Mathematics-II

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Module - 2
Lecture - 3
Determinants Part – 1

Good morning, viewers! Today's topic is determinants. I will start this topic by giving you first the basic concepts regarding determinants. I will introduce the concept and then we will talk about the properties of determinants. To start with, we will consider a system of 2 equations in 2 variables x and y.

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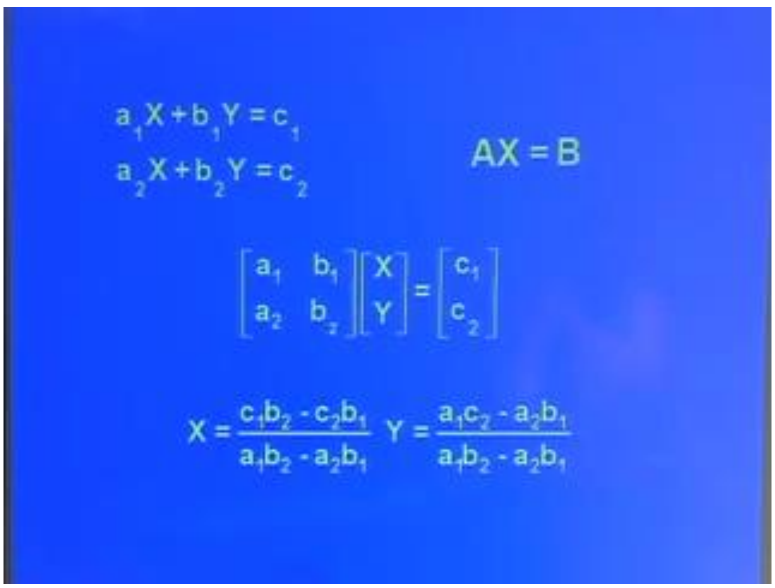

The two equations are a 1 X plus b 1 Y is equal to c 1 a 2 X plus b 2 Y is equal to c 2. These two equations can be written in the determinant form as we have discussed in my earlier lectures, as AX is equal to B where A is a 2 by 2 matrix and X Y is a column vector. While on the right hand side we have the column matrix c 1 c 2. We know how to solve these 2 equations; from our school days algebra, one can very easily obtain X is equal to c 1 b 2 minus c 2 b 1, divided by a 1 b 2 minus a 2 b 1while Y can be simplified as a 1 c 2 minus a 2 b 1, divided by a 1 b 2 minus a 2 b 1.

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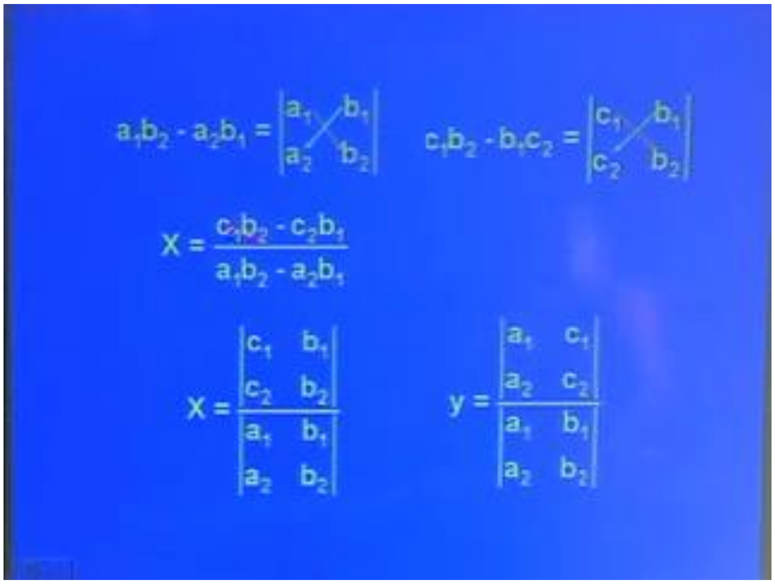

Now we try to represent the expression a 1 b 2 minus a 2 b 1 in this form and on the right hand side what I have written is called a determinant consisting of the matrix a 1 b 1 in the first row, and a 2 b 2 in the second row. This is enclosed by a pair of vertical lines; we call this is a determinant. This determinant is equal to the left hand side because one can very easily check that this left hand side is equal to a 1 b 2 - this product - minus the product b 1 a 2.

Similarly 1 can write down the expression: c 1 b 2 minus b 1 c 2 is equal to c 1 b 2 minus b 1 c 2. Then the expression X, which we have obtained in my earlier slide, the expression X is equal to c 1 b 2 minus c 2 b 1 divided by a 1 b 2 minus a 2 b 1; as obtained in my earlier slide is this can be written as the matrix c 1 b 1 c 2 b 2; then we are enclosing vertical lines, divided by the matrix a 1 b 1 a 2 b 2, vertical lines we call this as a determinant. The expression X is equal to c 1 b 2 minus c 2 b 1 divided by a 1 b 2 minus a 2 b 1 can be written as X is equal to c 1 b 1 divided c 2 b 2 divided by a 1 b 1 a 2 b 2. Now here we have used the notation which we have used here. We called this as a determinant of the matrix a 1 b 1 a 2 b 2. Using this notation, we write down Y in this particular form; we have already written this expression as we have already written this expression as this here, similarly we can obtain a 1 c 2 minus c 1 a 2 as this determinant.

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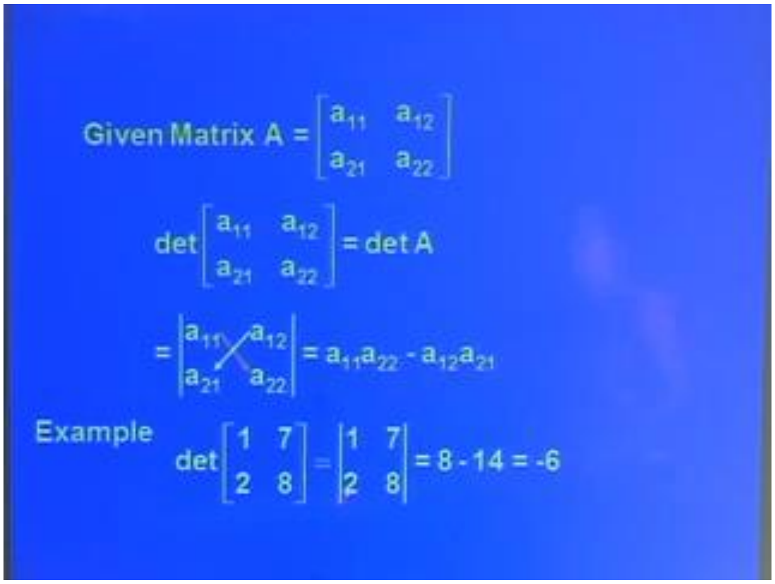

Now given this matrix A as a 1 1 a 1 2 in the first row, a 2 1 a 2 2 in the second row, one can define the determinant of the matrix as - or we denote it by determinant A - we can define it as a 1 1 a 2 2 minus a 1 2 a 2 1. This is as a 1 1 a 2 2 - the first term - and a 1 2 a 2 1 - the second term - with a minus sign in between. Now the example is if I have given a matrix 1 7 2 8, then its determinant can be written as this, which can be computed as 1 into 8 is 8 minus, 7 2 - 7 into 2 is fourteen - and the result is minus 6. Now this is the way we define matrixes; for this this is the way we define determinants for 2 by 2 matrices. But to define determinants in more general form, I will now give you some basic concepts.

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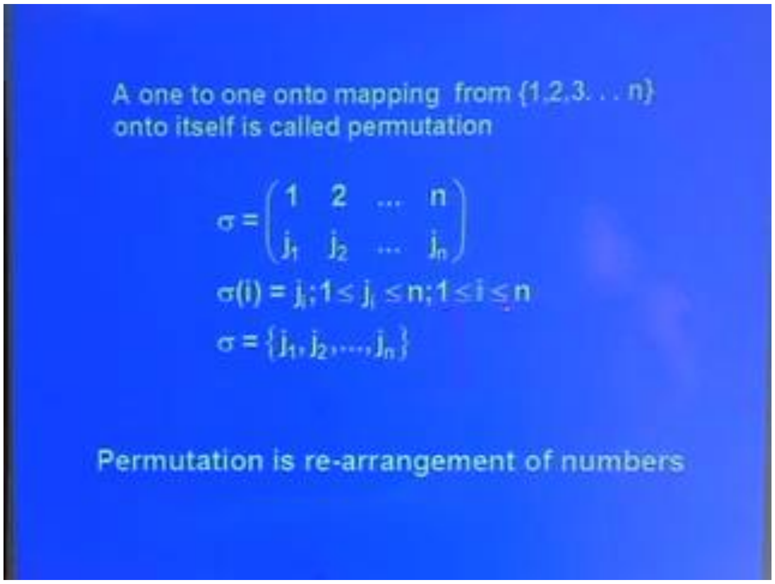

First thing is to define a permutation. A permutation is one to one onto mapping from the set 1, 2, 3 n onto itself. For example, if I have been given a set of first n natural numbers then a sigma - is denoted as the permutation - can be defined as this here: the number 1 is mapping to j 1, 2 is mapping to j 2 and n is mapping to jn. What are j 1, j 2, jn? They are simple the rearrangement of numbers 1, 2 to n. This is 1 one and onto mapping, so no number will be repeated while writing down j 1, j 2, jn and no number will be dropped off. We can write down the determinant; we can write down the permutation in a in form like sigma i is equal to ji, where ji lies between 1 to n and i lies between 1 and n; or the other way of writing this permutation is sigma is equal to j 1 j 2 to jn. So we have understood that permutation is simply a rearrangement of numbers.

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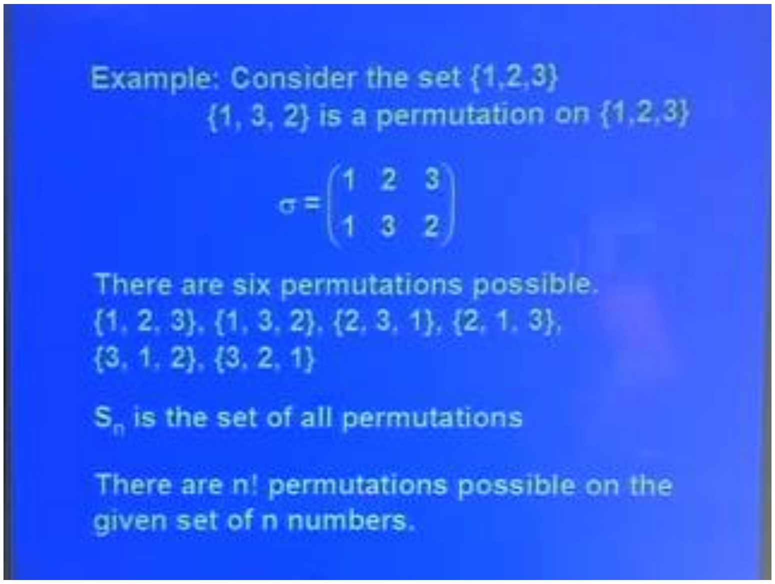

For example, if we consider this set of first 3 natural numbers 1 2 3, then 1 3 2 is a permutation on this set 1 2 3. We denote this permutation as sigma is equal to 1 2 3. This is the set and 1 is mapping to 1, 2 is mapping to 3, 3 is mapping to 2. So this way, for a given set of 3 natural numbers, there will be 6 permutations possible and they are 1 2 3 - the first permutation - then the next may be 1 3 2, next 2 3 1, 2 1 3, 3 1 2 and 3 2 1. So, in all, there are 6 permutation possible. So if Sn is a set of all permutations, then we say that there are factorial n permutations possible on the given set of n numbers.

