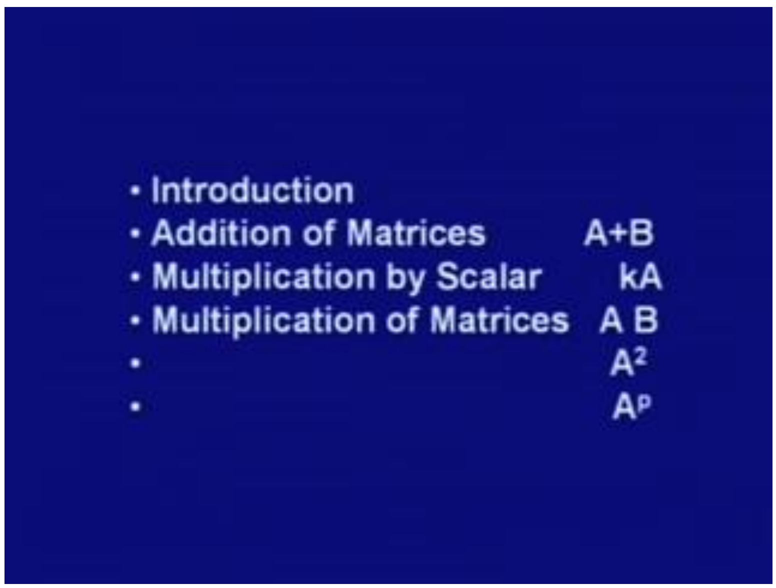
Mathematics II
Prof. Sunita Gakkhar
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 2
Matrix Algebra Part- 2

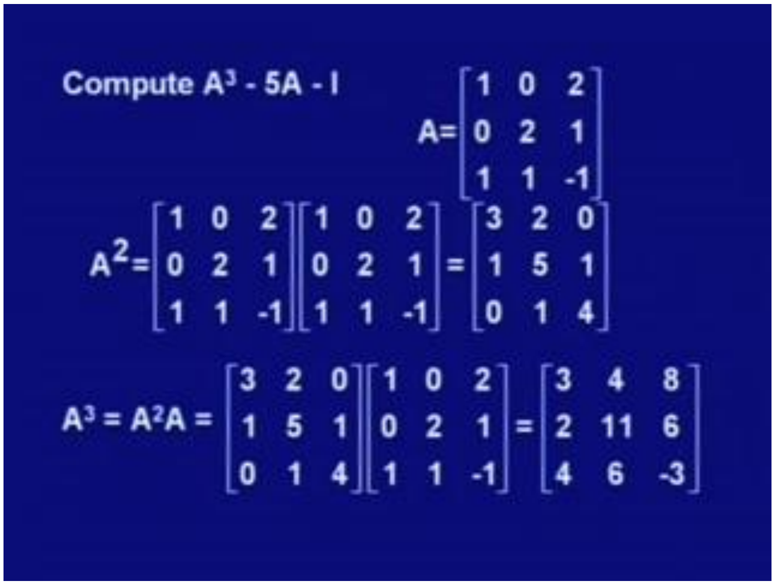
Welcome viewers! This is my second lecture on matrix algebra. In my first lecture on matrix algebra, I have introduced various concepts related to matrices. I have defined matrices; I have introduced various operations on matrices.

(Refer Slide Time: 00:45)



I have introduced addition of 2 matrices and its various properties. After that, I have introduced multiplication of matrix by a scalar quantity. Then multiplication of 2 matrices has been defined and we have discussed that matrix multiplication is not commutative. We have discussed the associative property and distributive property of matrices. We have introduced how we multiply a matrix by itself and what is the meaning of A square. I have given A raise to power p. After introducing all these things, now we are in a position to discuss expressions of the form A square plus 5 A minus 3 and so on. So to start with, let us discuss what's the meaning of A cube minus 5 A minus I.

(Refer Slide Time: 01:49)

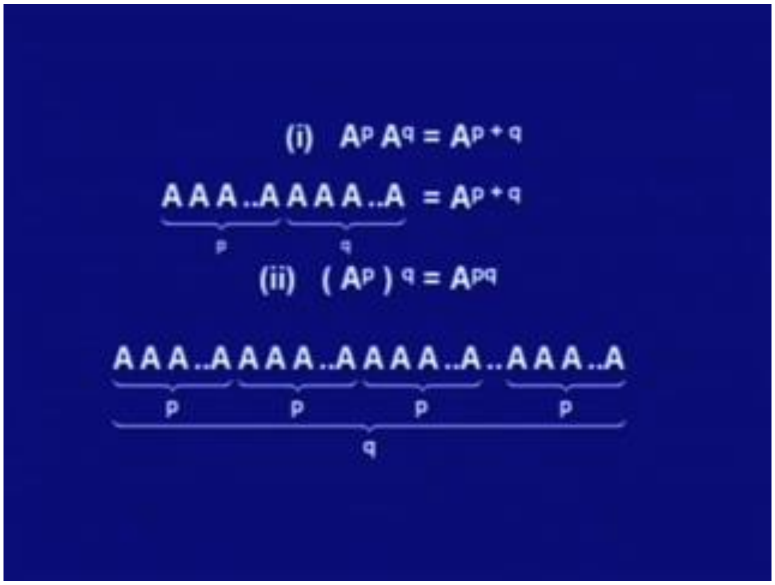


I happens to be the identity matrix. A has to be a square matrix; so that this expression becomes meaningful. The order of A and the identity matrix I must be the same. Let us compute this for the matrix A. Now to compute A cube, one has to first compute the square A square as this. Now to compute this, if you multiply the first row and the first column of A, we get the element 3 and when the second row is multiplied by the second column, we get the element 5 as the element in the product matrix and the next element third row third column gives 4. Similarly, other elements can be computed. Once we get A square, we can compute A cube which is A square multiplied by A. Now the order of A is not important. Here, we can multiply A square by A or we multiply A by A square because this is associative and this gives me the result as this matrix.

(Refer Slide Time: 03:32)

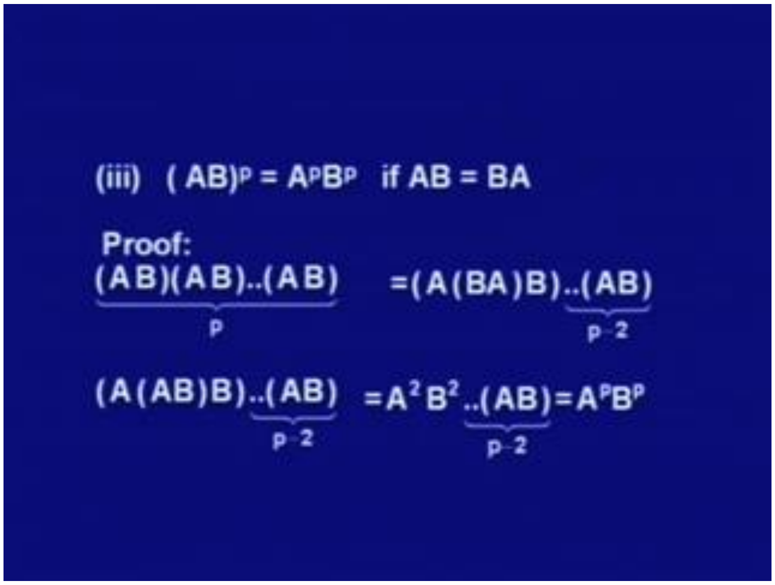
Once I have computed A cube, the values can be substituted in this. So I have 2 times the value of A cube minus 5 times A and the identity matrix of order three. 2 multiplied by A can be computed as 6 8 16 in the first row; 4, 22 and 12 in the second row; 8, 12 and minus 6 in the third row and this minus 5 can be taken inside and this matrix is nothing but minus 5 A, and then this matrix. and What we have is the final result as this matrix. So if i had A as the 3 by 3 matrix, then 2 A cube minus 5 A minus I is also a 3 by 3 matrix and it is computed as this.

(Refer Slide Time: 04:30)



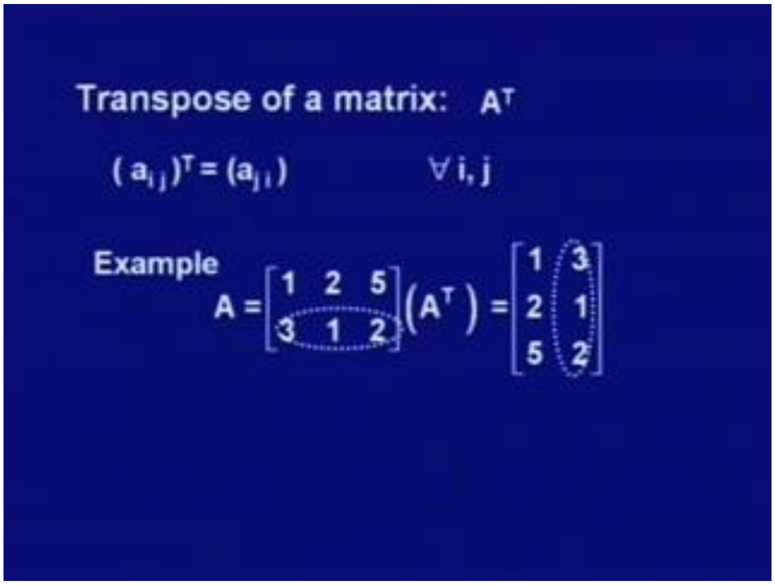
Now I can introduce the meaning for A p multiplied by A q; by A p i mean to say A multiplied p times and A q means A multiplied q times. So if i multiply A p and A q, I have a matrix A raise to power p plus q. Now this can be proved easily. Like the first expression, A p is this - that is A multiplied p times - and this is A q, that is A multiplied q times. Now you can notice that total number of A's they are p plus q and that is why we have A raised to power p plus q on the right hand side. Similarly A p when multiplied q times, we have a matrix A raised to power p q and this can be proved as this. Here A is p times, this is AP; this is another A p and this is to be multiplied q times. So and these are q such expressions and what we have on the right hand side, A raised to power p q.

(Refer Slide Time: 00:05:45 min)



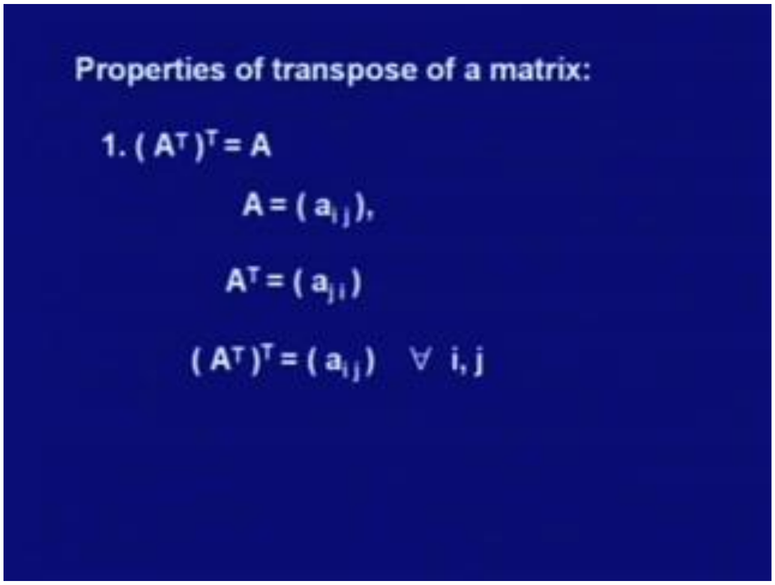
Now AB raised to power p can also be computed as A raised to power p, B raised to power p, provided AB is equal to BA. Now that means the matrix A and B must be commutative. Now to prove this, I have to multiply AB p times; so these are AB matrices p times. Now what I can do is I can combine the first two terms and then these are remaining p minus 2 terms; I am writing the first two terms in this particular form, the reason being that matrix multiplication is associative. So I can very easily write down first two terms in this form. Then since matrix multiplication is commutative - that is what i have assumed here - so I can write down this BA as AB, and this gives me the result as A square B square. So this is written as A square B square and rest of the p minus 2 terms, same thing can be applied and finally what we have is A p multiplied by B p.

