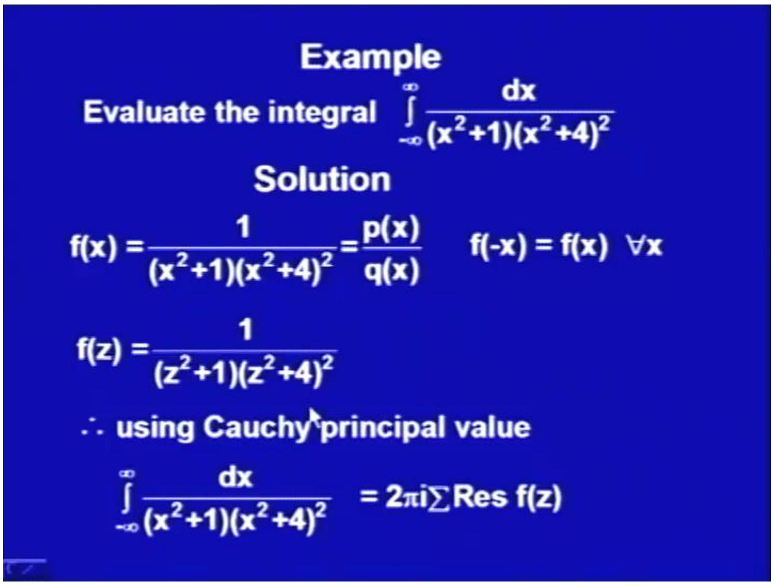
# Mathematics - II

**Prof. Tanuja Srivastava**
**Department of Mathematics**
**Indian Institute of Technology, Roorkee**

**Module - 1**
**Lecture - 14**
**Evaluation of Real Integrals – Revision**

Welcome to the lecture series on complex analysis for undergraduate students, today's lecture is the Revision of Evaluation of Real Integrals. We had learnt the complex integrals and there we had learnt one theory called residue theory and we had used it for evaluation of real integrals. Today we would summarize we had learnt many methods, today we will summarize those methods with help of some examples, that is which method would be applicable, so let us see the first example.

(Refer Slide Time: 01:04)




Evaluate the integral minus infinity to plus infinity d x upon x square plus 1 x square plus 4 whole square, now we see this function first this integral is improper integral, second the function which we have to evaluate that is 1 upon x square plus 1 into x square plus 4 whole square. If we just go with the function we see is that this we could say is a rational function, where the numerator is 1 and the denominator is a polynomial of degree 6.

Moreover with the denominator if we do see, the first part of this denominator that is first factor x square plus 1, it will have factors as x plus i and x minus i. The second factor which is x square plus 4 it is whole square, x square plus 4 again has the factors as x minus 2 i and x plus 2 i and both are the powers 2. What it says is that this denominator has all has no real zero, it has all the zeros which are in the complex plane.

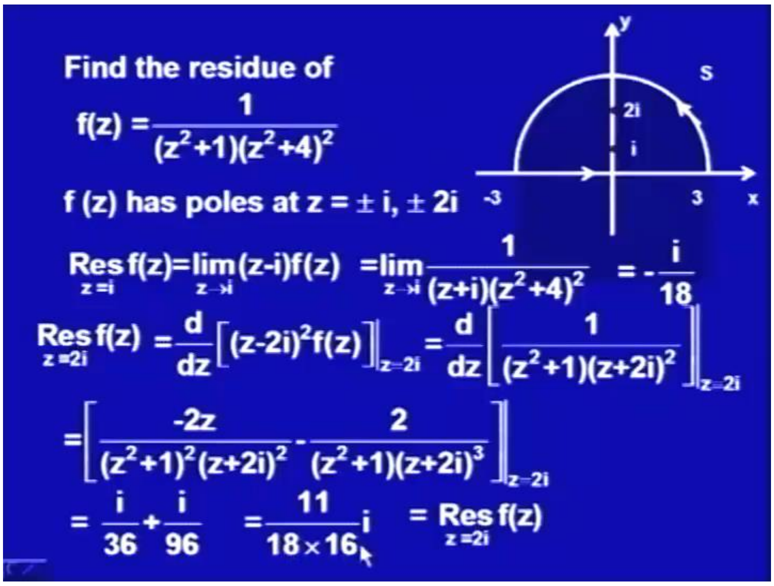
That says is the method which we had learnt for evaluation of improper integrals where our f x the function is a rational function of the form of p x upon q x and p x both are polynomials. Then the degree of the numerator and denominator the difference has to be at least 2, that is the degree of the denominator has to be at least 2 degrees higher than the degree of the numerator.

Here we see the numerator is the constant, so degree is 0, the denominator has the degree 6, so we do have that is portion is also getting satisfied. So, let us try it how we are going to do is we will apply the residue theory and we will consider the complex function moreover this integral is also this function is also an even function. So, the corresponding complex function we would consider as 1 upon z square plus 1 into z square plus 4 whole square.

Now, what we do have here is that is it would have the denominator would have a zeros at plus i minus i plus 2 i and minus 2 i, where plus i and minus i are the simple zeros, while as 2 i and minus 2 i they are your zeros of the second order. So, we will use the Cauchy principle value and that says is that integral minus infinity to plus infinity d x upon x square plus 1 into x square plus 4 whole square.

The formula is equal to 2 pi i summation residue of f z, where this summation is over all those poles which are in the upper half plane. So, let us just move to find out the residues of this function f z, where f z is this function.

(Refer Slide Time: 04:02)




Let us see if I do have this function which we had find out that the poles are i and 2 i minus i and minus 2 i, we are interested only in the poles in the upper half plane that is i and 2 i both are here. So, we are taking just a circle, semicircle that is this s from minus r to plus r, so here if i take it from minus 3 to plus 3, it would include both my simple poles that is both my poles of the upper half plane.

So, this r we could use this 1, but of course here we have to take this r to be a varying 1, so now find the residue of this function 1 upon z is square plus 1 and z is square plus 4 whole square, we do have that this has poles at plus minus i and plus minus 2 i. The poles at plus minus i that is a simple pole, but pole are 2 i and minus 2 i both are the poles of the order two.

So, first we would find out that simple pole, for simple pole we will use the simple formula that is residue at simple pole z naught is limit, z is approaching to z naught z minus z naught of f z. Since, we do have that function is more easy if we make the factors over here and multiply z minus z naught with the f z that would be much easy to evaluate rather then,using the other method of finding out the simple pole p z upon q dash at z naught.

So, z minus i first we are calculating it at z minus i, here we would get is z plus i into z minus i, so the z plus i that factor would remain here and z is square plus 4 whole square would be there. So, we get limit as z is approaching to i 1 upon z plus i into z square plus

4 whole square, now evaluate it at z is equal to i, z is equal to i this z square is minus 1 minus 1 plus 4 that is 3, 3 square is 9.

