Mathematics-II
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Module - 2
Lecture - 4
Determinants Part -2

Good morning, viewers. I will be continuing with the determinants.

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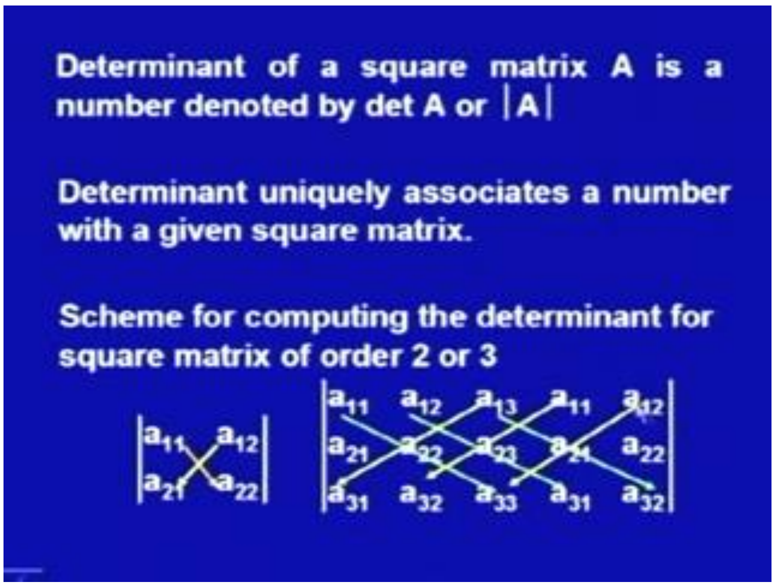

As a recap to my last lecture, we define the determinants first. Determinant of a square matrix A is a number denoted by determinant A. Actually we associate a number with a given matrix, especially a square matrix; then, we say that determinant uniquely associates a number with a square matrix. We have discussed some schemes for computing determinants of square matrices of order or three. Like, if I have a second order matrix then its determinant is computed as a 1 1 a 2 2 minus a 1 2 a 2 1, while for a third order, determinant is calculated as product of a 1 1 a 2 2 a 3 3 a 1 2 a 2 a 3 3 a 3 1 a 1 3 a 2 1 a 3 2 - these are summed up - and then we calculate the sum of terms a 1 3, a 3

2, a 3 1, a 1 1, a 2 3, a 3 2, a 1 2, a 2 1, a 3 3 and take the difference; but this type of definition cannot be continued for high order determinant.

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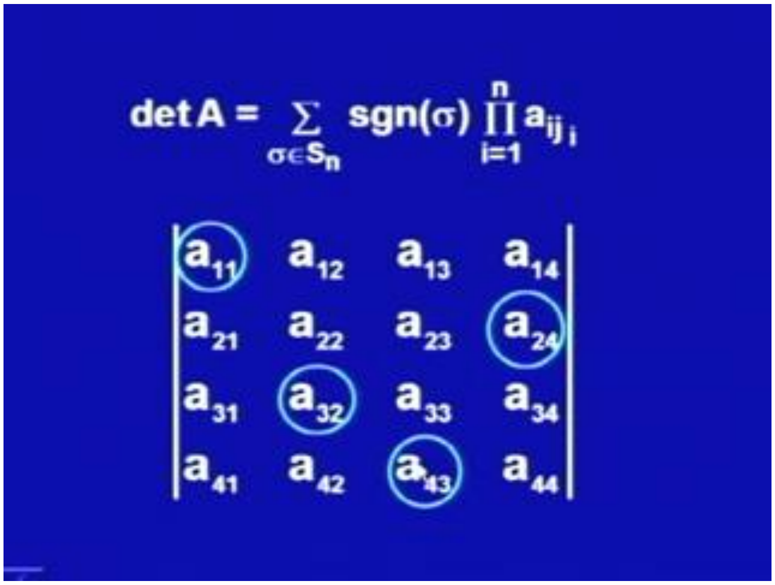

So we define determinant for n th order square matrix as determinant A is equal to signum of the permutation sigma and the product of i is equal to 1 2 n of aij i. Now this is explained with a 4 by 4 matrix and we say that we calculate the product of terms like a 1 1 from the first row first column, a 2 4 from the second row fourth column, a 3 2 from the third row second column, a 4 3 from the fourth row third column. The idea is we will take only one term from each row and column. This way - factorial n permutations are possible. We calculate this product of all such terms and then take the sum.

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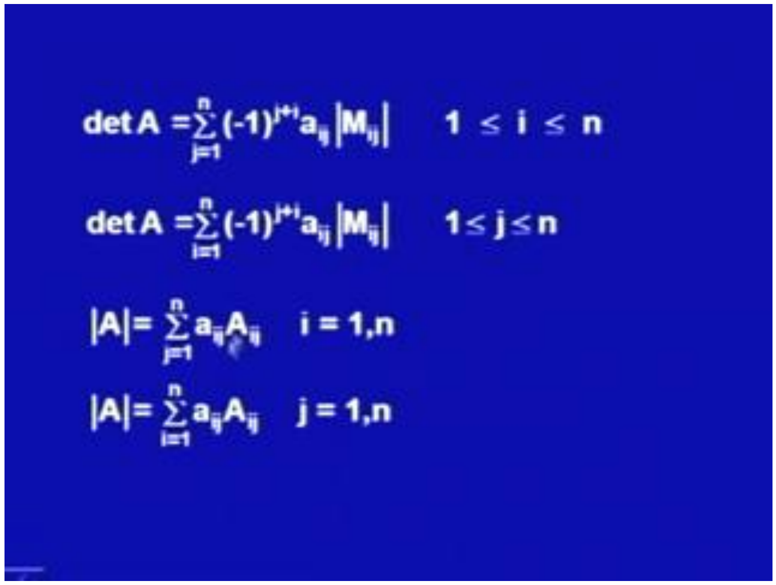

But then we define determinant A as summation of minus 1 raise to power j plus i aij minor Mij. Mij this actually denotes the minor – this, we have defined in the last one - and this definition can be applied to any row or any column. This way, we define a determinant in terms of lower order matrix determinants. This is true for i th row, this the definition can be applied to any row while this definition can be applied to any column; the result is determinant A. Now we have further defined cofactors aij for the idea is here we are having minus 1 j raised to power j plus i Mij. So we observe this term with this minor and we define Aij is the cofactor of aij th element of the matrix A. So we define determinant A in this order - in this in this particular manner - where i varies from 1 to n and if we consider column wise expansion, then we can take take this definition and j varies from 1 to n.

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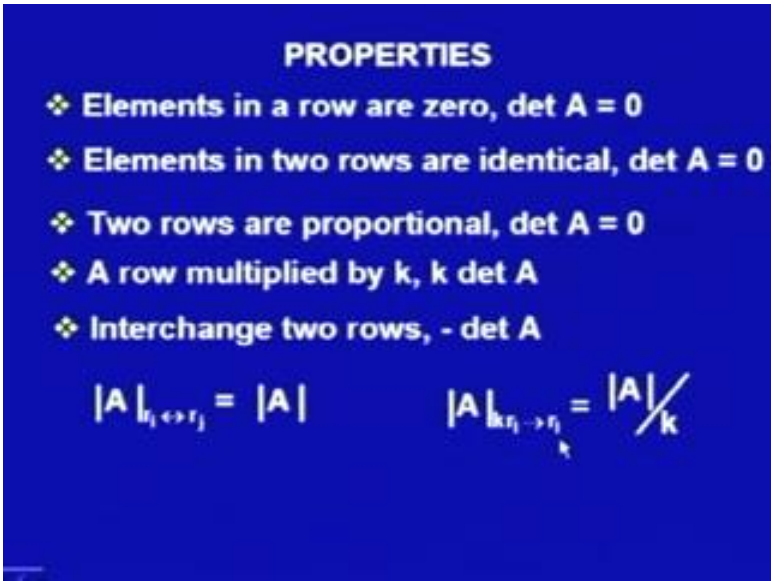

After that, we have discussed some properties and we have proved these properties these properties actually help us in evaluating determinants more conveniently like if the elements of a row are 0 then determinant A is 0 ; we need not compute all the terms which we have discussed. We simply say determinant A is 0, if any row or column becomes 0. Similarly, if the 2 rows are identical then determinant A is 0; you don't have to calculate all the minus. If 2 rows are proportional, then determinant A is 0 and if 2 rows are multiplied by k, then if a row is multiplied by k, then k times determinant A will be the value of new determinant. If interchange of two rows occurs, then determinant is multiplied by minus 1. So determinant A becomes minus determinant A after interchanging of 2 rows.

Similarly, we had applied elementary transformations to corresponding matrices and then we saw what the relationship is, between the determinant of the matrix, before applying the operation and after applying the operation. So before applying the operation it is determinant A but when we change it the determinant remains the same. But if we apply this elementary transformation, then determinant obtained after applying this transformation is divided by k. But this elementary transformation will not change the value of the determinant. Now these transformations actually help us in computing - in evaluating - the determinants more easily and with less computation, computational effort.

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Computational efforts

Cofactor expansion $N(n) \sim e\, n!$

n	Computational effort
5	0.0003 sec
10	10 sec
15	4×10^{6} sec $\approx$ x 40 days
20	7×10^{6} sec $\approx$ x 210,000 years
25	4×10^{19} sec $\approx$ x 10^{12} years

Now let us try to have some idea about the computation effort involved in the matrices because matrices are used extensively in engineering and scientific applications. So let us have some idea about the computational effort involved in it. Cofactor If we use the cofactor expansion to evaluate the determinant, then the computational effort is of the order of factorial n. This factorial n is coming because we have seen that for determinant they are factorial n product type of products possible; they have to be multiplied, they have to be added and subtracted.

So the computational effort the computational effort employed will be of order of factorial n. Now you will be having a better idea if you try to see it on a system, when n is equal to 1 2 and so on. So if n is equal to 5 and we are evaluating a determinant on a machine which can compute 10 raise to power 3 operations per second, then for n is equal to 5, the computational time will be of order point 0 0 0 3 seconds. If we increase the matrix size to be 10 by 10, then the computational effort required by the machine will be 10 seconds and if you further increase the size of the matrix to 15, then the determinant will require 4 into 10 raised to 6 seconds of computer time and this amounts to be forty days. If you further increase the matrix size to twenty, then huge effort is required and the computer time will be 7 into 10 per 6seconds which comes out to be these many years and if you further increase it, say to 25, then we require approximately 10 raised to power 12 years to compute a matrix - to compute the determinant of a matrix of size 25. Now engineering applications require large matrices to be handled and we have to require we have to we have to be compute determinants of higher orders. So if this is the scenario, then it will be difficult to use determinants in practice. So we will not be using cofactor expansion; rather, than we will be using other method which is called the triangularization method for computing determinants.