(Refer Slide Time: 06:59)



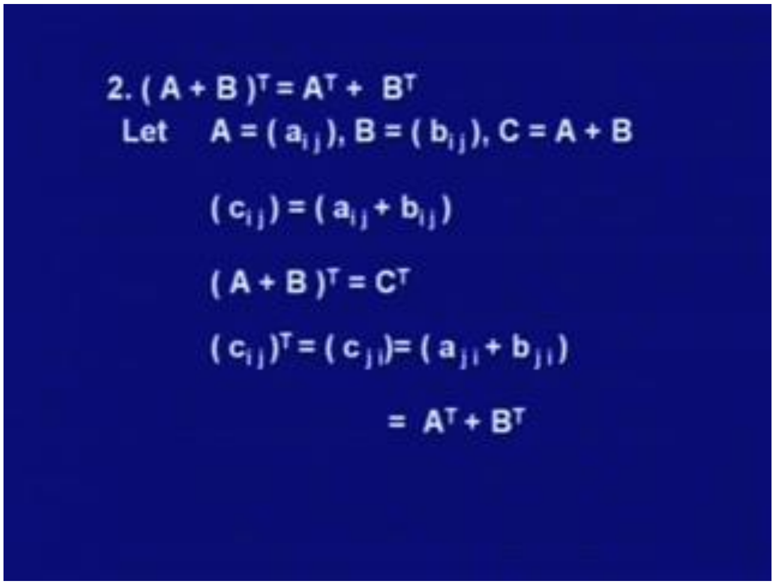
After this, we can introduce the transpose of a square matrix. The transpose of a square matrix is denoted by A raised to power A T and if aij is the typical element of the square matrix A, then its transpose is aji. That means, the row element is written as the column elements. So, rows and columns are interchanged. Now this is aij T is equal to aji for all values of i and j and what we get is a transpose matrix. For example, if i have a matrix A as 2 by 3 matrix, then its transpose will be 3 by 2 matrix. What I have done is the first row of A becomes first column of A transpose, while the second row of A become second column of A transpose. So rows and columns are interchanged; rows of A become columns of A transpose. Similarly, the columns of A become rows of A transpose.

(Refer Slide Time: 08:16)



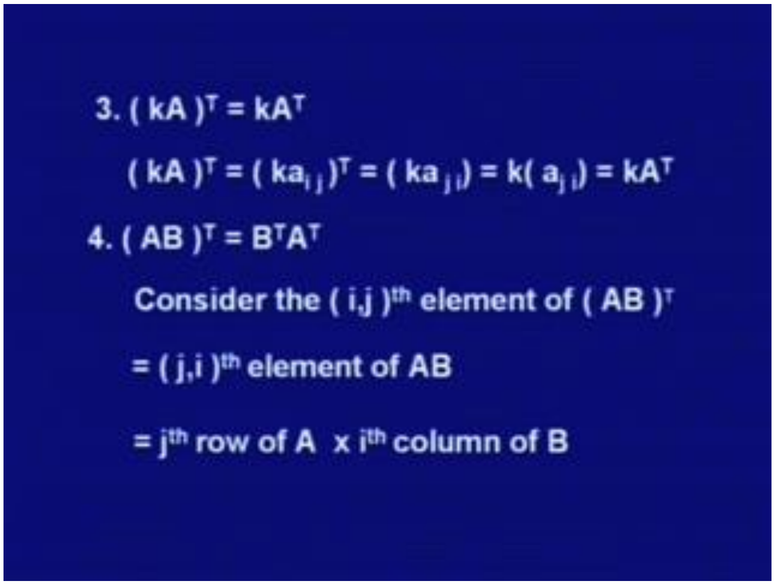
Let us now discuss some properties of transpose of a matrix. We start with the first property A transpose and its transposes A. So if A is aij, then A transpose is aji as we have defined. Then A transpose and its transpose will again become aij and this is true for all i and j. So we have proved that A transpose transposes A.

(Refer Slide Time: 08:49)



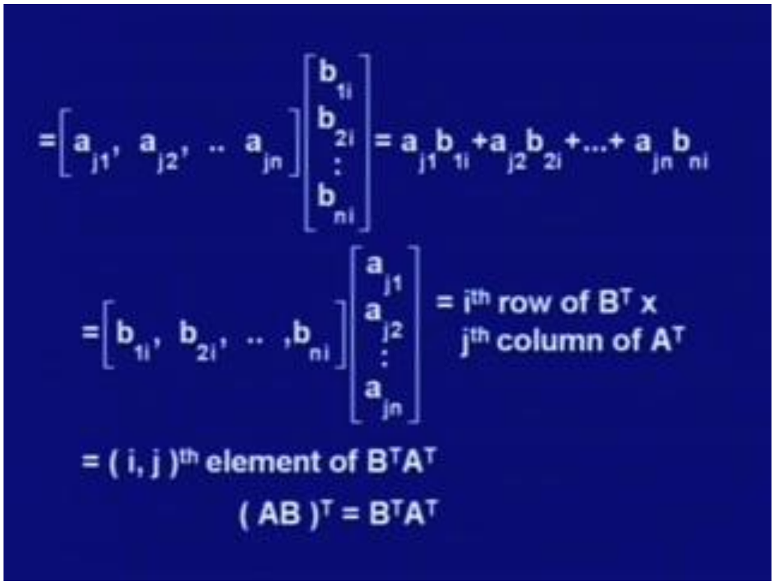
Now the second property is that A plus B transpose is A transpose plus B transpose. For this, let us consider A as aij, matrix B as bij matrix, then C is sum of A and B. So cij - the typical element of C matrix - will be aij plus bij. Then the transpose of A plus B is C transpose and cij transpose will be cji. Columns and js are interchanged; now cji will be aji plus bji by definition. Then what is aji? aji is A transposition because A is defined as aij and bji is B transpose by this; so we have proved that C transpose is equal to A transpose plus B transpose, that is, A plus B transpose is equal to A transpose pus B transpose.

(Refer Slide Time: 09:47)



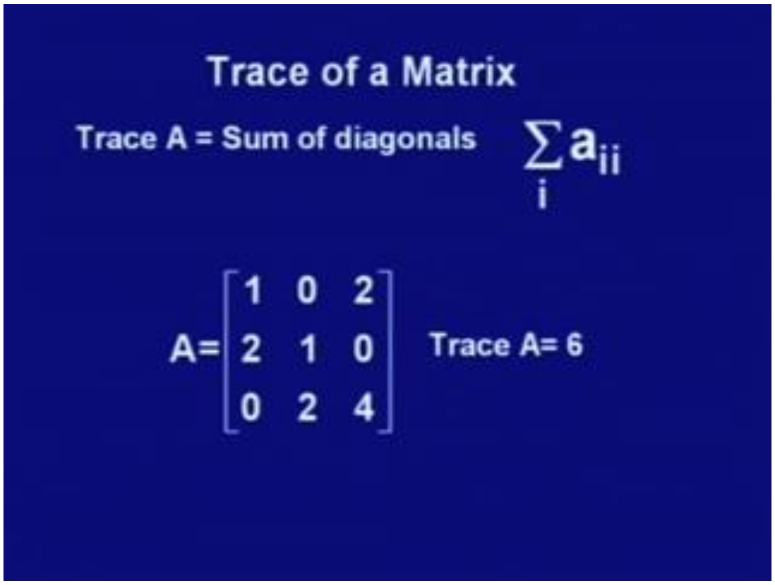
Next property is that scalar multiplication by k, and then the matrix A is transpose is k times A transpose. To prove this, I write down kA transpose as having a typical element k aij, because every element of A is to be multiplied by k by definition of this multiplication; then k aij transpose is equal to k - k is a scalar - so nothing will happen to this, but aij will become aji as we take the transpose; and this can also be written as k multiplied by aji. I am taking k outside, because this is common for all the elements; so finally, what we have is kA transpose. Next is AB transpose, that is multiply the 2 matrices A and B and then take the transpose - is the same as you take the transpose of B and then multiply it by A transpose. I consider the ijth element of AB transpose; A, B multiplied and then T . Basically A B transpose is nothing but ji’th element of A B. i’th element of AB transpose is ji’th element of A B. Now by definition of multiplication, we can say that ji’th element of A B is nothing but j’th row of A is multiplied by i’th column of B, because we said that matrix multiplication is row multiplied by column.

(Refer Slide Time: 11:27)



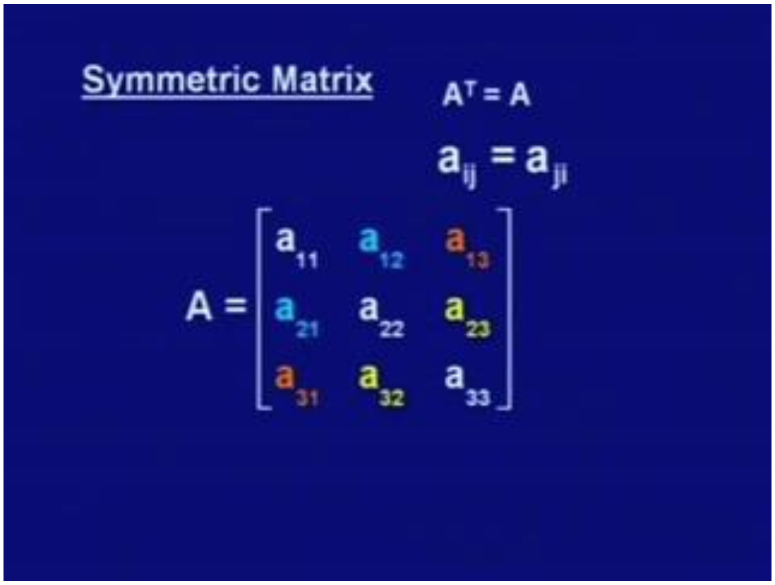
So I can write this down in this manner. This is the j’th row and this is the i’th column of the matrix. So if I multiply a j 1 by b 1 j, aj 2 b 2 j, ajn bnj then what we have is this product. This is the typical element; now this term is rearranged in this particular manner. So this becomes this- I am writing as a row - and this these terms - I have arranged in columns. So this product is same as this product. But one can very easily observe that this is nothing but the i’th row of B transpose, while this is nothing but the j’the column of A transpose. That means this product represents the ij’th element of B transpose A transpose, as by the definition of matrix multiplication. So we have proved that A B transpose is B transpose and A transpose; this is important because A B multiplication is not commutative. So if we have to take transpose of A B, then B we have to first take the transpose of B and then multiplied by A transpose. So the order has changed if you take the transpose. So this is important.

