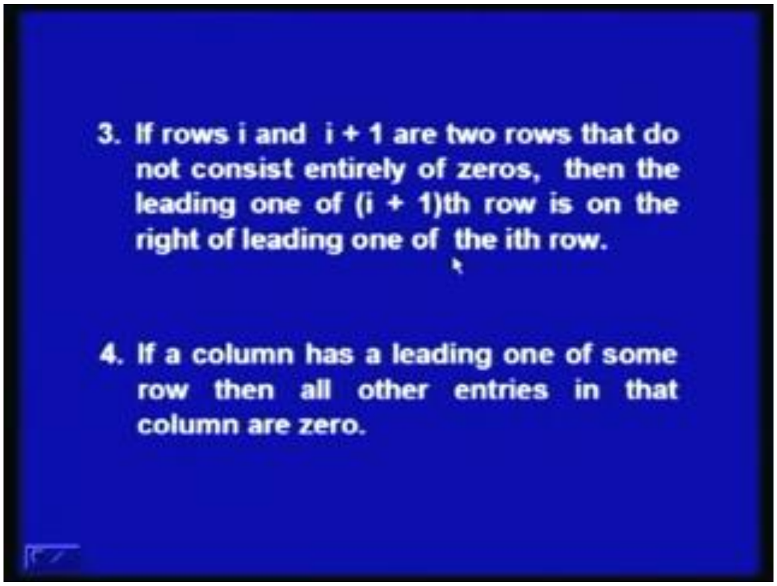

The next property is if there are 2 rows having nonzero elements - that means the rows which are not consisting of entirely zeros, that is, if the rows are i and i plus 1 - then the leading one of i plus 1 th row is on the right of leading one of the i th row. I will explain this with the example. And the fourth property is that if a column has a leading one of some row then all other entries in that column are zero. So let us see this example.

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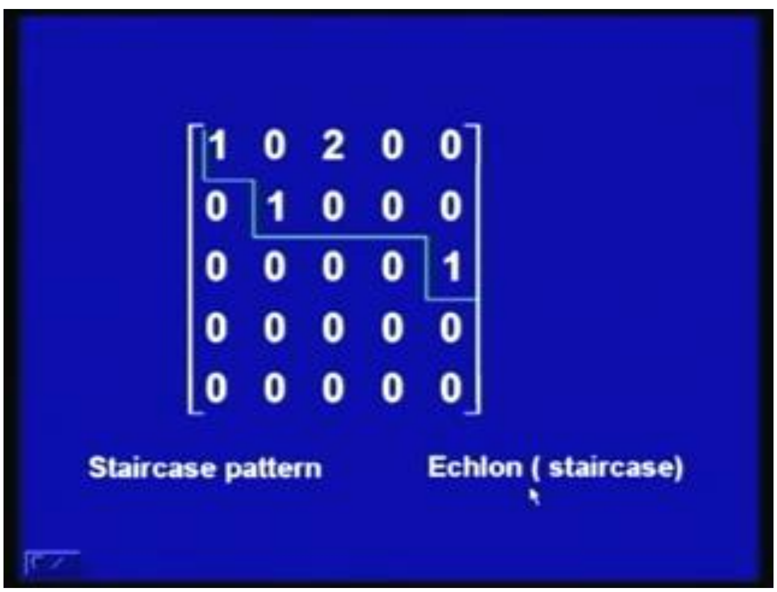

Here we are having 0 rows, these are nonzero rows, these are leading ones. Now this is nonzero row - it has a leading 1 here - and this is a nonzero row - it has a leading 1 here. Now this is the i th row, this is the i plus 1 th row, the leading one in this row has to be on this side and i plus 1 th row will be on the right side of this leading one. Similarly, this one and this row - these 2 rows - they are having non 0 elements and this is the this the leading elements of this row, this is the leading element of this row - this will be on the right of this. So this satisfies all the properties of row reduced form. So we say this matrix is in row reduces form. Now one may notice that this is the boundary where we have non 0 elements - rest of the elements here are zeros. Now this pattern is a sort of staircase pattern and that is why we call this matrix as an Echlon matrix or a staircase matrix. So row reduced form is - one may notice that all elements below this boundary are 0 while the nonzero elements are on the upper side of this boundary. This boundary forms a staircase pattern and that is why we call it an Echlon form. So this row reduced matrix is also called in the Echlon form.

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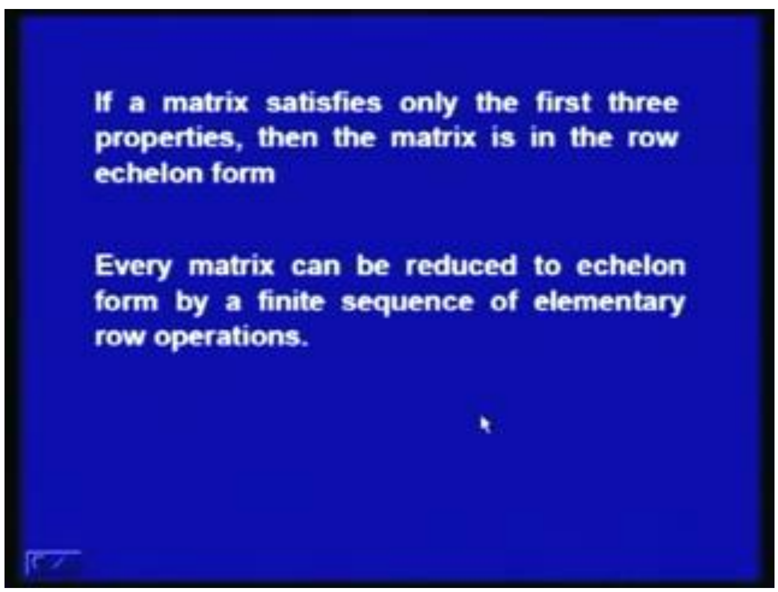
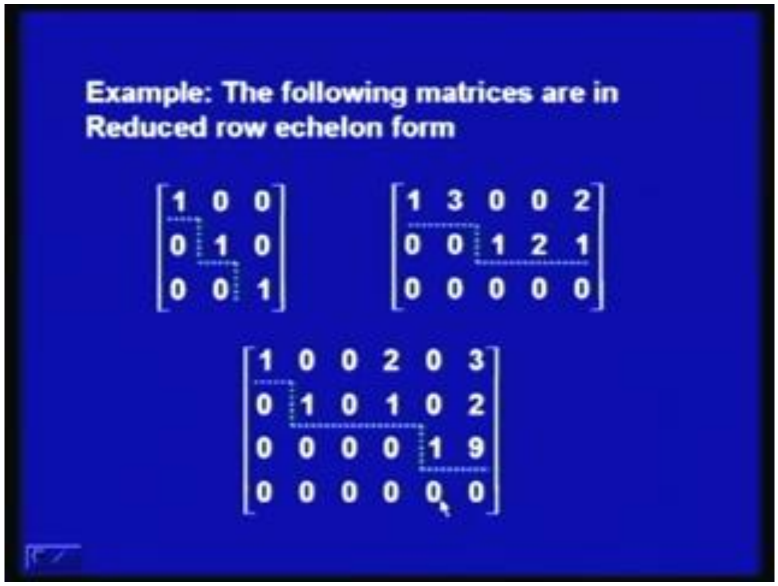


Now many times, a matrix satisfy only the first 3 properties; then the matrix is not in the row echelon form. I will explain this with example.



Like this matrix, a 3 by 3 matrix, is in row reduced echelon form as it satisfies all the properties - the staircase pattern. This matrix is again row reduced form; again all 0 rows here, leading rows ones and then all the properties are satisfied. This matrix - all zeros on the bound on the bottom of the matrix; this is again a staircase pattern. So this is again in reduced row echelon form.