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Let us first compare let us see how the computational effort in this case will be. So if we In this particular case the computational effort is of order 2 n cube divided by 3; so this is the order of the computational effort required in computing determinants using this method. So this is the table; for I am giving the values for 20 because if we are using triangularization method, then for n is equal to twenty, the computational effort will reduce to point 005 seconds. If you compare it with the earlier case, then this is a very small and then we increase n to be 25; then 0 point 0 1 second will be required to compute the determinant by this particular method. So although the method is the same definition - cofactor definition remains the same - but triangularization method reduces the computational effort tremendously. For 4 or 5 order matrices, the effort may not be very may not be compared or if we are doing hand computation, then maybe we find that triangularization method is not so is is messy. But for higher order matrices, things are really more comfortable when we use triangularization method. Now after this, we will be discussing some more properties of determinants; they will be helpful in applying determinants and matrices to engineering problems.

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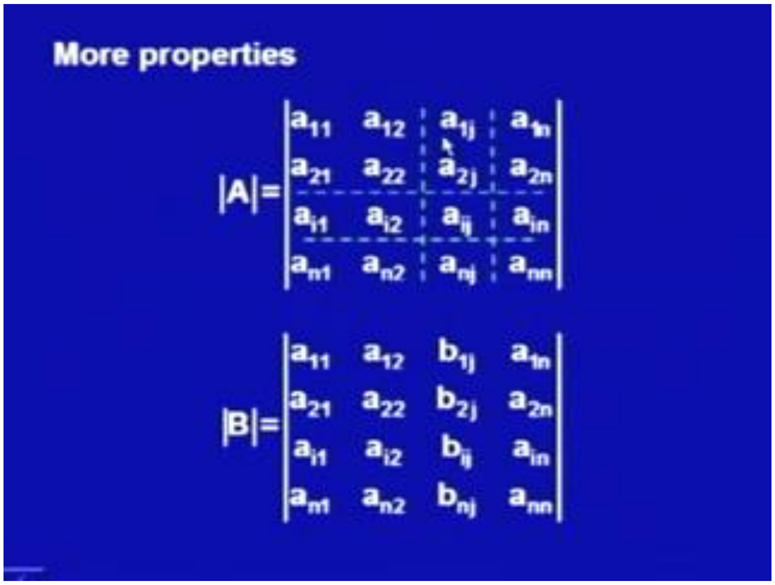

Now for this, let us consider a matrix n by n square matrix consisting of a 1 1 a 1 2 a 1 j a 1 n elements in the first row, a 2 1 a 2 two a 2 j a 2 n in the second row; this is the i th row the matrix while this is the last row. This indicates that there maybe number of rows in between and this also indicates there maybe number of columns in between here or there. Now the matrix - the determinant B - is the determinant of this matrix; this matrix is the same as this matrix. Only difference is in the j th column. So a 1 1 in the B matrix is the same as here; so all rows and columns are same except this particular column.

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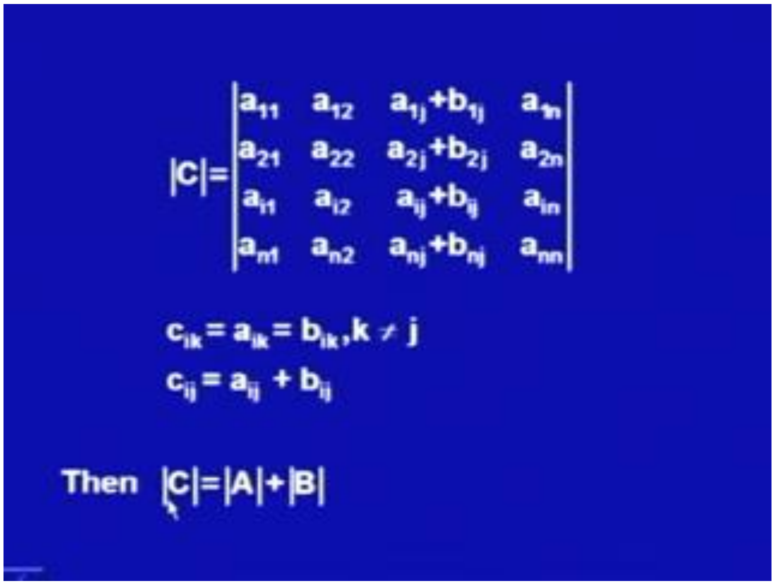

So the determinant C is the same as determinant A , but the difference is in the j th row. The j th row is having a 1 j and b 1 j - the sum of the two; in fact, the j th column of a and j th column of B are added to give the j th column of C matrix. So if we define C matrix in this particular manner, then cik is equal to aik is equal to bik for all k not equal to j and cij is equal to aij plus bij for j th column. Then we have a result which says that the determinant of C is equal to determinant of A plus determinant of B. Let us prove this result.

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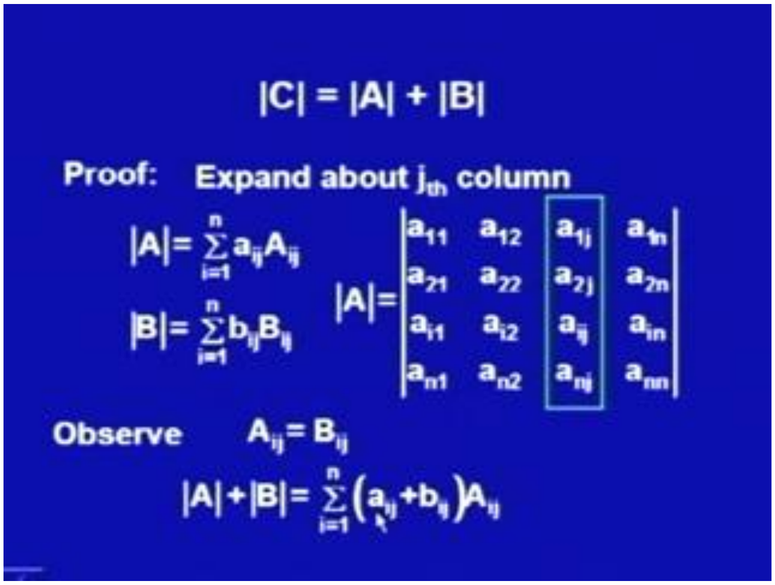

We can expand the matrix about j th column to get the determinant A. So this way, we write down determinant A as summation - i is equal 1 to n - aij multiplied by Aij -Aij being the cofactor of aji. That is, this is the matrix A; so we have obtained determinant A about this column; same thing can be done for B. Now one can observe that Aij is equal to Bij. Why? because Aij, the cofactor of aij th element, is obtained by deleting this column and this row; Bij is the cofactor in the B matrix by eliminating this column and this row. But this is the only difference in A and B. So this cofactor Aij will be equal to Bij and this means A plus B determinant A plus determinant B is equal to aij plus bij - this is taken and Aij is common.

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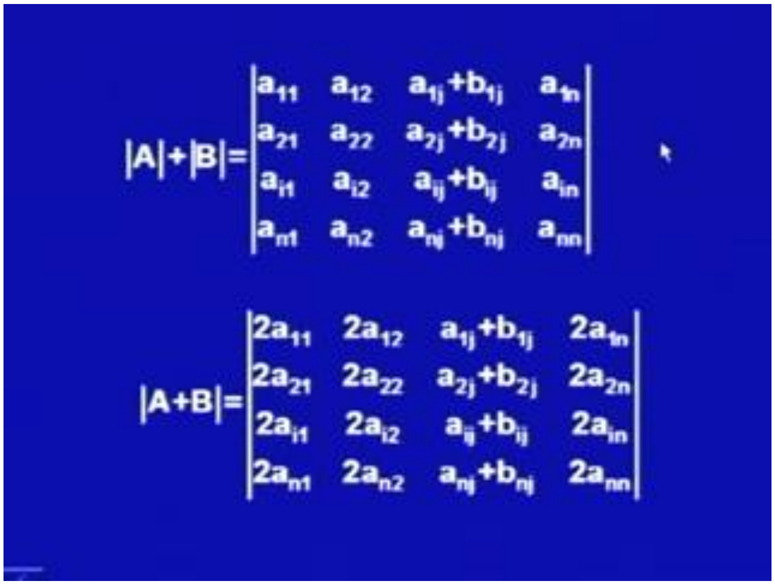

So this is equal to aij plus bij multiplied by Aij and this simply means that determinant A plus determinant B is equal to this determinant where the j th element is the sum of the two. Now this is nothing but what we have, that is the C matrix. This simply means the determinant of A plus determinant of B is equal to determinant of C. Now it gives me a feeling the determinant of A plus determinant of B will be equal to determinant of A plus B but actually it is not, because if I add two determinants A and B, the result will be 2 times a 1 1 2 times a 1 2 but this will be a 1 j plus b 1 j and this will also be 2 a 1 n. So that means A plus B is not equal to determinant of A plus B; that means this is the this what do we mean by this the simply means that we first add the matrices and then take the determinant. So we first add the matrices; so this is the resultant matrix and taking the determinant, we will be getting this. Since the elements are different, then this that means A plus B is not equal to determinant of A plus B.

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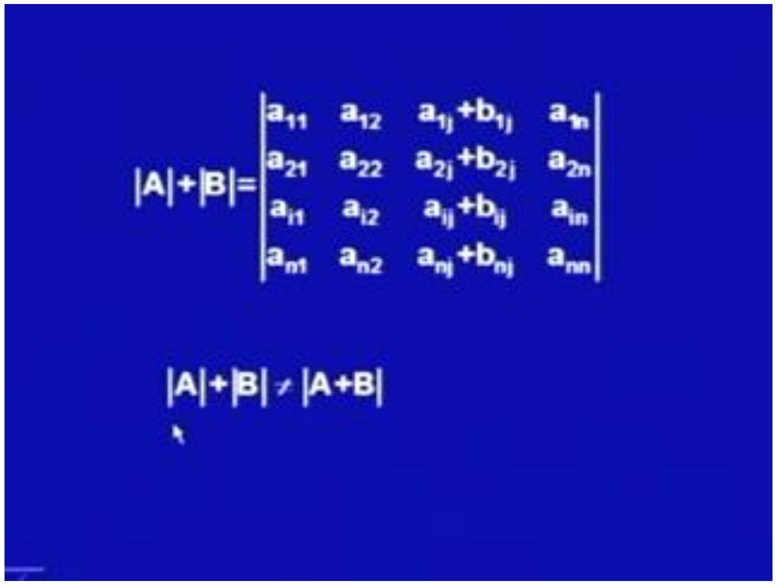

So this is what we have: determinant of A plus determinant of B is not equal to determinant of A plus B. Or this I can explain I can verify this with an example also.