(Refer Slide Time: 12:54)



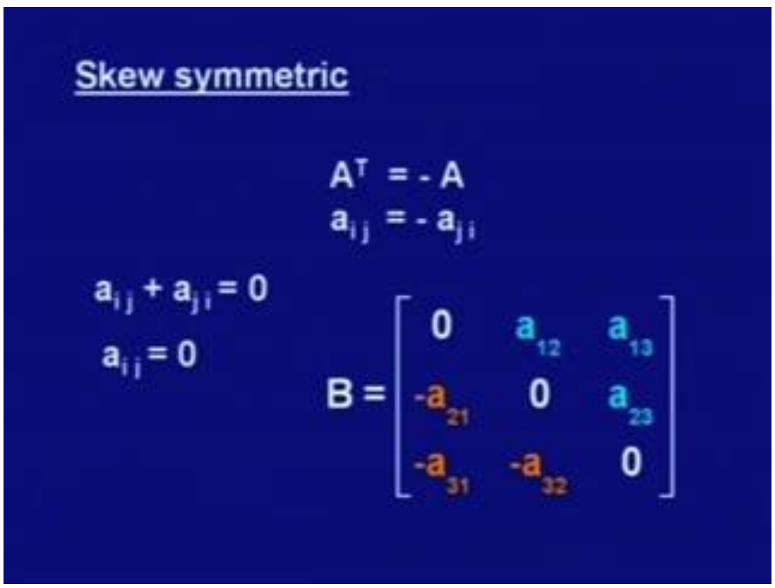
Now, next concept is trace of a matrix. Now, trace of a given matrix A is defined as sum of its diagonal elements. Now Diagonal elements are identified by equal indices; so tray sum of diagonal elements is nothing but aii summed over the index i. That means, if the matrix A is given as this 3 by 3 matrix, then trace of A will be the sum of the diagonal elements this, this and this. That means trace of A is 6. Now we define some special type of matrices.

(Refer Slide Time: 13:41)



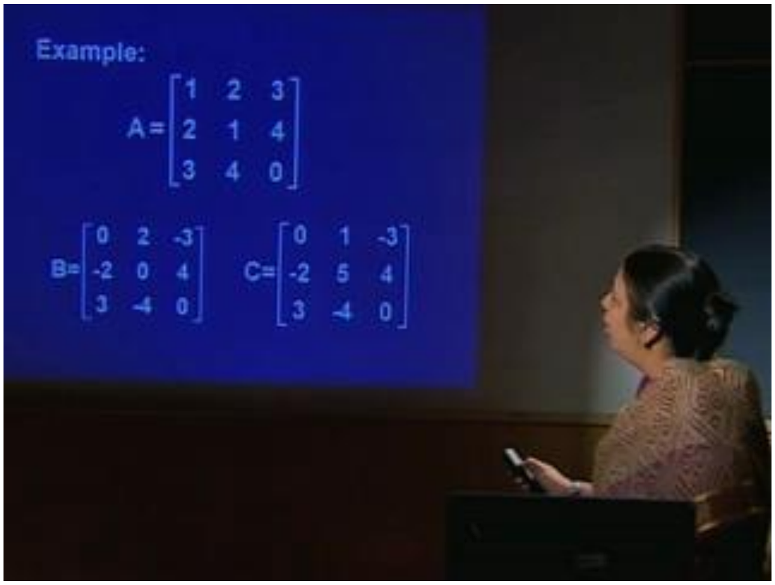
The first is the symmetric matrix. The symmetric matrix is the matrix which is having same which is same as its transpose. That is, A transpose is equal to A for a symmetric matrix and by this, I mean that ij'th element is same as ji'th element in a symmetric matrix. So if I have a square matrix A, then this matrix will be symmetric if a 2 1 is a same as a 1 2 and a 3 1 is a same as a 1 3 by the definition, and a 3 2 is same as a 2 3; so this is the diagonal and these are the elements below the diagonal and these are the elements above the diagonal. So the elements are mirror images - this element is same as this, this elements is same as this, this element is a same as this. So this is the typical property of symmetric matrices.

(Refer Slide Time: 14:47)



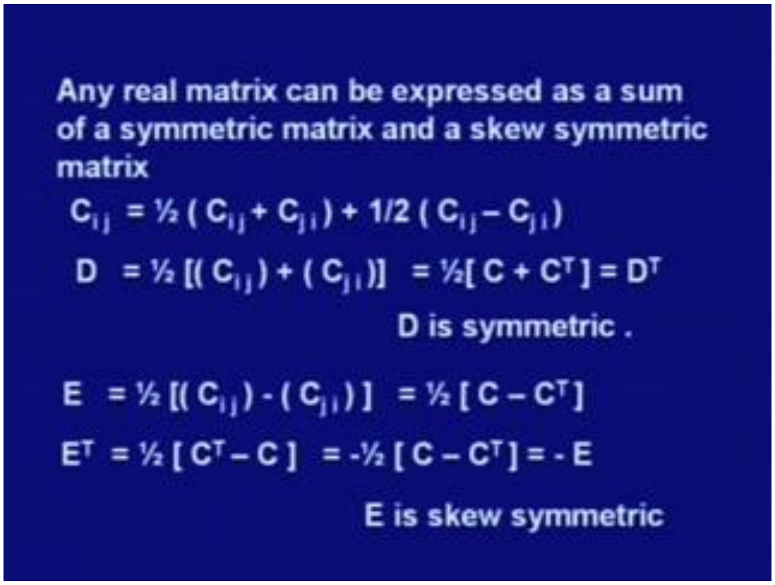
In skew symmetric matrices, we define A transpose as minus A. That is, the ij'th element of A is the same as negative of aji. That is, if I simplify then aij plus aji is 0. So in a skew symmetric matrix aij plus aji is 0 and aij is equal to minus aji. That means, the i'th element aii will be 0. Only then this is possible. Like this matrix B is skew symmetric; a 1 2 is a same as a 2 1 with the negative sign; a 1 3 and a 31 they differ with negative sign; a 2 3 and a 3 2 they differ with negative sign, while on the diagonal all the elements are 0. So, this is a skew symmetric matrix.

(Refer Slide Time: 15:50)



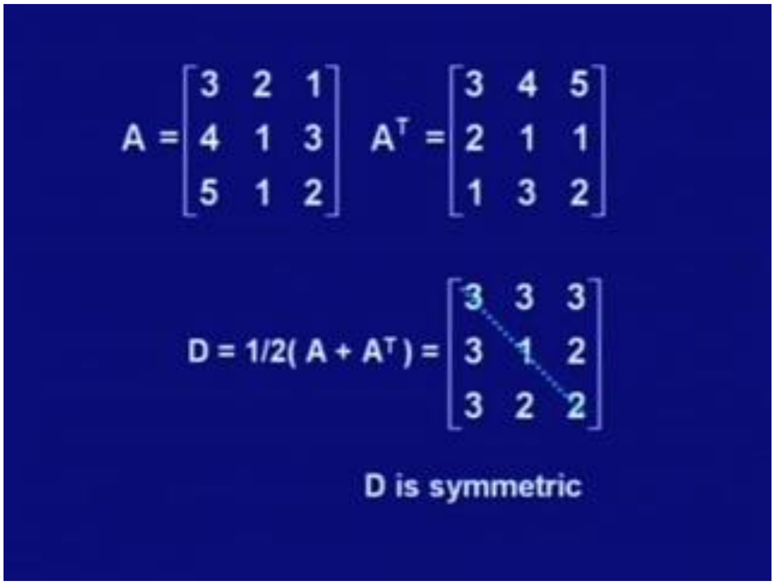
This is an example - a 3 by 3 matrix; this matrix is symmetric matrix. One can see that these are the diagonal elements, and the upper side is the upper side is the mirror image is the image of the lower side. Here we have 2 - here also we have 2; this 3 is same as this element; this element is same as this element. So this matrix happens to be a symmetric matrix. Here aij is equal to aji for all the elements. While this matrix is a skew symmetric matrix, look at the diagonal element; all elements are 0. This is 2, so here we have minus 2. If this element is minus 3, this will this element is 3 and if this is 4, this is minus 4. So this is the diagonal element and this is the mirror image on this side. While this matrix C - we have this diagonal. This element is not the same as this; this element is not the same; this element is not the same. So this this matrix is not skew symmetric because this is now 0. Although this element is this and this element are rim mirror images, this is 4 and this is minus 4; but because this is minus 2 and this is 1, so this matrix is not skew symmetric. This matrix is not symmetric because this is 1 and here we have minus 2; so this is neither symmetric matrix nor skew symmetric matrix.

(Refer Slide Time: 17:35)



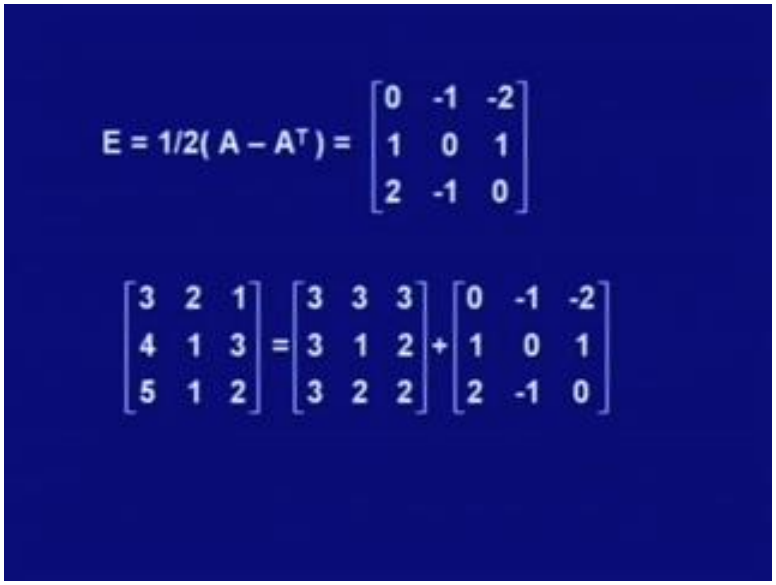
Now this is an important result. It states that any real matrix can be expressed as a sum of a symmetric matrix and a skew symmetric matrix. Now this can be proved easily. What we do is a typical element Cij of the matrix given matrix can be written as half Cij plus Cji, plus half Cij minus Cji. That means what I have done is I have added and subtracted Cji and the result is Cij. So i have manipulated Cij in this particular manner. Now I say that the D matrix is given by this half Cij plus Cji. One can notice that D is a symmetric matrix; the reason is if I take that transpose of D, D transpose, then this becomes Cji this becomes Cij and sum is commutative. So this is nothing but C plus C transpose so this is D transpose. So D and B transpose are same; so D is a symmetric matrix. And this matrix C and a half Cij minus Cji - I denoted by E - and I will prove that this E is half C minus C transpose. This is C and this is C transpose, but if I take the E transpose then half C transpose minus C - this becomes transpose; C transpose transpose is C. So what I have is minus half outside and then we will have C minus C transpose and the result is E with the minus sign with the minus sign. So E is skew symmetric matrix. So the matrix C is represented as a sum of symmetric matrix and a skew symmetric matrix.