And here is i plus i that is 2 i, so we wound get 9 into 2 i r minus 1 i upon 18, because i if I am taking up, then it would be multiplying it by minus i, so it is minus i upon 18. Now, 2 i is your pole of second order that says is this formula will not be applicable, rather we would use the formula for the second order pole, that residue at z is equal to z naught for n th of pole is 1 upon factorial m minus 1, the m minus 1 th derivative of the z minus z naught into f z.

So, here z minus z naught, because we want the pole is at 2 i we want the residue at 2 i, so z minus 2 i multiply with f z and we take the derivative, because it is a the second order of pole, so m would be 2, that is m minus 1 th derivative that is first derivative and 1 upon factor 1 that is 1. So, we would get is d by d z of z minus 2 i whole square, because the m th order pole, so there is a z minus z naught to the power m into f z evaluated at z is equal to 2 i.

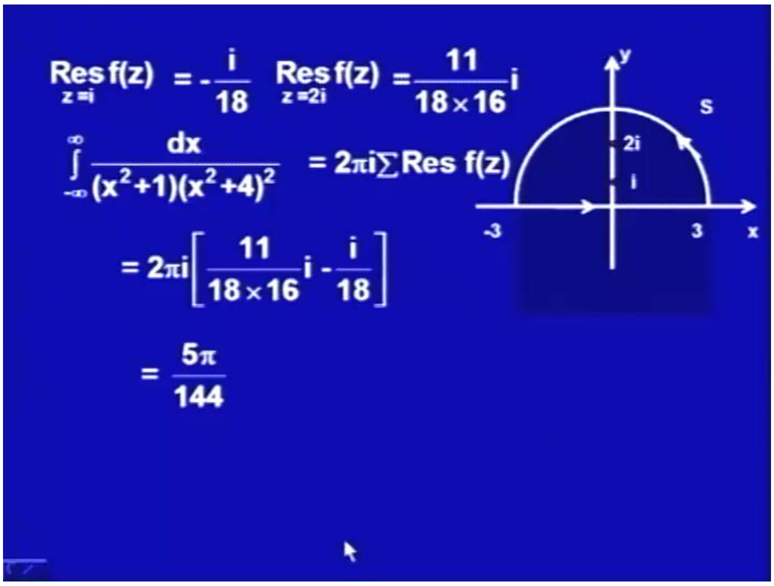
Now, even we are writing it the second factor, if I make it would be z plus 2 i whole square into z minus 2 i whole square, z minus 2 i whole square we are multiplying, so what we would be left is 1 upon z square plus 1 as such and here z plus 2 i whole square. We have to differentiate it with respect to z once and then,evaluate that derivative at z is equal to 2 i, so first differentiate it with respect to z, we would treat it as 1 upon z square plus 1 as 1 function another function as 1 upon z plus 2 i whole square.

So, by multiplication rule we would just use it, here we would get it is minus 2 z upon z square plus 1 whole square into z minus 2 i whole square as such. Then, plus the first function into the derivative of the second function derivative of the second function, because it is 1 upon z plus 2 i whole square, it would be minus 2 upon z plus 2 i whole cube. So, we do get is minus 2 upon z plus 2 i whole cube and the first function 1 upon z square plus 1 as such.

Now, evaluate it at z is equal 2 i, when we are writing z is equal to 2 i, here i would get it minus 4 i, z is equal to 2 i that says is z square would be minus 4, minus 4 plus 1 is minus 3 whole square that is 9 2 i into i that is 4 i, so here what we are getting is minus 4 i 9 into 4 i whole square. And here what we would getting is again z square plus 1, that is your minus 3 and z plus 2 i whole cube that is 4 i cube.

So, now simply it, the first one we would get i upon 36, while the other one we would get i upon 96, adding it up we do get is 11 upon 18 into 16 i. So, now we have got the residue at i, which is a simple pole that is we had calculated as minus i upon 18 and at 2 i, which is a second order pole as 11 by 18 into 16 i.

(Refer Slide Time: 09:29)




Now, apply our theorem which says is that, residue of this we had calculated, so now applying this result that is minus infinity to plus infinity this integral of 1 upon x square plus 1 into x square plus 4 whole square with respect to x. This could be given as the 2 pi i times summation residue of f z, where summation is at all the poles in the upper half plane that is at these two ones.

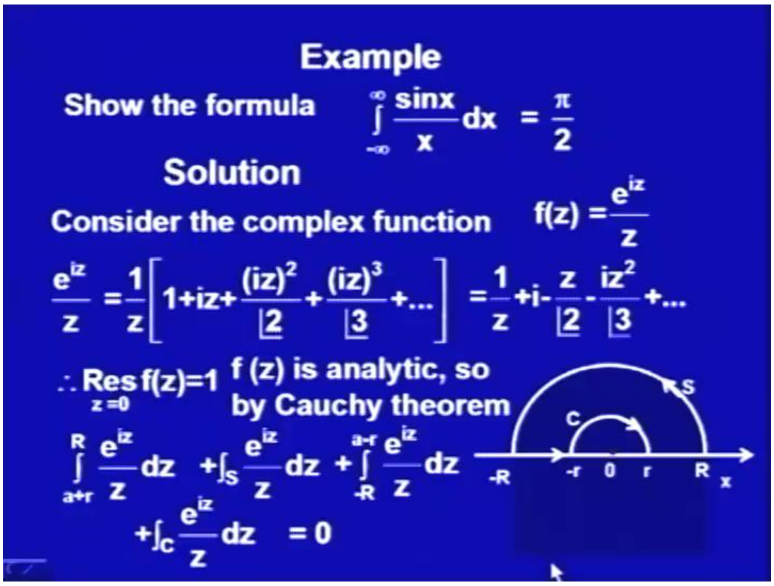
So, we just add upon these two residues multiplied with 2 pi i, so 2 pi i that is a here in the numerator we would get it actually i square that is minus 1, so that is also we are taking outside 2 pi i 11 upon 18 into 16 i minus i upon 18, take i upon 18 common. And we would get it 11 by 16 minus 1 that is minus 5 by 16 and thus we would get is 2 pi i then,i that is minus 1 and i square is minus 1, minus 5 pi 5 upon 16 was there and 11 by 18 was there, so we 1 upon 18 was there.

So, we have simplified and you will get that 5 pi upon 144, so you had find out that is here we had applied for evaluation improper integral, the first method which we had learn. When f x is a rational function both the numerator and denominators are polynomials, denominator does not have any real 0 and the degree condition that is the degree of the denominator is at least 2 degrees higher than the degree of the numerator, those are satisfied that method we had applied.