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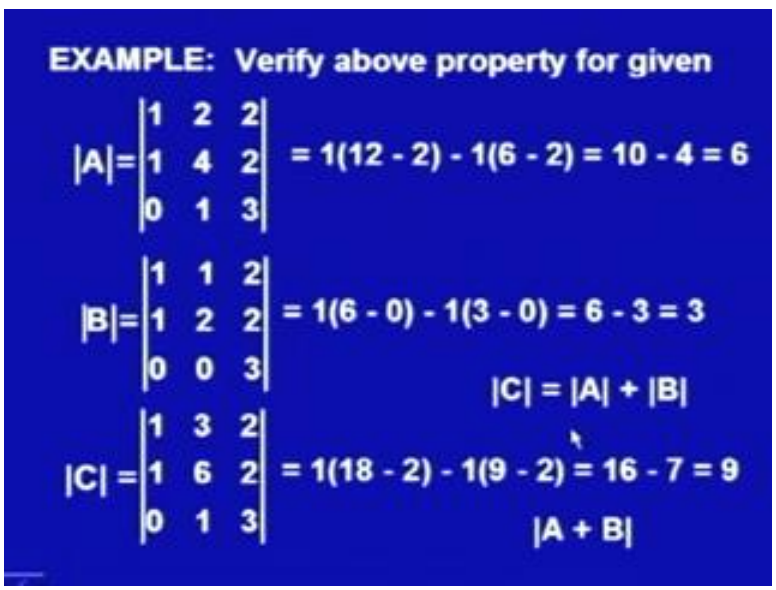

So, if I have a determinant A as 1 2 2 1 4 2 0 1 3 and determinant B as 1 1 0 1 2 0 2 2 3, then the determinant C will be: first column is copied as such, third column is copied as such but the second column is 2 plus 1 3, 4 plus 2 6,1 plus 0 1; so determinant C is this. So let us calculate the determinants. The determinants determinant A is first column first element - first 1 into 12 minus 2 – minus - I am expanding about this – 6 minus 2 and I actually I can expand about any row or any column; but since this particular matrix has 1 element 0 in the first column, so I will prefer to compute determinant by expanding about the first column. The idea is this way, I will be performing less computation as compared to this column this row or any other column. So, I will be computing only these 2 terms, not the third term; so this gives me the result as 6. As far as the determinant of B is concerned, I will use the determinant same column by evaluating determinant and it comes out to be 1 into 6 minus 0 minus 1 and it is 3 minus 0; so the final result is 3. Similarly the third determinant: 1 into 18 minus 2 minus 1 into 9 minus 2 and that gives me 9. and One can check that determinant C which is 9 is equal to determinant A plus determinant B which is 6 plus 3. So we have verified the result, but if you compute A plus B determinant, then it will not come out to be same as A plus B. Let us see this.

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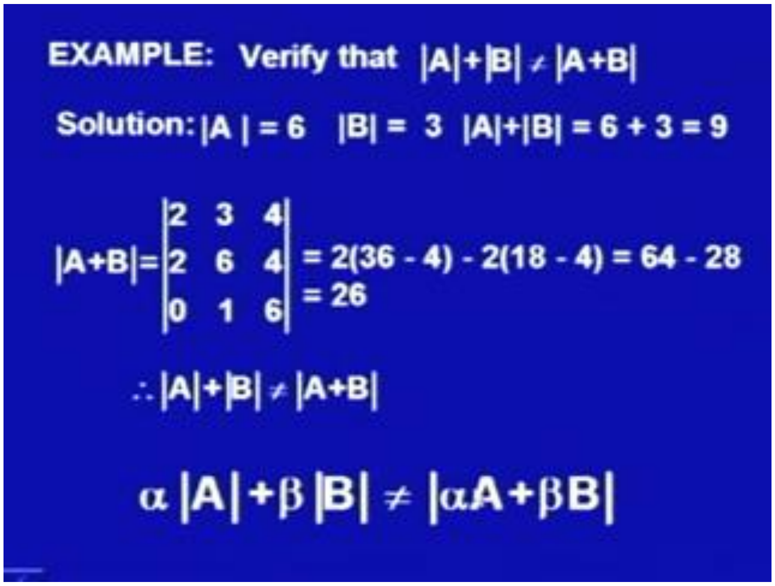

We have already computed A is equal to 6; we have computed determinant B as 3; we have computed A plus B as 9. But if you have to compute A plus B determinant, then this comes out to be 2 into 36 minus 4 minus 2 into 18 minus 4 - that gives me sixty 4 minus twenty 8 and finally A plus B is 26. Clearly A plus B determinant is not the same as A determinant plus B determinant and that proves. So in general, this result is not true. In fact, if we have to add this and if the some rows or some columns are identical except one particular row, then we have the result given by earlier theorem. If you can generalize this, then alpha A plus beta B is not equal to alpha A plus beta B; actually it is very surprising that this result doesn't hold for addition. So this is a linear property; this must hold for determinants, but we have verified that this is not true for determinants. This is very surprising and one has to be very careful about it that alpha A plus beta B is not equal to alpha A plus beta B. So that is the basic difference between determinants and the numbers.

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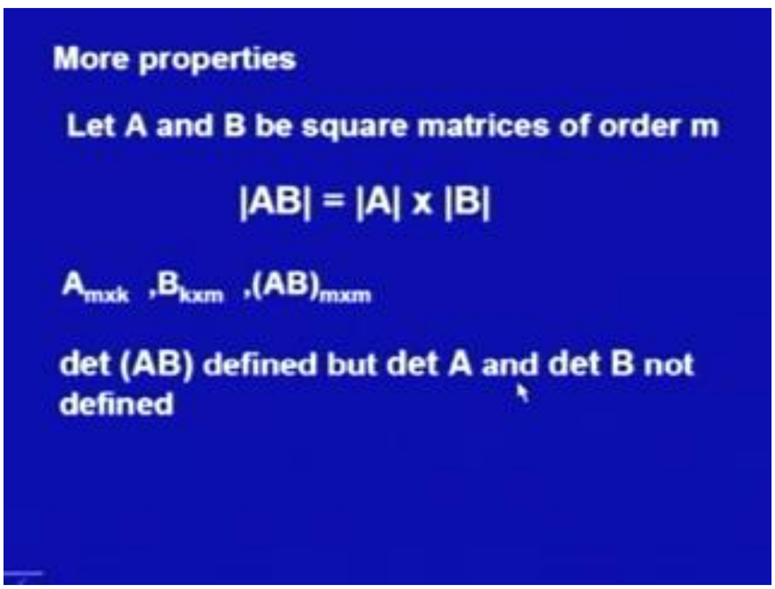

But if we have 2 matrices A and B and of order they are of order m, then the product AB and this determinant is equal to determinant of A into determinant of B. So this is not true; this property doesn't hold for addition, but this property holds for determinants. So here, one has to be very careful - the determinants are determinants are applicable only to square matrices. So It may happen that AB is a square matrix but A and B are not square matrices. So in that case, determinant of like In this case, A is m by k, B is k by m, AB will be m by m - so AB is a square matrix, its determinant is possible. But if you want to apply this, then determinant of Amk is not defined, determinant of B is not defined, so this will not be defined for this set of matrices. This is this property is true only when A and B are square matrices of same order, say m. So in this case, determinant AB is defined but determinant A and determinants B are not defined; so this property is invalid.

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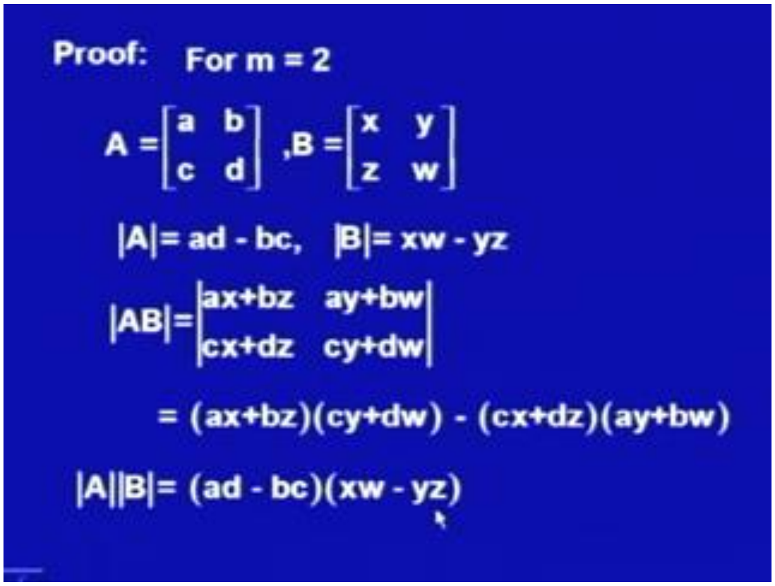

now to prove this Actually the proof of this property is really very complex. I will first try to prove it for m is equal to 2. So for m is equal to 2, let us consider 2 square matrices of order 2 A as a b c d and B as x y z w. Then determinant A is equal to ad minus bc while determinant B is xw minus yz. However, the determinant AB will be: first take the product and then take the determinant; this means the product of these 2 matrices. The first row first column element will be ax bz; this element is obtained by multiplying ab by yw and this element is cx dz and finally the last element will be cy dw; so this is determinant AB. So let us see what A multiplied by B is. If we expand AB, then it will be ax plus bz multiplied by cy plus dw minus ay plus bw into ay plus bw. If you further expand, then AB is equal to ad minus bc multiplied by xw minus yz. Now we will try to match the 2 things.