(Refer Slide Time: 19:32)



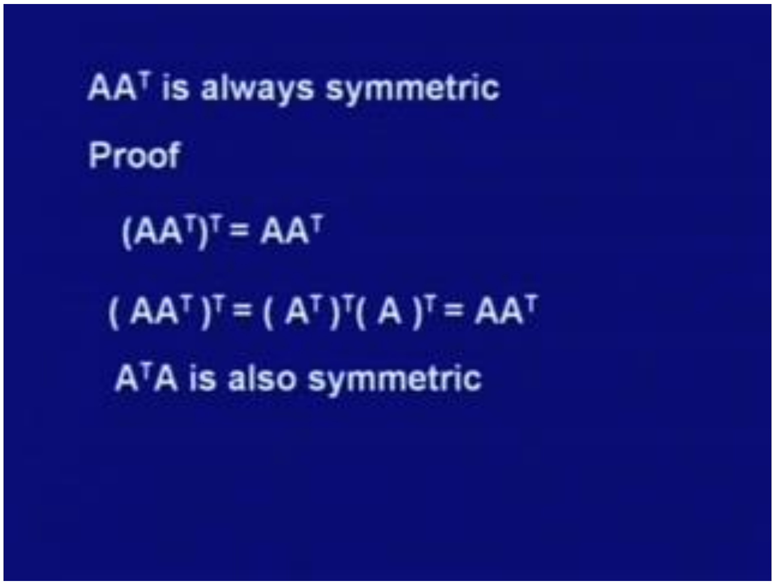
Now this I illustrate with an example. So if I have A matrix as 3 2 1 4 1 3 5 1 2 or 3 by 3 matrix, then it’s a transpose will be 3 4 5 in the first row; first column becomes first row, second column become second row, third column becomes third row in A transpose matrix. Then D is half A plus A transpose; if you sum this up, 3 plus 3 by 2 is 3, 2 plus 4 by 2 is 3, 1 plus 5 divided by 2 is 3. Similarly other elements can be computed; so this is D and one can notice that this matrix is symmetric. This 3 and this 3 are the same, this 3 and this 3 are the same, this 2 and this 2 are the same; so this is symmetric matrix.

(Refer Slide Time: 20:28)



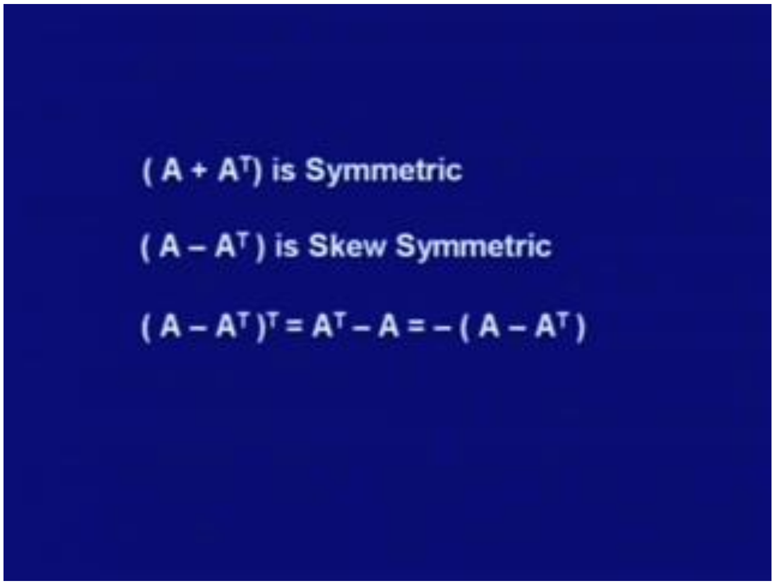
Now if you compute, the matrix E is half of A minus A transpose for the given matrix. Then it comes out to be this 3 by 3 matrix and one can note down that this matrix is symmetric - a skew symmetric; this is minus 1 so this is 1, this is minus 2 so this is 2 this is 1 and this is minus 1 and the diagonal elements all are 0. So this is a skew symmetric matrix. So the I have written this square matrix A as this symmetric matrix plus this and a skew symmetric. So any square matrix can be written as a sum of a symmetric matrix and a skew symmetric matrix.

(Refer Slide Time: 21:10)



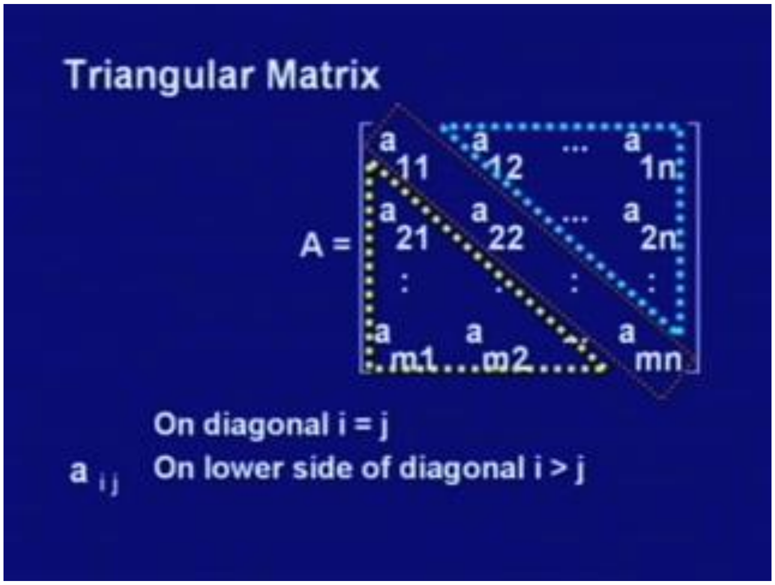
Then one can prove that A multiplied by A transpose is always symmetric. To prove this, we start with AA transpose and its transpose. If this comes out to be AA transpose then we have proved the result. So to prove this, I start with AA transpose transpose as A transpose transpose. This is the B matrix, this is AB; so this is B transpose A transpose. So I had A transpose transpose and A transpose but i have already proved that A transpose transpose is nothing but A, so this multiplication is nothing but A A transpose. So AA transpose transpose is the same as AA transpose and that is a basic property of symmetric matrices; Enhance We can say that AA transposes a symmetric matrix.

(Refer Slide Time: 22:06)



Now A plus A transpose is always symmetric. Similarly A minus A transpose is skew symmetric. These are the properties one can very easily prove; then the next property is - this this can be easily proved - A minus A transpose transpose is equal to A transpose minus A transpose transpose I have written as A is equal to - I have taken minus outside - so the A minus A transpose. So what I have proved is A minus A transpose transpose as minus times A minus A transpose and that proves the skew symmetric property of A minus A transpose.

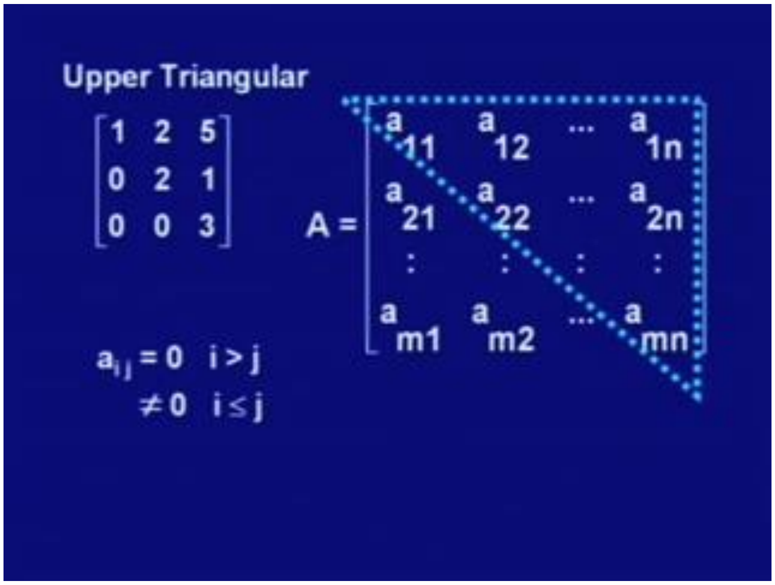
(Refer Slide Time: 22:51)



Now for triangular matrices, I consider a square matrix A of size m by n. Now in this square matrix, one may observe that the elements which appear in the diagonal they all have row index equal to column index. Like the elements which I have shown in this box, the first element is a 1 1, its row index and column index they are equal to 1; the second element a 2 2, row index, column index they are same and the last element amn - here also the row index and column index are the same. So this is the typical characteristic diagonal elements that i and j are equal, but if you consider this set of elements in the matrix then one may easily notice that the row index is bigger than the column index for each and every element here.

Like if we consider the element a 2 1, then the row index 2 is bigger than the column index 1. Similarly for this, n is bigger than 1; same thing happens here and is this is true for all elements appearing in this. So we say that aij - if it appears in the lower side of the diagonal then i is greater than j. Similarly, if you consider the upper side of the diagonal then all elements have row index smaller than j like a 1 2. The row index 1 is smaller than the its column index 2. Same things happens for a 1 and a 2 and and so on. So for all the elements which appear in the upper side of the diagonal i is less than j.

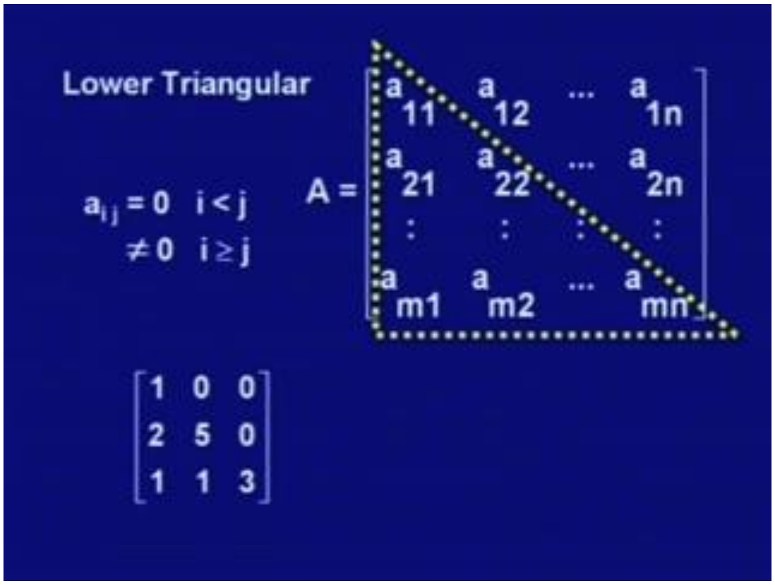
(Refer Slide Time: 24:51)



Now on the basis of this, I can define upper triangular matrix. Now in this square matrix, if we consider the element shown in this triangular box, all elements either they are on the upper side of the diagonal or they are on the diagonal. So I define an upper triangular matrix is the one in which all elements which appear on the lower side of the diagonal that is aij is 0 - i greater than j, i greater than j, i is greater than j, i is greater than j; for all these elements they are 0 and a non 0 elements appear only in the upper side of the diagonal right. So the upper triangular matrix is one where non 0 elements appear on the upper side of the diagonal, that is diagonal including and the triangle.

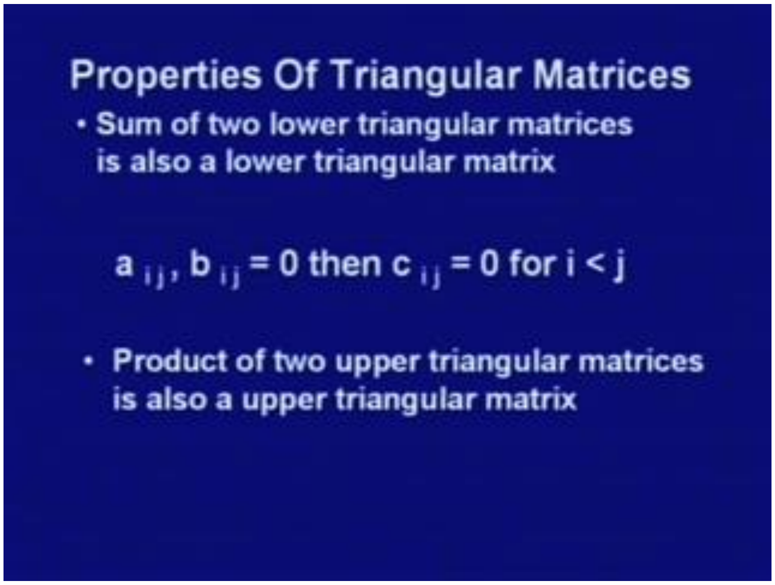
For example in this square matrix, all the elements which appear on the lower side of the diagonal they are 0. aij is 0 when i is greater than j and all the elements which are either on the diagonal or on the upper side they are non 0. Like aij is not 0 when i is less than equal to j, this equality is taken for the diagonal elements. So this matrix is an upper triangular matrix; so elements on the upper side they are non 0.