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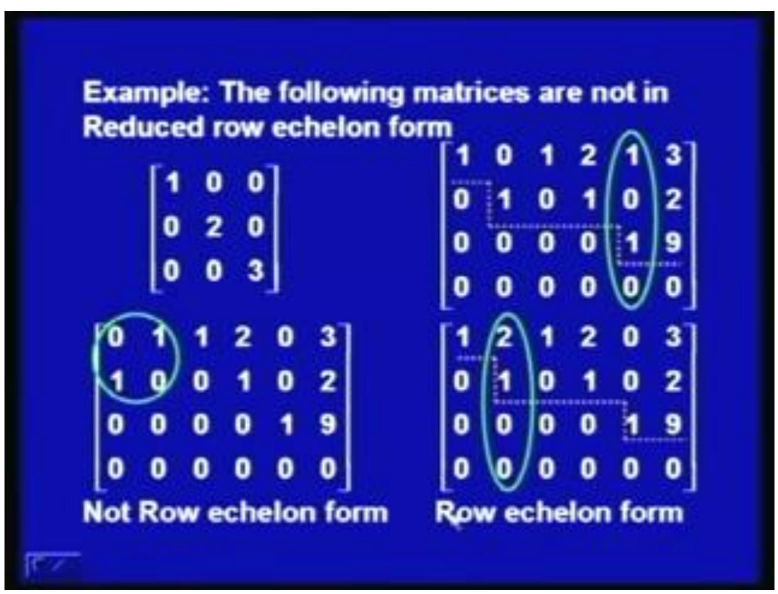

However, this matrix is not in reduced row echelon form because this is a leading one but this value is two. This should be the leading on in this row but the value is 3 here; so this is not in reduced row echelon form. In this case - and this is all 0- is the staircase pattern is still there, but if you look at this particular column this has a leading one but this has this should have all column all entries in the column should be 0. That is the fourth property. But this property is not satisfied. That is why it is not in reduced row echelon form; this is nonzero, this is 1, so this is not in reduced row echelon form. The same thing is here: this particular pattern is against the form because this is a leading one in this row; this is leading one in this row. But this leading one is on the right hand side; this should be on this side and this should be on this side. Only then it will be in row reduced form; so this is not in reduced row echelon form. Now in this case again, the pattern is there - staircase pattern is there - but this particular column makes it different, then row reduced echelon form because this is the leading 1 but this has another entry which is nonzero. So this again is not in reduced row echelon form. So they are not in row But they as they they can be in row echelon form.

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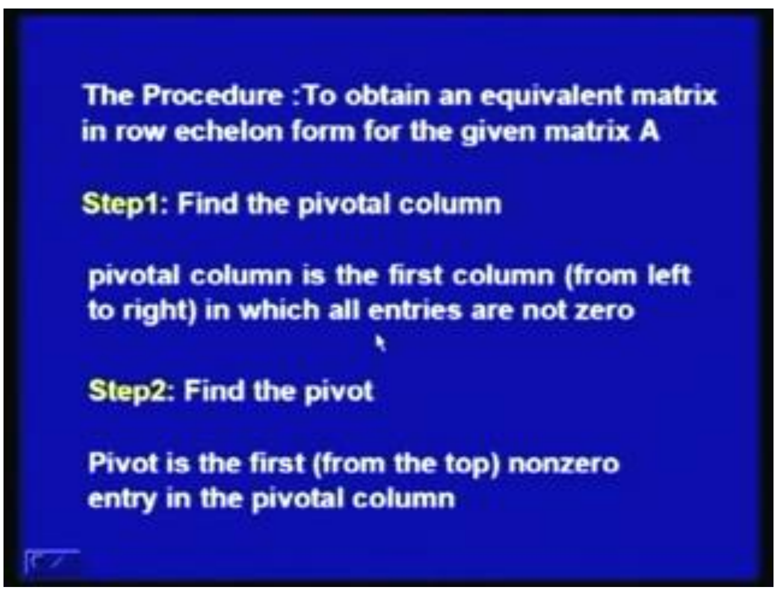

The Now we will use we will describe a procedure to obtain an equivalent matrix in row echelon form for the given matrix. Now this particular form is very suitable for solving to system of equations. So first, we will describe a procedure through which we can reduce a given matrix in row echelon form and then we will see how the solution can be obtained easily. So the step one is that you have to first find the pivotal column -that means, the column in which we have a pivotal element. Then Pivotal column is the first column from left to right in which all entries are not zero; that is the meaning of pivotal column. So first, we have to find the pivotal column in the given matrix.

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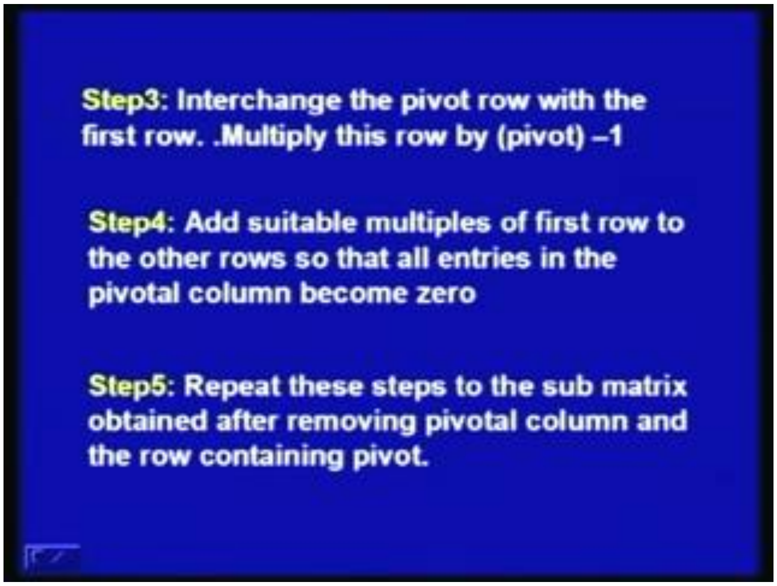

Then find out its pivot and the next thing is interchange the pivot row with the first row; multiply this row by pivot minus one and finally add suitable multiple of first row to the other rows so that all entries in the pivotal column becomes zero. And Then we can apply this method to sub matrices after removing pivotal column and the row containing the pivot. Let us see this with this illustration.

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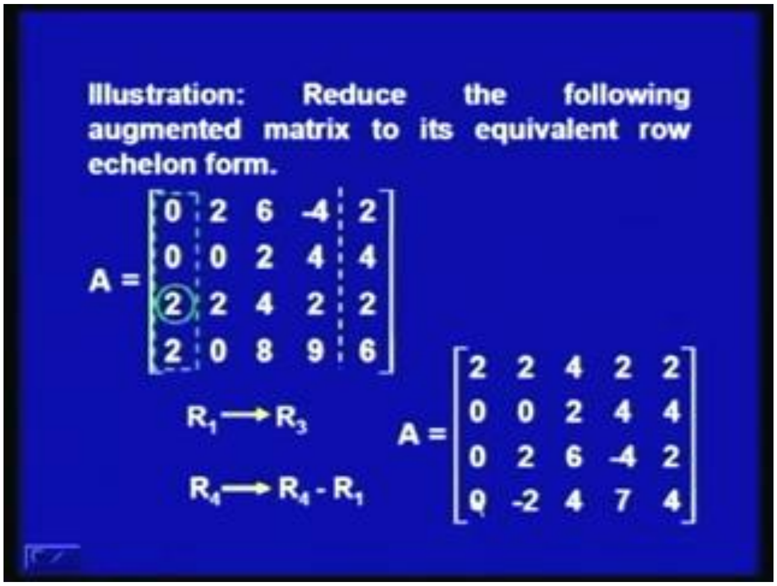

So we have been given a matrix A. I call this matrix A as in augmented matrix; that means, it actually represent a system of equations with this as the coefficient matrix and this on the right hand side. This is A matrix x multiplied by the column vector B. So this is B, this is A. Now first thing is you may notice that this is the first element which is nonzero. So we have to first make this pivot as 1. So the first step is you apply R 1 to R 3. So first this make this this particular row here bring this particular row here and then R 4 is R 4 minus R one. So once this becomes 1 R 1, then R 4 minus R 1 will make this element zero. So if we apply these 2 transformations, then what we have is this matrix. So It is 2 here 0 0 0 - this 0 0 are already there, this is been interchanged and this is subtracted from this, so we will have this matrix.