Because, that is guaranteeing us that is when we are writing this closed contour that is s and the part of the real line from minus r to plus r, that takes that is all the residues are the isolated singularities inside the or interior to the contour and using the residue theorem we have got this result. And then,we are evaluating it into breaking it into two integrals, one is integral on the semicircle, another is integral from minus r to plus r.

Showing that integral on this semicircle s as r is approaching to infinity is approaching to 0 that require the degree condition, since that was satisfied, so that we would go to 0. And this integral from minus r to plus r on the real line will approach to minus infinity to plus infinity that is how we had got the Cauchy principle value. And since the function is even and satisfying all those condition integral is existing, so Cauchy principle value would be equal to value of the integral. So, we have got this result, that 5 pi upon 144 is the value of this integral, so this is one method which we had learnt for this one, now let us do some more examples.

(Refer Slide Time: 12:29)




Show the formula minus infinity to plus infinity sin x upon x d x is pi by 2, now let us see this function, again this integral is improper integral the function we are having is of the form 1 upon x into sin x. Now, upon x is a rational function that is we are having of the form minus infinity to plus infinity f x sin s x d x kind of function, again this is another improper integral which was having the rational functions with sin and cosine functions, so my rational function here is 1 upon x.

If I see this rational function 1 upon x, certainly it is the rational function, where the numerator is 1 and denominator is x, but the degree condition here is not satisfied, moreover x is having 0, that is x is equal to 0 this function is vanishing. So, we are having actually an improper integral, improper in the sense that is the limits are infinite as well as we are having is the function is also vanishing at x is equal to 0; at x is equal to 0, we are getting is that this function 1 upon x would be approaching to infinity.

Certainly sin x upon x we could say is, if you do remember that this has essential singularity at x is equal to 0 and that can be removed if we are defining the function separately, but that condition is not being given. So, it has two kind of improprieties that is one improper integral is because of the limits, another is because of this function is undefined at x is equal to 0.

Let us see is how we are going to solve or how we are going to find out this formula, first we will do is that is we would we are interested in applying the residue theory, for applying the residue theory we require the complex function, so we will consider the complex function e to the power i z upon z. Now, see this function e to the power i z is entire function and z has a 0 at z at the origin, so this f z would have a simple pole at origin.

Because, we are interested in applying the residue theory, so I am first finding out the residue of this function and then,we will see that is how this integral we would be doing. For finding out the residue I would simply use our first method is that is of Laurent series, for writing the Laurent series for this function it is easy, if I write the Maclaurin series for e to the power i z.

So, 1 upon z and Maclaurin expansion of e to the power i z is 1 plus i z plus i z whole square upon factorial 2 plus i z of cube upon factorial 3 and so on, multiplied with 1 by z we would get 1 by z plus i minus z upon factorial 2 minus i is times z square upon factorial 3 and so on. So, now what we are getting is, the Laurent series is having the terms, from here the terms are a naught plus a 1 z a 2 z square and so on, and this is the term of the your principle part is containing only one term 1 upon z.

That says is again it a confirms that is your z is equal to 0 is your simple pole and the residue at z is equal to 0 is the coefficient of 1 upon z that is 1, so we have got the it has only 1 simple pole at z is equal to 0 the function and residue at that point is 1. Now, let us see is where is this pole, if I just go the use of finding out that improper integral of the form integral from minus infinity to plus infinity f x d x, we use to take this semicircle from minus R to plus R and then,the portion of the real line from minus R to plus R.

And then,we use the residue theory, using it that the all the poles are inside the or interior to this close contour, but here in this particular example what we are having is that our pole is on the real line that is it is lying on the contour. So, this particular contour that is this S and the portion of this real line from minus R to plus R, this is not going to serve our purpose, because our singularity is lying on the contour.

What we require that close contour should not have singularities or the singularities should be interior to the close contour C, so let us redefine our close contour. Let us say is that is our contour is starting from here, where we have taken one more semicircle around this singularity 0 of the radius is small r. This semicircle we are taking again in the upper half portion and the orientation you see is in the negative sense, that is it is in the clockwise manner.

Now, let us a start from this point r, a small r from this is small r to capital R, then this upper semicircle or the bigger semicircle, then the portion of real line from minus R to minus of small r and then,this smaller semicircle in this orientation. So, now we had made our close contour and in this close contour, we are having is that the function f z is analytic.

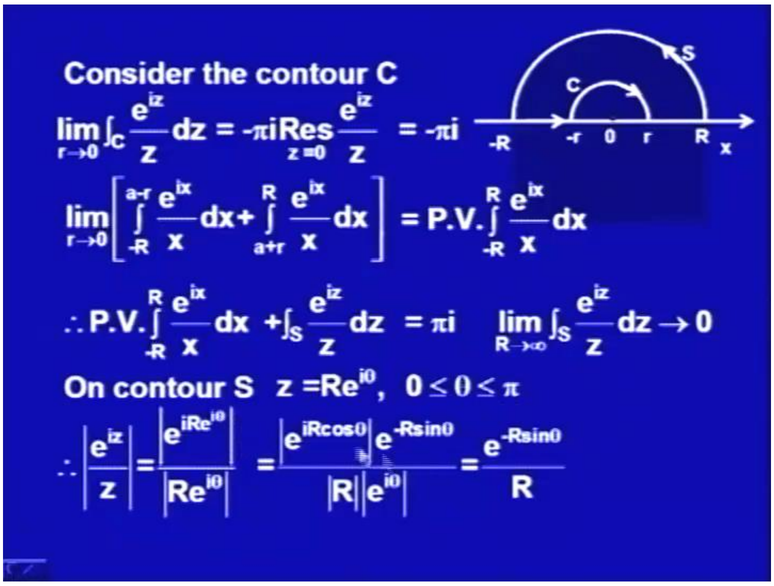
Actually we are not having this e to the power i z upon z, this is analytic in this close contour that is on this close contour and inside or interior to this one, that says is with the Cauchy integral theorem, we could apply which says is that integral along this close contour must be 0. Because, if the function is analytic on and interior or inside to a close contour C, then the integral along that close contour if the function f is 0.

So, what is this close contour close contour is your r, so here it is written a plus r, so it is 0, a is your 0, so r to capital R then integral along this semicircle in the counter clockwise manner from r to minus R. And then,minus R to this minus r and then,your C in the clockwise manner all this should sum up to be 0. Now, let us take this a smaller semicircle C, what now we have to evaluate, actually we have to find out the integral from minus infinity to plus infinity of e to the power i z rather you could say.

So, first thing is that is we are interested, first we will find out the improper integral of the form that is where the limits are finite and the function is vanishing or function is approaching to infinite, at some point of function is having singularity in the on the real line. In that method what we used to take is, that we use to take a smaller neighborhood around that singularity and we use to push this r using that improper integral is defined that is, that r is approaching to 0.