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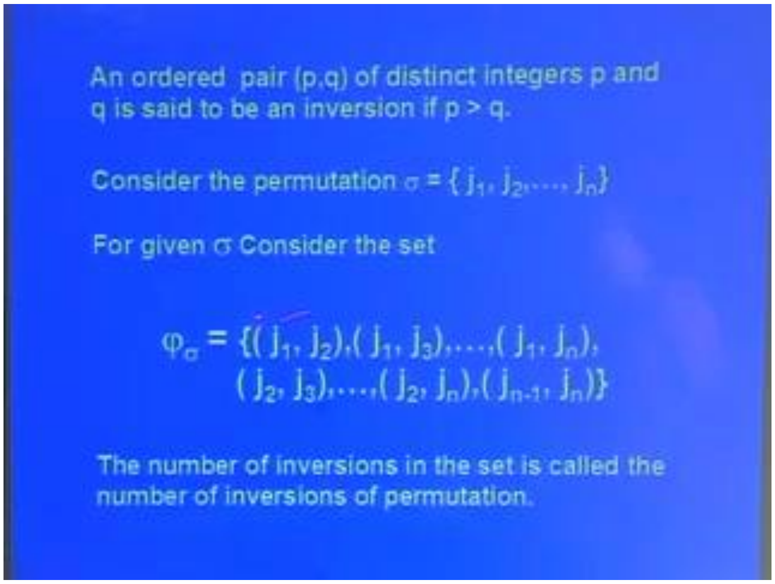

We further define an ordered pair p q of distinct integers p and q; then q is said to be an inversion if p is greater than q. So, if we consider the permutation sigma is equal to j 1, j 2, jn, then for given sigma we consider the set phi sigma which is a set of ordered pairs - j 1, then we consider j 2, j 1 j j 1 next number j 3, next is order next ordered pair is j 1, jn; then we start with j 2 j 2 j 3, j 3 j 2 j 4, it will go up to j2, jn and finally we will have jn minus 1 and jn; so we call this set as phi sigma. Now in this set we consider how many inversions are there. Let us say we consider this ordered pair - 1 greater than 2 - not true; so it is not an inversion. 1 greater than 3 is not an inversion. The number of inversions in the set is called the number of inversions of the permutations. I will take i will explain this with an example; next slide.

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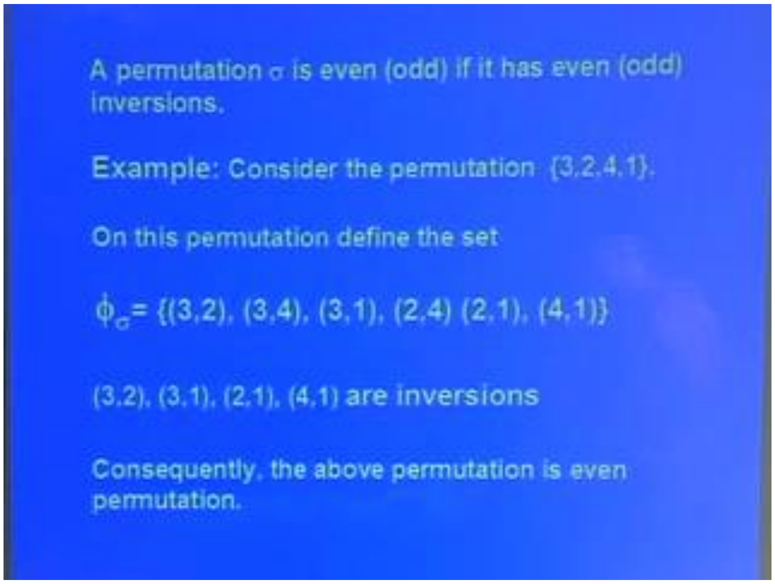

A permutation is sigma is even if it has even number of inversions and we call the permutation as odd if it has odd number of inversions. So for this, let us consider the permutation 3 2 4 1 on the set of first 4 natural numbers. On this permutation we define the set phi sigma as 3 2, 3 4, 3 1 - these are the first three; then we start with 2 4, 2 1 - these are the ordered pairs; then we start with 4 four and 1. So in all, we are having 6 ordered pairs in phi sigma. Now if we consider the number of inversions in this, then 3 2 is an inversion because p is greater than q; 3 4 is not an inversion; 3 1 is an inversion because 3 is greater than 1. Similarly, we will have 2 1 2 1 and 4 1 are inversion. So in this set, we are in all have 4 inversions and consequently the above permutation is an even permutation.

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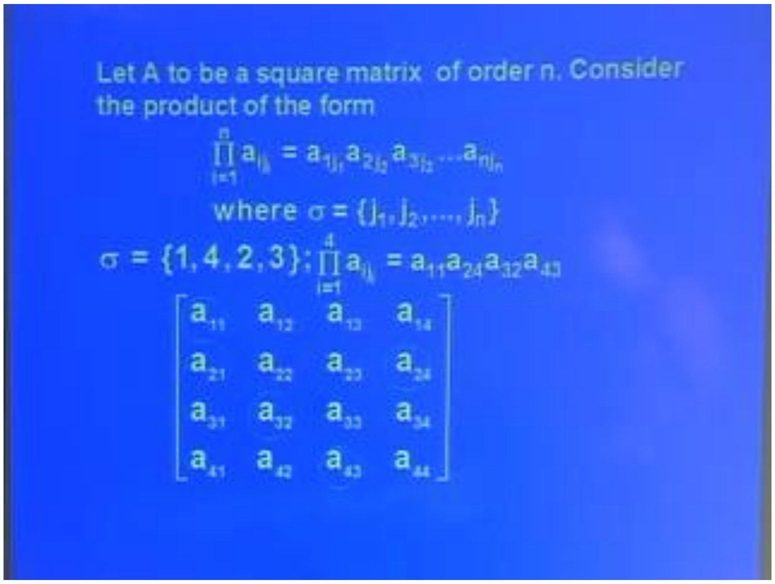

Now we consider a square matrix of order n and we try to form the product of n numbers. Then these numbers - this phi is denoting the product phi aiji; we write it as a 1 j 1 a 2 j 2 a 3 j 3 and anjn. No, if we consider, for example, sigma as this permutation then this product i is equal to 1 to 4; aiji is will be a 1 1, a 2 4, a 3 2 and a 4 3. So if we consider this product and consider this square matrix, then this product basically means that we are considering this term multiplied by this term, this term and this term. Now one can very easily observe that while formulating this product we are considering only one term from a corresponding row and column. So in the first row and column, we will have only this term; from the second, we are having this term; from the third row, we are considering this term and the fourth row, I am considering this term. So if I am considering this term, no other column is selected; no other column element is selected if I am considering this term - no other column element is selected. So I am selecting 1 term from each column and row and forming a product. So this is one product, but actually there will be factorial 3 factorial 4 such products possible.

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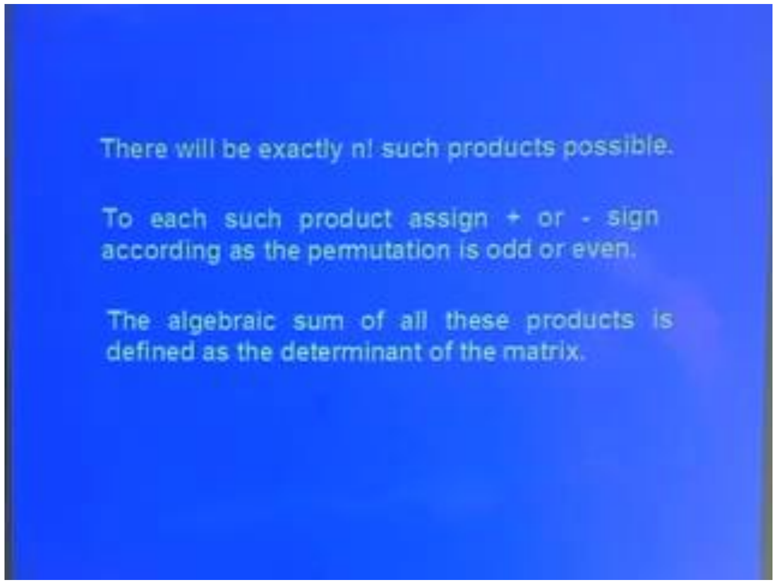

So if we are having n square matrix, then there will be exactly factorial n such products. Now to each such product, if we are assign plus or minus sign according as the permutation is even or odd, then the algebraic sum of all these product is defined as a determinant of the matrix.

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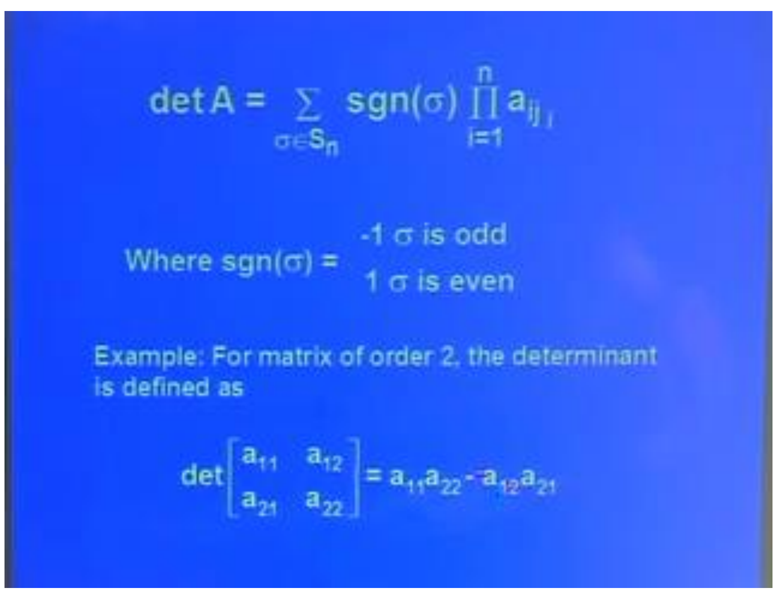

Now this way, we define determinant A as summation - sigma - over the set of all permutations. signum sigma determines the sign and this is the product. This signum sigma is defined as minus 1 if sigma is odd and it is 1 if sigma is even. Now let us see how what is the meaning of this in contacts with a 2 by 2 matrix for which we have all ready introduced the determinant. So if we give this matrix - 2 by 2 matrix - then its determinant is a 1 1 a 2 2 minus a 1 2 a 2 1. Now one can see that if we consider this a 1 1, then we cannot consider this element and this element; only this product is possible so we will have a 1 1 a 2 2 minus - the second row, I am considering this element and this element - if i am considering - then these 2 elements cannot become considered, so we have only option left with a 1 1; so we will have a 1 2 a 2. Now this is coming with negative sign, because this combination - this 1 2 2 1. this permutation - is an odd permutation; so we will have a negative sign. So we will be applying this definition; we are forming product of 2 terms and then this we use this concept and what we have is the determinant of this 2 by 2 matrix; so this is what we have already defined. Now the similar concept is used for defining determinant of third order matrix.

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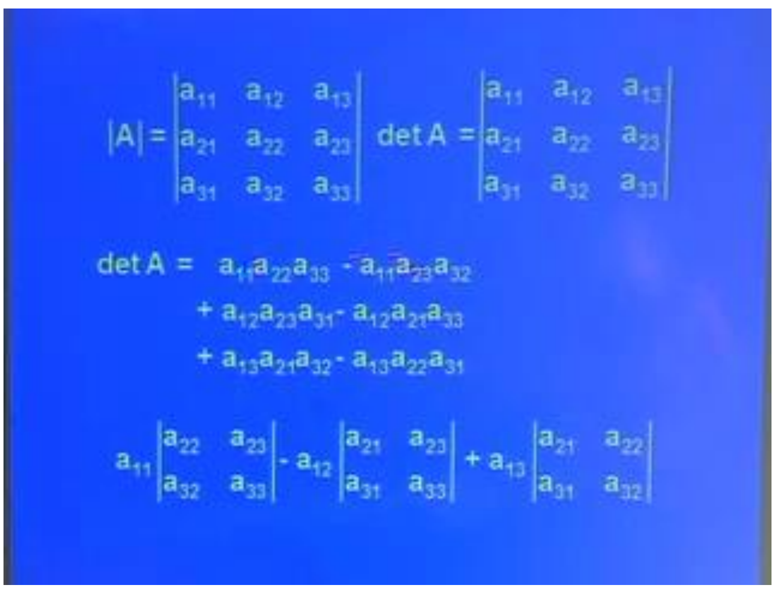

So if I have third order matrix as a 1 1 a 1 2 a 1 3 a 2 1 a 2 2 a 2 3 a 3 1 a 3 2 a 3 3, then we can denote it by this symbol, where a is the 3 by 3 matrix, or I can write it as determinant a or I write it like this. Now this - a 3 by 3 matrix; so there will be factorial 3 terms possible; so totally are 6 terms which are possible and each term is actually a product of 3 terms : a 1 1, a 2 2, a 3 3; and all these are different. Every time, I am considering only 1 term from a particular row and column; one can check this in each and every term. Now what if we consider these 2 terms, then one can very easily see that a 1 1 can is common from these two terms and what remains is a 2 2 a 3 3 minus a 2 3 a 3 2 and these two can be written down in the form of a determinant; this is a 2 2 a 3 3 minus a 2 3 a 3 2. Similarly one can write down an a 1 2 can be taken outside from these 2 terms and a 1 3 can be taken out from these 2 terms and then determinant a can be written in this particular form. I have taken the terms a 1 one a 1 2 a 1 3 outside and then I am multiplying by a 2 by 2 determinant. Now this a 1 1, a 1 2, a 1 3 happen to be the elements in the first row; so that means, I am defying the determinant in terms of first row. But this is not all. We can define determinant in terms of any rows or columns like what we can see in the next slide.