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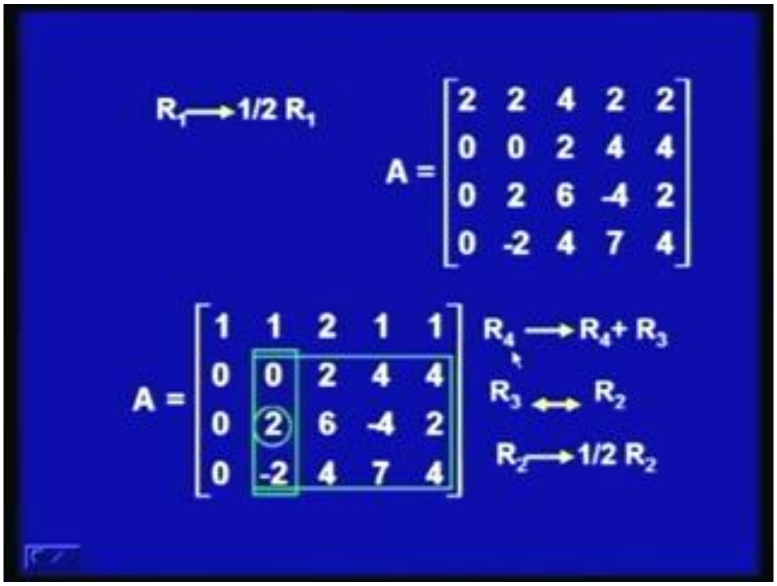

Now Then once we obtain this matrix, we write down R 1 is half times R 1; that makes it a pivot. So we have this matrix. So One may notice that only in this part there is a problem but as far as this is concerned, this is perfectly in line with the row reduced echelon form. So we have to make this as row reduced echelon form and then the final result will be in row reduced form. So That is why we now concentrate on this particular sub matrix. When we apply the same procedure here again, we find the pivotal column. So this is the pivotal pivotal column. We apply a transformation R 4 is R 4 plus R 3 - that will make this 0. R 3 and R 2 are interchange - so these two are interchanged; then this 2 will come here. Then R 2 is half of R 2, so that this would become one.

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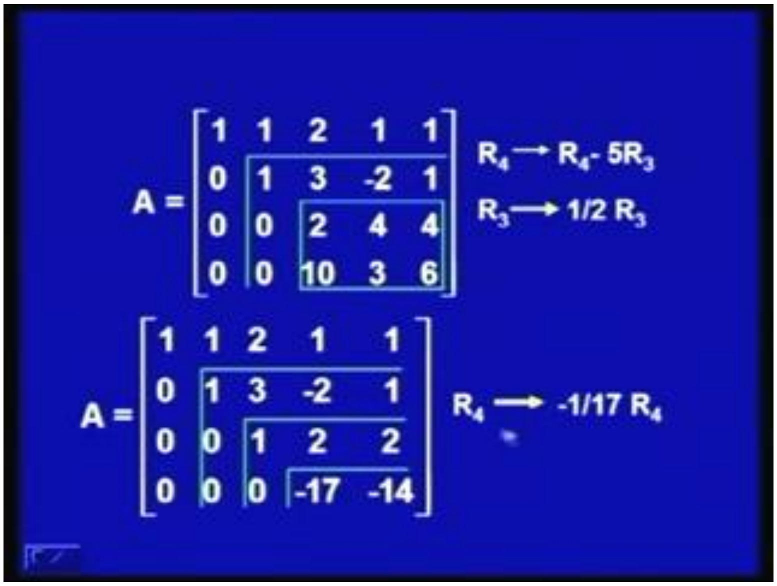

So if you apply this series of operations, we will have a matrix as this this becomes 1 and these two will become 0. So this We have started with the matrix: this is the leading row having 1 here, this has the leading row - 1 being here this is the pivotal element. Now we try to apply the same procedure on this sub matrix. Now for this, we first apply R 4 is R 4 minus 5 R 3 - that we make this to be 0 and then R 3 is half of R 3. So this will become 1. So if I apply these 2 operations, I will get the matrix as this. So this is 2 2 minus 17 and 14 because of this operation. Now This - again pivotal element; now this sub matrix is to be reduced. So I apply this operation and this operation will give me this matrix in this form.

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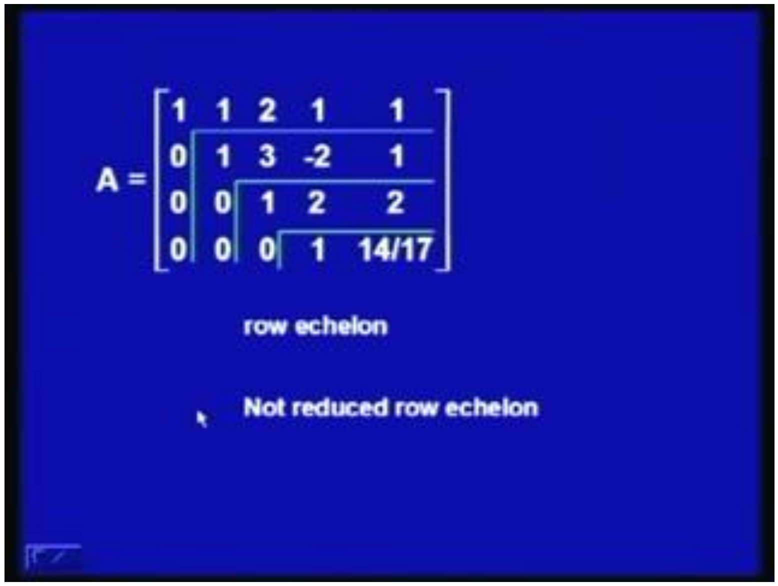

So this is first row - first pivot; this is second pivot, this is third pivot and there is a forth pivot. But this is not enough. It is not in row reduced form because of this non 0 element here. Right? But this is in row echelon form; it is not in row reduced echelon form, it is simply row echelon form.

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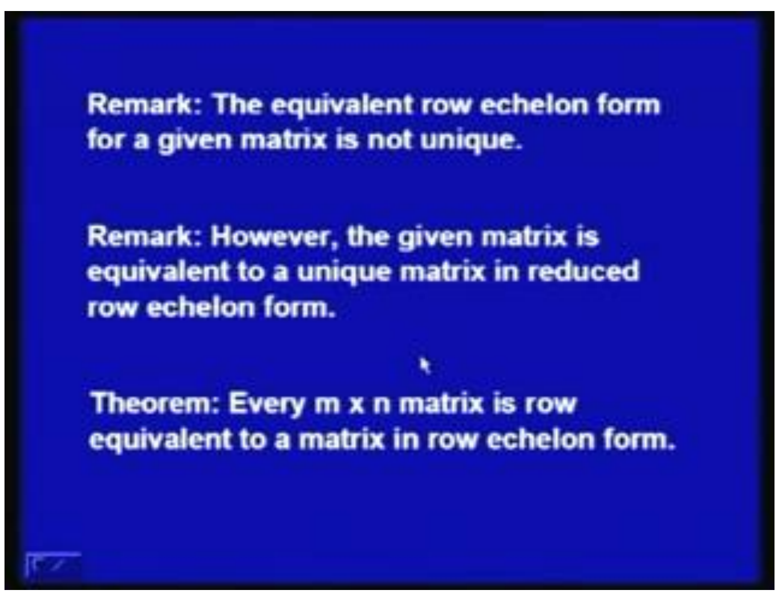

Now the equivalent row echelon form for a given matrix is not unique. This is the remark, because I have applied some set of transformation. You can get to a row reduced echelon form which will have different coefficients, because you have applied different operations in different order. So this form is not unique; but if you want to transform into a reduced row echelon form, then it will be unique. So that is the basic difference between row reduce a row echelon form and row reduced echelon form. Now we have a result which says that every m by n matrix is row equivalent to a matrix in row echelon form. You can apply a transformation. So that will be reduced to in row echelon form.