So, we first make here the limit as a small r is approaching to 0, in that case what will happen we have done one result, if you do remember, which said is that if there is a singularity at a and we do have a semicircle, then that integral along that semicircle was approaching to 0 as r is approaching to 0.

(Refer Slide Time: 20:32)




Let see that is how we are doing it, so this if we are considering this contour C, then we had got one result if you do remember done, which said is that limit r approaching to 0, integral along this close this semicircle c e of the function e i z upon z would be minus pi i residue at z is equal to 0 e i z upon z. That says is we required only the residue of the function e i z upon z at z is equal to 0, which we had already calculated, so this is we had calculated as 1, so it is minus pi i.

Now, limit as r is approaching to 0 minus R to this a minus r, that is 0 minus r minus r e to the power i x upon x and integral from r to R capital R e to the power i x upon x as r approaches to 0. It is actually Cauchy principle value of integral from minus R to plus R e to the power i x upon x d x, so what we have got if I am taking the limit as r is approaching to 0 in our this close contour, over this close contour the integral was 0.

So, there if I am taking the limit as r is approaching to 0, the right hand side would remain as 0, the left hand side the two terms we had or rather three times we had made, that is the two terms which are on the real line, those we have got as principle value of integral from minus R to plus R e to the power i x upon x. And the third term that is when limit as r is approaching to 0 integral along this semicircle, a smaller semicircle c is minus pi i.

That says is that principle value from minus R to plus R e to the power i x d x plus the integral along this bigger semicircle S of e i z upon z d z is equal to pi i, that minus pi i I am taking out this right hand side to the 0 side. Now, we want the integral from minus infinity to plus infinity, that says is that improper integral we would be defining or the principle value we would be defining if I take the limit as R is approaching to, capital R is approaching to plus infinity.

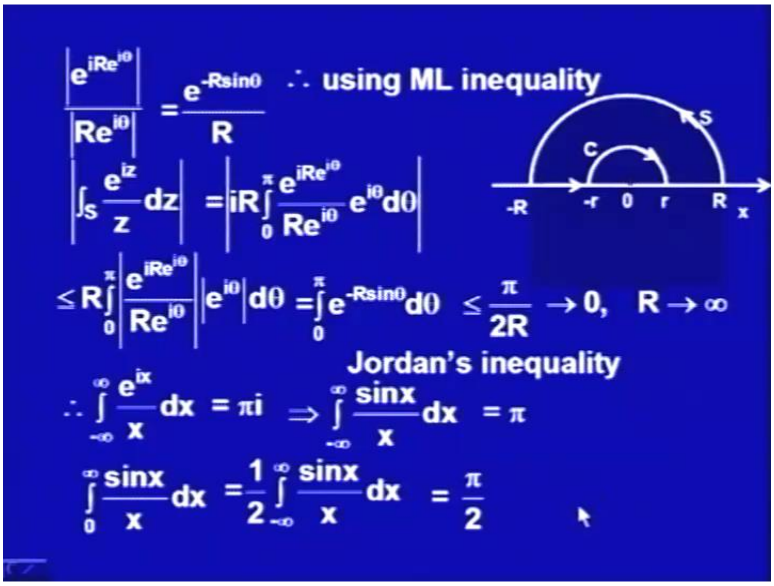
In that case I have to first find out what this second integral is approaching, because the first integral will approach to the principle value of minus infinity to plus infinity e to the power ix upon x d x. But, what this integral along the semicircle S will be, we say is that limit as R is approaching to infinity integral along the semicircle of the function e to the power i z upon z would be approaching to 0, see that is how it is approaching to 0.

If I take this contour S, this is a semicircle centered at 0 we could write this parametric equation for this semicircle as capital R e to the power i theta, for theta is varying from 0 to pi. And what will be our this function e to the power i z upon z, because on this contour S z is R e to the power i theta, so I am replacing z with r e to the power i theta, so what we get e to the power i R times e to the power i theta upon R e to the power i theta.

Now, this numerator i R e to the power i theta, e to the power i theta, if I use the Euler's formula which says e to the power i theta is cos theta plus i sin theta, what I would get is e to the power i R cos theta into e to the power minus R sin theta. So, the absolute value or the modulus of this, we would be having e to the power i R cos theta into e to the power minus R sin theta, the denominator we would get mod R and mod of e to the power i theta.

Mod of e to the power i theta and mod of e to the power i R cos theta both are 1, so what we had and the R is positive, because we have taken R to be the positive R and minus R like this. So, what we have got that absolute value of e to the power i z upon z is actually e to the power minus R sin theta upon R, so this is true for all z, when z is on this semicircle s, now write this integral along this semicircle.

(Refer Slide Time: 25:20)




So, if I use this ML inequality along the semicircle S e to the power i z upon z d z, this rather than ML inequality, we are writing it as first the line integral along this semicircle that is the contour integration. We are changing z to the R e to the power i R theta that says is that we write it as f theta and then,z dash theta d theta, so function e to the power i z, we could write at e to the power i R e to the power i theta, z R times e to the power i theta, d z would be i R e to the power i theta d theta.

This absolute value of this integral, we could write it as absolute value of i R that is R and the absolute value of the integral 0 to pi and all this inside thing, that would always be smaller than the integral 0 to pi absolute value of the integrant. This is again we are using the line integral property or you could say definite integrals property, so it is less than or equal to R times integral 0 to pi e to the power i R into e to the power i theta upon R e i theta into e to the power i theta d theta.

We do know that absolute value of e to the power i theta is 1, the absolute value of this we had all ready calculated that it is e to the power minus R sin theta upon R, so upon R that is constant that we can take it out. So, what we would get is actually it is integral 0 to pi e to the power minus R sin theta d theta. Now, you do remember that one result we have got that we called Jordan's inequality, which said is that this integral 0 to pi e to the power minus R sin theta d theta is less than pi upon 2 R, this was the Jordan's inequality.

So, now what we have got that integral of function f z that is e to the power i z upon z along the semicircle S, absolute value of this is bounded by pi upon 2 R this depending upon R. So, now if I take the limit as R is approaching to infinity this function, this pi upon 2 or this would approach to that as R is increasing, pi upon 2 R would be decreasing to 0; since this is absolute value this has to be positive that says is that the value itself must be 0.

So, what we have got is that integral along the semicircle S of the function e to the power i z upon z d z as R is approaching to infinity is approaching to 0 that we had shown. So, now if I take the limit as R is approaching to infinity in the last expression, which we have got that principle value of minus r to plus r e to the power i x upon x d x plus integral along the S e to the power i z upon z d z is equal to pi i; so this would be pi i, because the second one part is 0.

Now, this e to the power i x is again using Euler’s formula, we could write as cos x plus i sin x, that says is this integral we could write as two integrals, one integral is cos x upon x d x, another is i times integral minus infinity to plus infinity sin x upon x d x. Equating the real and imaginary parts, we would get it that sin x upon x the integral from minus infinity to plus infinity with respect to x is pi, so this is for how we had obtained a formula showing this one.