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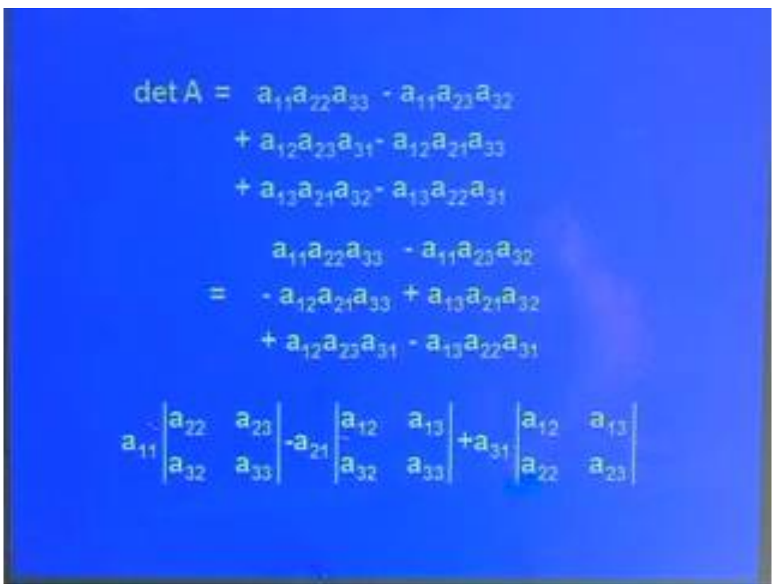

So if we start with determinant A is equal to this, then we can rearrange the terms in a different manner so that we can now have a 1 one a 2 1 and a 3 1 and they are multiplied by corresponding 2 by 2 determinants. So this is not all; one can take out, one can expand in terms of any rows or any columns.

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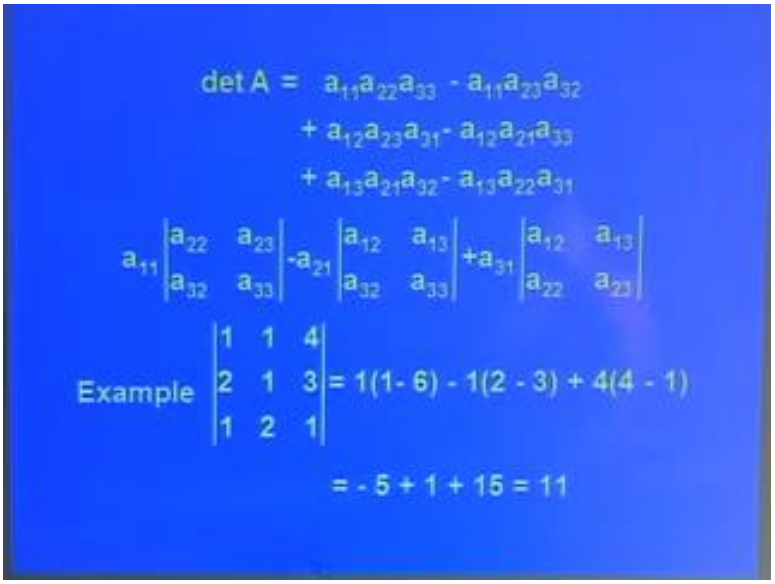

We have already discussed this. Now we will take the example; so if we have to obtain the determinant of this 3 by 3 matrix, then I am expanding about the first row. Then 1 is taken out then, we talk about we evaluate this - determinant 1 one minus 3 2 minus, this is with negative sign; then we will talk about 2 1 product minus 3 1. So the next is 4 multiplied by 4 minus 1 and the final result is eleven. What we have done for 2 by 2 matrix can we do it for a 3 by 3 matrix?

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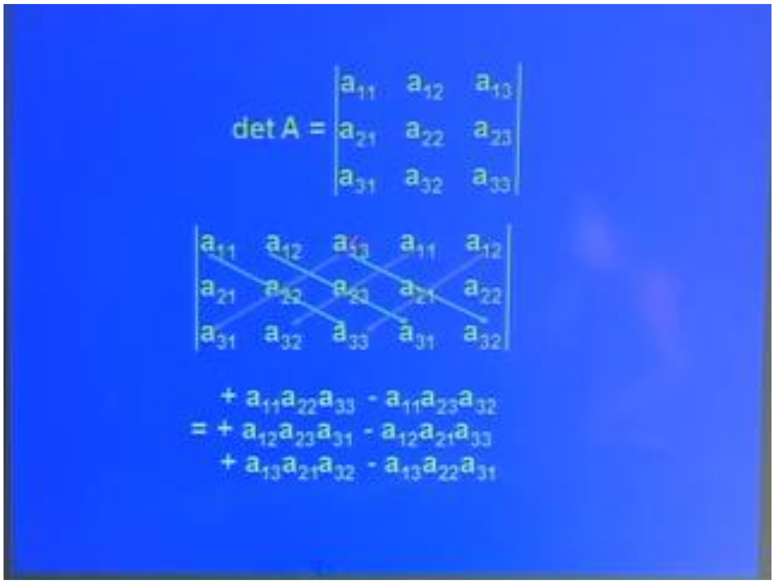

For this, let us consider the matrix 3 by 3 matrix - a 1 one a 1 2 a 1 3 a 2 1 a 2 2 a 2 3 a 3 1 a 3 2 a 3 3. To evaluate this determinant, what one can do is can repeat the first column here and can repeat this second column here. Then form the product a 1 1 a 2 2 a 3 3, a 1 2 a 2 3 a 3 1 and a 1 3 a 2 1 a 3 2. These products are written down here and then we take the product of these terms like a 1 3 a 2 3 a 3, 1 a 1 1 a 2 3 a 3 and 2 a 1 2 a 2 1 a 3 3 and these terms are written with a negative sign. So that is how you evaluate a 3 by 3 determinant.

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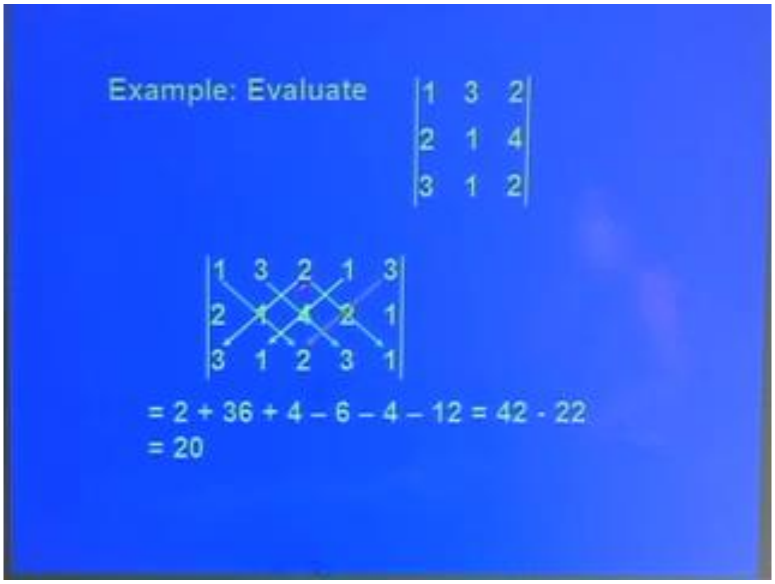

So if we illustrate this with this example, then I have copied down 1 2 3 here and this column is copied down here. Then I obtain this product, this product and this product minus this product, this product and this product and finally what i get is this 2, 36, 4, 6, 4 and 12 - this comes out to be 20. Now this is the way we have evaluated third order determinant, but if you keep on applying this method for evaluating determinants then things will become very tedious. So we can evaluate high order determinants in terms of low order determinants; I have explained this also with a 3 by 3 determinant. But now, I will give you more basic concepts related to this.

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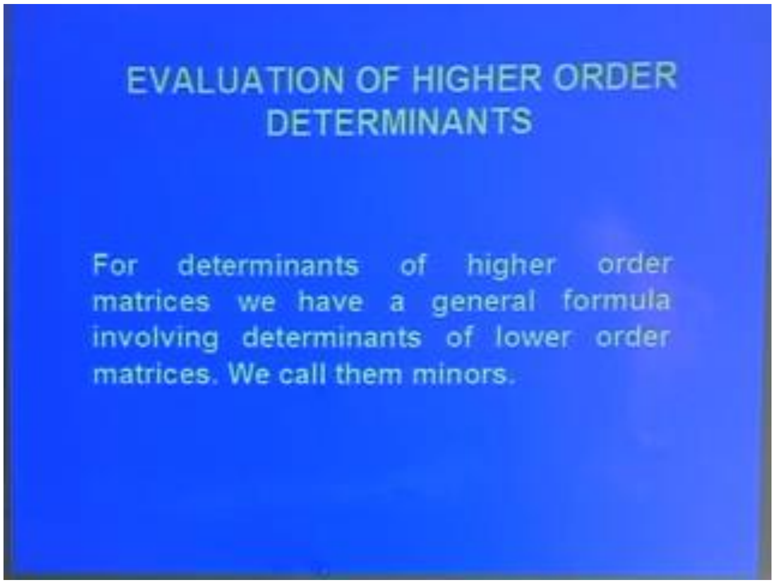

So for determinants for higher order matrices, we have a general formula involving determinants of lower order matrices; this is called minors.

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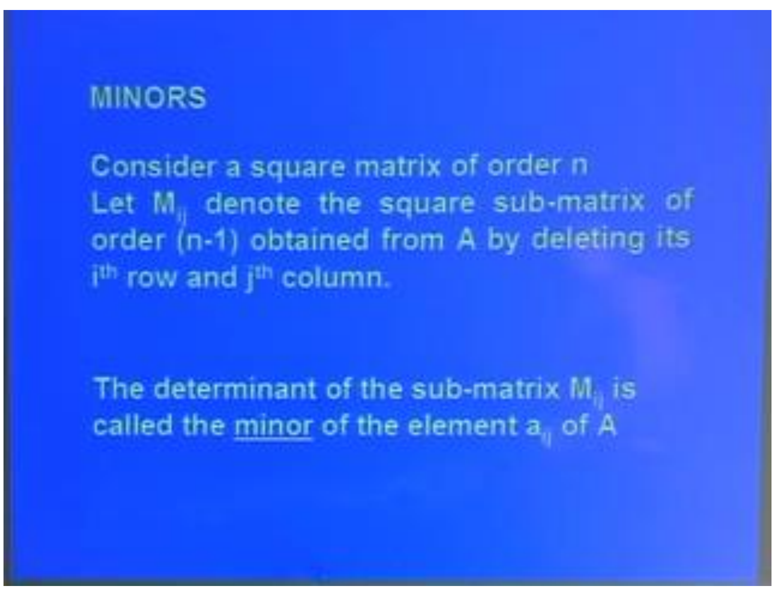

Now minor is defined as a Mij th is noted is as a Mij. it's a sub It is a sub matrix of order n minus 1. It is a square sub matrix of order n minus 1 which is obtained from a by deleting the i th row, that is this i th row and j the column. The determinant of the sub matrix mij is called the minor of the element aij of A.