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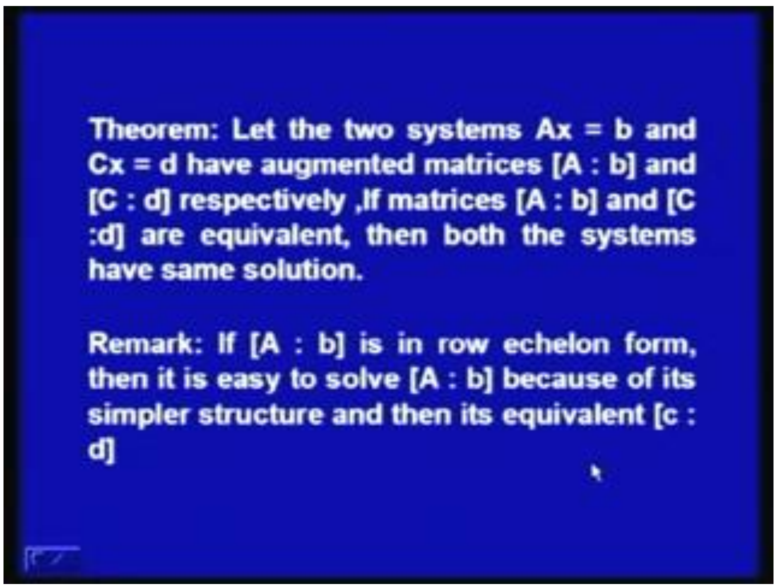

Another theorem says that the 2 systems Ax is equal to b and Cx is equal to d have augmented matrices A b and C and d respectively; if matrices A b and C d are equivalent then both the systems have same solutions. Now this is an important result because you can apply different type of transformation; you may get 2 different types of matrices - equivalent matrices - but both of them will have the same solution. So if A b is in row echelon form, then it is easy to solve A b because of its simpler structure and then it is equivalent to c d. So this is system is simpler and that is why it is better to reduce the system of equation into row echelon form. So this is the basic concept behind the elimination method we will be which we will be using.

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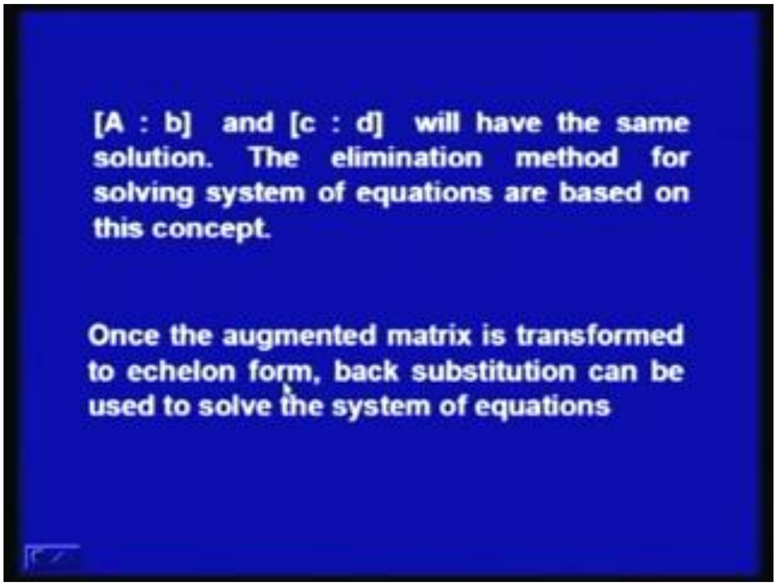

So what we do is we consider the augmented matrix and transform into echelon form.

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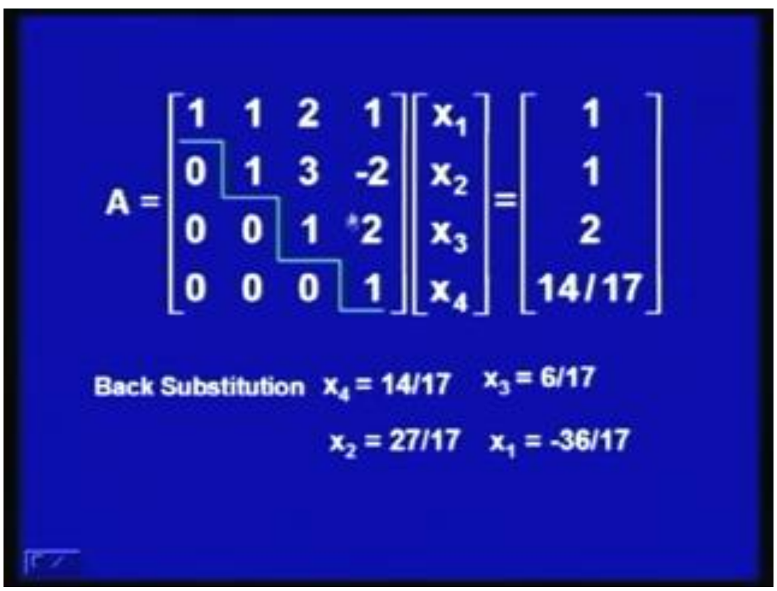

Once it is transformed into echelon form, then we use back substitution and get the solution of the equation; like in this case, we have been given a system of equations and these equations are: this matrix is converted into this form and from here we can apply back substitution, like this in echelon form. We apply back substitution: if you consider the last equation then x 1- 0, x 2 is not contributing, x 3 is not contributing, x 4 is equal to fourteen by seventeen. So once you get x 4 is equal to 14 by 17, now you apply the same theory to this. That is x 1, this will not contribute, it is x 3 plus 2 -x 4 - is equal to 2, from here we already obtain x 4; so we solve it for x 3 and x 3 comes out to be 6 by 17. So once you solve for x 4 and x 3, now we form this third equation. So x 1 is not there but x 3 and x 4 - we have already computed. So we can straight away get the value for x 2. Then finally we use the last equation: this is 1 x 1, 2 x 2 - x 2 is already known, 2 x 3 - x 3 is already known, x 4 is already known - is equal to 1. So we substitute in this equation and we will finally get the value of x 1. So this is a solution for the given system of equations.

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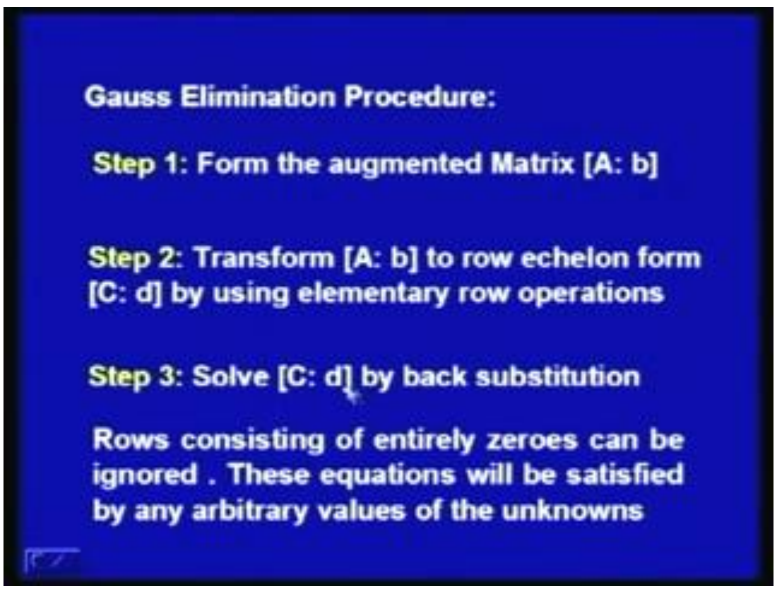

So The Gauss elimination procedure: this procedure was given by Gauss. This uses this concept. So what the first step is form the augmented matrix from the given system of equations; second step is transform A is b augmented matrix to row echelon form C d by using elementary row transformation and here we have actually used that theorem that after transforming this A b to C d in a simpler form the solution of the 2 systems remains the same. So this transformation will not affect the solution of this system but the system will be simpler. Once we get the echelon form - row echelon form - then we use the back substitution to get the final solution. Now it may happen when you apply this transformation, that some of the rows become 0 because this is the property of echelon form - that sum of the rows may be consisting of entirely zeros; then those rows can be ignored. And in fact if there are some unknowns involved in it, then they may take any arbitrary value and then equation will still be satisfied or those equation will still be satisfied for arbitrary values of unknowns and the unknowns will be obtained from a nonzero elements present in the system of equations.