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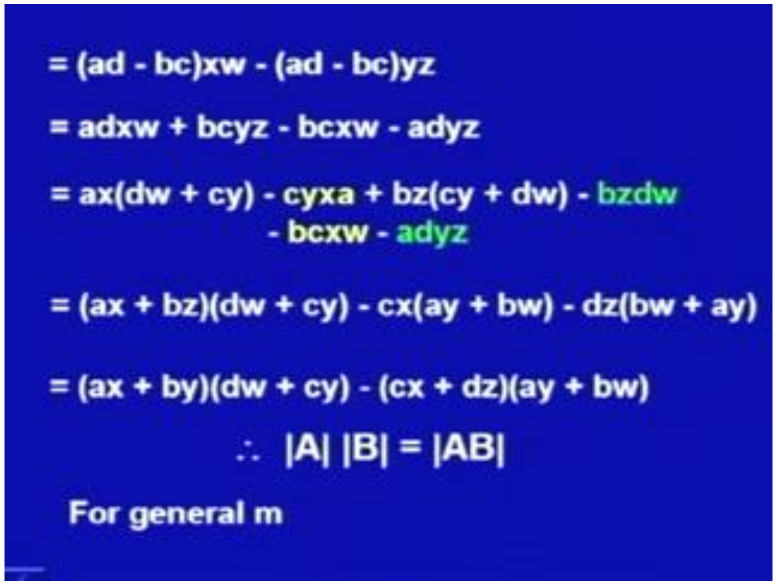

So to start with, ad minus bc xw minus ad minus bc y yz by expansion, and then adxw plus bcyz minus bcxw minus adyz. And Here, I am trying to add and subtract certain things, so that the final result can be obtained easily. So ax dw is with me; I will add cy ax and subtract cyxa. Similarly I write down bz cy plus dw .bz cy is with me; I subtract bzdw and add bzdw and then these two terms. Now we can adjust these quantities and we can write down ax plus bz into dw plus cy minus cx ay plus bw minus dz bw plus ay

and that gives me the final result as this. So determinant of products is equal to product of determinants. Now as for as this proof for general m, that is that cannot be done in half an hour; so this requires lot of computational effort. So I leave this as an exercise for the viewers. You just try and check that this result hold for other values of m also.

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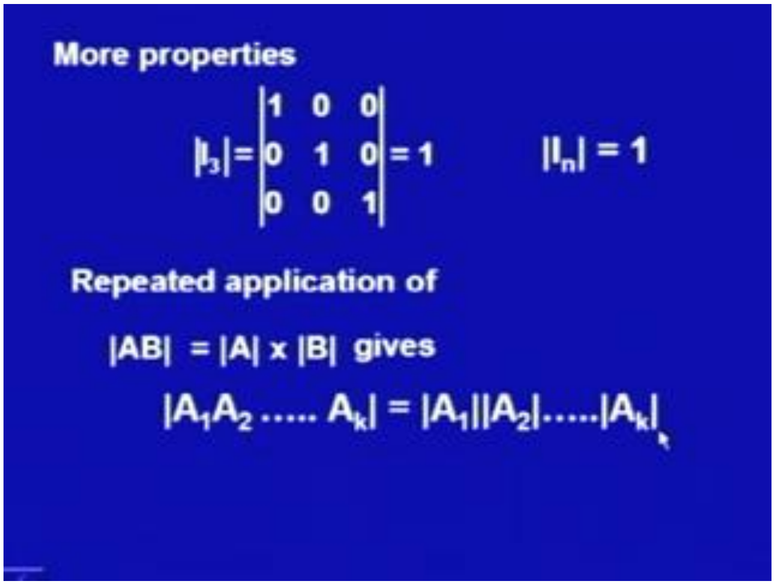

Now we have computed determinants of number of matrices, some results maybe of worth noting. That is, the determinant of identity matrix is 1; this can be easily obtained for I 3 - in general, I n is equal to 1. This identity matrix happens to be a diagonal matrix and all the diagonal elements are 1; so I n will be 1.and Then of course, we have already proved the result that AB is equal to determinant of A into B; so we can use it repeatedly to get the result as the product of k matrices -of course, they have to be square matrices and they have to be of same order. Then this is equal to product of k determinants. One has to be sure that all the determinants are of same order. All the matrices are of same order and they are of they are square; only then, this result is valid. What we can do is we first take this term multiplied by A 1, so it is A 1 into determinant of this; then we can apply the same result to this, canceling A 2, and rest of the terms can be taken together. So we will be having A 1 A 2 and the rest of the terms and this can be repeatedly applied to give the final result.

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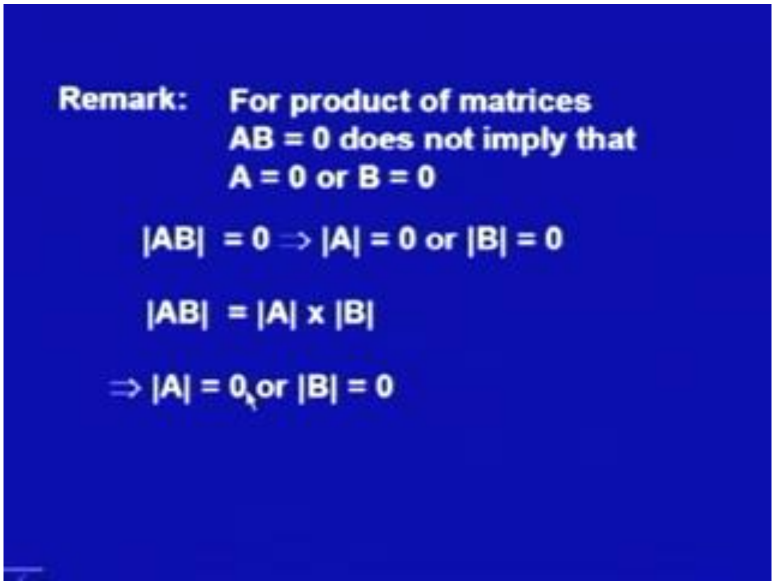

There are certain remarks. One thing is that for product of matrices AB is equal to 0 does not imply that A is equal to 0 or B is equal to 0. This is the result which we have seen when we were discussing determinants that it is quite possible that A and B - none of them - is 0 but their product AB is 0. This is not true for determinants; like if I have determinant of a product as 0, this simply means that either determinant A is 0 or determinant B is 0; this can be proved very easily. Like You can say that determinant of this product is equal to determinant A into the determinant of B determinant Band this simply means if determinant A is 0 then what are determinant A and determinant B? They are simply numbers; so one of the two numbers have to be 0. So either we will have determinant A is 0 or determinant B is 0. So this result was not in general true for matrices but this result is true for their determinants.

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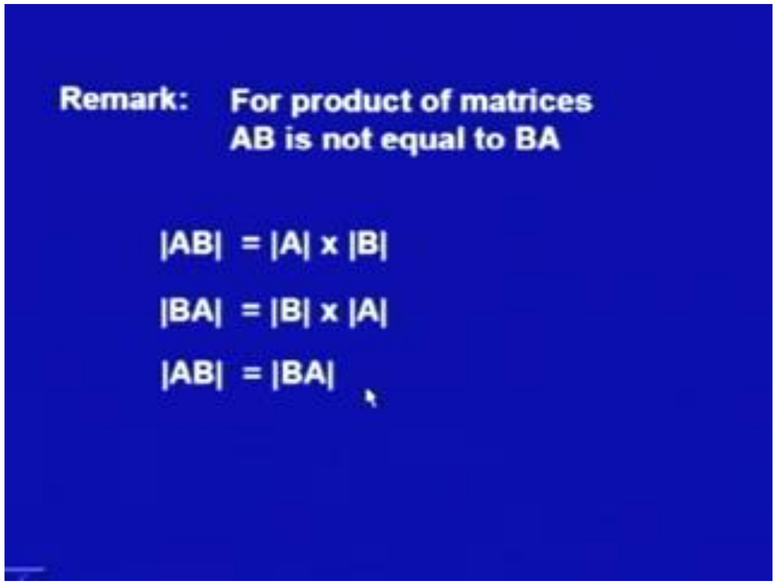

Similarly this is another remark. The product of matrices AB is not equal to BA. We have proved that the product of matrices is not commutative, that is AB is not equal to BA.Of course it is important that AB and BA should be of same order and these 2 products 2 are possible. But if you consider the determinants AB of a square matrix - A and B of order m - then determinant AB is equal to determinant of A into determinant B. The same way, determinant of BA is equal to determinant B into determinant A. But what are determinant AB? Determinant B and A - they are simply numbers and the numbers can be commutative. So determinant B into determinant A can written as determinant A into determinant B and that makes determinant of AB is equal to determinant of BA. So this is again in contrast with theory of matrices, where AB is not equal to BA but for determinants, AB is equal to BA. We have seen that if the matrix is a diagonal matrix, then one can very easily calculate the determinant by simply multiplying the diagonal elements.

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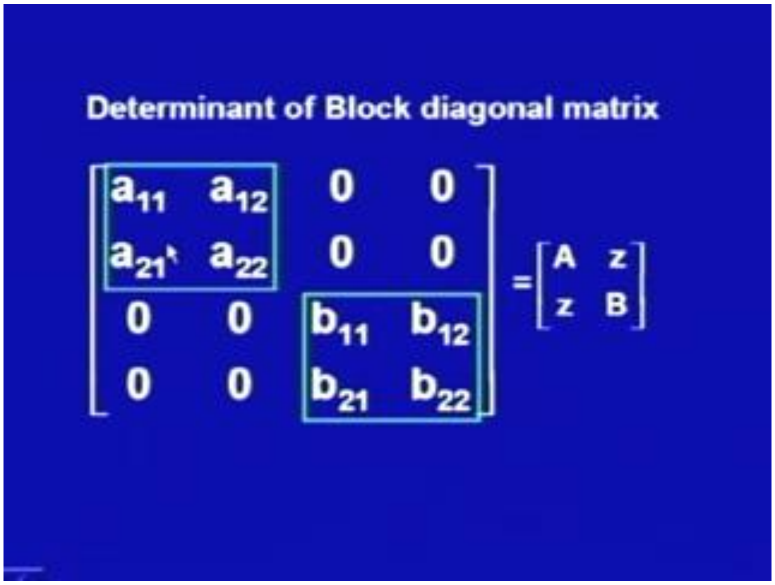

Let us see what happens if the matrix is a block diagonal matrix. By this, I mean to say that I have 1 matrix of order say 4 by 4 and this can be partitioned into sub matrices like this. This can be partitioned and this is a sub matrix of order 2 by 2; this is a sub matrix of order 2 by 2; they lie in the diagonal and these are the matrices, which are null matrices. I denote them by z. So let us say this is A matrix; this is B matrix and this is null matrix; so we will have A, B, z and z. So it is a block diagonal matrix; then how can we calculate the determinant? So if let us say we have 2 by 2 matrix and it's a diagonal matrix then what will be A? A is a simple single term; B will be single term; z will be 0 z will be 0 and determinant of this matrix will be simply be the product of these two terms, that is, a into b. But in this case, I am having not a single term but I am having a matrix of order 2 by 2 or a matrix of order 2 by 2 as the B matrix. So how we can calculate the determinant of this matrix? Then we noted that - this I have taken for example, it may happen that that this is of order 3 by 3 and this is of order 2 by 2. So these null matrices may be rectangular but these matrices have to be square matrices. Like we will have a 5 by 5 matrix so we maybe having 3 by 3 matrix and 2 by 2 matrix here; so this will be a rectangular matrix and this will be a rectangular matrix. So we will be having a block diagonal matrix AB zz. So how can the determinant of this be calculated?