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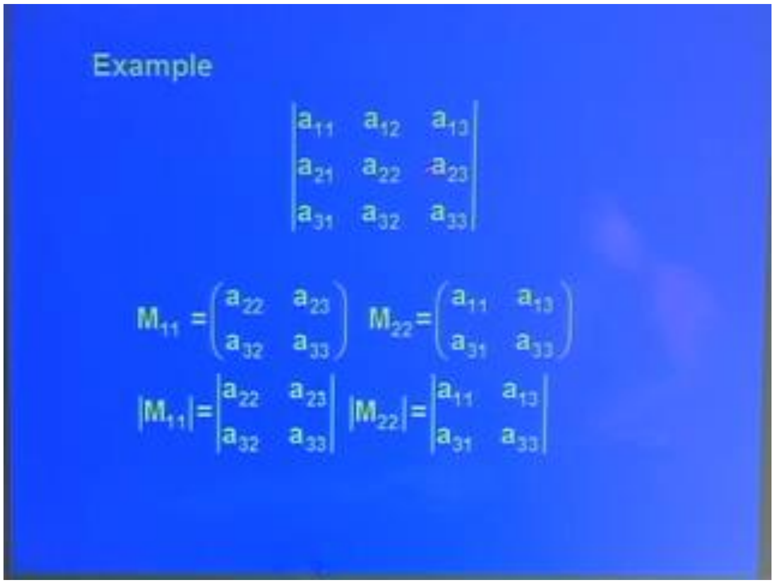

For example, if I have this 3 by 3 matrix, then the determinant of a 1 1 - if we denote it by M 1 1 - then it is obtained by the determinant of this 2 by 2 square sub matrix. While M 2 2 M 2 2 is this element is the is the minor of this element and this this sub matrix is obtained by deleting this row and this column, and what we have is this minor. Now M 1 1, this is sub matrix - M 1 1 and M 2 two are sub matrices - M 1 1 is the minor of M 1 1 which is the determinant of this, while M 2 2 is the determinant of this sub matrix a 1 one a 1 3 a 3 1 and a 3 3.

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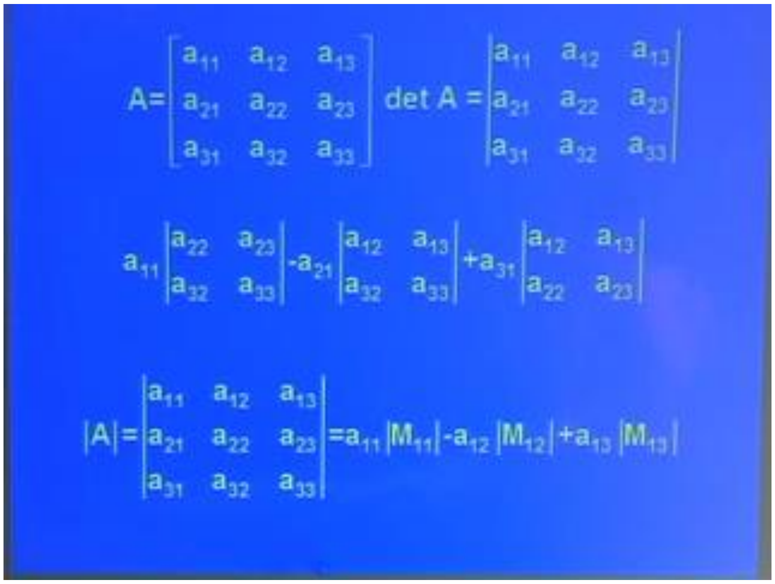

So for given matrix A, the determinant A is this and is computed as a 1 1 multiplied by a 2 2 a 2 3 a 3 2 a 3 3, that is this, minus a 2 1 – this - multiplied by a second order determinant and then a 3 1 multiplied by this. Actually this formula is already being obtained in my earlier slides. Using this concept, one can write down determinant A as a 1 1 minor M 1 1 minus a 1 2 multiplied by M 1 2 plus a 1 3 multiplied by M 1 3.

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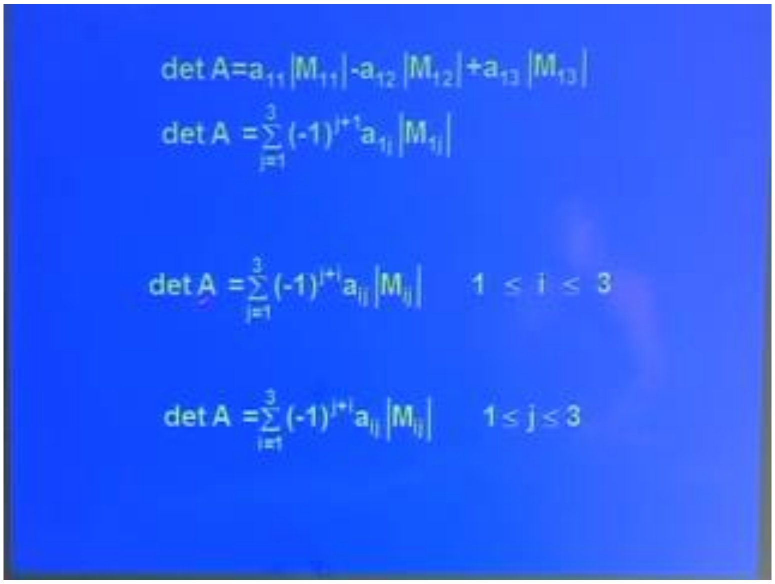

This minus and plus is coming because we have to have alternate signs and this formula can be used for these signs. Now we say a 1 j M 1 j, the sign which is to be taken is minus 1 raise to power j plus i and this is when I expand this around the first row but as I told you, this can be done for each and every row or column. So if I take any row from 1 to 3, then this formula can be used for evaluating determinant and this formula can be used for evaluating determinant when I am considering columns. Now I have introduced the concept of minors, but the problem with minors is one has to be remember the sign minus, plus - which term is to be taken with minus sign, which term is to be taken by with plus sign.

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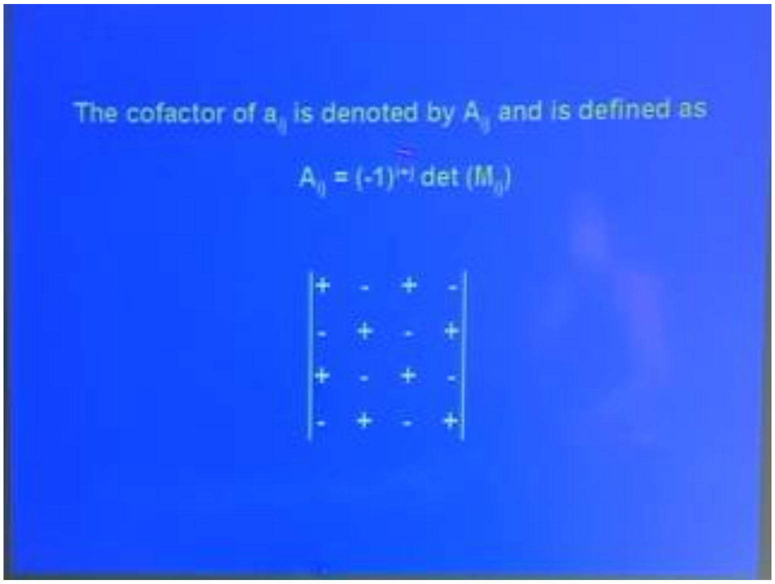

So what I do is what we do is we introduce another term called cofactor associated with the element aij - we denote it by Aij. It is defined as Aij is equal to minus 1 raised to power i plus j determinant Mij so that minus 1 i plus j is taken with the cofactor and in fact we don't have to remember this formula. One can use this checked pattern to associate sign with appropriate Mij. The first term starts with plus, then next term is minus, then plus, minus and so on; so this checked pattern can be used for evaluating determinants.

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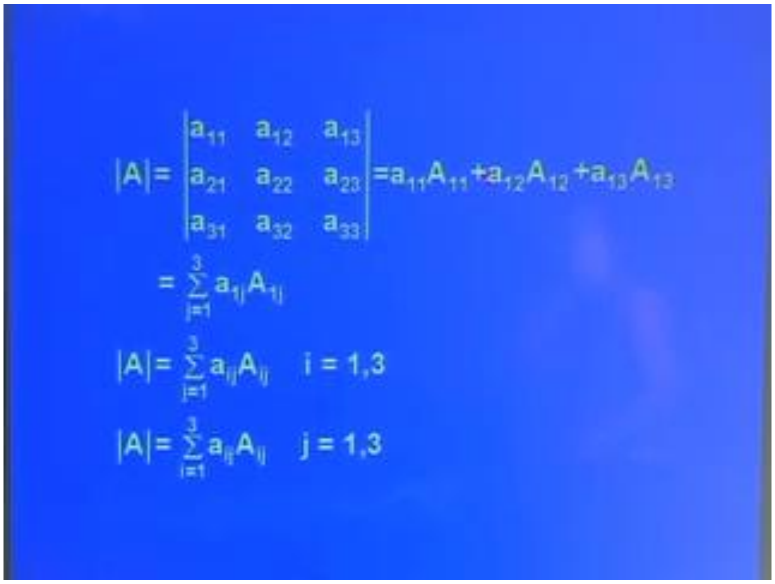

Using this, one can write down the determinant A as a 1 1 a 1 1 plus a 1 2 a 1 2, a 1 3 a 1 3. Here a 1 1 is the cofactor of a 1 1; a 1 2 is the cofactor of a 1 2; a 1 3 is a cofactor of a 1 3. So in short we can write down this expression as summation - j is equal to 1 to 3 - a 1 j A 1 j; this is cofactor. Then determinant A can also be written in this. I can take any value from 1 to 3; that means it can be expanded about any row and this means j is taking values from 1 to 3; that means it can be expanded about any column.

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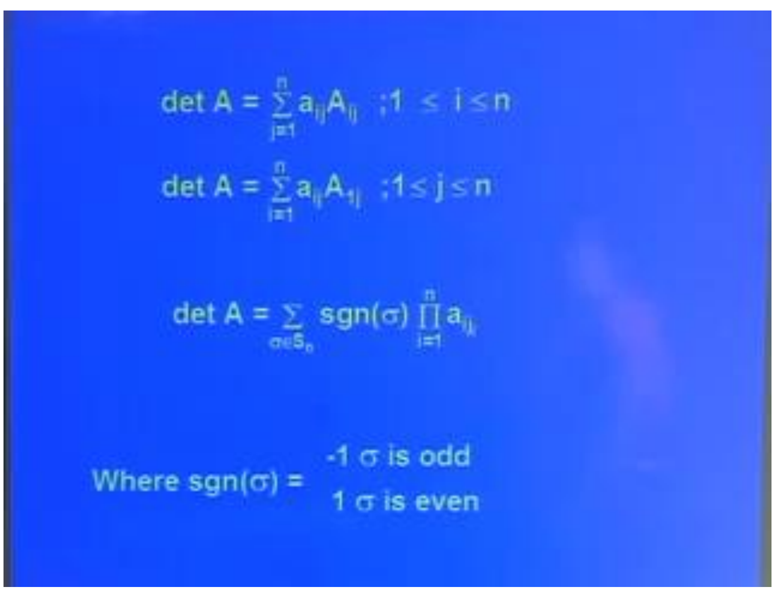

For general square matrix of order A, the formula can be written in this particular form where i will take values from 1 to n and j will take values from 1 to n. This is what we have obtained earlier and determinant A is equal to sigma belonging to Sn; so that is how this definition is being derived from the definition which I have introduced in the beginning. Now i illustrate this with the help of an example.