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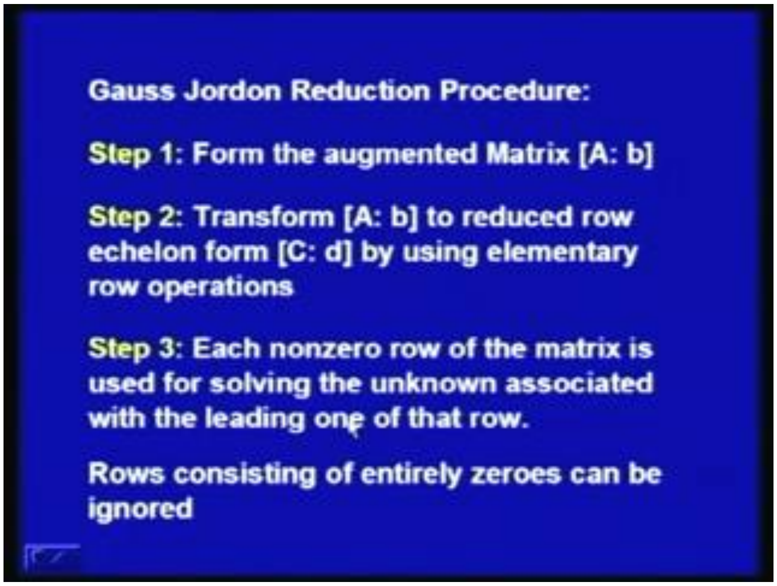

Now Gauss Jordon reduction procedure is likely a the slightly as the slight variation from this in the sense that first step remains the same that we have to form the first the augmented matrix, the step 2 is transform A b to reduced row echelon form by using elementary row transformation - that is the basic difference between the methods: Gauss Jordon reduction procedure and the method which we have discussed just now. That method transforms the equation into row echelon form but not reduced row echelon form. So the idea is this will be this matrix C d is unique in this case. In the previous case it may vary depending upon the type operations you tried up to apply. Then once we obtain the reduce row echelon form, then this can be used for solving system of equations. Let us see how we apply this.

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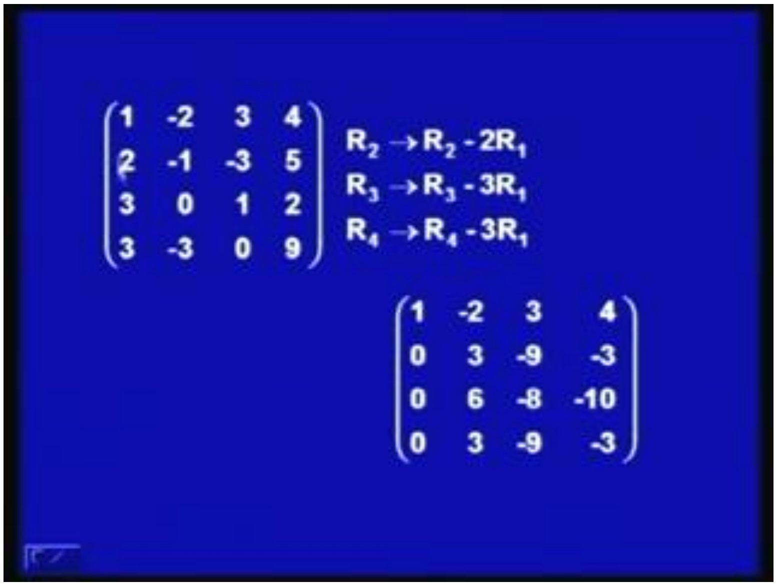

We apply this for the system of equa. So the first step is to write the augmented matrix. Since the equations are 4 in number in 3 unknowns x y and z, so we will be getting a 4 by 4 matrix. This is the coefficient matrix and this is the right hand side. The second step is to reduce this into echelon form. So to start with, I will have to perform series of row operations. First operation is: this is already 1, so these of the elements have to be 0, so I apply the sequence of operations R 2 is R 2 minus 2 R 1; so that this become zero.

R 3 is R 3 minus 3 R one; so that this become 0. R 4 is R 4 minus 3 R 1; so that this becomes 0. So if I apply these operations, then I will get to this system.

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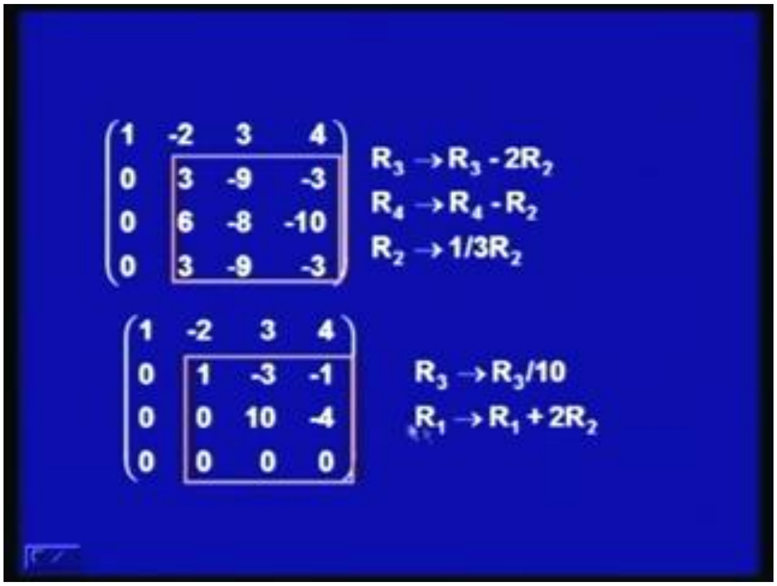

After applying these operations, one may notice that only this part is to be made is this part is to be reduced. So to reduce this part, I will first write R 3 is R 3 minus 2 R 2, so that this becomes 0; R 4 is R 4 minus R 2, so that this becomes 0 and then I say R 2 is 1 by 3 of R 2 that we make this as 1. So by applying these operations, I will come back to I will go to this form. In this particular form, only this mats sub matrix is to be reduced. So I say R 3 is R 3 by 10; so this becomes 1 and this will be 0. This will be minus 4 by 10. Now you may notice that this is 1, but his is not 0. This also has to be 0, this also has to be 0. So to apply this, I will say R 1 is R 1 plus 2 R 2 so that this element will become 0.