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$$|C| = \begin{vmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{vmatrix} = a_{11}A_{11} + a_{21}A_{21}$$

$$= a_{11}a_{22}\,|B| - a_{21}a_{12}\,|B| = (a_{11}a_{22} - a_{21}a_{12})|B|$$

$$= |A|\,|B|$$

$$|C| = |D_1| \cdot |D_2| \ldots |D_K|$$

This - I will show for C matrix which is the determinant of a 1 1 a 1 2 0 0 a 2 1 a 2 2 0 0 0 0 b 1 1 b 1 2 0 0 b 2 1 b 2 2. Now the determinant of this can be easily calculated. What I can do is I can expand it about the first column. So it will be a 1 1 multiplied by its cofactor plus a 2 1 multiplied by its cofactor. This term will not contribute, this term will not contribute; that's a advantage of having 0 s in the particular row or column. In fact, we select the particular row or column which will have some of the terms 0 s. Now in this case, every column or row can will be having 2 0 terms; so it doesn't make much difference whether you use this column or this column. But for convenience, I am using this particular column.

Now what is A 1 1? A 1 1 is a cofactor of a 1 1. How did we obtain a 1 1? A 1 1 is obtained by neglecting this row and this column - that means, this particular determinant. So this particular to evaluate this particular determinant, I will again expand by a cofactor; so for this, I will consider this column. So it is a 2 2 multiplied by this determinant. This determinant I know is b and this term and this term will not contribute. So a 1 1 is nothing but a 2 2 B. Similarly a 2 1 a 2 a 2 1 A 2 1 that is cofactor a 2 1, will be a 2 1 into a 1 2. Let us see if we calculate the cofactor of a 2 1; then this row and this column has to be deleted. So the sub matrix will be having a 1 2 0 0 0 b 1 1 b 1 2 0 b 2 1 b 2 2 and by again by cofactor expansion this will come out to be a 2 1 a 1 2 determinant B. So determinant B is common in both the terms; so B can be taken out and what we have is a 1 1 a 2 2 minus a 2 1 a 1 2. What is this? This is nothing but the determinant of A matrix - a 1 1 a 2 2 minus a 1 2 a 2 1; that is, noting but determinant A multiplied by determinant B. So what we can say is if we have block diagonal matrix consisting of A matrix and B matrix in the diagonal, then the determinant of block diagonal matrix is nothing but the determinant A and determinant B. That means, if i have a matrix C having k block diagonal matrices or k block sub diagonal matrices, then we it will having determinant C as determinant D 1 into determinant B 2 into determinant Dk. So one can generalize this result and this will help this will be helpful in evaluating determinants of higher order matrices.

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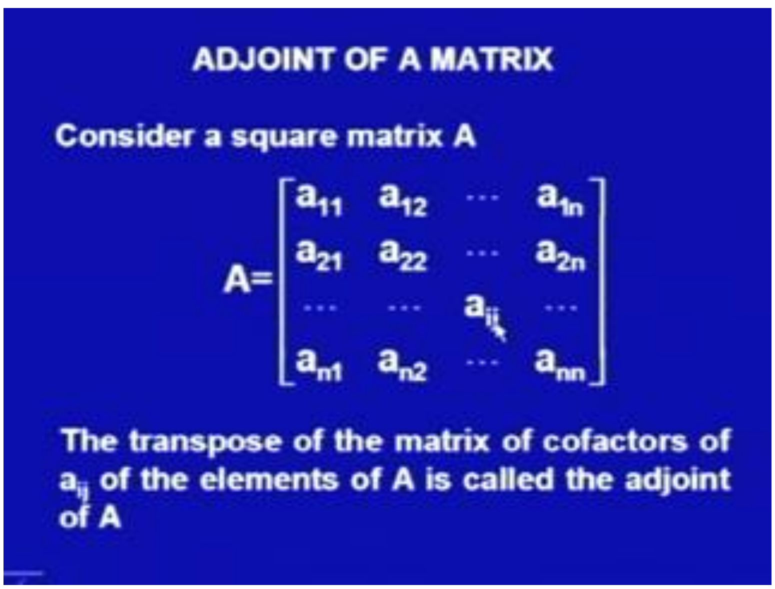

Transpose of the matrix of cofactors of aij of the elements of A is called adjoint of A.

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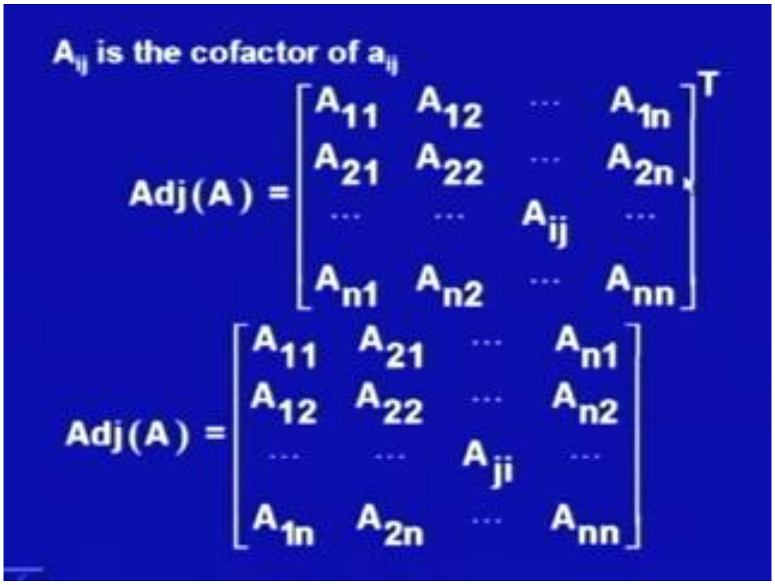

By this, I mean to say that if Aji is the cofactor or aij; here the difference is here is the capital Aij, small aij is the element, capital Aij represents the cofactor of the ij th element of the matrix. That is, adjoint A is A 1 1, the cofactor of element A 1 one in the matrix A; A 1 2 is the cofactor of first row second column element of the matrix A; A 1 n. And Then we take cofactor of each and every element and then we take the transpose of it; that defines the a joint of A. Or adjoint of A is - transpose simply means the rows and columns are interchanged - so i am writing A 1 1 A 1 2 A 1 n the first row as the first column A 1 1 A 1 2 A 1 n and this is the second row second column which is here the second row, and the last row here will become the last column here.

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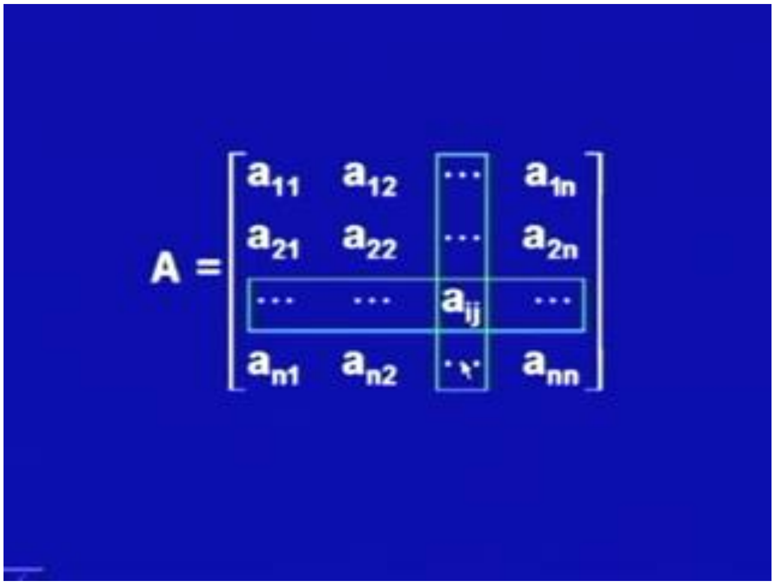

So if I have the matrix A as this, then cofactor aij is obtained by neglecting by considering a sub matrix deleting the i th row and j th column and what we have is an n minus 1 by n minus 1 sub matrix. Take its determinant and then multiply it by minus 1 raised to power i plus j; that becomes the determinant of, that becomes the cofactor of aij.

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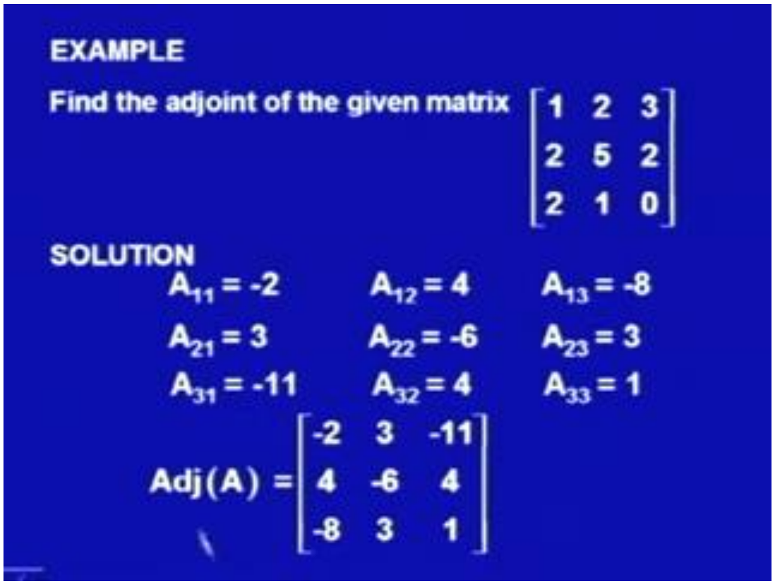

For example, if you have to find the determinant of the matrix 1 2 3 2 5 2 2 1 0, a 3 by 3 matrix, then we have to find out the have to find out the cofactors of elements of this matrix. I call A 1 1 as the cofactor of first element in the first row and first column. I obtain this by neglecting this row and this column and forming the determinant of this sub matrix. Determinant of this sub matrix is 5 into 0 minus 2 into 1 - that means minus 2 - so A 1 1 comes out to be minus 2. A 1 2 is this cofactor of this element which is obtained by neglecting this and this column and what we have is 2 into 0 minus 2 into 4 - that comes to be A 1 2 into 4; but actually, this is determinant is minus 4 and I use the second party for getting the sign of the cofactor. The sign of cofactor here is plus here and minus here. So this minus multiplied by this minus makes it 4; so A 1 2 comes to be 4. Now A 1 3 is this determinant with plus sign; so it is 2 minus 10 it comes put to be A 1 3.