(Refer Slide Time: 26:19)



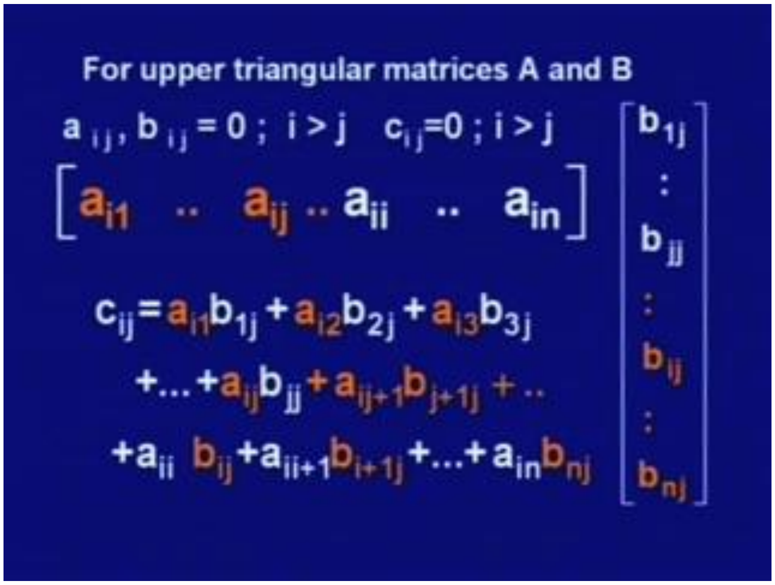
Similarly one can define a lower triangular matrix. So if in the square matrix A, if the non 0 elements appear in this triangular box only and the outside all these the elements are 0; then that matrix is called a lower triangular matrix. By this, I mean to say that aij is 0 whenever i is less than j. You can notice that these are the elements which have to be 0. For them i is less than j, 1 is less than 2, 1 is less than n. So all these elements have to be 0 for a lower triangular matrix but the elements which are inside, for them i is either equal to j for the diagonal or i is greater than j and these elements are non 0. For example, we have this lower triangular matrix here all these elements are 0 and non 0 elements appear in the lower triangle.

(Refer Slide Time: 27:21)



Now these triangular matrices appear in many engineering applications and various properties of these matrices are useful. For that purpose, we will discussing properties of triangular matrices. Now the first property we will be like discuss is that sum of 2 lower triangular matrices is also a lower triangular matrix. Now this, I can prove by simple fact that aij and bij are 0 if they are at the lower side. That means i is less than j; then this aij and bij are 0 for lower triangular matrix whenever i is less than j. Then so will be cij and that proves that the sum matrix will also be a lower triangular matrix. Now similar result is obtained for the upper triangular matrix and that can be proved on the same lines and the result is that sum of two upper triangular matrices is also an upper triangular matrix. Now this is regarding the sum of 2 triangular matrices. Now this result is that product of 2 upper triangular matrices is also an upper triangular matrix.

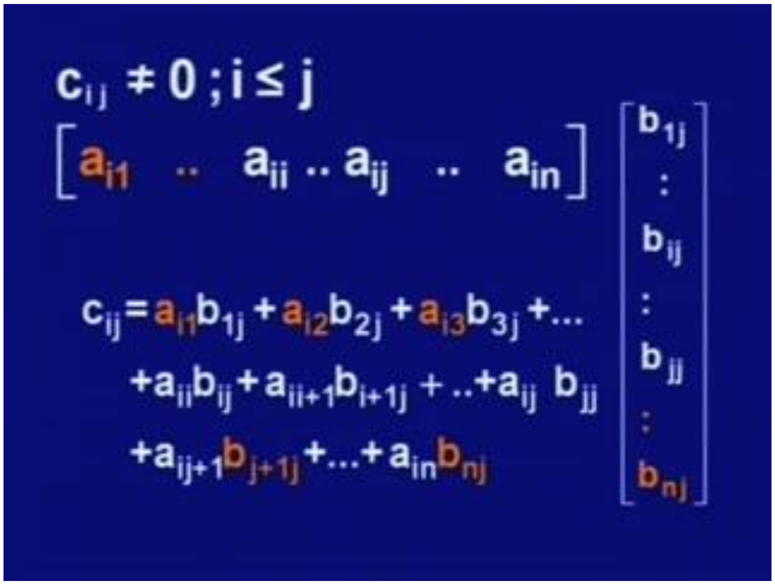
(Refer Slide Time: 28:40)



Now to prove this result, I consider 2 upper triangular matrices - A and B. Now since they are upper triangular matrices, so we assume that aij and bij are 0 whenever i is greater than j. Now we will prove that the product matrix will also be an upper triangular matrix; that means the product element cij will be 0 whenever i is greater than j and non 0 whenever i is less than equal to j. For this purpose, I consider a typical element cij of the matrix. Now this cij element is obtained actually from the product; so to obtain cij i have to multiply the i'th row of the matrix A by the j'th column of matrix B.

So if I write down the i'th element, i'th row of a matrix A - and this is the j'th column of matrix B - notice that some of the elements in this mat in this row matrix they are orange and some of the elements here are also orange. Actually they denote that these are the elements which are 0. They are 0 because for all these elements - i less than j - they are 0. While i greater than j, they are non 0. Same thing is applicable here. Now if I multiply them, then the product is obtained as cij is equal to ai 1 b 1 j, ai 2 b 2 j, aij bjj. aij bjj Then we have this element multiplied by this element and so on. ain multiplied by bnj will be the last element. Now one can notice that in this, every term involves an orange element that means 0. So this is 0 because of this 0; this is 0 because of this 0; this term is 0 because this term is 0 and here both of them are 0. Same thing happens here and that means cij is 0 whenever i is greater than j. Now we have to prove that on the upper triangle all the elements need not be 0.

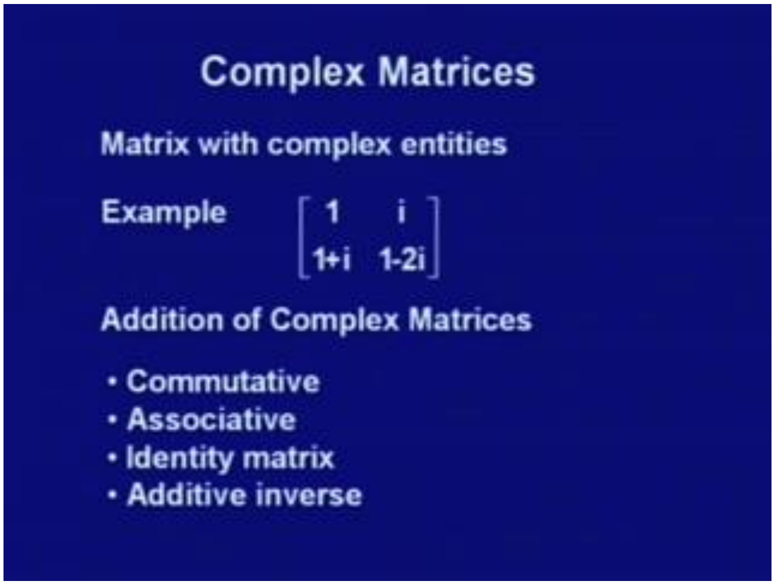
(Refer Slide Time: 31:24)



So for this purpose, to show that the ah elements cij appearing on the upper side of the diagonal cij is not 0 whenever i is less than equal to j. We consider this element cij whenever i is less than equal to j here since i is less than j. So one may notice that aii comes first, then aij because i is less than equal to j. So all these elements are 0 but these elements are not 0. The same thing appears here: j comes first - bij a comes first - and bjj comes next. So these elements are non 0. So when you take the product cij, ai 1 multiplied by b 1 j - it is 0, because the i 1 is 0 - ai 2 multiplied by b 2 j - it is 0 because this is 0 - ai 3 b 3 j is 0, but aii bij this is not 0. Similarly ai plus 1 ai - ai plus 1 bi plus 1 j is not 0; the next element is not 0 and up to bjj all elements will be contributing while after this these terms will be 0. So what we have is bij plus 1 bj plus 1 j - this is 0 - and the same thing happens with the last element ain bnj.

So all these terms will not be contributing - these terms will not be contributing; cij is not 0 because sum of these terms will not be 0. So cij is not 0 when i is less than equal to j; that proves our result that sum of 2 upper triangular matrices is also an upper triangular matrix. Now the similar result can be obtained for lower triangular matrices, that means sum of 2 lower triangular matrix is a lower triangular matrix and product of 2 lower triangular matrix is also a lower triangular matrix. After this we come to the next concept complex matrices/.

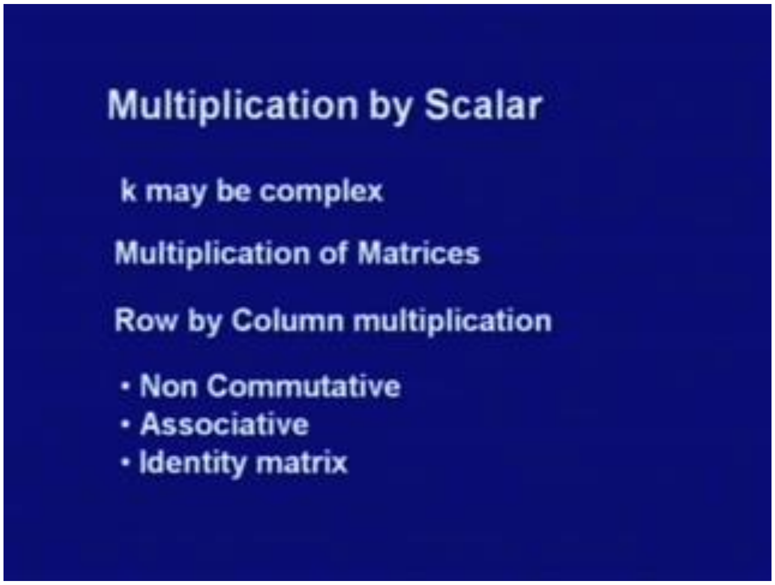
(Refer Slide Time: 33:05)



Viewers, so far we were discussing matrices which have real elements. But they can be complex also and such matrices come under complex matrices. So if we have a matrix with complex entities then we have a complex matrix. For example, this 2 by 2 matrix its ah first row second element is I - the initial number - while the second row has 1 plus i and 2 plus I - they are complex numbers; so this a matrix having complex entities. So it is a complex matrix. Now we have established number of results for real matrices. Similar results can be obtained for complex matrices. We can add 2 complex matrices and one can prove on the same lines that addition of complex matrices is commutative and proof is not difficult because real numbers are commutative.