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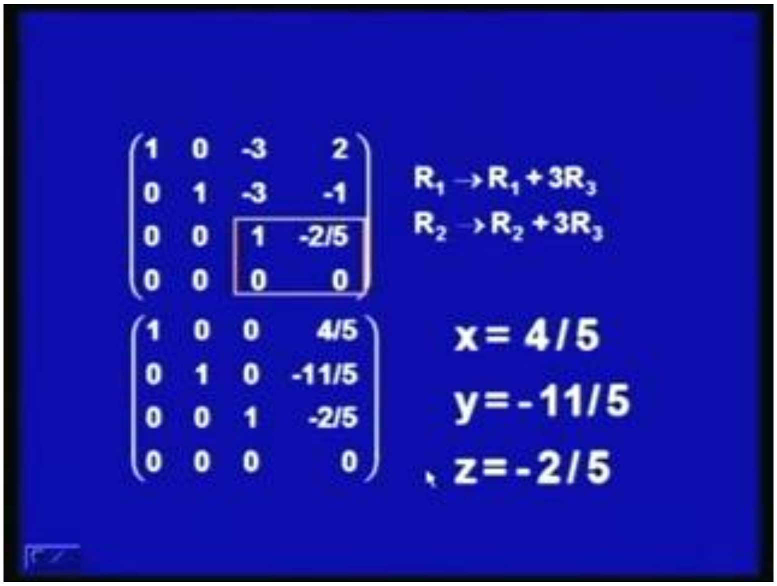

So the result is this becomes zero and we will have this sub matrix. Now in this sub matrix, I apply R 1 is R 1 plus 3 R 3 so that this will become 0 and R 2 is R 2 plus 3 R 3 so that this becomes 0. So After applying these 2 operations, I will be having this echelon form in this; this is the coefficient matrix, this is the right hand side. We will have 1, the leading element, here in the third row; all these elements - 0 here; this is the second row and this is the leading 1 in the third row. Now this finding the solution is straight forward in this case. This element corresponds to z; so z is equal to minus 2 by 5. This leading 1 corresponds to y; so y is equal to minus 11 by 5. This leading 1 corresponds to x; so x is equal to 4 by 5. So Once the equation is reduced into echelon form, then finding the solution is simpler. So, that is the basic difference between the first method and the second method which we have discussed.

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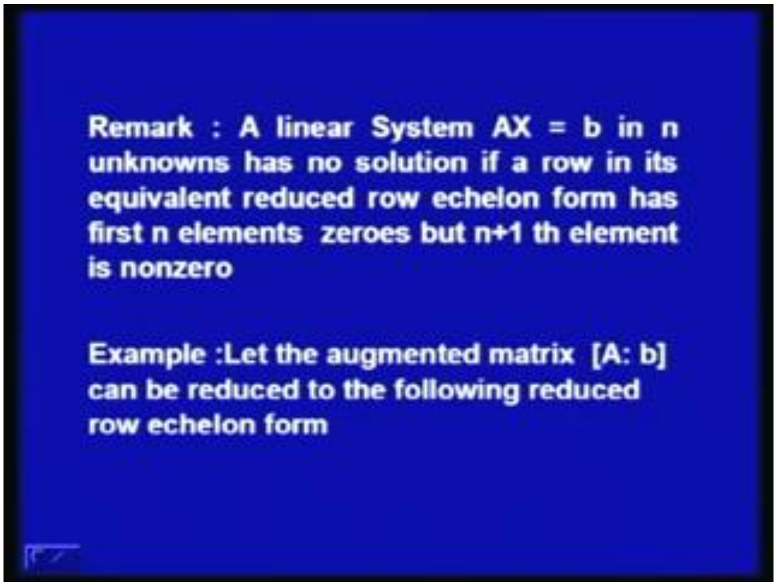

Now number of remarks here: a linear system AX is equal to b in n unknowns no solution if a row in its equivalent reduced row echelon form has first n elements zeros but n plus 1 th element is nonzero. So if this is a situation that all the elements in a particular row are 0 except the last one, then the system will not have any solution. Example is let the augmented matrix A b can be reduced to the following reduced row echelon form as this.

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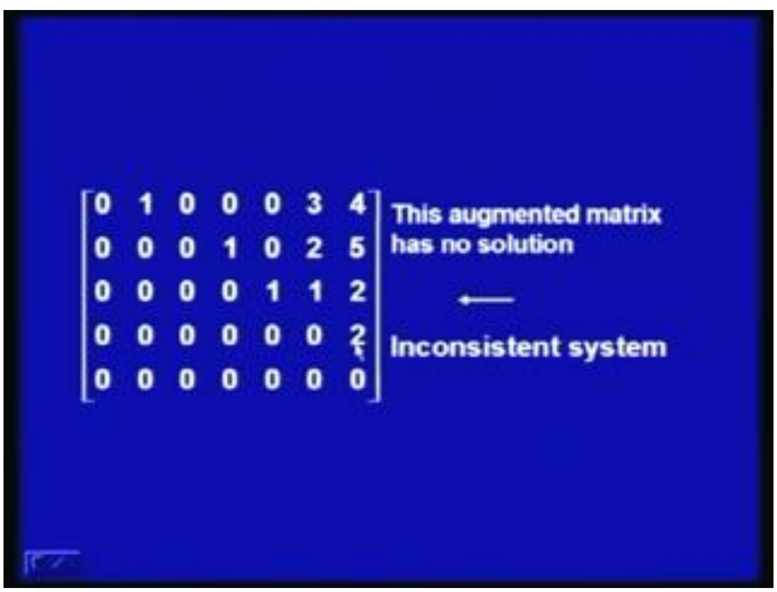

Now you may notice that this particular this particular row has all zeros except the last column as 2. Now this is an inconsistent system because whatever be the values of x 1 x 2 xn you can see there, that will not be equal to two. So this system will not have any solution; so this augmented matrix has no solution and we say such a system is inconsistent system.

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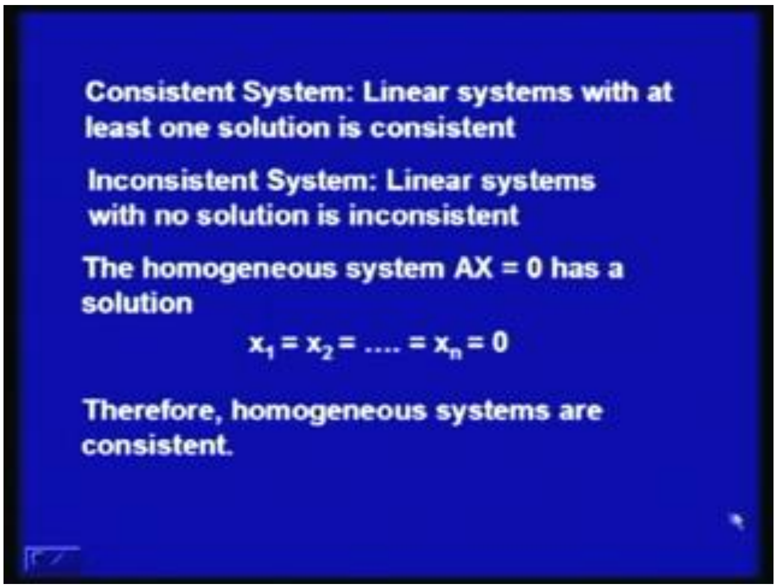

So what is the consistent system? The consistent system is one which the linear system having at least one solution; so a linear system with the at least 1 solution is consistent. It may have more than one solution but one solution has to be there; and inconsistent system is a linear system with no solution. Now If you have a homogeneous system, where the right hand side is 0, then it will always has a solution because x is equal to 0 is a solution of this equation. x is equal to 0 means all its components are 0. So they whatever be the matrix A, AX will be 0 in this case. So this system will always have a solution and that solution is a 0 solution. It may have other solutions but it has a 0 solution and this system will always be consistent. So x 1 x 2 xn is equal to 0 will be a solution for this. So there Therefore we say that homogeneous systems are consistent.