Similarly, one can compute A 2 1 as 3 - A 2 1 means cofactor of this element, which is 2 into 0 minus 3 into 1, multiplied by minus 1 that makes it 3. A 2 2 is minus 6 which is 1 into 0 minus 3 into 2 - it is minus 6 - and cofactor of this element A 2 3 is 3 1 minus 4 with a negative sign here. Similarly A 3 1 is minus 11; A 3 1 is this element - it is 4 minus 15, so it is minus 11. A 3 2 is 2 minus 6 with a negative sign - so it is 4; A 3 3 is 5 minus 4 - comes out to be 1. So what is adjoint A? A 1 1 A 2 1 A 3 1 - these are elements which are in the first column. But when we are talking of when we are talking about adjoint A, they have they will become in first row because it is a transpose of matrix of cofactors. So what we have is minus 2 3 minus 11 is the first row, 4 minus 64 as a second row, minus 8 3 1 as the third row. So adjoint of A is equal to this matrix -this 3 by 3 matrix.

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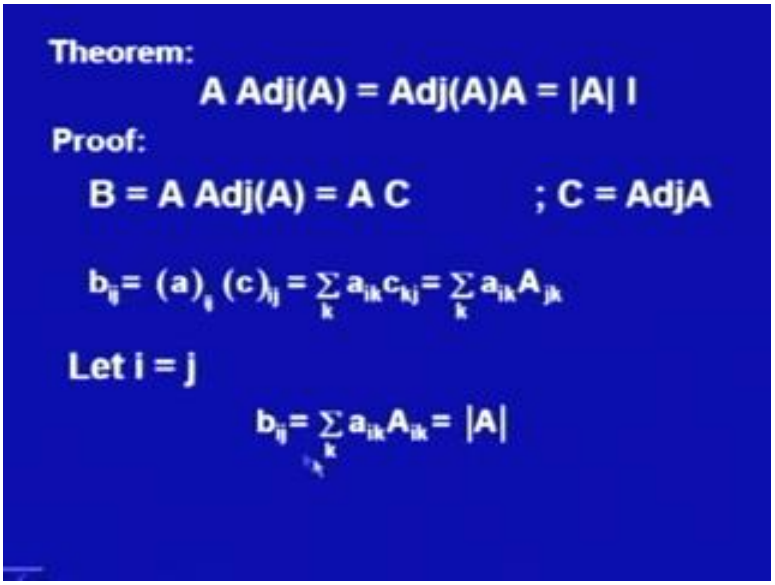

Now on the basis of this, we have a theorem which states that A multiplied by adjoint of A is equal to adjoint of A multiplied by A - it is commutative -and not only this is equal to determinant of A multiplied by identity. This is an important result; so let us try to prove this. For this purpose, let us consider the product B as A into adjoint of A. Now, I denote adjoint of A as C; so this B is equal to AC. To calculate B is equal to AC, what I have to do is I have to calculate the elements of the matrix B - which is a product of these 2 matrices. So bij a typical element of this matrix B - is the product of this. So what we have is aij multiplied by cij; this is equal to summation over k aik ckj. Now this is from the definition of product of matrices. Now what is ckj? cjk is a adjoint of A; so this is nothing but- adjoint is C is equal to adjoint of A - so it is nothing but cofactor of aik and its transpose; so it becomes Ajk. So ckj is nothing but Ajk. So this means bij is summation of aik Ajk over k. Now to compute this, let us consider a case when i is equal to j. When we consider i is equal to j, then bij is equal to summation over k aik Aik , because j is equal to i and we have already proved that this nothing but the determinant of A. So when i equal to j, bij is nothing but determinant of A and this is true for all values of i from 1 2 to n.

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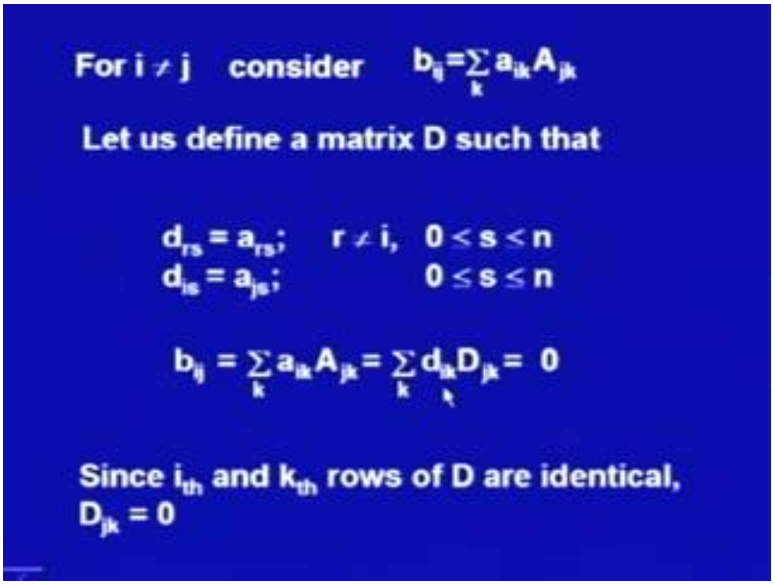

But what happens when i is not equal to j? To prove this result let us consider bij is equal to aik Ajk, summed over k. Now we define a new matrix D such that drs is equal to ars, when r is not is equal to I, for all values of s lying between 0 and n but dis is equal to ajs for all values of n. That means the i th i th row of D matrix is the same as j th row of A matrix; that is how, we define a new matrix D. Now this means that bij is equal to aij Ajk; this is what we have here - summed over k, is equal to dik. Why am I using dik? Because aik is equal to dik by the definition here. into Djk Ajk is equal to Djk. What is Ajk? Ajk is cofactor matrix; so this is the same as Djk. Now one may notice that since i th and k th rows of D are identical, so all the determinants in the all the cofactors djk will be 0 because i th and k th rows of D are identical and that simply means that dij is 0, whenever i is not equal to j.

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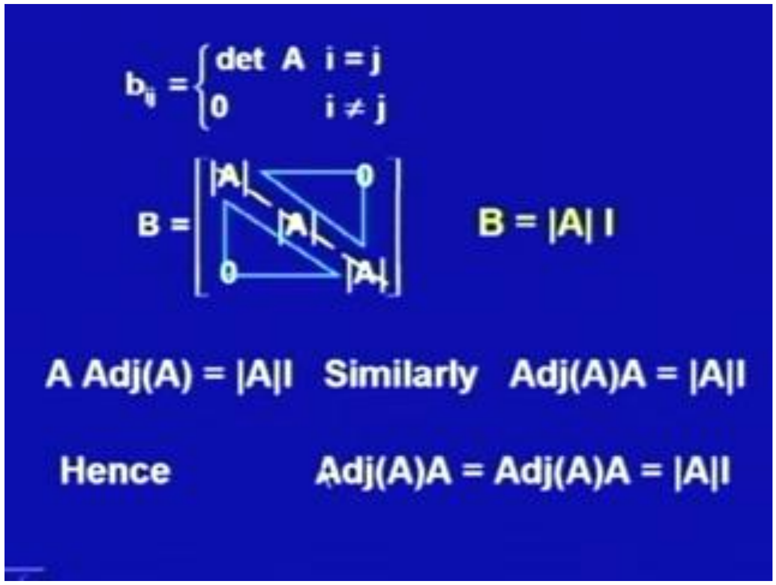

And This simply means that bij is equal to determinant a when i is equal to j and it is 0 when i is not equal to j. By this, I mean to say that the determinant B matrix will look like as this where their determinant A will appear in the diagonal because for all the diagonal elements, i is equal to j; and all the elements which are lying on the upper side of the diagonal or on the lower side of the diagonal they are 0 because these are the elements where i is not equal to j; so basically B is a diagonal matrix. So if B is a diagonal matrix, then one can take determinant A outside - so number only - we can take it outside and what remains here is nothing but the identity matrix and that proves the result that A determinant adjoint A is determinant A times I. Similarly one can easily prove adjoint of A is equal to A as same as determinant AI. Now this - I leave with an exercise for the viewers to verify this. Hence we can say that adjoint of A multiplied by A is equal to adjoint of A multiplied by A is the same as determinant of A multiplied by identity matrix. Now on the basis of this, one can get some useful results regarding non singular matrices; so let me introduce non - singular matrix.

