Mathematics - II
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Lecture - 13
Evaluation of Real Improper Integrals – 4

Welcome to the lecture series on complex analysis. Today’s lecture is Evaluation of Improper Integrals. We are talking about evaluation of improper integrals of the real integrals. Let us talk about another kind of improper integral.

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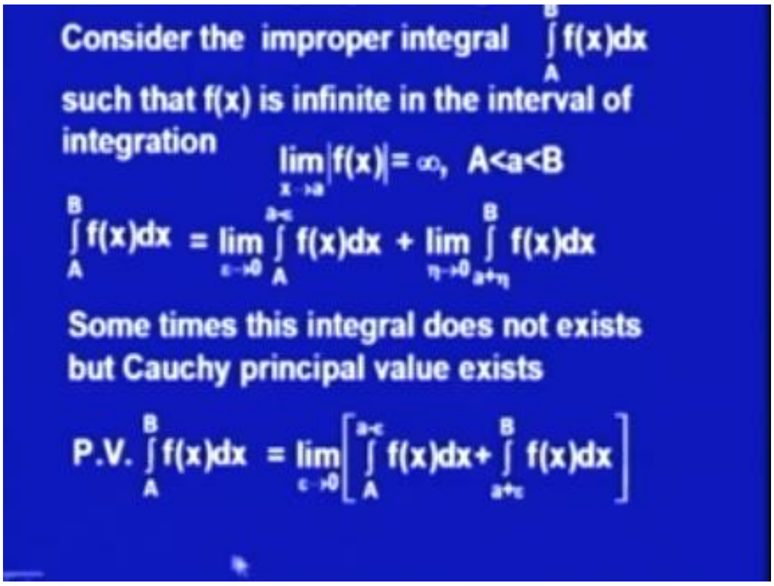

Say, the integral the limit is finite A to B of a function f x integrating in the interval A to B with respect to x. But, the thing is that the function f x is infinite in the interval of integration. That is say, there is a point A in the interval of integration from A to B a small a. Such that, limit as x is approaching to a of mod f x that becomes infinite. Now, how can we evaluate this kind of improper integral, using the residue theory.

These kinds of integrals are being defined as the integral. That limit as Epsilon approaches to 0 A to a minus Epsilon f x d x plus limit as eta is approaching to 0 a plus eta to B of f x d x. This is the definition of this improper integral. We say that this improper integral exist if both these limits are existing for different values of Epsilon and eta.

Both Epsilon and eta independently are approaching towards 0. They are not related with the each other. And they are approaching towards 0 from the positive side on both these limits. If both these limits are existing, we say this improper integral does exist. But, sometimes it happens that, these limits do not exist, when we use this Epsilon and eta as independent. Or this independently, when we vary this Epsilon and eta, this limits may not exist and this integral may not exist.

But, if we assign another value as limit Epsilon approaching to 0 from A to small a minus Epsilon f x d x plus integral a plus Epsilon to B of f x d x, this limit is existing. So, sometimes it may happen, that when Epsilon in eta we are taking as approaching towards 0 independently. The limit may not be existing of either of this integrals or one of the this integrals.

But, this limit is existing, then this limit is called the principle value of this improper integral. And we simply say is Cauchy principle value. That is of A to B f x d x is this one. When the integral is existing and then this Cauchy principle value would be actually equal to this a value of the integral. But, even if this integral is not existing, we do have that Cauchy principle value may exist. And that is some finite value.

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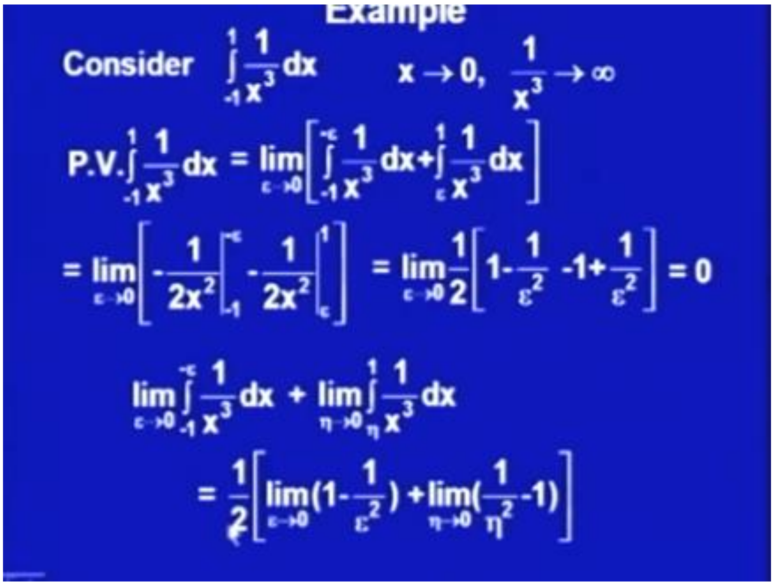

Say for example, if I consider this integral from minus 1 to plus 1 1 upon x cube d x. We do know that 1 upon x cube at x is equal to 0 is infinite or that is it is undefined. So, we are having that the function is approaching towards infinite, in the interval of integration.