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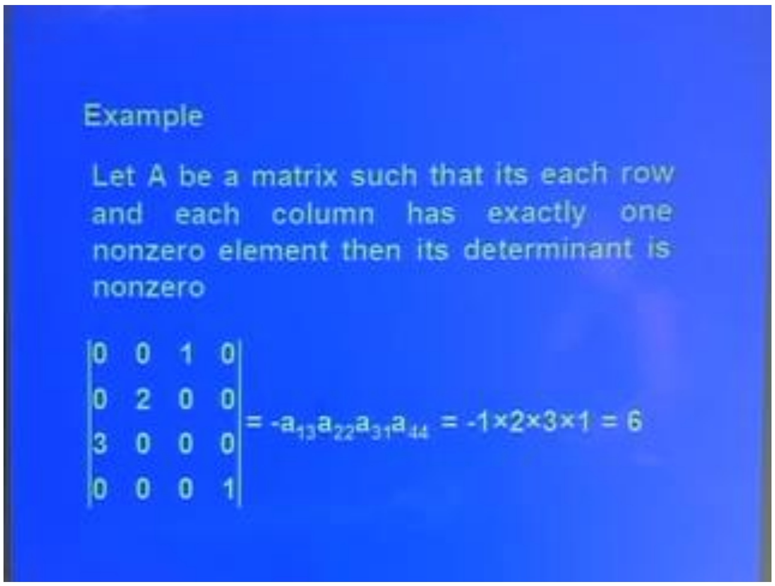

Let us say A be a matrix such that its each row and each column has exactly 1 nonzero element; then its determinant is nonzero. See if I have I am taking here a 4 by 4 determinant; then every row or column has 1 only 1 entry nonzero. So when you form this determinant and form factorial 4 terms, then you will notice that only 1 term will remain; rest of the term will be rest of terms will be 0 and what is that term? that term is a 1 3, a 1 3 a 2 2 - this term - a 3 1 - this term and a 4 4 - this term; rest of the terms will be 0 and we are we are putting this with negative sign because this is of order this if of inversion 1. So it is an odd inversion; that is why we are having a negative sign. So what is the determinant of this matrix? That is 1 into 2 into 3 into 1 and the final result is 6. Now after defining the determinants, I will discuss the properties of determinants. These properties are important because they can be used to evaluate this this can be used to compute the determinant in a convenient manner. Let us see the properties one by one.

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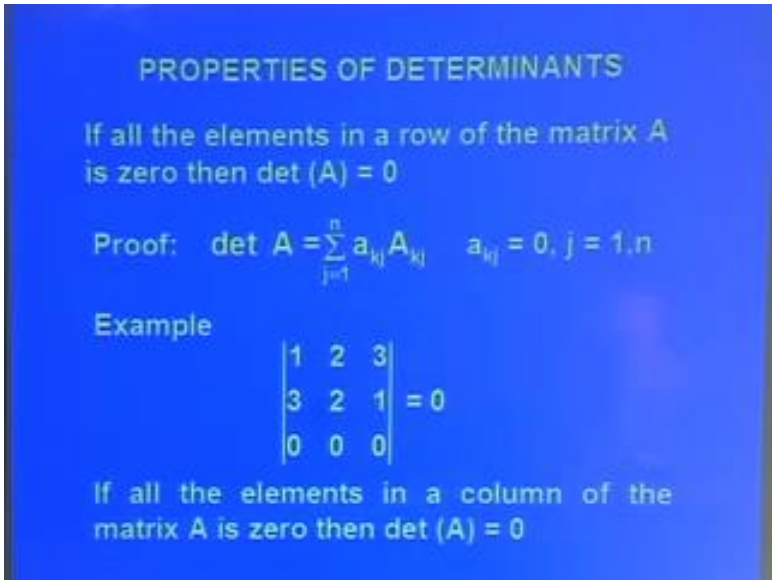

If all the elements in a row of the matrix A is zero then determinant A is zero. Now first, let me give you the proof of this. Determinant A, as I told you, can be written is j is equal to 1 to n akj Akj - this akj is a cofactor of akj. Now if all the elements in a row of the matrix A is 0, then you can use that particular row for expanding the determinants. So let us say k is the row in which we will have all elements zero. Then we can use this definition; since akj is zero so for all values of j 1 to n, so everything will be 0. For example, if you have this determinant in which the third row has all elements 0, then the final result will be 0. One can very easily check start with 0; whatever be the determinant this term will not contribute. Same thing will happen with this; this multiplied by its cofactor will not contribute anything; this multiplied by its cofactor will also not contribute anything and that is what we are having zero. Now this proof I have given then the row of a matrix has all 0 elements; but it may happen that a column of a matrix has all zero elements; the proof will be on the same lines and the result will be that if all the elements in a column of a matrix A is 0, then determinant A is 0.

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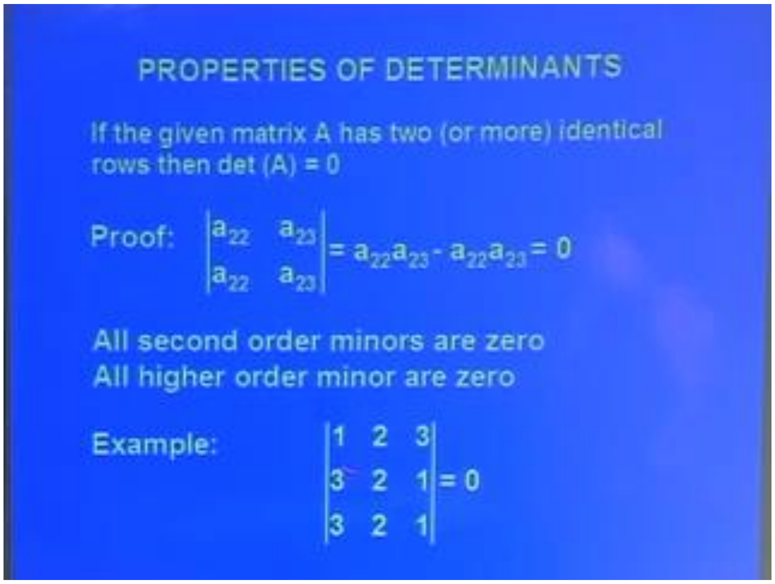

Now the next property says that if the given matrix A has 2 or more identical rows then still then still the determinant A will be 0. Now to prove this result, I will start with a 2 by 2 matrix; and in this case a 2 2 and the first row is identical with the second row so a 2 2 a 2 3 minus a 2 3 a 2 2; this is 0. Now this can this concept can be generalized for n th order determinant because n th order determinant can be written in terms of n minus 1 one th order determinant. This n minus 1 th order determinant will be written in terms of n minus with order determinant and ultimately every term will be written in terms of second order determinant. and Since all the second order determinants are 0 here, so the final result will be 0. So all second order minors are 0 in this case and all high order minors are also 0 because of this, and that is how the result is proved. For example in this particular matrix, the second row and third row are identical and that is why if you evaluate it, the final result will be zero.

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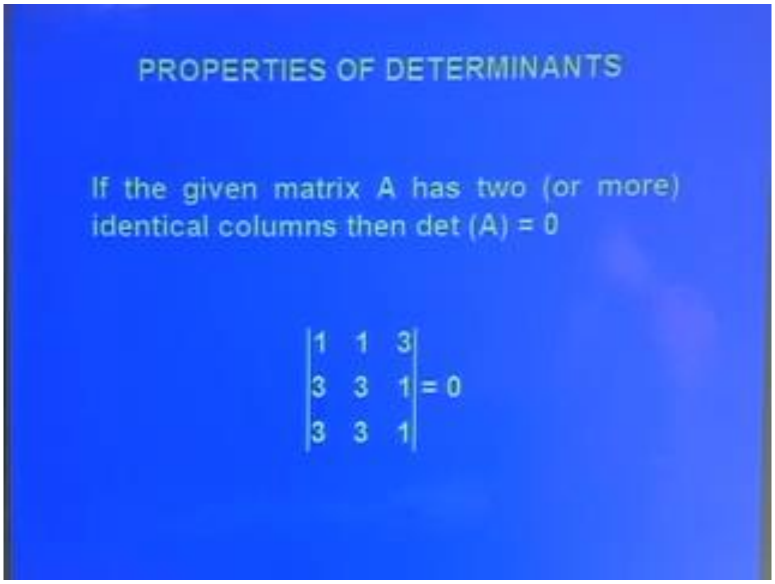

The next property says that if the given matrix A has 2 or more identical columns then determinant A is zero. So earlier, we have proved with rows and this, we have proved for columns; the proof will go in the exactly on the same lines as we have done earlier.

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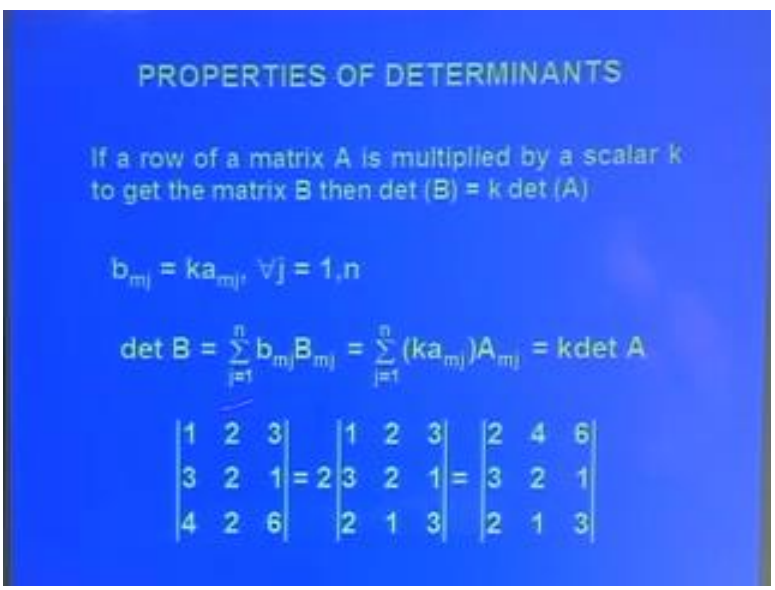

Now this property says that if matrix A is multiplied by a scalar k to get the matrix B, then determinant B is equal to k determinant A. For example, let us say we have a matrix

A with elements aij; if in the m th row we have k times amj for all values of 1 to n, that means that if a row of a matrix A is multiplied by scalar k - so let us say, I am multiplying m th row of the matrix by k and i am getting a new matrix bmj - then determinant B will be k times determinant A. The proof of this is simple. I am writing determinant B; by definition it is j is equal to 1 to n summation bmj and its cofactor Bmj. is equal to Now for bmj, I have k times amj so this - kamj into Amj. Now k can be taken outside so what remains is amj into Amj and that is the determinant A; so determinant B is k times determinant A. So this is the result is proved and we illustrate this result by this example, where we are having a matrix 1 2 3 3 2 1 4 2 6 - 3 by 3 matrix. Now this particular row is 2 times this particular row; so if i take 2 outside then this row becomes 2 1 3 and this is the same as 2 this is the same as if this row is multiplied by 2. So it is 2 4 6 3 2 1 2 1 3; so this determinant is equal to this determinant. Now in the next property, we discuss what happens when the 2 rows of a matrix are identical. What happens to its determinant?

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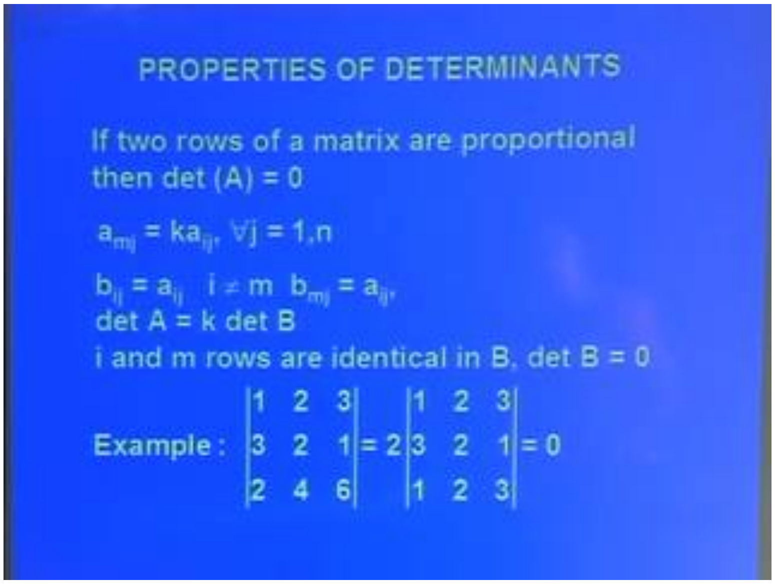

So if 2 rows of a matrix are proportional, then determinant A is equal to 0. Now to prove this, we assume that m th and i th of the given matrix A are proportional; that means amj is equal to kij for all values of j between 1 to n. Then one can find one can determine a matrix B, which is same as bij is equal to aij for all values of i not equal to m; but that means all the rows of B and A are identical except the m th row. In The m th row of the matrix B is given by bmj is equal to aij; that means it is identical with the ij th row. Now that means one can take k outside from the B matrix and what we have is determinant A is equal to k times determinant B. This is a result which we have proved in my earlier property. Now since determinant B has 2 identical rows namely m th row and i th row, that means determinant B is 0. So i and m rows are identical in B; so determinant B is zero and that means determinant A is also zero. Now look at this example. Here we are having 1 2 3 , 3 2 1, 2 4 6, that is, the first row and the third row - they are proportional. So one can write down it is 2 times this; so this is my B matrix. Now in this B matrix, 1 2 3 and 1 2 3 in the first row and the third row are identical. So this determinant is zero and that proves that this 3 by 3 determinant has value zero.