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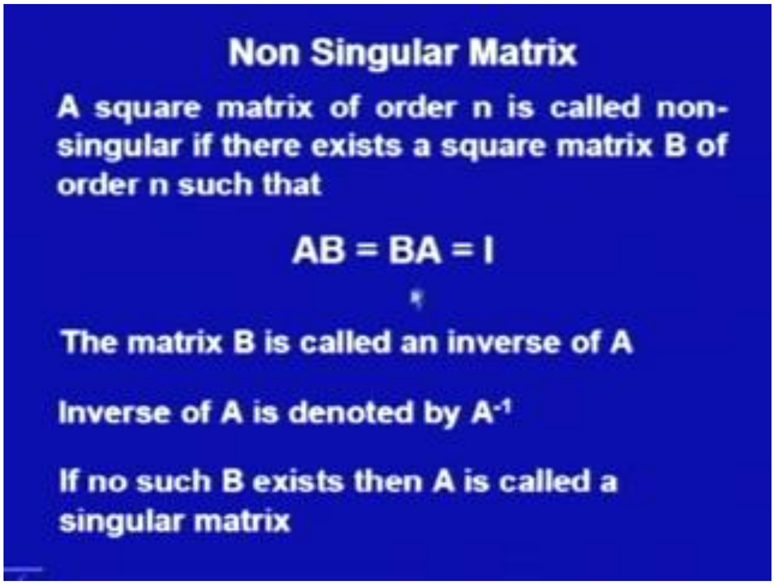

So of we say a non a square matrix of order n is called non singular if there exists another square matrix B of order n such that AB even multiplied by the A multiplied by B is the same as B multiplied by A, which gives me an identity. In that case i say that matrix B is an inverse of matrix A. So if such a matrix B can be found, then the matrix A is called a non singular matrix. The inverse of A is denoted by A inverse. A inverse So we say B is equal to A inverse or we can say A inverse is equal to B. If no such B exist, then we say the matrix A is singular matrix. One can notice that this product is commutative.

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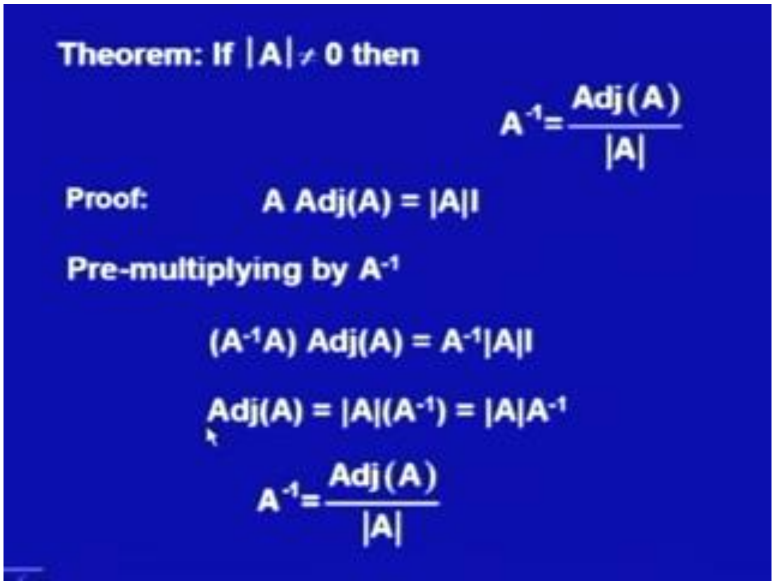

Now we have a result which says that if determinant A is not 0, then A inverse is equal to adjoint A divided by determinant of A. Now this, actually later on once we prove this, then it becomes a formula for finding out determinant of A inverse. That is the importance of adjoint of A. Now to prove this, let us consider the product A adjoint A and we will prove that this is determinant AI. So what we can do is we multiply - this result, we have already established - so what we can do is we pre-multiply it by A inverse. So A inverse A is equal to adjoint A is equal to: this becomes identity and adjoint A becomes determinant A - this this can be taken outside - determinant A is equal to A inverse, so what we have is adjoint A is equal to determinant A into A inverse or A inverse, because determinant A is simply a number; you can always divide it by determinant A - both the sides. So A inverse is equal to adjoint A divided by determinant A.

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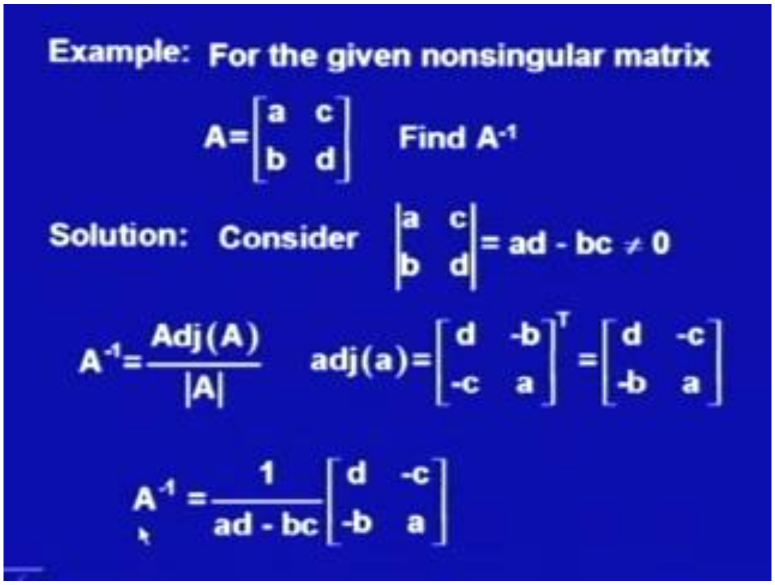

Let us take this example A is equal to 2 by 2 matrix ac bd. Then you have to find the inverse of given matrix A. For this we consider first the determinant ad minus bc. So if ad minus bc is not 0 - this is required, so this condition is true ad minus bc is not 0 - only then we can find out the determinant; only then, we can apply this formula because if ad minus bc becomes 0 then this is not possible and inverse will not exist. So A inverse is equal to adjoint A divided by determinant A and this gives adjoint of a as: d - adjoint of cofactor of a is d; cofactor of c is minus b; cofactor of b is minus c; cofactor of d is a. Take the transpose and what we have is d minus c minus b a and then A inverse is equal to determinant A in the denominator; so ad minus bc 1 upon ad minus bc multiplied by d minus c minus b a becomes the inverse of given matrix A. Let us verify that this actually is an inverse.

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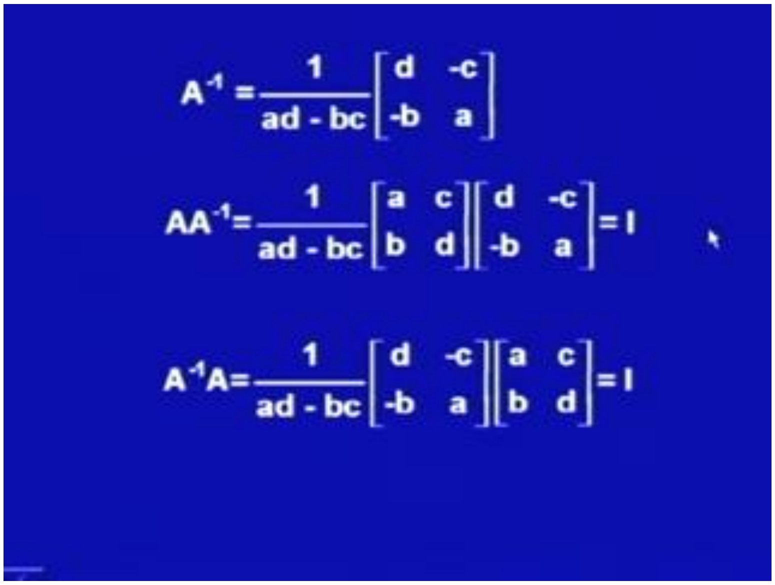

So what we can do is we we multiply with A and let us see if it comes out to be an identity. So if we multiply A by A inverse we will have this product and let us see this product. It is ad minus bc which is 1 so we will be having 1 here; but ac, the second element : if we consider it, is a my ac is negative sign, then ca they will cancel out and what we have is 0 on the first row second column. Similarly bd minus bd that gives me 0 as the element in the second row first column - the right hand side - and a then b minus c and da that give me 1 when divided by ad minus bc. So what we have on the right hand side is the identity matrix. Similarly one can prove that A inverse A is also identity matrix. So user can viewer can actually perform this multiplication and get convinced that this is actually comes out to be an identity matrix; so the result is verified. Now I have done this for a 2 by 2 matrix; little more effort is required for a 3 by 3 matrix.

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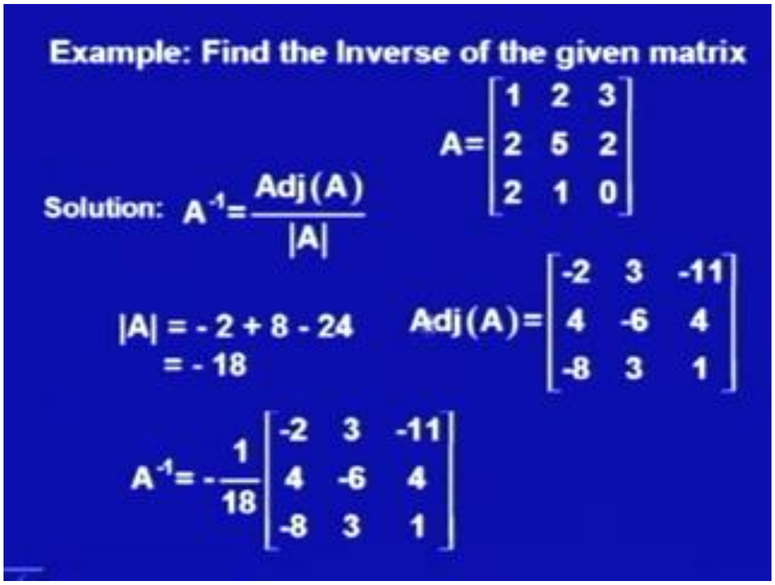

So, if I have a 3 by 3 matrix A as 1 2 3 2 5 2 2 1 0, then its determinant is calculated as minus 18. 1 into this, 2 into this and finally 3 into this - that comes out to be minus 18. Adjoint A is - this I have already done in my one of the earlier exercises; so I have already computed this. So simply A inverse is 1 upon eighteen 1 upon minus 18 as adjoint of A; so that is A inverse. So given A and A inverse, which I have computed, one can verify that this comes out to be identity.