So, here what we had used again we had use the residue theory, but that residue theory rather we had use not only the residue theory, we had use the Cauchy integral theorem as well. Because, that function was having only one isolated singularity and that isolated singularity was at origin that is on the real line.

And the function of the rational form, but it was not satisfying our degree condition, but the second method which said is that degree condition we could use when the difference is only of at least one. And the function is of the form, integral is of the form f x into sin x or cosine x and integrating from minus infinity to plus infinity, in that method we had used, so what we have got since there was no other singularity in this region.

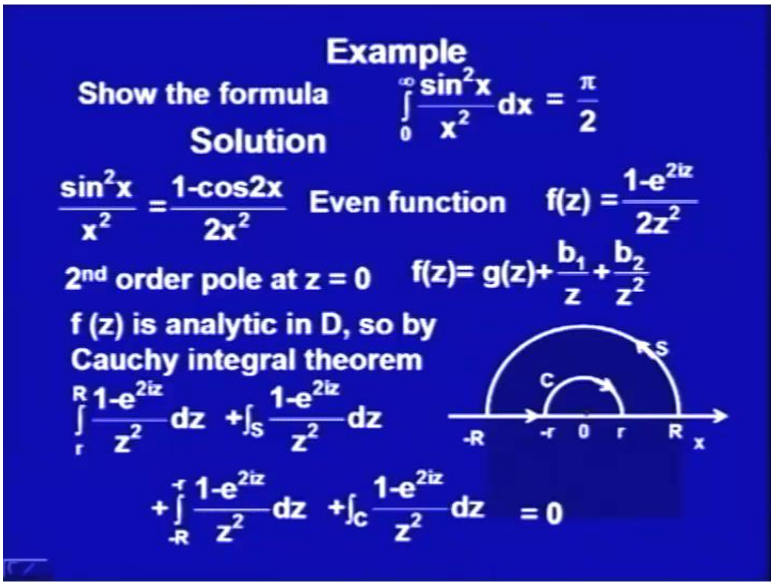
So, rather than using for this region, the region between these two semicircles we have got the function is analytic, and thus using Cauchy integral theorem on the boundary of this one, we Cauchy integral theorem we got that this integral was 0. And from there we had, now break that integral into the parts that is one part upper semicircle that is bigger semicircle, then the smaller semicircle and then,the two portions of the real line.

When we are talking about this smaller semicircle, this is smaller semicircle when we are talking about with this a portion of real line, which is involving this singularity at the origin. We had one result already proved, which said is that the integral along this semicircle would be residue pi i times, the residue of the function f z complex function f z at the point of singularity.

And that point of singularity comes out to be here a pole, a simple pole that we could calculate very easily. And then,for the bigger circle, semicircle S on this portion we could show that, because f is of the form and that we could show using with the help of Jordan's inequality. That it is approaching to 0 as R is approaching to infinity thus we could evaluate this formula or we could find out this formula.

So, this improper integral we are not able to evaluate directly, we are using it with the complex 1 it is much easy to evaluate and this we are using as a formula in the real integrals, let us have one more formula. So, here we could have that is, since this function if I do take sin x upon x, the function is even function, because sin of minus x is minus sin x and minus x, so we would get again f of minus x is as f of x. That says is if I have to evaluate the formula 0 to infinity sin x upon x d x, that should be half of minus infinity to plus infinity sin x upon x d x, so it should be pi by 2, so one more over here. let us move little bit one more the different kind of example.

(Refer Slide Time: 32:39)




So, consider one more example, show the formula integral 0 to infinity sin square x upon x square d x is equal to pi by 2, so again here is we have to show this formula or rather you could say is we have to evaluate this integral. And we are being given the value of this integral is pi by 2, so we have to reach to this value, let us see this integral once more.

Integral is again of the form of improper integral where the one limit is approaching to infinity, the function is sin square x upon x square, so we get that here we are involving sin square term. So, it is neither f x times sin x cos x terms or it is neither we are going to use it as polynomial a rational function sin square x upon x square, we would like to convert it into the form of f x cos x or sin x kind of thing.

We do know that sin square x is 1 minus cos 2 x upon 2, so sin square x upon x square we can write as 1 minus cos 2 x upon 2 x square. Now, we are having is the function one, function you could say is 1 upon 2 x square and another as 1 upon 2 x square, the two integrals one is 1 upon 2 x square and minus integral 1 upon 2 x square cos 2 x. So, one function you could say is your 1 upon 2 x square that is the rational function another is rational function multiplied with cos is x sin is x kind of thing.

Or now let us see rather than evaluating it as two integrals I would evaluate it actually as one integral that is integral of the function f x, my f x is rational p x upon q x, p x is 1 minus cos 2 x and my q x is 2 x square. So, we do have that the degree of this one we would take this cos 2 x that is as a single function, single degree, so we would take it as that degree condition is being satisfied.

Or rather basically degree condition we require to show that the integral along bigger semicircle or somewhere this integral if you are using this one it should be 0, this function is all right is of the rational function p x upon q x. But, the condition that q x should not have a real 0 that is being failing here, we do have that x is equal to 0 is the 0; now we have done the method which said is that when the function is having a real 0, whether it is single zero or more than one zero on the real line, but all those has to be the simple zeros.

What here we are having is this zero of denominator 2 x square it is the second order 0, so now the method whichever the formulae's we have done or the methods we had explained, they are not directly applicable. Rather we would move in the same manner and try to see is that is can we still reach to this formula using the residue theory, another thing which we are finding it out is that this function is an even function.

Because, sin square x as x is positive or negative, it is again the same thing and the x square is also positive and negative same thing that is f of minus x same as f of x. Even function says is that I can move to the integral evaluate the Cauchy principle value for minus infinity to plus infinity sin square x upon x square and from there with the half of that we could evaluate this integral.

Consider the complex function we are intended to apply our residue theory, so consider the complex function 1 minus e to the power 2 i z, you see that is the function I am considering is e to the power 2 i z upon 2 z square. If I write the Euler's formula for e to the power 2 i z, I would get it is cos 2 z plus i sin 2 z that is 1 minus cos 2 z minus i times sin 2 z. So, we would get it actually this a function that is which we have to evaluate this one, that is the real part of this complex function.

So, that portion is all right, that is we would evaluate the integral of this and whatever we would be getting the integral of this along the some contour C, we have to change it to the real part of that. So, the integral from minus infinity to plus infinity that is on the real line of 1 minus cos 2 x upon 2 x square, we would like to evaluate the integral along a close contour, which is containing the portion of real line.