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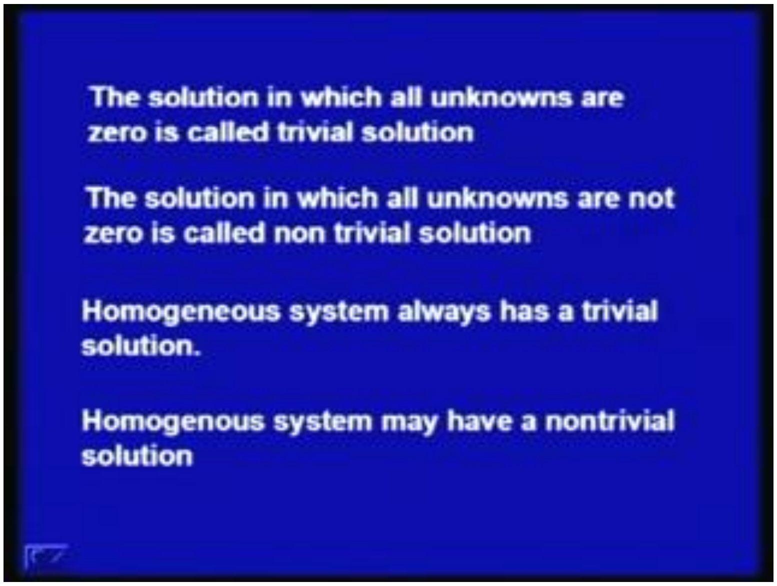

The solution in which all unknowns are 0 is called trivial solution. So homogeneous system will always have a trivial solution; that is why it is consistent. And the solution in which all unknowns are not 0 - some of them as some of them may be 0 - but at least there may be 1, which is non-zero, then that solution is called a non trivial solution. So a homogeneous system always has a trivial solution; it may have a non trivial solution or may not have a non trivial solution.

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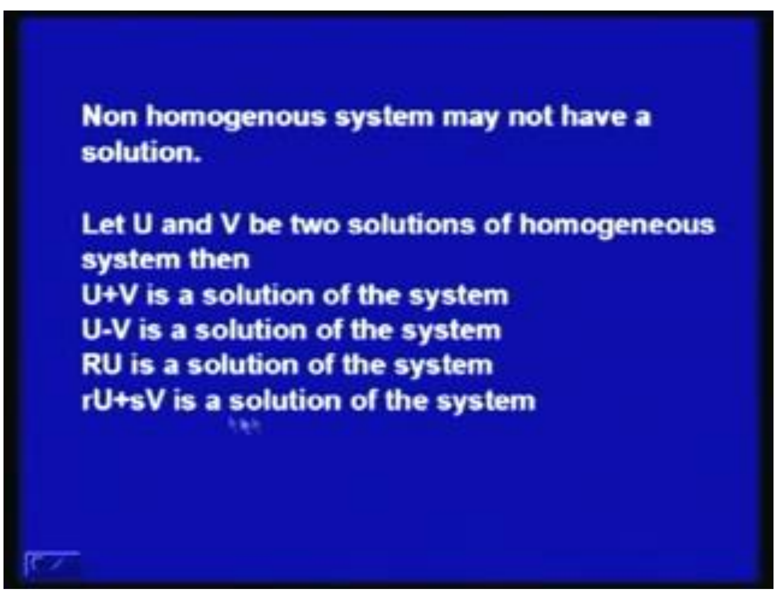

However if we have a non homogeneous system, then it may not have a solution; it may be inconsistent. Now we have some results which say that if let U and V be two solutions of homogeneous system then U plus V is a solution of the system. Then if U and V are solutions of the system then U minus V is also solution of the homogeneous system. Similarly if U is a solution of system, then up it up constant R times U is also solution of the system. Not only this, but there is the linear combination of U and V is also solution of the system. That is, r times U plus s times V is a solution of the homogeneous system. These properties are not true for non homogeneous system but these are the properties which are true for homogeneous system. Now trivial solution is not the solution of non homogeneous system. This is obvious because the non homogeneous system will have at least 1 equation having a non 0 entry on the right hand side; so that particular equation will not be satisfied with the trivial solution.

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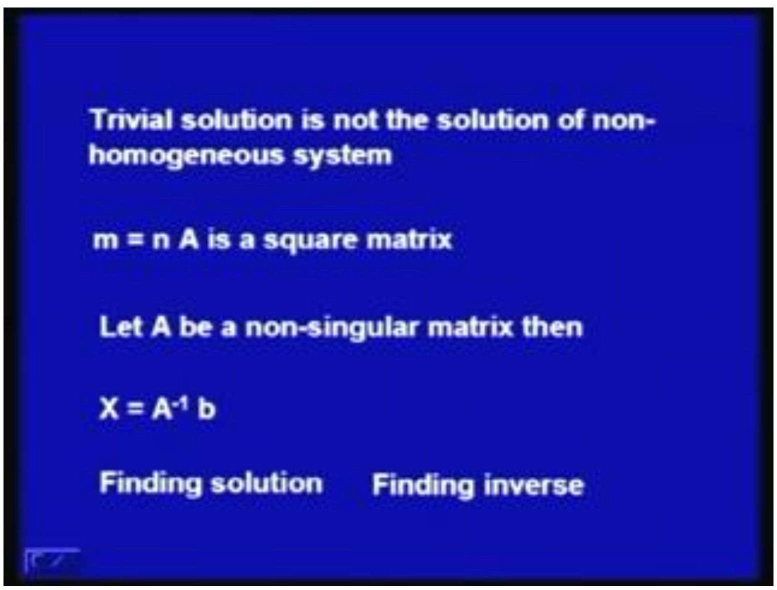

Now for a non homogeneous system we have row solution also. So now will consider at special case where the number of unknowns and the number of equations are equal. So In that case, A the coefficient matrix will become a square matrix. Now, I am considering another case when we have a square matrix as the coefficient matrix and this matrix is non singular. Then it means A inverse exist and that means the equations Ax is equal to b can be written as x is equal to A inverse b. That means if you can find out the inverse of the given matrix A, then the solution matrix x can be obtained as A inverse b. So A inverse is to be multiplied by the right hand side and we will get the solution. That means that finding solution of given system means finding inverse of the given matrix; but the inverse may not exist, as the matrix may be singular also.

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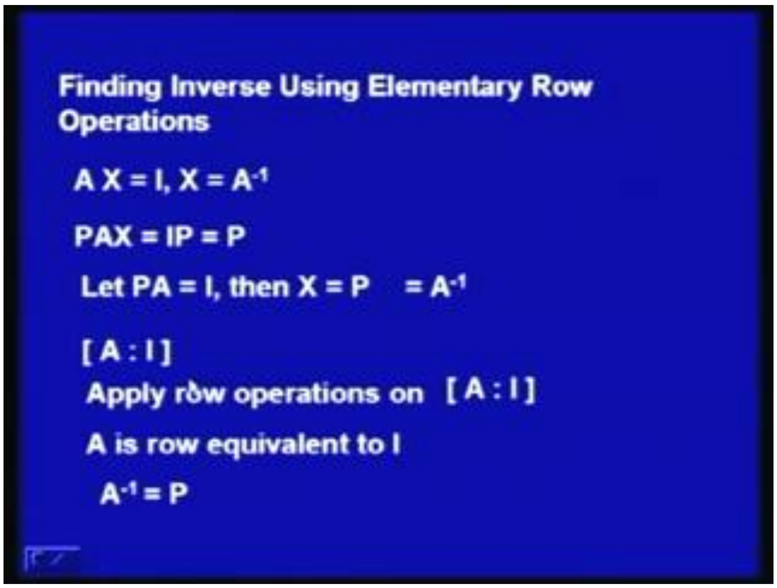

So Let us discuss a method for finding inverse of given square matrix if it exists. We have already discuss one method of finding inverse, but today, here, we are discussing a method which uses elementary row operations to obtain the inverse of a matrix. So let us say we have to find the solution of AX is equal to identity matrix or we have to find X is equal to A inverse. What we can do is we can apply row transformations. That means we have to find out a matrix P, that is, we apply the successive row transformation to the system AX; then PX is equal to I time P or simply P. Now if we apply row operations in such a manner that PA is equal to I, then X becomes P or it becomes A inverse. So what we do is we apply row transformation on the augmented matrix A colon I. So instead of having a colon matrix b in the case of system of equations, we are having now an identify matrix of appropriate order; or we apply row transformations row operations on A and we try to reduce this A into identity matrix, and the same operations are applied on I and we will see that by the time A is transformed into I, I will be transformed into A inverse. So A inverse becomes P. Now Let us illustrate with example.