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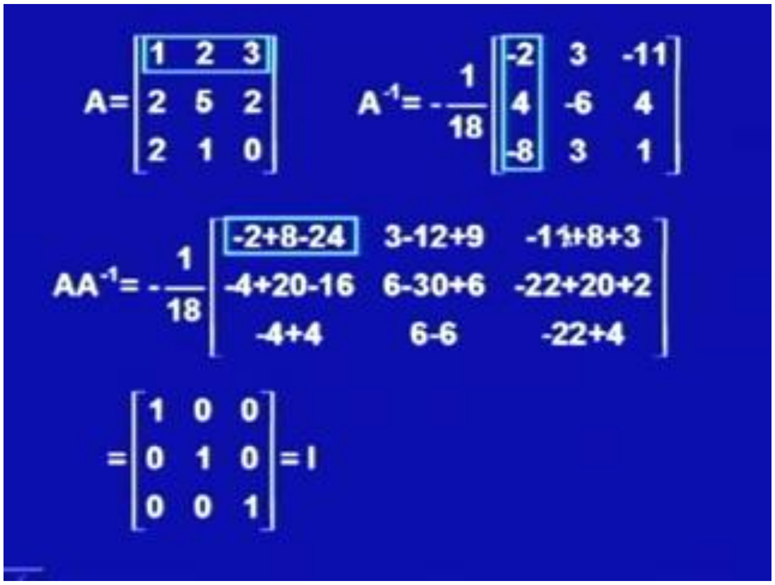

So if this row is multiplied by this, we will be having minus 2 8 and minus 24; this comes to be minus eighteen, so the result is 1 - this is 1- and you can similarly calculate other terms and this comes out to be an identity matrix of order 3.

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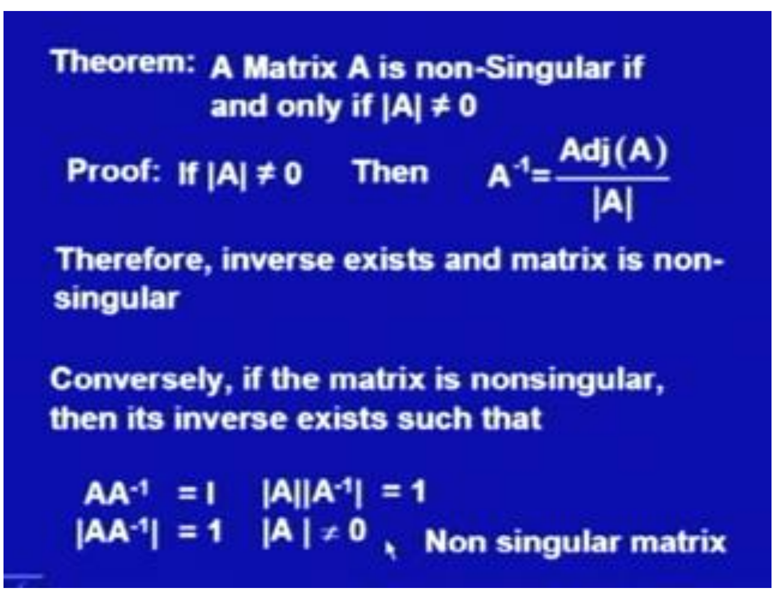

Now this is an important theorem which says that if a matrix is non-singular and if determinant A is not 0. The proof of this is that if determinant A is not 0 then the matrix is non-singular. This has actually 2 parts. If the determinant A is not 0, then determinant A inverse exists and if determinant A inverse exists, then determinant A is not 0. Now, if determinant A is not 0, then one can find the determinant from the previous result that A inverse exists because adjoint A can very easily be obtained - is a matrix cofactors. Determinant A is not 0 - so we can divide it by determinant A - and in the earlier theorem we have proved that an A inverse is given by this result; so if determinant A is not 0, then A inverse exist and the matrix is non-singular. However if the matrix is non-singular, then its inverse exists such that AA inverse is 1. So if the matrix is non-singular and its inverse exists, then we have to show that its determinant will not be 0. So let us consider AA inverse is equal to identity. AA inverse is equal to identity, because inverse exists and this is the very definition of inverse.

So let us take the determinant of this product. So determinant of this product comes out to be 1, because determinant of identity is proved to be 1. So we apply the product determinant A into A inverse which comes out to be 1, and this simply means determinant of 2 is equal to determinant of 2 matrices its result is 1. So determinant A cannot be 0; that simply means that for a non-singular matrix, determinant A is not 0. Now after proving this, one can take this result as a definition for a non-singular matrix. Previously, we were defining that non-singular matrix is the one which has which has inverse, but if now we say that if the matrix is having non zero determinant, then it is non-singular. So this can be taken as a definition of non-singular matrices.

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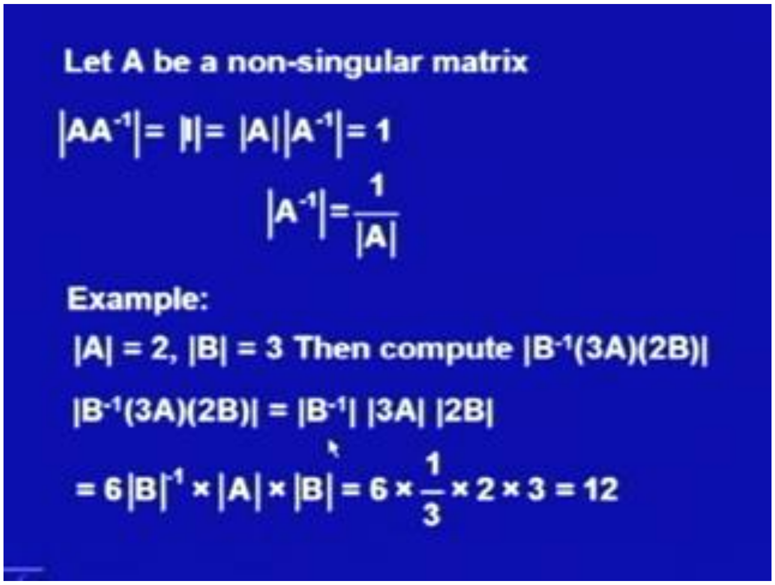

So Let A be non-singular matrix. Then AA inverse is identity or AA inverse is equal to 1. So A inverse is actually 1 upon determinant A. So If determinant A is not 0, then A inverse is 1 upon determinant of A. So, if we know that determinant of A, one can very easily find out the determinant of its inverse. So this result is helpful in that sense. So Let us illustrate this with this example. We have a determinant A as 2, determinant B as 3; then you have to compute the determinant of B inverse multiplied by 3 A into 2 B. So what we can do is we can write down the determinant of this product as determinant of B inverse into determinant of 3 A into determinant of 2 B. Now determinant of B inverse is the same as 1 upon determinant of B; so 6 into B inverse is nothing but 6 into 1 by 3 because determinant B is 3. Then Determinant of 3 A is 3 times determinant of A, determinant of 2 B is 2 times determinant of B; that is how this 6 is coming - so 3 and 2 are taken out. So this result is equal to 6 times B inverse into A - determinant of A - into determinant of B; that is 6 into 1 by 3 into determinant of A is 2 into determinant of 3, which is 3 and final result come out to be 12. So if determinants are given, then one can compute this type of product.

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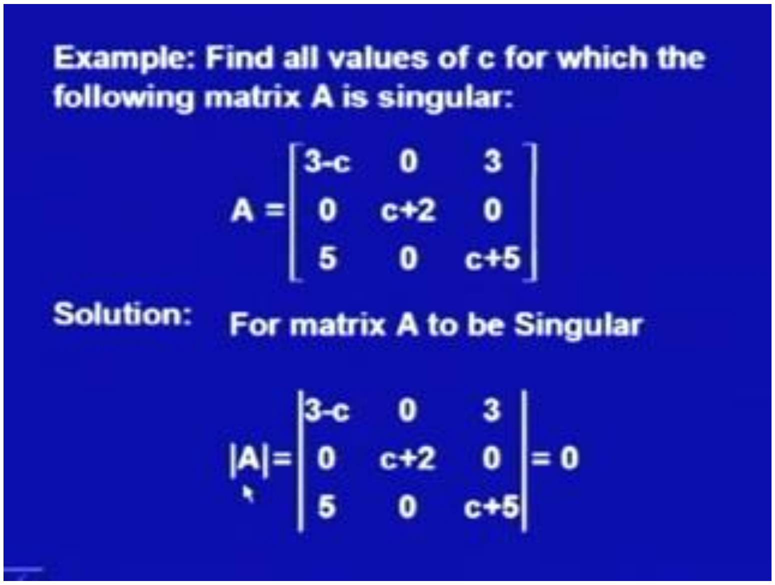

Now, I am considering the example in which you have to find out the value c such that the matrix A is singular. So Matrix A is given as 3 minus c 0 3, 0 c plus 2 0, 5 0 c plus 5. Now I will be using the theorem, which I have just now proved, that the matrix A is singular if its determinant is 0. So determinant A is equal to 3 minus c 0 3, 0 c plus 2 0, 5 0 c plus 5 0. So Basically this matrix determinant is to be evaluated.

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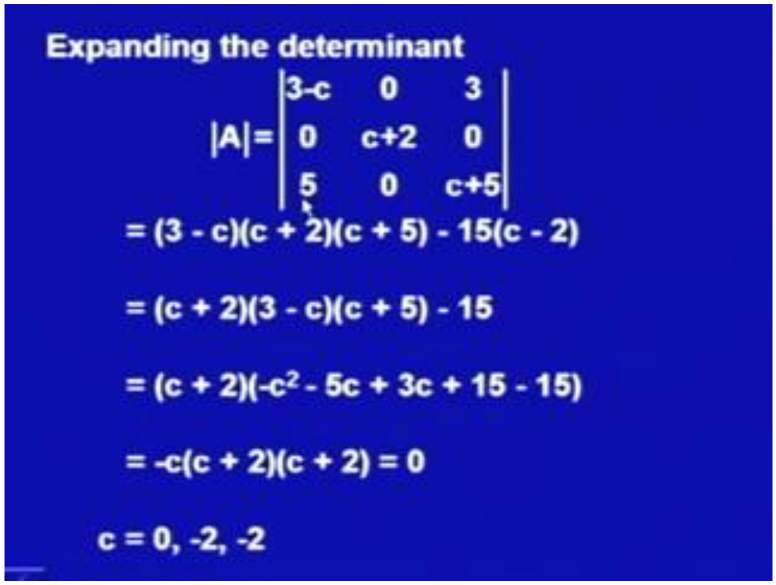

To evaluate the determinant, one can expand it about this particular row; so it is 3 minus c into c plus 2 into c plus 5 minus 0 into 3 times 5 -that is 3 into c plus 2 times 5; that is, this term. And you simplify it; it comes out to be minus c c plus 2 into c plus 2 is equal to 0 and that simply means either c is equal to 0 or c is equal to minus 2 or c is equal to minus 2. So for these values of c, the matrix A is singular; this matrix A is singular. Viewers, today we have discussed some interesting properties of determinants and we have defined adjoint of a matrix and we have seen how this can be used in evaluating inverse of a square matrix. Thank you.