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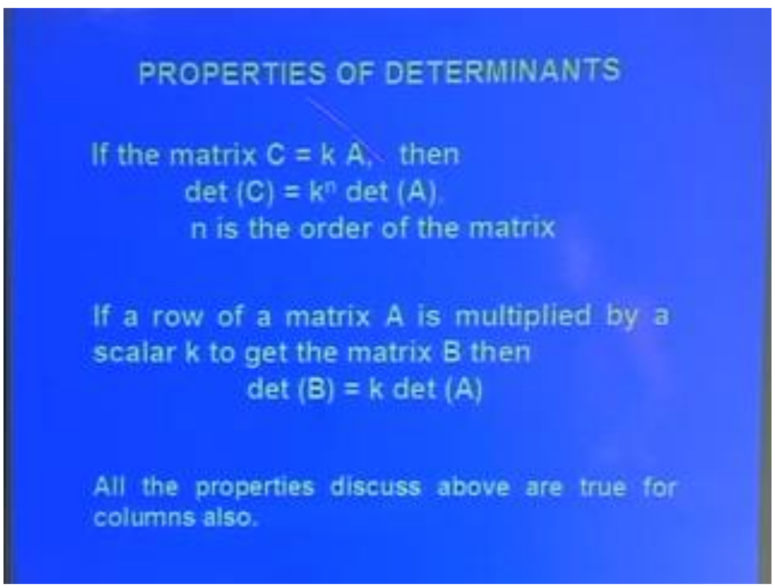

Now if a matrix C is multiplied by a constant k, that means each and every row of the matrix A is multiplied by k, then determinant C is equal to k raised to power n determinant A, where n is the order of the matrix. The idea is each row is from each row, one can take k outside and that way, there will be n such ks taken out and the final result is k determinant C is equal to kn determinant A.

Now if a row of a matrix A is multiplied by a scalar k to get the matrix B, then determinant B is equal to k determinant A; this is what we have proved in my earlier result and if you apply it for each and every row, then we will having this. Now, the properties which we have discussed so far, they are applicable to rows; we have proved them for rows, but they are applicable to columns also. So, all the properties are applicable to rows as well as columns.

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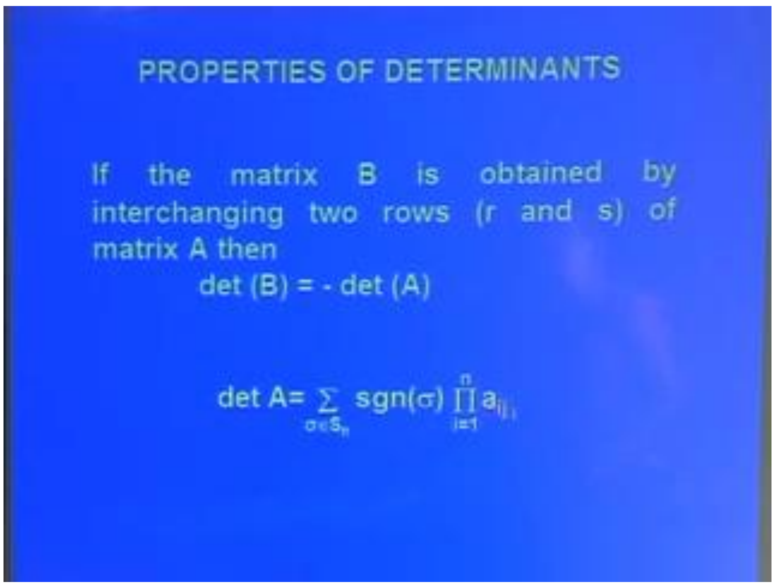

Now if the matrix B is obtained by interchanging 2 rows, namely r and s of the matrix A, then determinant B is the negative of determinant A. That means, when you interchange 2 rows of a matrix, then this minus sign will come in between. Now the proof of this the proof of this thing can be proved very easily from this, because if you are changing 2 rows, then actually the signum sigma will change; the inversion will change and 1 minus sign will be coming because of this. That is why determinant B is equal to minus determinant A.

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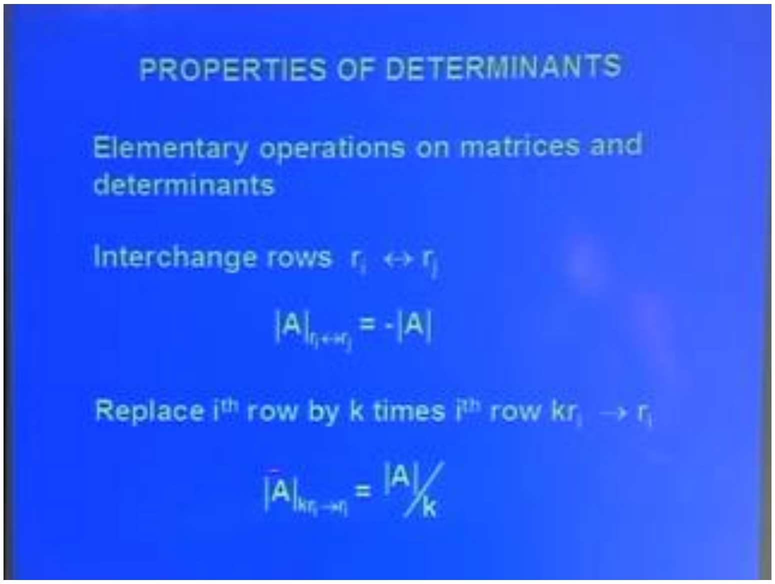

Now in the next set of properties, we will see the effect of elementary transformat elementary operations on matrices and we will see what their effect on determinants is. So we have first, elementary operations where we interchange i th row with j th row of the matrix. Now if we change the i th row and j th row, we denote it by determinant A; ri and rj are interchanged; this is used as a suffix; then this is equal to minus time A. This, I have proved in my earlier slide. Now if you replace i th row by k times i th row, then ki will go to ri and in that case - if we use our earlier results - then determinant of A, by interchanging kri to ri, is equal to determinant A divided by k. I will explain this with examples.

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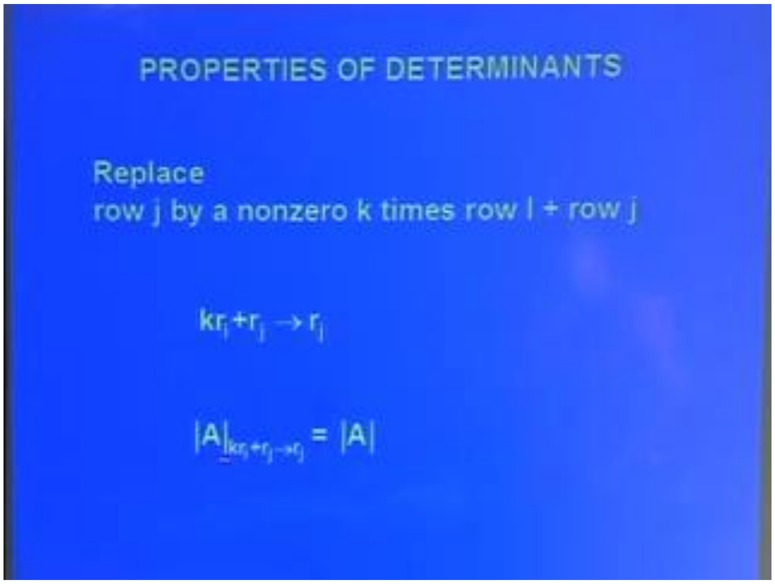

The third elementary operation is when we replace row j by a nonzero k times row plus row j. So kri plus rj will be replaced by rj; so that means determinant of A after replacing this by rj is equal to A. So Let me explain this with example.

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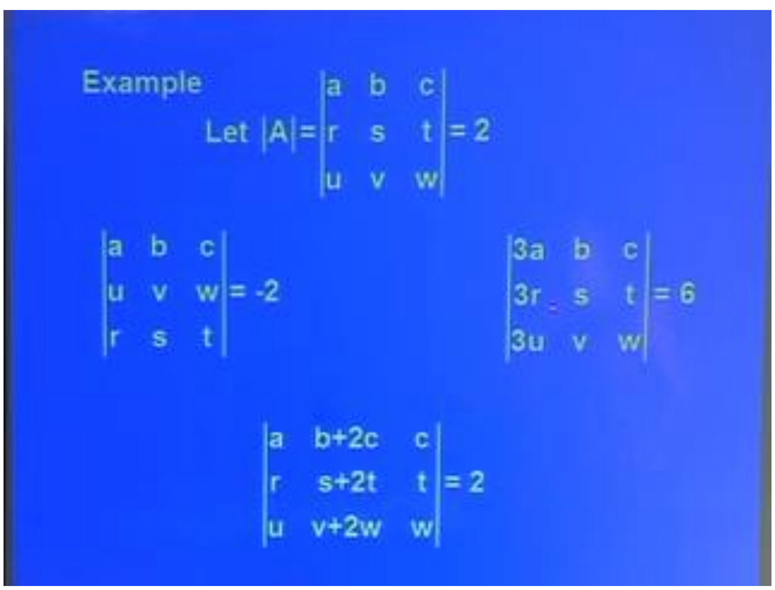

So let us say I have matrix A as a b c in the first row, r s t in the second row, u v w in the third row and its value is 2. Let us say this the a b c are such that this determinant is equal to 2. Then a b c u v w r s t - this determinant - is actually obtained from this determinant, by interchanging the second row and third row. So because we are changing 2 rows, so the minus sign will come; since this this determinant is 2, so this determinant will be minus 2. Now all these elementary row operations are all the - all the results related with rows - are applicable to columns also. So if I write down this determinant as 3 a b c 3 r s t 3 u v w, that simply means that I am multiplying this column by 3 and I am getting this this 3 times a 3 times - this is this is actually related with this, not with this; so I have to i have change it it is not with this I have to show this reference. This determinant is appear This determinant is obtained from this determinant by multiplying this column by 3. So this column is 3 a 3 r 3 u; rest of the things remains the same - b s v c t w. Since this is this is multiplied by 3, so the effect of this is that is 3 times 2 and the final result is 6. Now this matrix is obtained by replacing the second column as b plus 2 c s plus 2 t v plus 2 w. This is actually obtained by adding this plus 2 times this column; so it is the second column c 2 multiplied by 2 times third column. So each element of second row will second row will become b plus 2 c s plus 2 t v plus 2 w and the result will not be change; it will remain as 2 only. So this determinant and this determinant both have values 2.