That is complete real line minus infinity to plus infinity for 1 minus e to the power 2 i x upon 2 x square and it is real part would give me the integral whatever we are requiring. So, the complex function we have taken this one, this function is having, the denominator is having 0 at origin and that 0 is of the second order, that says is this function has the second order pole at z is equal to 0 using that result of a relating with zeros and pole when they are polynomials.

Now, what we are having is, now the contour which we would consider, it is a if I just consider the line from R to plus R, then the a singularity is at the contour, so we have to skip this singularity at 0. So, we would consider the contour from a small r that is around this origin we have taken a smaller semicircle with radius a small r, so from this point a small r, so my contour is consisting of the portion of the real line here from small r to capital R.

Then, the semicircle S of the radius R, then the portion of real line from minus R to minus of a small r and then,this is a small semicircle C with the radius a small r around the origin 0. Both the semicircles are centered at origin and the radius of the larger semicircle capital S is your capital R, while is for this smaller semicircle C is small r, function has a pole a second order of pole at z is equal to 0.

And the function inside this close contour which just now I had described, the function is analytic inside and on this close contour just now which I have been described. Since it is having a pole of the second order, the function f z we can write g z plus b 1 upon z plus b 2 upon z square, we do know that the if the function is having isolated singularity we can write the its Laurent series.

And since it is a pole the Laurent series, the principle would terminate at the finite number of times that is what is the order of the pole at that point, so because it is a second order pole that say is in the principle part we would have only two terms. Since the pole is at z is equal to 0, so z minus z naught is the simply z, we would get f z as g z plus b 1 upon z plus b 2 upon z square.

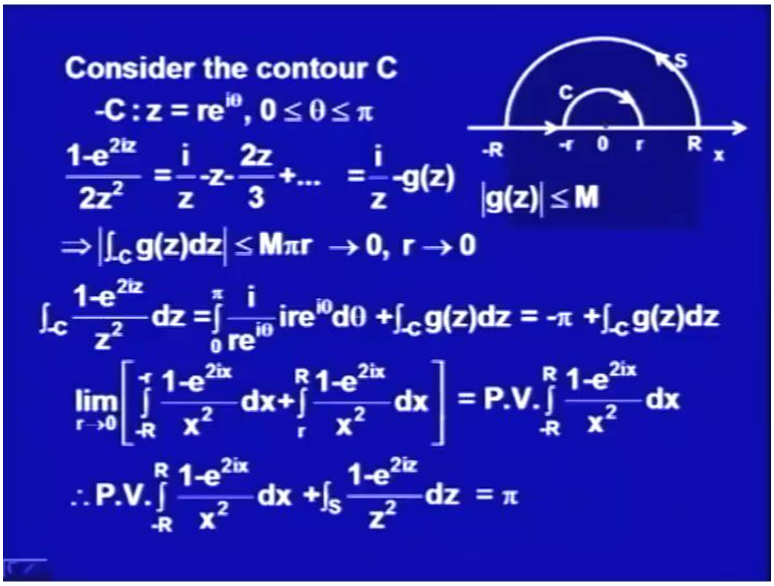
Now, let us see that is in this region that is, if I take the d region as containing this the close contour which I have defined and the interior to it, then this function f z is analytic inside and on that close contour. So, you could say in the domain D, simply connected domain D, so using the Cauchy integral theorem, the integral along this close contour would be 0.

So, we would get the integral from r to capital R of the function 1 minus e to the power 2 i z upon z square d z plus the integral along this contour, that is the semicircle s of the function 1 minus e to the power 2 i z upon z square dz. Plus the integral along this portion of the real line minus R 2 minus a small r 1 minus e to the power 2 i z upon z square d z plus the integral along this semicircle C from minus r to plus r that is from pi to 0.

1 minus e to the power 2 i z upon z square d z this would be equal to 0, according to the Cauchy integral theorem, because the function is analytic inside and on this close contour C. So, now from here we are interested in actually finding out this formula or rather we are interested in evaluation of the integral of this function from minus infinity to plus infinity.

We would be using it as the two improper integrals, because the this singularity is on the real line, so we would be using is that, once we would put this limit a small r is approaching to 0 and try to evaluate this integral along this a smaller semicircle. And then,once that has been evaluated, then we move to the outer circle and we will try to put the limit as capital R is approaching to infinite. Thus this whole region we would be covering and then,we will try to show that this portion of that integral along this semicircle is approaching to 0, hence what is going to remain is that the principle value on the real line, so let us start one by one.

(Refer Slide Time: 43:16)




First we will consider the contour C, so let us write this in the anti-clock manner that is the minus C, for that z is equal to r e to the power i theta, theta ranging from 0 to pi, because if I take this orientation in this manner, then the my theta 0 would be coming this side, so minus C I am taking it, so this is it is going theta is 0 to i pi. And along this contour if I take my function 1 minus e to the power 2 i z upon 2 z square again write a the Maclaurin expansion for e to the power 2 i z.

We would get it as a 1 plus 2 i z plus 2 i z whole square upon factorial 2 and so on terms, so what we would be getting is the term constant 1 minus 1 that would be canceling it out. We would be getting the first term as 2 i z upon factorial 1 with the minus sign and divided by 2 z square i would getting it as a i upon z and so on, we would be getting the terms. What it says is that, I would be getting actually all those terms would be involving the positive powers of z, so those z to the power n is always an entire function.

So, we do have is that function we are writing as g z and this function the part which is principle part that we are writing as i upon z, so what we could say is that this function we could write as i upon z minus g z. Certainly here what we are getting is that the residue if I try to find out that residue of this function, we are not using the residue result over here. Rather what we are doing is that we are writing this function into two portion, one portion is analytic another portion is having a part where the singularity is there.

Since g z is analytic, so if I consider g z on this contour C, C is the contour which is a semicircle around this origin 0 with a small radius r, that says is we are and g z is analytic, analytic at z is equal to 0, use g z is analytic at z is equal to 0 that says is g z is continuous and z is equal to 0. By the definition of continuity if I take any z in the small neighborhood of 0, then g z minus g 0 that is 0 is g z minus 0 that would be less than epsilon for all z such that, z minus 0 mod of z minus 0 is less than r.

So, we have got that mod of g z that would be bounded by some constant capital M say for all z on this contour C, or rather you could say is more easily, since it is continuous let us take 1 z which is at the maximum distance from this 0. Since this is a semicircle we do have that every 0 would have equal kind of distance and the value, so what we say is that is suppose M is the value which is taken by this function g on this contour, so let say that M is the maximum of g z on this contour.

Then, we do have that mod of g z would be less than or equal to M for all and that will happen because, we do have it is a continuous function, so we would find it out that it will have some value at which is maximum for all the z on this contour C, so we have got a bound for this one. Now, let us a use the integral of this function along this contour C, so rather than contour C I am considering the contour minus C, the change in the C and minus C is that is the change of orientation only.