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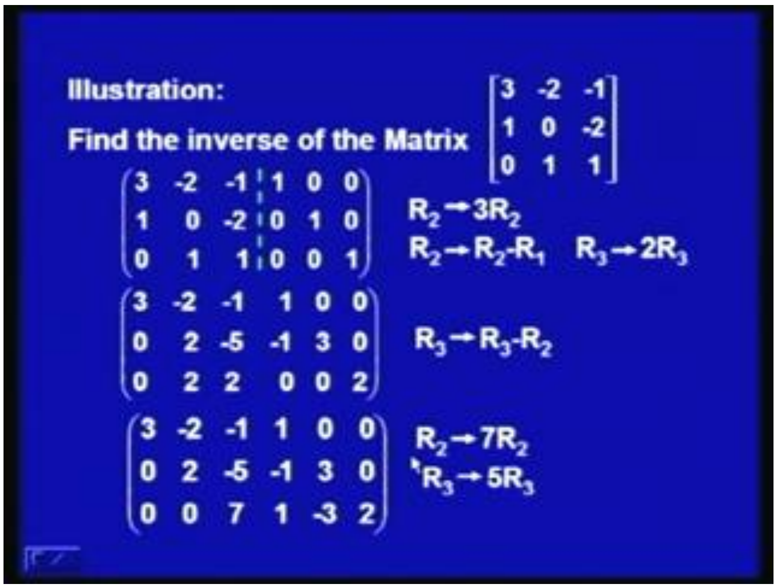

So let us find out the inverse of the matrix a 3 by 3 matrix as this. Then we consider the augmented matrix, as this matrix on the right hand side will be an identity matrix of order 3. What we have to do is we have to transform this matrix into identity matrix and by the time it is identity matrix, this will become the inverse of this matrix. Now to convert this into identity matrix, first thing is that this should be 1 and this should be 0. So to perform this, we will write down R 2 as 3 R 2 and then we subtract R 2 from R 1 and then R 3 is 2 times R 3; so the these transformations give us this matrix. So This is now subtracted R 3 is R 2 R 3 minus R 2; so even this will become 0 and then this has to be made 0 and this has to be made 0.

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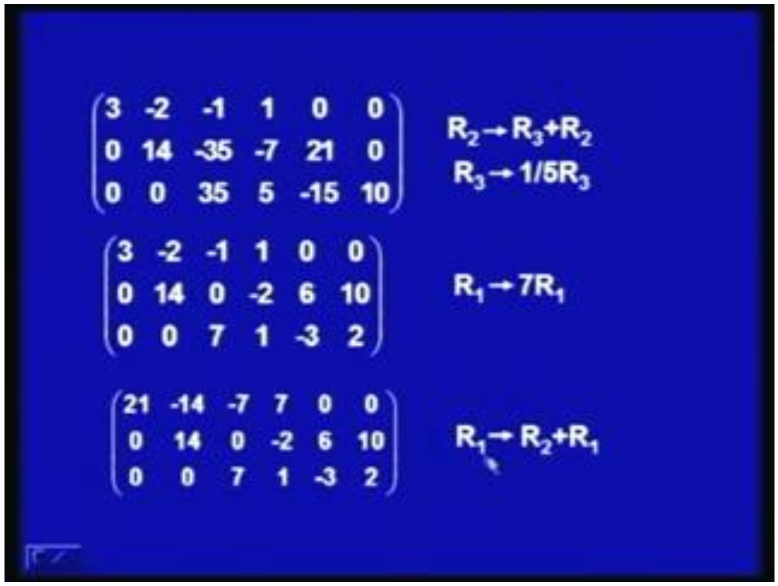

So R 2 is 7 times R 2 and R 3 is 5 times R 3 and that will make the 2 coefficients equal. So add the 2, this will become 0 and to make it - R 3 is again 1 by 5 R 3, so that becomes 7 and we apply another transformation: R 1 is 7 times R 1 so that this is becomes 0. In the next one, i am we add R 2 and R 1 so this will become zero.

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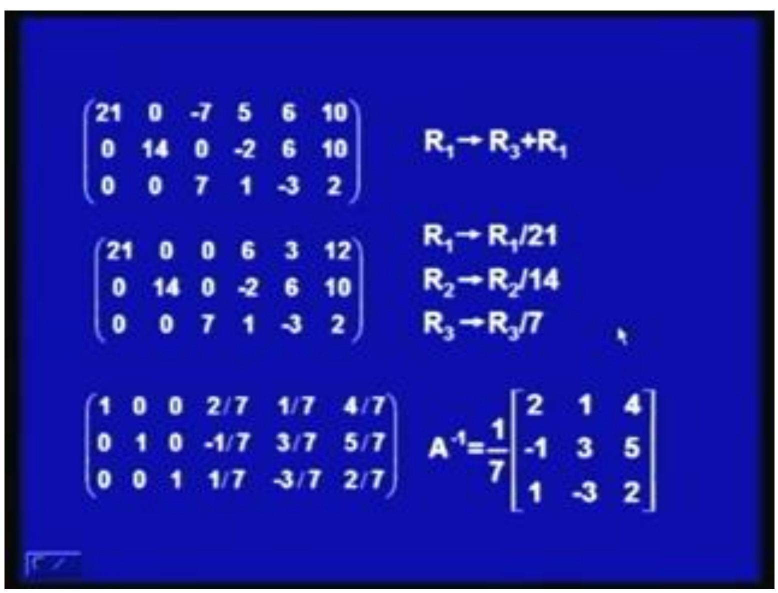

And finally R 1 and R 3 plus R 1- so this term will also become 0. Then these have to be make 1 these diagonals have to be 1. So R 1 is R 1 is by 21; R 2 is R 2 by 14; R 3 is the R 3 by 7 and what we have is an identity matrix here. So we say this is nothing but the inverse of the given matrix. So A inverse is 1 by 7 - 1 by 7 i have taken out. So it is 2 1 4 minus 1 3 5 1 minus 3 2 as inverse of given matrix. That is how we use those transformations to obtain the inverse of a given matrix.

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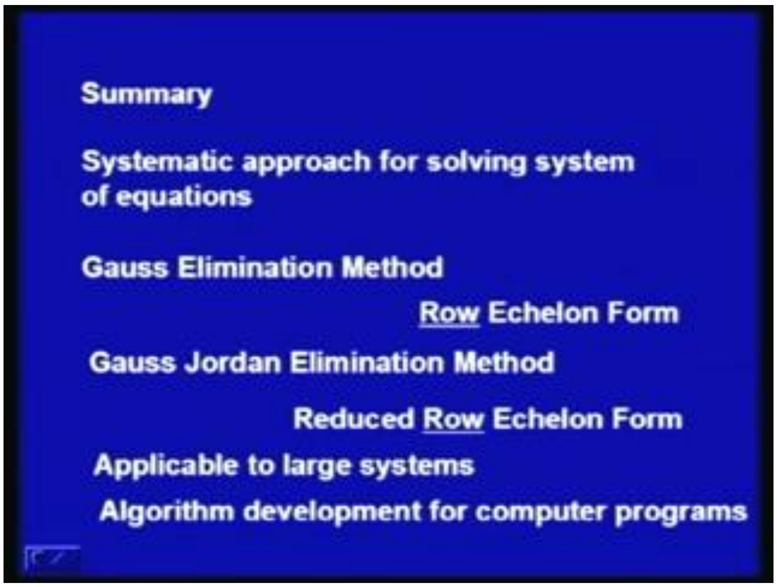

So Today we have we have seen how to solve system of equations. We have obtained systematic approach for solving system of equations. We have obtained 2 methods: one is the Gauss elimination method in which we transform our augmented matrix into a row echelon form and the other method is Gauss Jordan elimination method in which we reduce the augmented matrix into reduced row echelon form; basically we have to apply a row operations to transform the augmented matrix. This is important because only row operations have to be applied; this gives us a method for solving large system of equations. It is the systematic approach and these these methods can be used for a algorithm development so that the system of equations can be solved with the help of computers. Thank you, viewers.