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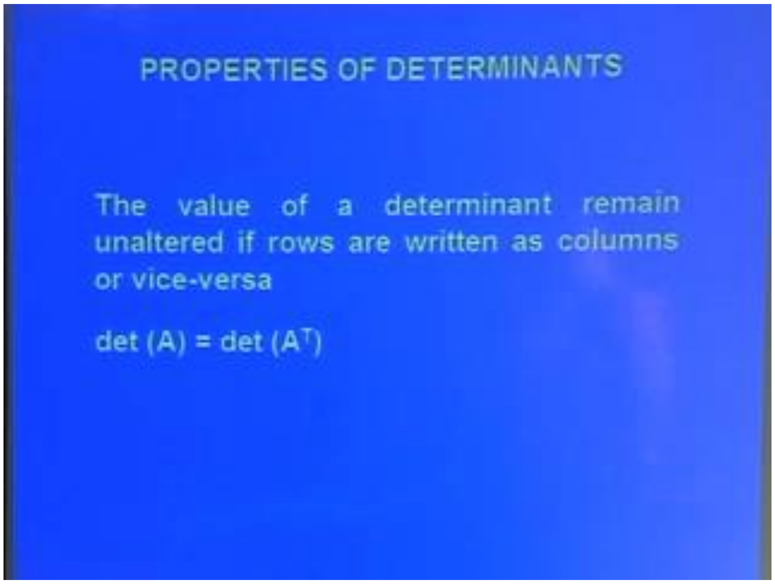

Now the value of the determinant remains unaltered if rows are written as columns or vice versa because all the properties which we have discussed so far there are applicable to rows as well as to columns and with this we have another property; that is, determinant A is equal to determinant of A transpose because the transpose A is actually obtained from the matrix A by writing rows as columns. So that is why determinant A is equal to determinant of A transpose. The things will be further simplified, if we talk about some special matrices like diagonal matrix.

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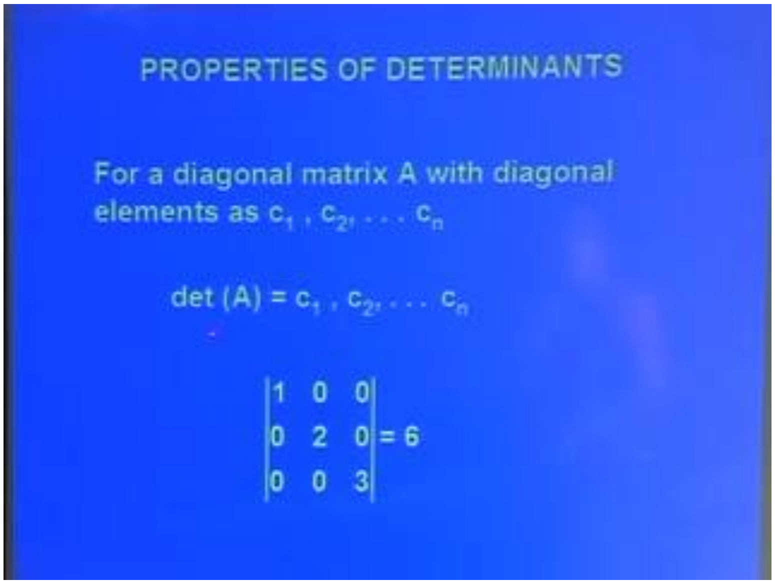

Now is we have a diagonal matrix A having elements as c 1, c 2, cn. The diagonal matrix A means it has only the diagonal elements nonzero; rest of the elements are 0. So the determinant of A in that case will be c 1 c 2 cn, because this is actually true from the result which we have discussed in my first property. So if we have this diagonal element diagonal matrix, then its determinant will be 1 2 and 3, that is 6. So we have to simple multiply the diagonal elements and that will be the value of determinant. Now this result can be generalized to a triangular matrix.

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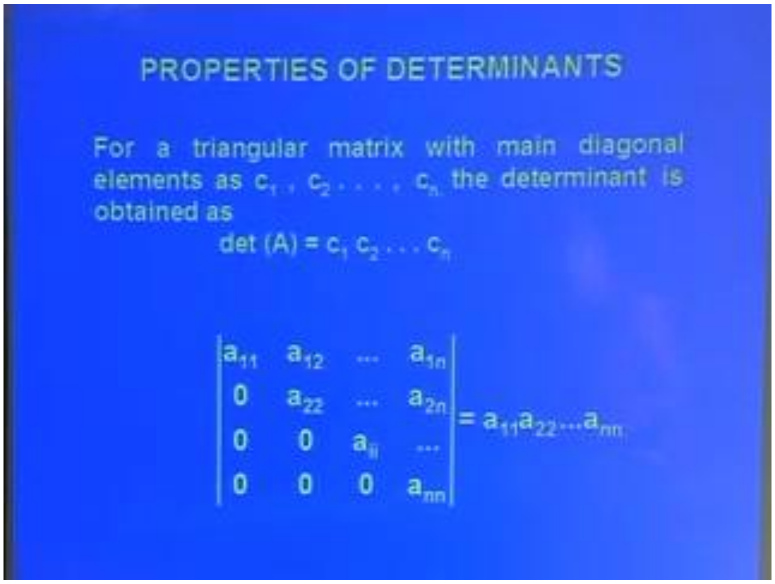

So if we have a triangular matrix with main diagonal elements as c 1 c 2 cn, then the determinant is obtained as the product of diagonal elements. Let us see that is proof for this. So Let us consider this triangular matrix. This is actually an upper triangular matrix in which all the elements shown on the upper side of the diagonal they are nonzero while all the elements on the lower side of the diagonal are 0. So if this determinant is to be evaluated this, this determinant comes out to be the product of diagonal elements. The proof is simple; what we have to do is we will consider this a 1 we will we can expand this determinant along this first column. So we will having a 1 1 multiplied by this minor plus 0 multiplied by its corresponding minor - will not contribute anything; 0 multiplied by its minor will not contribute anything, 0 multiplied by this minor will not contribute anything; so we will have only a 1 1 multiplied by this minor. Now for this minor, we evaluate along this particular column; so a 2 2 multiplied by this minor will contribute but rest of the terms we will not contribute. Similarly we will go to the next lower lower order minor and this way we will continue and finally we will have only this term left with us.So what we have is the product a 1 1 a 2 2 ann as the value of this lower triangular matrix.

Now this is the case when we are having lower triangular matrix. If we will have an upper triangular matrix in which only the these terms in the lower diagonal they are nonzero and these terms are 0, so the result can be proved on the same very lines only. Thing is Instead of taking this particular row, one has to take this particularly column in that case and determinant A will be simply the product of diagonal terms. So whether we have a triangular matrix or we have diagonal matrix, its determinant will be product of diagonal terms.

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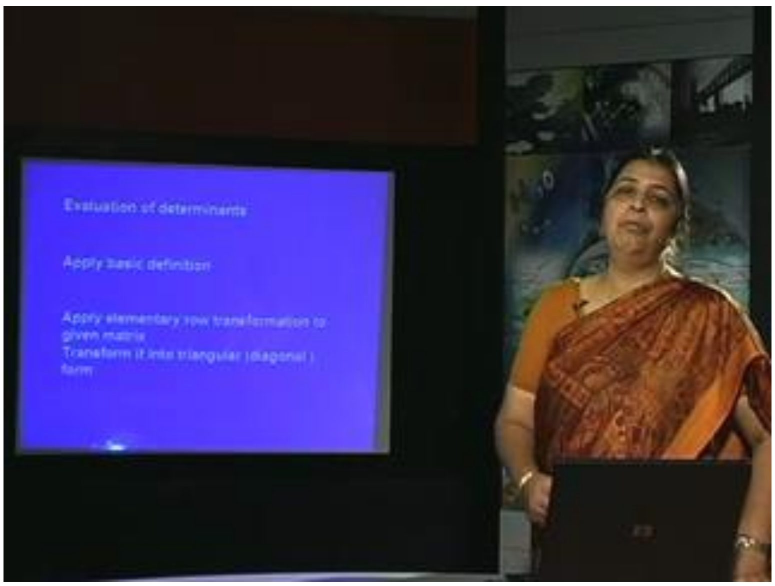

Now we have seen number of properties for evaluating determinants. If we have to apply these properties, then things will be very complicated especially when we have higher order determinants, because the computation effort required will be much more. So what we do is we we can apply basic definition, but the lot of computation effort will be required for that purpose. But if we use elementary transformations to a given matrix and simplify it to some suitable form and then one can very easily evaluate the determinant with least effort. So what we can do is we can try to transform our matrix may be to triangular matrix; so that some effort is required to reduce it into triangular matrix. but then evaluation of determinant will be simple because in that case you have to simply get the product of diagonal terms. So that is the way to evaluate the diagonals; let me explain this with an example.

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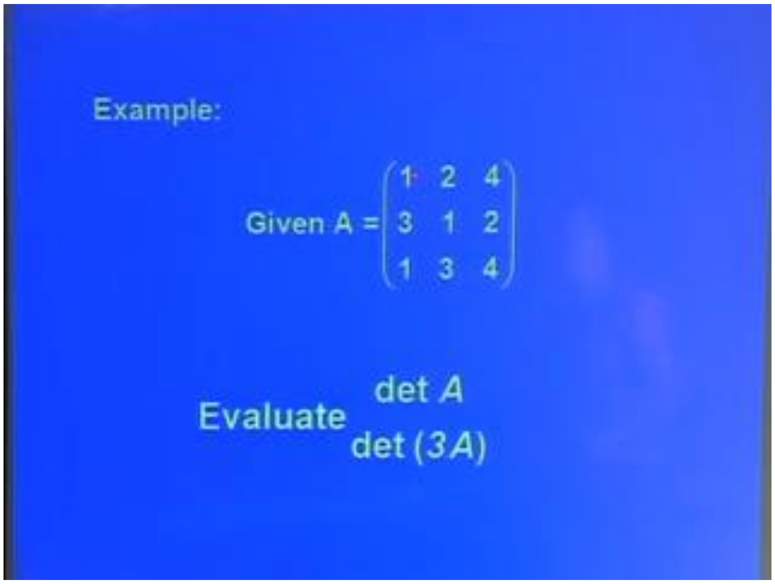

So let us take a given matrix is A.

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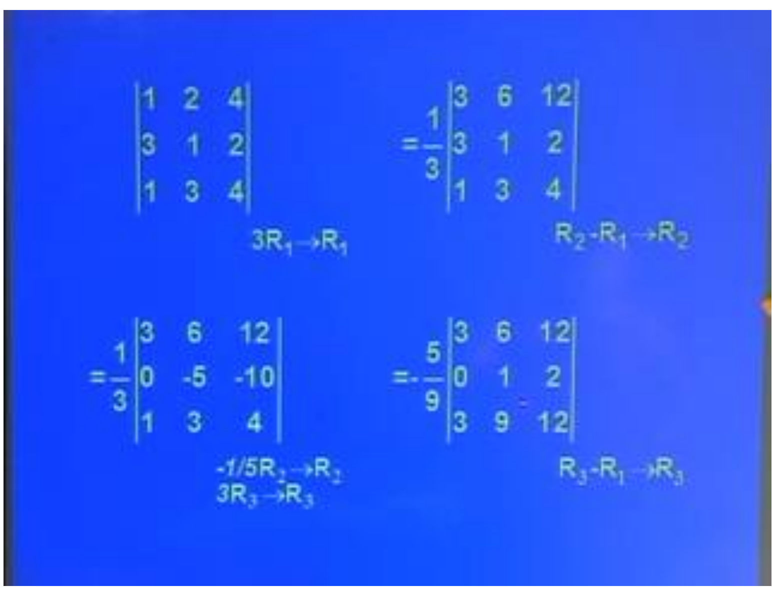

If you have to evaluate this 3 by 3 matrix then what you have to do is: one way may be just evaluate this multiplied by this minor minus this multiplied by its corresponding minor plus this multiplied by its corresponding minor, but that will required lot of computation effort. Now we will try to make this matrix as a triangular matrix so that we can get the determinant by simply evaluating the product. Now for this, if we apply the we multiply the first row by 3, that is, this row is multiplied by 3 so 3 into 6 into 12; since I am changing the row by a multiple, so these two will be equal, if we will behaving 1 by 3 on this side. So the after applying this operation simply means 1 by 3 of this.

Now since these two are identical, so what I can do is R 2 minus R 1 is replaced by R 2. So second row is replaced by R 2 minus R 1; so this is my new second row; this is this minus this; so it is zero. 1 minus 6 is minus 5, 2 minus 12 is minus 10 and third row remains the same. Now on this determinant, I will apply this transformation - this elementary operation - that is, second row is minus 1 by 5 times R 2. So if I apply this row, then since I am multiplying by minus 1 by 5, so minus 5 will appear here and this second row will become 0 1, because minus 5 has been taken out and minus 2 n we will have 2 here. Now for the third row: this row is changed here. How have I obtained this row? I have multiplied this row by 3. So we will have 3 here, 9 here and 12 here, and since I am multiplying it by 3, so I am dividing by 3 here; so the factor is minus 5 by nine; so this matrix is obtained. Now again, I can apply this operation. Third row is changed by R 3 minus R 1 because these rows are identical.