And we do know by the contour integration that if we are changing the orientation, then the integral along that contour minus C is that is nothing but, minus of that one. And since we are talking about the absolute value of this minus it is not going to make that much difference or we could say is that is it should be simply minus would be taken out. Use this ML inequality, since g z is bounded by M this absolute value of this integral would be less than or equal to M times the length of the contour.

What will be the length of the semicircle pi times r, where the r is the radius of the semicircle, now as r is approaching to 0 certainly this would approach to 0. Since the absolute value of this integral along this contour minus C is approaching to 0, that says is the integral itself must approach to 0 along this contour minus C, since along the contour minus C it is approaching to 0. So, along contour C also that is minus of that 1 minus of 0 is 0 that would also be 0, so what we had obtained that as r is approaching to 0, this integral would be 0.

Now, what we are writing this function, the integral of this function 1 minus e to the power 2 i z upon 2 z square along this contour minus C, 1 minus 2 i z upon z square d z, this is integral 0 to pi i upon r. We are writing this contour integration that is the formula, that is if I am representing it by the parametric equation, then we write it f of z of t into z dash t d t is it all right, so t is here theta.

So, this is i upon, this is what is the function i upon z minus g z like that, so i upon z we would write r e to the power i theta z dash theta d theta i r e to the power i theta d theta plus this integral along this contour minus C of g z dz. Just now we had shown that this integral is approaching to 0 as r is approaching to 0, and what will happen to this integral, r e to the power i theta and r e to the power i theta that is a vanishing out the terms it being canceled out, what we are being left is i square that is minus 1.

So, integral 0 to pi of minus 1 with respect to theta is minus pi, so what we are getting is that limit this as minus pi plus this one and now we take the limit as r is approaching to 0, what we do get is a this integral would be minus pi only and because this would be approaching to 0. And as r is approaching to 0, the integral from minus capital R to minus small r 1 minus e to the power 2 i x upon x square, and from small r to capital R 1 minus e to the power 2 i x upon x square d x.

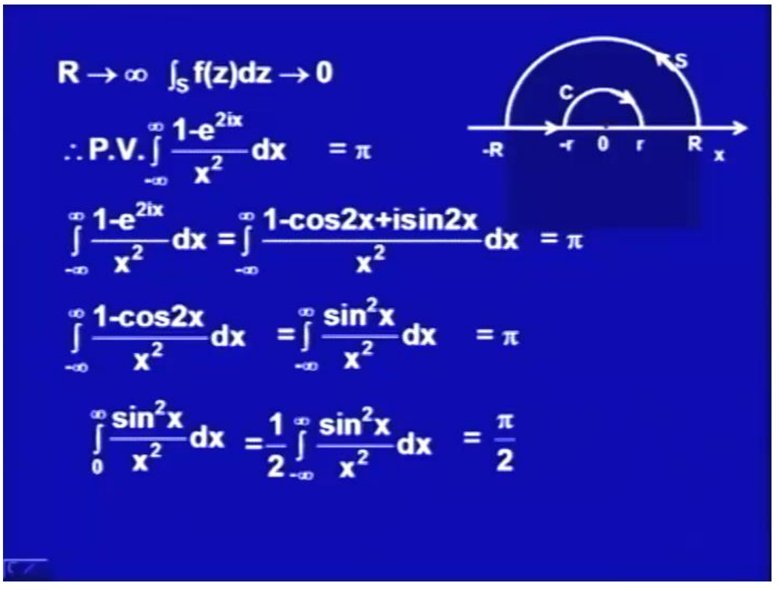
This is nothing but, the definition of improper integral minus capital R to plus capital R of or that principle value of this minus R to plus R 1 minus e to the power 2 i x upon x square d x, so use this limit what we get is this principle value is equal to plus. Now, we are going on our integral the along this close contour was 0, now we are putting this limit r is equal to, small r is approaching to 0 along this whole close contour which is equal to 0.

So, we are getting is this one, since because of these two integrals on the real line, they are approaching to the principle value from of the integral minus R to plus R, plus integral along this bigger semicircle of this function. And plus this integral along this a smaller semicircle, that says is now it is minus pi plus this one and as a small r is approaching to 0, this approaches to 0, so it is minus pi which is equal to 0.

So, this minus pi I am taking that side with on the right hand side as plus pi, now the thing remaining is to show that this integral moves to 0 as your capital R approaches to 0. That you could show, we could write this integral as the form of two integrals, one is 1 upon z square another is e to the power 2 i z upon z square and then,we can show because the condition you could see is that is we are having is one function as 1 upon z square.

It is rational function the degree differences here is the constant and here is 2 degree differences 2, here if I take then the function I will take is 1 upon z square into e to the power i as z kind of thing. And there also the f z is having the degree condition is satisfied 1 upon z square that is 0 and 2, the difference is more than 2 or the is at least 2. So, both the conditions are satisfied that says is that limit conditions would be satisfying and we would get it as approaching to 0 as z is approaching as capital R is approaching to infinity.

(Refer Slide Time: 52:50)



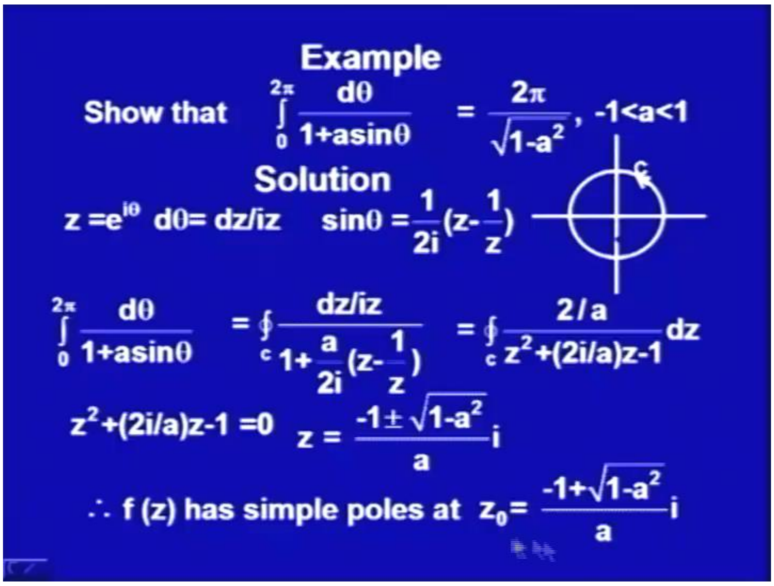

Thus as R approaches to infinity this integral would approach to 0, hence from the previous formula, we would get principle value from of minus R to plus R, now that would approach to minus infinity to plus infinity, 1 minus e to the power 2 i x upon x square d x is equal to pi. Now, this left hand side 1 minus e to the power 2 i x, if I write it as cos 2 x plus i sin 2 x, what I would get it as integral minus infinity to plus infinity 1 minus cos 2 x plus or minus i sin 2 x upon x square d x.