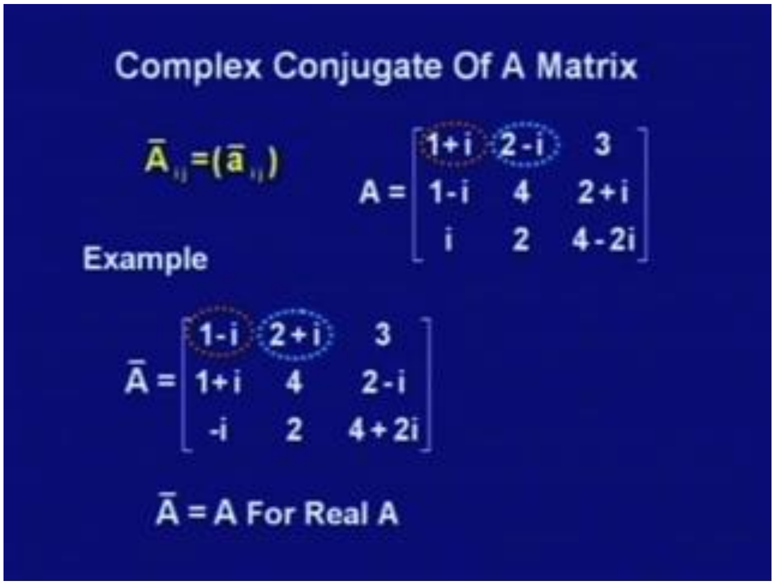
So every element of real symmetric real matrix is a commutative; same thing happens with complex matrices. Since we complex numbers are commutative in nature and so are the complex matrices, similarly one can prove the associativity of addition of matrices if the real matrices are associative. Sum of two sum of real matrices are associative; so is sum of complex matrices are also associative. As far as identity matrices are concerned we can define identity matrix as the same additive identity; this works for complex number also. Additive inverse can also be defined on the same lines, that is additive inverse of a given matrix is the one which gives on addition the null matrix. So all these concepts are in line.

(Refer Slide Time: 35:38)



Now when it comes to multiplication by scalar then k may be this scalar k may be complex or it may be real and when we multiply k by a complex matrix then the result will be a complex matrix. Whether it is real or not, the result will be complex. Now when it comes to multiplication of matrices, then again it is row multiplied by column multiplication as usual and again multiplication of 2 matrices will again be complex. So all those properties which we have proved for real matrices they are applicable; they are non commutative as we have proved for real matrices; they are associated because complex numbers are associative - you can add them in any order. Similarly the existence of identity matrix - the identity matrix which works for matrix multiplication for real matrices - that works for complex matrices also.

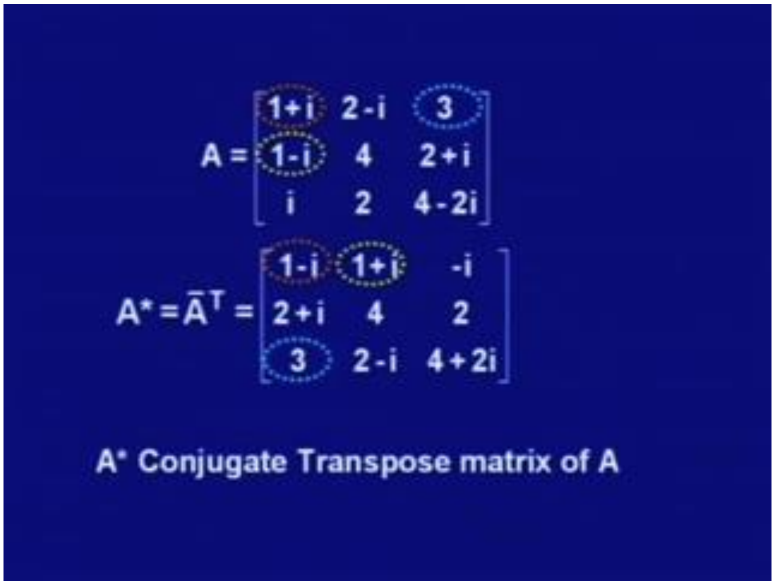
(Refer Slide Time: 36:41)



So we define a complex conjugate of a matrix. So if we have a matrix Aij, then complex conjugate of a matrix is denoted by a bar ij and its element is aij bar; that means the element is the complex conjugate of the element Aij. So the elements of complex conjugate aij bar is the complex conjugate of corresponding matrix. Like if I have a matrix A consisting of complex numbers, then its conjugate A bar will be given by this matrix; you may notice that 1 plus i is the first element in the first row and 1 minus i is the is the complex conjugate of 1 plus i appearing in the first row first column. 2 minus i -its complex conjugate is 2 plus I; conjugate means the imaginary part will be multiplied by minus 1.

So 2 minus i and here we have 2 plus i and this is actually true for each and every element. When it comes to the next element 3, 3 is a real number - its complex conjugate remains the same. So this is not affected but 1 minus i here we have 1 plus I, 4 is real it is not affected. 2 minus i is the complex conjugate of 2 plus; here we have I, here we have minus i. So that way this element - this matrix – is being given as A, then A bar - the complex conjugate - will be in this matrix and one may notice that A bar is equal to A for real matrix A. So we can extend this concept to real matrices also, but in that case a complex conjugate remains the same.

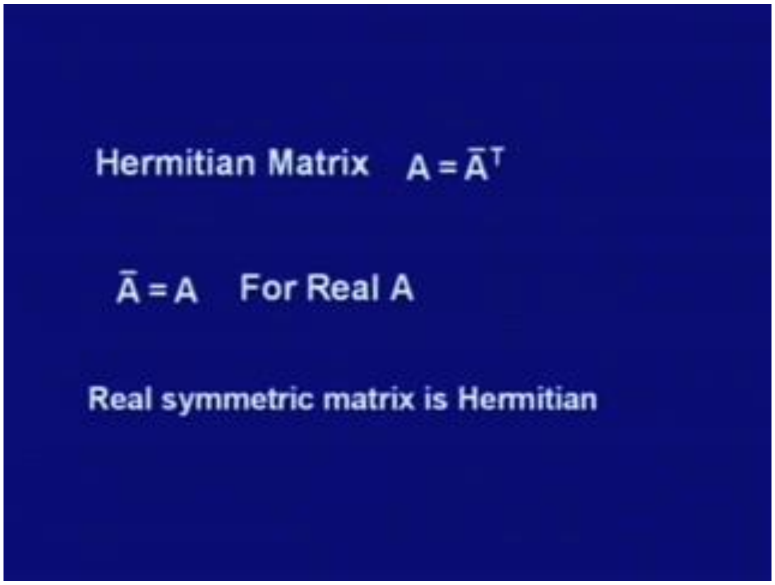
(Refer Slide Time: 38:41)



Now if given matrix is A as this, then we define A star as complex transpose matrix of A; complex transpose matrix is the one. It is the combination of 2 operations; we take the conjugate of individual item and then take its transpose. For example if A is this, then 1 plus i is 1 minus i. This is appearing in the first row first column. But 2 minus i is the element appearing in the second row of the first column; its conjugate 2 plus i appears in the second row first column.

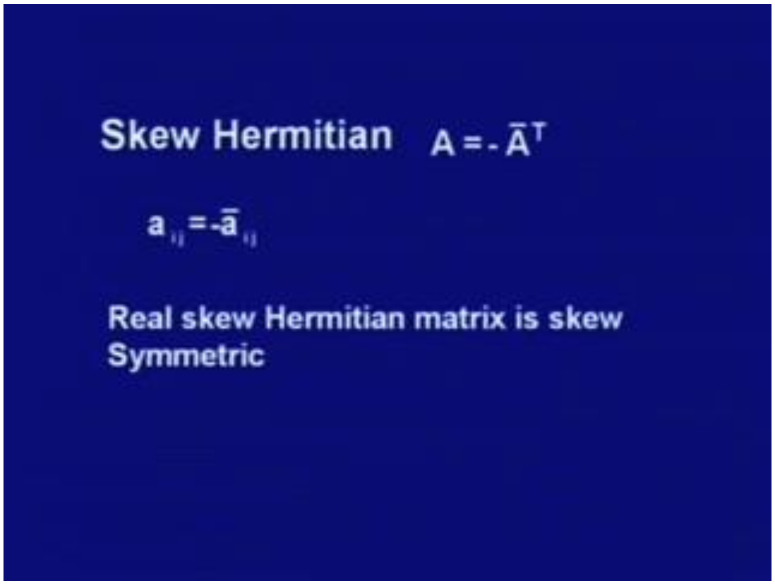
So this row, this column and this row - they are been interchanged. That is why it is A star; so we denoted it by A bar transpose. Two operations are taking together: bar and transpose. This can be checked for other elements also. It is 1 minus I - 1 plus i - complex conjugate of this. This appears in the first row second row first column and this appears in the first row second column and one can check it for each and every element here. This 3 - its conjugate is 3; this appears in first row third column, this appears in third row first column.

(Refer Slide Time: 00:40:03 min)



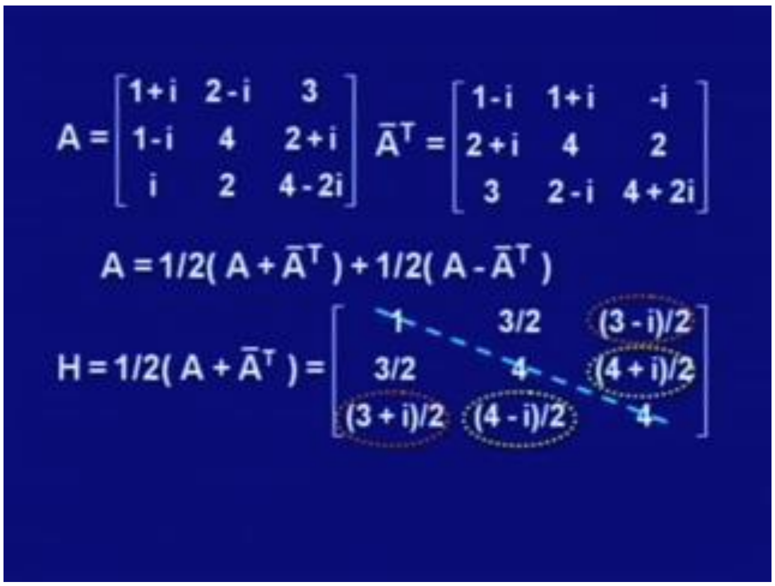
Now on the basis of that, we can introduce Hermitian matrices. A Hermitian matrix is a square matrix in which the matrix A is same as its complex conjugate transpose. So A bar transpose is same as A for a Hermitian matrix. As I told you earlier, that A bar is equal to A for real matrix; so if a real symmetric matrix is Hermitian also because the real symmetric matrix will satisfy this property trivially. So a real symmetric matrix is a Hermitian matrix, but this is not true for complex matrices.