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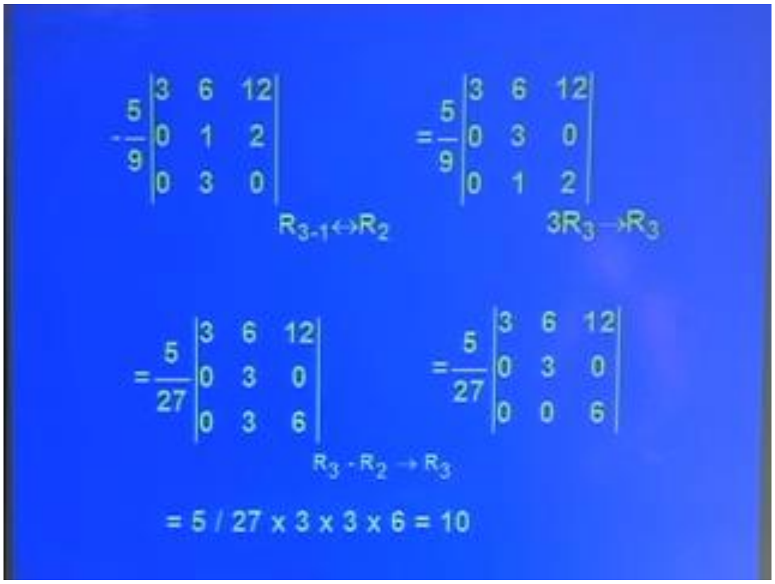

Now so you can the this element is identical; so if I subtract, I will get 0, then 1, 2, 0, 3 and 0; it is close triangular matrix but still there are certain terms which will be not make it to be a diagonal element; so I will write down as R 3. I am interchanging 2 rows now so 1 and 2 will be here and it is zero 3 0 3 0 is appearing here; since I am interchanging two rows, so this minus sign will be absorbed here. Now it is 3 R 3 is replaced by R 3; that means, third row is replaced by 3 times this. So this 3 times this means 3 means here and 2 into 3 is 6 and since I am multiplying by 3, so we will be having 5 by twenty seven here. So now these two become identical.

So I will again apply this elementary transformation and what I have is simply this. This and this will get cancelled; so zero 3 zero will remain the same, but this 0 3 minus 3 0 6 minus 0 is 6. Now This is now a triangular matrix and its determinant will simply be the product of these - this term, this term and this term - and what I have is this term. Now in the beginning, it may appear that we are doing lot of effort in reducing a given matrix into a triangular matrix because because lot of elementary operations have to done - but if for 3 by 3 matrix it may be true - but when the number of as the size of matrix is increasing, then the effort involved in reducing the triangular matrix is much small as compared to evaluating factorial n products and simplifying the determinant. So this effort is much less as can be seen from this 3 by 3 matrix.

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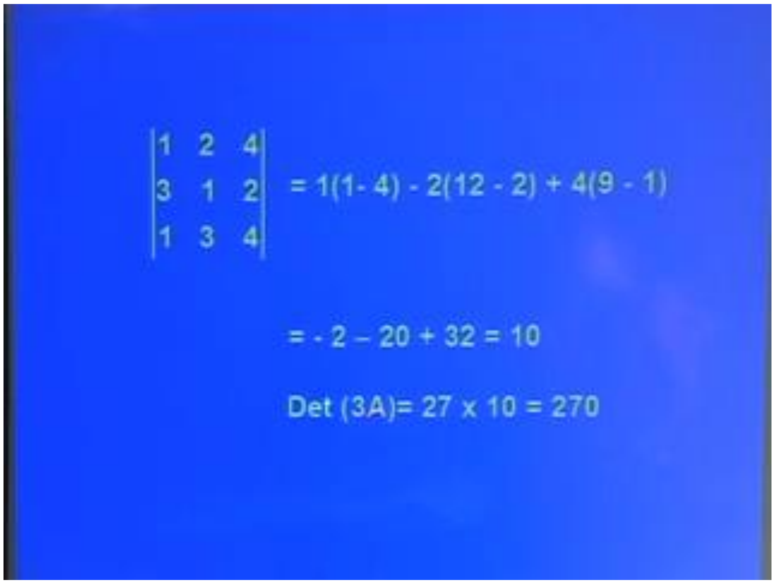

Now we can see that if we evaluate this determinant by a normal definition, which I have given you earlier, then this comes out to be 10; so the result is the same. So the determinant evaluated by the usual definition also comes out to be the same as we have obtained in the case of triangular reduction. Finally the determinant of 3 A is 3 into 3 because A is a 3 by 3 matrix; so the determinant for A is known and since I am multiplying 3 A means I am multiplying each and every row by 3; so it is 3 raised to power 3, that is, 27 into 10. So determinant of 3 A comes out to be 270.

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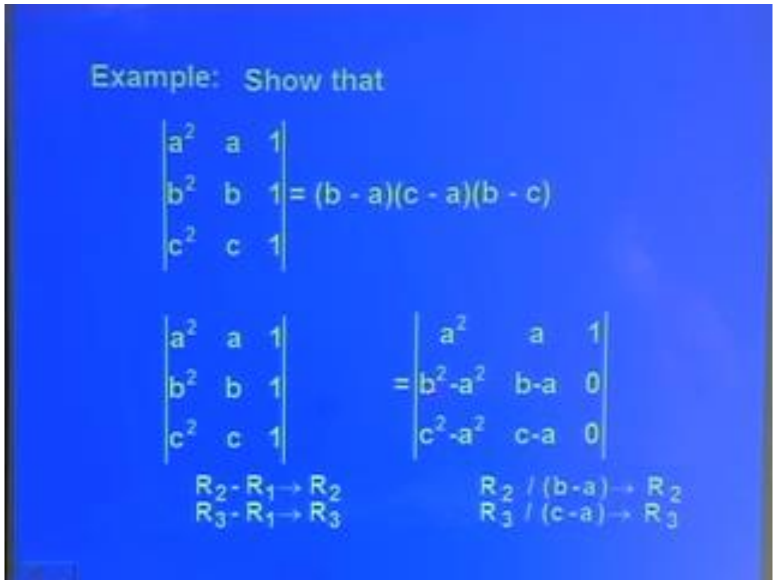

Now in this example, we will try to evaluate this determinant by using properties of the determinants which we have discussed so far. In this example, we are trying to show that the value of this determinant is equal to b minus a into c minus a into b minus c. Now this determinant is evaluated using elementary row operations. We will try to reduce this determinant into a triangular matrix. Now to reduce it into a triangular matrix, we apply a series of elementary transformations. The first transformation is that the second row is replaced by R 2 minus R 1. So here, I am writing the new second row - it is R 2 minus R 1 of this; so it is b square minus a square, b minus a and 1 minus 1 is zero. Simultaneously I apply R 3 minus R 1 to get new R 3; so that means c square minus a square, c minus a and 0. So this matrix is obtained after applying these 2 elementary operations. Now on this, I apply these elementary operations. The idea is one can very easily see that b minus a is can be taken outside from this row and c minus a can be taken outside from this row. So i say R 2 divided by b minus a is R 2 and R 3 divided by c minus a is R 3.

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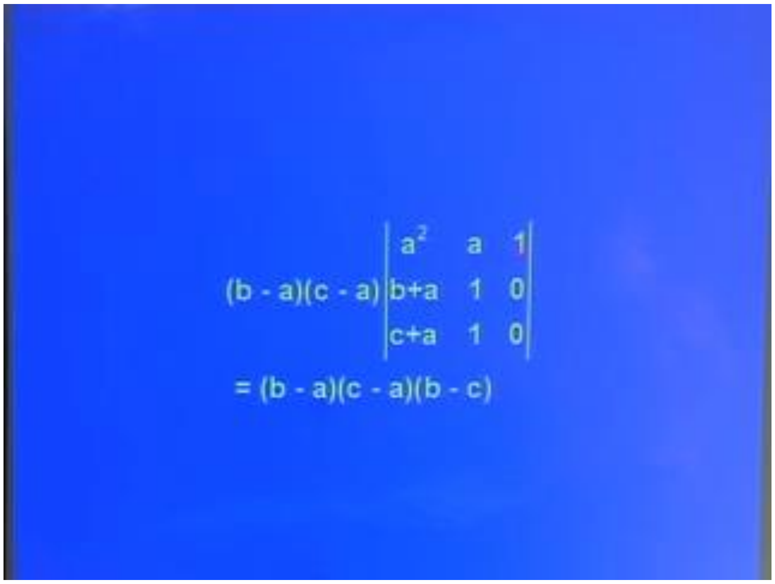

So the row will not be. So if I apply these operations on this matrix then what I get is b minus a into c minus a is taken outside; what we have is b plus a from here, 1 and 0 in the second row. c plus a will remain from the by c minus c plus a will be the remainder of b square minus a square when we take c minus a outside, then 1 and zero. Now in this, if you expand around this column, then it will be plus, minus and plus. So we will be having 1 into minor – this - and what is the minor? It is b plus a multiplied by c plus a. So these two remain as such, but when you simplify it is b plus a minus c plus a; so we will have b minus c. So this is evaluated as b minus a into c minus a into b minus c. In the next example we try to verify that this determinant is equal to this determinant.

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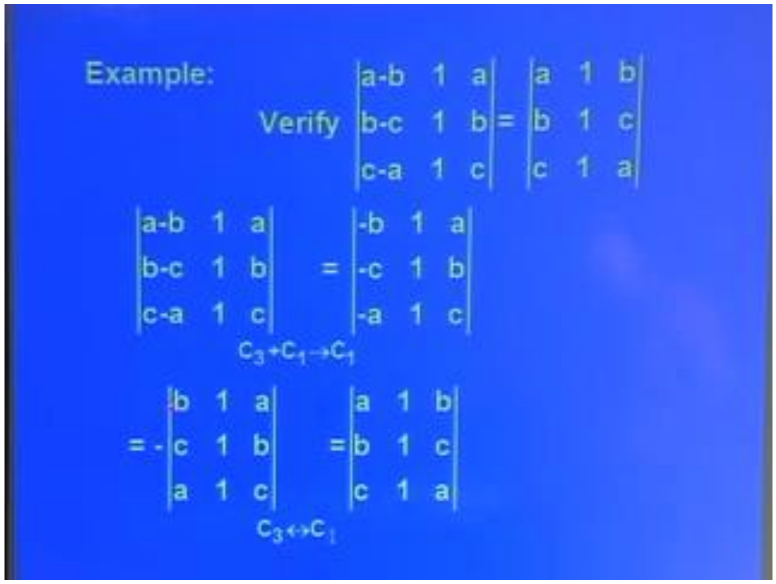

Now for this, we again apply a series of operations. c 3 plus c 1 is equal to c 1; so if I multiply, if I add the third column and the first column, then this a will be out -a will be canceled out; b minus c plus b will have only minus c; c minus a plus c will be having minus a. Now if you compare this matrix with the final objective, one can note down that the columns are actually interchanged. So to get the final result, we can interchange the column c 3 and c 1. So we apply this transformation; so c 3 becomes c 1 and c 1 becomes c 3 - this when when we apply, we get this result. and Since we are interchanging 2 rows, 2 columns here, so this minus sign and minus sign due to this interchange will be observed and what we have is the final result a 1 b b 1 c c 1 a which is the same as this.

Viewers, with this we have arrived at the end of this lecture on determinants. In this lecture, I have defined determinants. I have started with the 2 by 2 2 by 2 determinant. Then I have introduced how to evaluate a 3 by 3 determinant. Then we have given a general definition for n th order determinant. We have discussed some properties of the determinants and using those properties, we see how we can apply elementary transformation and the determinant will reduce to determinant of a triangular matrix and then the triangular matrix determinant can easily be computed. So that is all for today's lecture. Thank you.