Now, it has a one real part, another is complex part, we are interested in the real part, so integral minus infinity to plus infinity 1 minus cos 2 x upon x square would be pi what is this one, if you do remember this we have got as sin square x. So, this is integral minus infinity to plus infinity sin square x upon x square d x, which we had find out as pi, we are actually in the process of evaluation of the integral from 0 to infinity sin square x upon x square.

As we had noticed that this function is an even function, so we do have that integral 0 to infinity sin square x upon x square d x can be written as half of integral from minus infinity to plus infinity sin square x upon x square d x. So, the since this is even function, we do have that the Cauchy principle value is equal to the value of the integral and thus we are writing it as value of the integral this is equal to pi by 2.

So, this second formula also we had established, now here we had find it out that the all the methods which we have try to establish those were not satisfying. You are not actually finding out any result which we were doing, because the condition which we are having is that the pole on the real line and that is also not a simple pole. But, still we could apply the residue theory, because other conditions are getting satisfied and finally, using on the same lines we could establish this formula. We had use this residue theory in applying the or in solving the definite integrals also, let us just do one example over here.

(Refer Slide Time: 55:19)



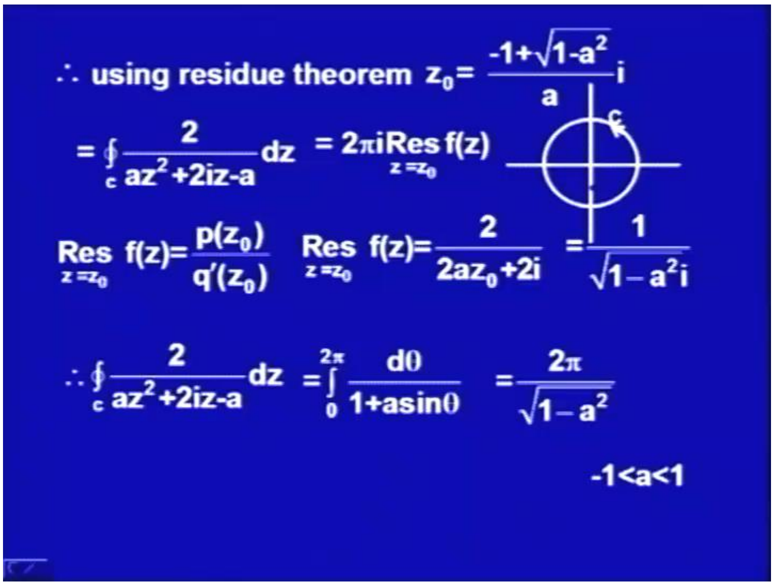

So, show that integral 0 to 2 pi d theta upon 1 plus a sin theta is equal to 2 pi upon square root of 1 minus a square for a lying between minus 1 and plus 1, so what we are here to show one more formula. Of course, we require the condition that a square should lie between a should lie between minus 1 to plus 1, so that this is a real number. At a is equal to 0, we do find out that the left hand side would be the integral of 0 to 2 pi d theta that is 2 pi and right hand side is also 2 pi, so that part we do not require to establish.

For others let us use this function if you do remember, we had used the residue theory first in evaluation of real definite integrals, where our function were of the form of function of cosine theta and sin theta. There we used to remember, if you do remember we used this transformation z as e to the power i theta, so d z would be d z upon i z. And sin theta we could write as e to the power i theta minus e to the power minus i theta upon 2 i that is 1 upon 2 i time z minus 1 by z, because e to the power minus i theta would be 1 upon z.

And then,if you are changing this integral, a definite integral along from 0 to 2 pi we are changing it to the close contour and that close contour is actually your unit circle, so let us transform this integral 0 to 2 pi d theta upon 1 plus a sin theta. D theta we would be writing as d z upon i z 1 plus a sin theta that is a upon 2 i z minus 1 upon z and now close contour C is my unit circle, so simplify this 1 we get 2 upon a z square plus 2 i upon a z minus 1.

Now, let us see this denominator, this denominator has zeros purely imaginary zeros, so this is what are the zeros minus 1 plus minus square root 1 minus a square upon a i, so if I take because a square is less than 1, so 1 minus a square root of 1 minus a square this is a real number. We are getting and this number would also be smaller than 1, so when we do have minus 1 minus of this thing that says is we are going outside this region or this contour unit circle. When we do have minus 1 plus square root of 1 minus a square upon a we are inside this close contour, so we are interested or what we have got that this function f z is having a simple pole at z naught. Let us say that z naught is minus 1 plus square root of 1 minus a square upon a i.

(Refer Slide Time: 58:13)




Now, we would use our residue theory, which says is that integral along the contour, close contour of any function f z d z is actually 2 pi times residue at z is equal to z naught f z. Now, here z naught is this one and f z is 2 upon a z square plus 2 i z minus z, now the function is again of the form of 1 upon f z and it has two roots z minus z naught and z minus z naught conjugate.

But, rather than using that formula I would like to use here the formula residue at z is equal to z naught as p z naught upon q dash z naught, what is q dash z here that would be 2 a z plus 2 i. So, we would get residue at z is equal to z naught, because numerator is only 2 it is constant, 2 a z naught plus 2 i, at z naught if I write 2 a z naught we would get is minus 2 plus square root of 2 times 1 minus a square i.

And 2 i, so minus 2 i plus 2 i will get cancel it out 2 and 2 would get cancel it out, what I would be getting is 1 upon square root 1 minus a square i or rather you could write minus i upon minus a square. A Square root of 1 minus a square, but that I would not be writing, because 2 pi i here we are having which we have to multiply. So, we see this integral along this close contour C of this function according to this formula would be 2 pi i.

And this one is actually we had transformed from here is coming as 2 pi and i is being canceled out upon square root of 1 minus a square and a is we have already taken as between minus 1 to plus 1. So, this formula also we had, we were able to establish very easily if we had use this residue method or residue integration method, so we had learnt that in complex integration, contour integration is something which is coming up like as a real integrations.

Then, we had tried to find it out that is its not only that complex functions when they take this, we could break into the two parts that real parts, we could write it as that two integrals on the real once. Then, we had moved to little bit more complex, once that is complex functions or into complex 1 and there we had used this complex integration theorem complex. And then,we have got Cauchy integral theorem, Cauchy principle, value Cauchy integral formula.

And moved over from here when we had singularities that function was not analytic with the singularities we had come upon more method, that is called residue integration methods. And using this residue integration methods we were able to solve many difficult real integrals with the help of that residue once or with the help of complex that Cauchy integral formula or with the help of Cauchy integral theorem. So, this is what we had learnt, today we had revised all those kind of methods which we had learnt for the complex integration, so that is all for today.

Thank you.