(Refer Slide Time: 40:54)



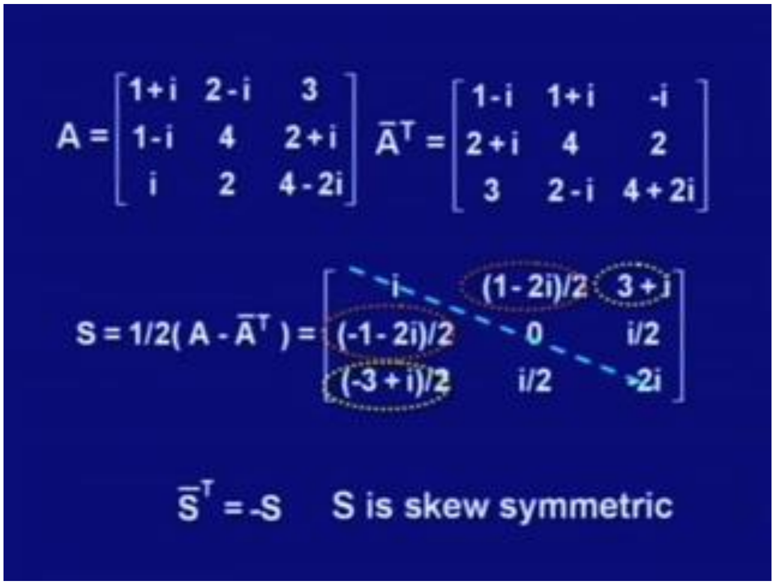
Now when it comes to Skew Hermitian matrix, then it is defined as a square matrix A which is equal to minus times A bar transpose; that means the element aij in A becomes minus aij bar in A bar transpose. So if this property is satisfied, then we say the matrix is a Skew Hermitian matrix. Now this is the result which can be said about real matrices that a real Skew Hermitian matrix is Skew symmetric, because the for a Skew symmetric matrix which is real, A bar transpose is equal to A bar n that gives us a Skew symmetric matrix.

(Refer Slide Time: 00:41:41)



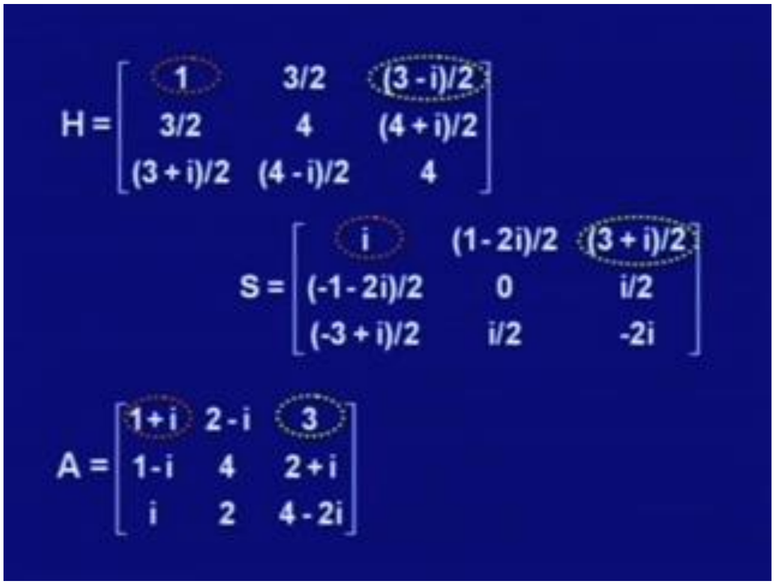
Now if we consider this 3 by 3 complex matrix, then its A bar transpose is 1 plus I, 1 minus I, 2 minus I, 2 plus I, 3 and we have 3 here; one minus I, 1 plus I, 4 and we have 4, 2 plus I, 2 minus I; Then we have i here minus I, 2 and 2, 4 minus 2 i and 4 plus 2 i. Now any matrix A can be written as this sum - like we have done for real matrices. A is written as half A plus A bar transpose plus half A minus A bar transpose. So what I have done is I have added and subtracted this from this - from the matrix A. Now one can prove that the first part - half A plus A bar transpose - is Hermitian, while this part is Skew Hermitian. So what I do is I take this part as H - half A plus A bar transpose; for given this A, I add A bar transpose and take half of it so I have this matrix. Now in this matrix - this is this matrix is Skew Hermitian - you notice that this is symmetric and these are the conjugates. This and this are same; 3 plus i by 2 is here 3 minus i by 2. I have taken the conjugate and then transpose. Then 4 minus i by 2 - here I have 4 plus i by 2. So this matrix is Hermitian. If you take its H bar transpose, you will have the same matrix. So this matrix is Hermitian. These elements are H bar transpose; they are equal; same thing happens here. It is the diagonal and that means H bar transpose is equal to H and this proves that A is Hermitian.

(Refer Slide Time: 43:52)



So with the given matrix A, A bar transpose is computed as this. Then the matrix S which is half of A minus A bar transpose is computed from these two matrices. So we have 1 plus i and 1 minus i - so it is subtraction - so it is i here; 2 minus i minus 1 plus i by 2 is this element; 3 and this minus i becomes plus I, so 3 plus i by 2. So that is how we calculate this matrix S and this is the diagonal element. Here we can notice that S bar transpose is equal to minus S and S is a Skew symmetric matrix on the spaces.

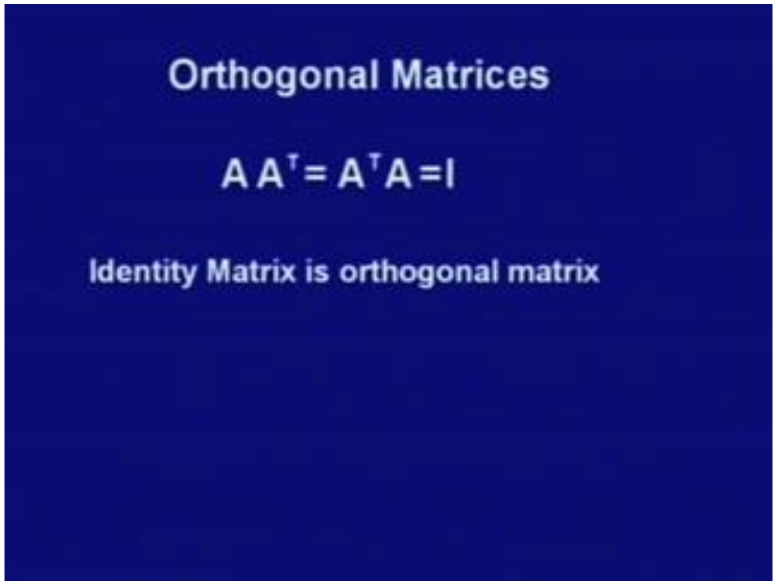
(Refer Slide Time: 44:44)



So what I have done is I have calculated H, I have calculated S. H is half of A plus A bar transpose, S is half of A minus A bar transpose and my claim is that H is Hermitian and S is Skew Hermitian and sum of these two matrices is the given matrix A. One can check: 1 plus i is 1 plus I; so H plus S is A. So any matrix which is complex can be written as sum of a Hermitian matrix and a Skew Hermitian matrix.

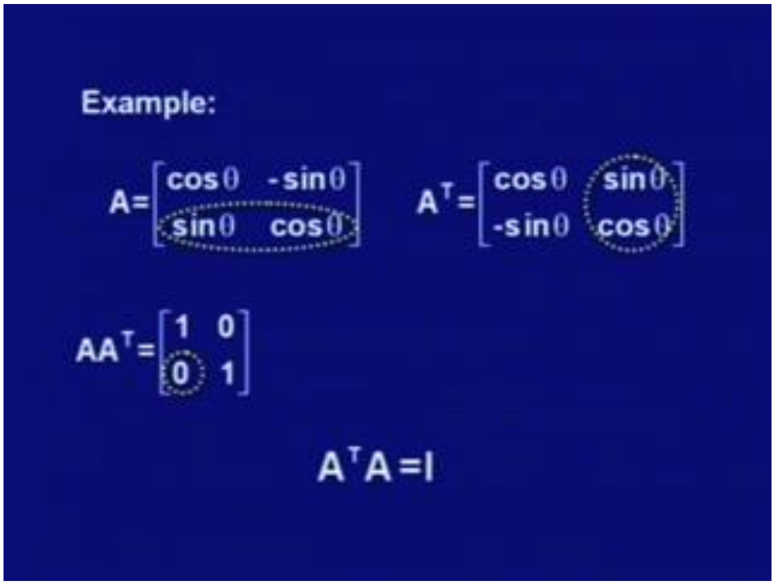
Now one can notice that in Hermitian matrix, the diagonal elements have to be real because if they are not real then they cannot satisfy the property that H is equal to H bar transpose. Similarly on this Skew symmetric matrix, these elements have to be imaginary and of course, they have to be complex conjugate. This and this - they have to be minus of complex conjugate, because S has to be Skews Hermitian. So this is being shown 1 plus i is 1 plus I; 3 minus i by 2 ,3 plus i by 2 is 3 and so on.

(Refer Slide Time: 46:03)



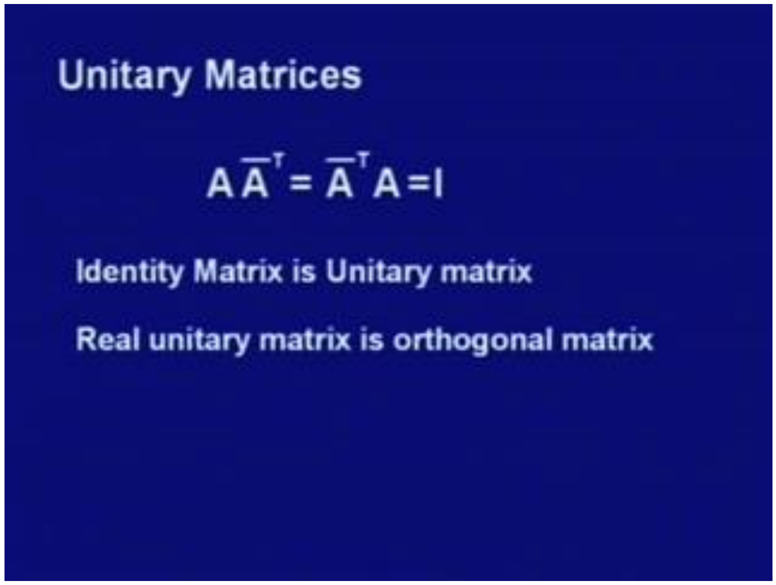
Now after this, I introduce orthogonal matrices. Orthogonal matrices are used extensively in mathematics and we define a square matrix to be orthogonal if it satisfies this property; that is, A and A transpose - if they multiply together - they give us identity matrix. That is A A transpose, is equal to A transpose A, is equal to I. This property is actually satisfied by identity matrix. One can actually work out A and A transpose; A happens to be identity matrix, its transpose is also identity matrix because identity matrix are diagonal matrix. So the transpose will also be the same and multiply identity matrix with itself is again an identity matrix; so identity matrix is a good example of orthogonal matrices.

(Refer Slide Time: 47:02)



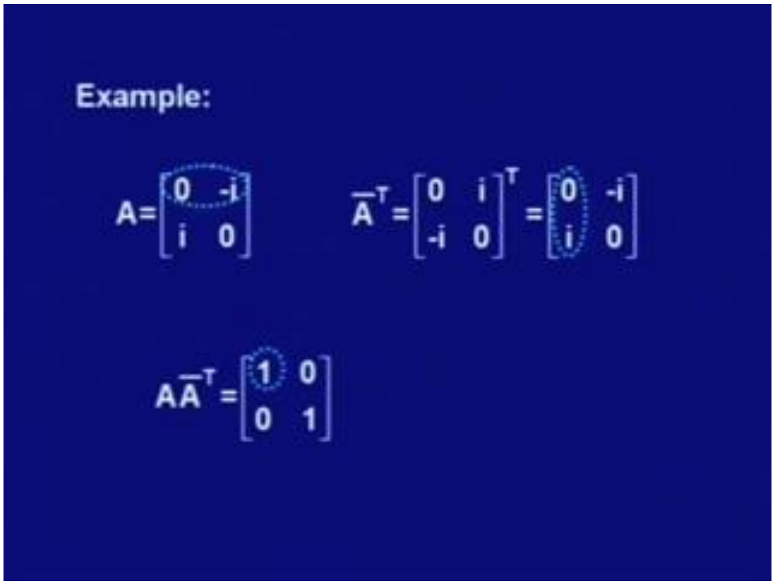
The other example is, we have A matrix 2 by 2; cos theta, minus sin theta in the first row; sin theta, cos theta in the second row. Then A transpose happens to be cos theta, sin theta; this column becomes this row and this column becomes this row. If you multiply the 2 cos square theta plus sin square theta, that is, 1 cos theta multiplied by sin theta minus sin theta multiplied by cos theta, they will cancel out; so this element is 0. Similarly if you multiply this sin theta cos theta by sin theta cos theta, this column will have 0 and finally when sin theta multiplied by sin theta - sin square theta- cos theta multiplied cos theta - cos square theta - and the result is one. So A A transpose is nothing but identity matrix; so we can say that the matrix A is an orthogonal matrix.

(Refer Slide Time: 48:16)



Now, unitary matrices are square matrices which satisfy this property: that is, A multiplied by A bar transpose is equal to A bar transpose A and the product is I. Now if the matrix A is complex, only then A bar transpose is a mini; if it is a real then A A bar transpose is nothing but A transpose itself and in that case it is A A transpose is equal to A transpose A is equal to I; that means, it is the same as orthogonal matrix. So a unitary matrix is a generalization of orthogonal matrices in complex numbers. In the same sense, identity matrix is a unitary matrix because identity matrix is a real matrix satisfying this property; so it is not only orthogonal, it is unitary also. Now as I told you, real unitary matrix is orthogonal matrix because for real matrix, A bar transpose is nothing but A transpose. So this property will be satisfied.

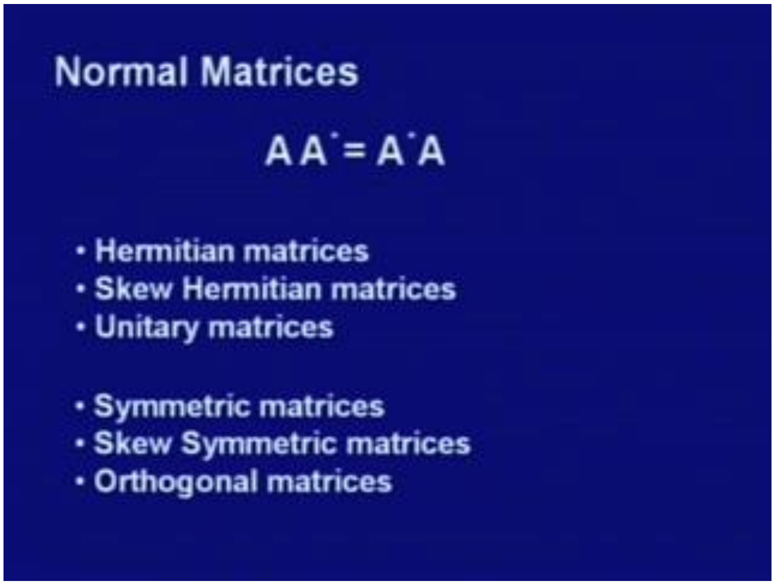
(Refer Slide Time: 49:30)



Now the example which I am considering here is a matrix A, complex matrix - 0 minus i, I, 0 - so 2 by 2 matrix. Then it is A bar transpose: this remains 0, but minus i becomes minus I, here minus i is i and minus i transpose is i. So this A bar transpose: for what I have done is I have first taken the conjugate - so conjugate of this is 0, minus i I, i minus I, 0 0 and then I take the transpose. So this row becomes this column and this row becomes this column; so this is A bar transpose. If I multiply the two – A, A bar transpose - then one can check that 0 multiplied by 0 and minus i multiplied by i is 1 and one can check other elements also. If we multiply first row by second column, this is 0. We multiply the second row by first column, this again is 0.

But if you multiply second row second column - i multiplied by minus i - is minus i square which is1. So we have 1. So this is an identity matrix. Now a matrix multiplication - so one can very easily check that A bar transpose multiplied by A; that means whether you multiply it on the left hand side or on the right hand side, the result will be the same. One can check it easily and on this basis, one can say that A is a unitary matrix.

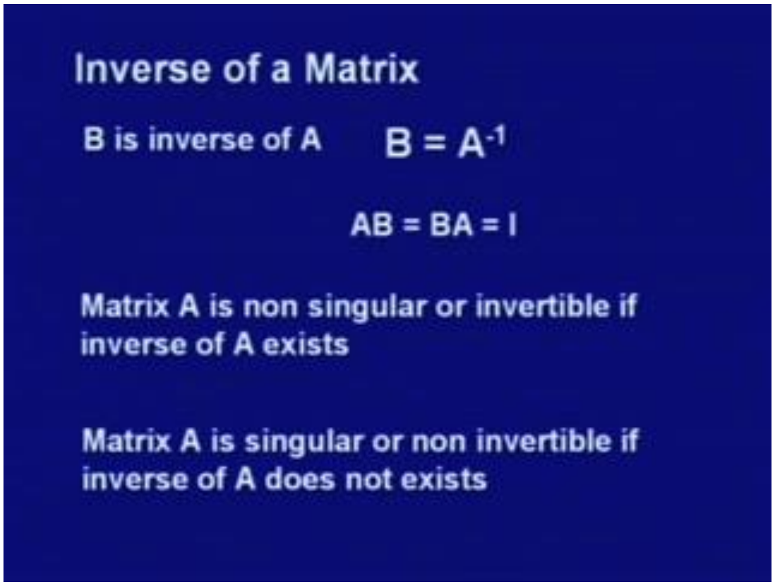
(Refer Slide Time: 51:24)



After this, I introduce normal matrices. Normal matrices are those matrices in which A A star - A star is A bar transpose - so A A star is the same as A star A. That means when you multiply A by its A star - whether left or right - they are the same. Now if it becomes 1, identity; if A A star becomes identity, it becomes unitary matrix. If A becomes real, then it becomes orthogonal matrix. So basically normal matrices are the generalization of orthogonal matrices and unitary matrices. So the examples of normal matrices: the Hermitian matrix. For Hermitian matrix A A bar, A A star - what is A star? A star is a same as A; so basically A A star for Hermitian matrix is A square and we know whether we post multiply or pre multiply, the result is the same.

So, Hermitian matrices are normal matrices. Similarly for skew symmetric matrices, this is A - A bar star is minus of A, so whether you multiplied post or ah post multiply or pre multiply the result will be the same. Only thing is difference of minus will come here as well as here. So, Skew symmetric matrices are also normal matrices. Similarly one can prove that unitary matrices are also normal matrices. In fact for unitary matrix, we have one more addition thing that A A bar A A star is equal to identity. Now this is the case when we are talking about complex matrices. But when we are having real matrices, then symmetric matrix A A bar A A transpose - this bar doesn't have any meaning in the case of symmetric in the real matrices - so A A bar A A transpose is equal to A transpose A and they are same. Similarly, one can prove this result that Skew symmetric matrices are normal matrices and orthogonal matrices are also normal matrices.

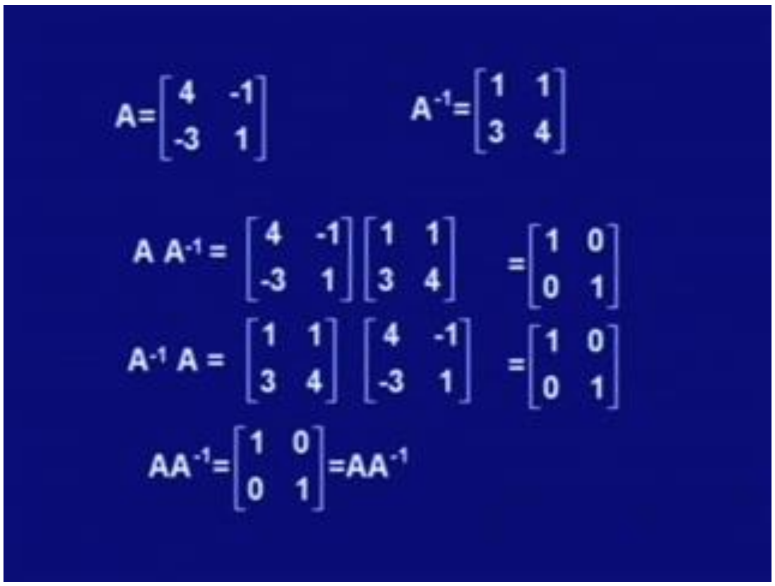
(Refer Slide Time: 53:50)



Then another important concept in matrices is the inverse of a matrix. See we have already defined additive inverse; that means, if I have a matrix A, then the matrix B is called the additive inverse of A. If the sum of A plus B is an identity - and in the case of additive identity - it is 0. So A plus B is equal to I, A plus B is equal to 0 - that is the additive inverse of the given matrix. But when it comes to multiplication, then we say that B is inverse of A. We write B as A inverse. If A B and B A are the same and they are equal to I - so if I multiply A B and it gives me I, whether pre multiply or post multiply it should give me I - so if we have such a matrix B, then we say B is inverse of A.

So with this, we have obtained the existence of multiplicative inverse of a given matrix. Now this is not as simple as we had in the case of addition of matrices. Now we say that matrix this is a very important property of matrices. We say a matrix A is non singular or we say it is invertible if inverse of A exists. It may not exist for all the matrices for all the given matrix A, because first thing is A has to be a square matrix only when A B and B A are possible. So it is doesn't exist for all matrices - it has to be square matrix first - and even if it is a square matrix, still and the product may not be possible. The product may not be possible or product may not be identity. So every matrix is for every matrix, you cannot find such an inverse B. So we say those class of matrices for which you can find inverse those class of matrices are invertible matrices or we say the matrix is a non singular matrix. On the other hand, we say a matrix A is singular or non invertible if inverse of A does not exist. So square matrix is singular or non invertible, if inverse of A does not exist. If the matrix is not a square matrix, then of course there is no question that inverse will exist.

(Refer Slide Time: 56:24)



Now, so for example we consider 2 by 2 matrix A as 4, minus 1 in the first row, minus 3, 1 in the second row; then my claim is that A inverse is this. How I find is that's a different issue, but at the moment my claim is that A inverse is the inverse of this. So Let us check whether my claim is correct or not. So what I do is I perform the multiplication A and A inverse; A is this and A inverse is this. If I multiply this first row by first column, the result is 1; first row second column the result is 0: 4 minus 4 - 0; minus 3 1, 1 3 – 0; minus 3 1 and 1 4 - the result is 1. So A A inverse is identity; but this not enough. One has to prove that A inverse A is also identity.

So, I again consider A as this and A inverse as this. Multiply them and one can prove that this is also identity matrix. So we can say A inverse is the same as A A inverse and both of them are identity matrices. So we say this matrix is the inverse of this matrix A or we say for this given matrix A, this is A inverse. So this matrix A is non singular or we say this matrix is invertible.

Towards the end of my lecture, I like to summarize what we have done today. Starting with the expression of the form A cube minus 5 A minus I, I have introduced i have introduced triangular matrices. Then, I have performed various operations on triangular matrices. I have introduced special types of matrices: the Hermitian matrix, the symmetric matrices, Skew symmetric matrix, Skew Hermitian matrix and then orthogonal matrix, unitary matrices and finally the normal matrices; and towards the end, I have introduced the inverse of a given matrix and I have introduced the concept of non singular matrices. All these concepts - you will find application in later part of your course. My next lecture will be on determinants and we will be seeing that how these concepts will be useful in solving equations and they will be useful in other branches of science, and generally. Thank you.