So, this is an improper integral. Now, try to define the Cauchy principle value, find out the Cauchy principle value of this integral.

Cauchy principle value of this integral would be limit as Epsilon approaching to 0 minus 1 to minus Epsilon 1 upon x cube d x, and then integral from Epsilon to 1 1 upon x cube d x. Now, if I consider this positive integral minus 1 to minus Epsilon, this function is finite one and this integral is existing. And the value of this integral is minus 1 upon 2 x square. So, we have to evaluate it from minus 1 to minus Epsilon.

Similarly, when we are taking this Epsilon to 1, because Epsilon is a positive quantity. This in second integral is also existing and the value of that integral is minus 1 upon 2 x square and that you have to evaluate from Epsilon to 1. So, let us write it out, we are getting it minus 1 upon 2 x square minus 1 to Epsilon minus 1 upon 2 x square Epsilon to 1.

So, minus 1 upon 2 or 1 upon 2 we can take common. And we are getting is limit as a Epsilon is approaching to 0 1 upon epsilon. This one would be 1 minus 1 upon Epsilon square and this would be minus 1 plus Epsilon square. Since, Epsilon is common over there we hear what we get 1 minus 1 is 0 minus 1 upon Epsilon square plus 1 upon Epsilon square this term is 0. So, this integral both these integrals, they are summing up to 0. And thus it is independent of Epsilon and the limit would be simply 0.

So, what we have got that Cauchy principle value for this integral minus 1 to plus 1 1 upon x square d x is existing and it is 0. But, see what is that integral. The integral of this is not existing, because if I take Epsilon and eta separately. That is, if I take Epsilon is approaching to 0 minus 1 to minus Epsilon 1 upon x cube plus limits as eta is approaching to 0 eta to 1 1 upon x cube d x.

Then, as usually here we would get is that in this interval, this function is integrable and the integral does exist and we can evaluate it. Similarly, in this interval of integration, the integral is existing and we can evaluate it. So, let us write out this evaluation, what we are getting is the first one would be 1 minus 1 upon Epsilon square. The second one would be 1 upon eta square minus 1.

Now, when I take the limit as Epsilon as approaching to 0 in the first integral, this limit is not existing, similarly the second limit is also not existing. So, what we are finding it out that, this improper integral does not exist. But, it is Cauchy principle value is existing. If the integral is also existing then of course, the Cauchy principle value would be equal to the value of the integral.

We are not interested here to at first place. That is whether integral is existing or not, we are just interested in the Cauchy principle value of this kind of improper integrals. This kind of improper integrals, what they are going to give us. Let us see, that is we are doing all these improper integrals, where we are talking about that my f x is a rational function.

So, here also we have taken this example, where I have got that f x is irrational function is a rational function. Since, we want that is it is in at some real point, it is going to be nonexistent or it is approaching towards infinite. So, we would again have this rational kind of functions.

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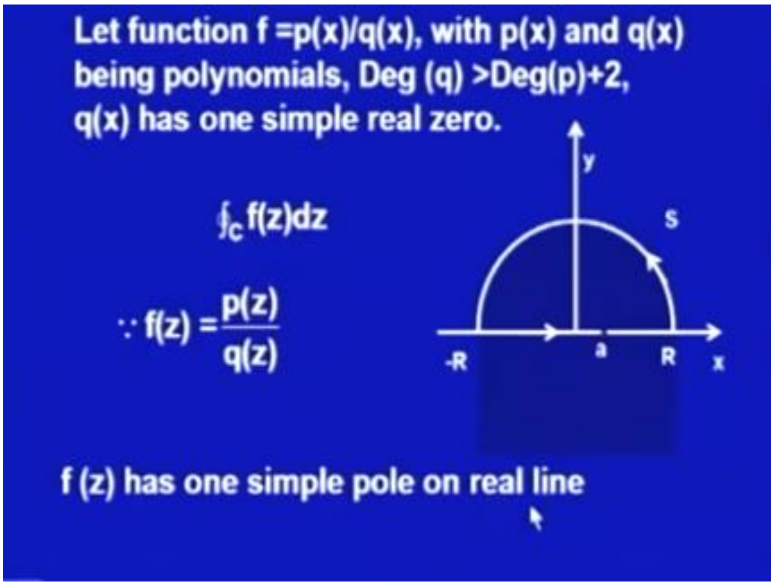

So, let us see this kind of integral or we are talking about another kind of improper integrals. Let this function f is again rational f x upon p x, where both p and q being polynomials. Degree condition is also satisfied, the degree as of the denominator is at least 2 degrees higher than the degree of the numerator. But, what we are now having is since we have got that our function f x is approaching towards, infinite at some point in the interval of integration.

Or in other words, what we are saying, because when we are having it is p x upon q x form, that q x is approaching to 0 or q x is 0 for some real number, that says q x is having one simple real 0. Now, here I had added this condition simple. So, we are defining the condition in the first place, we were using that our f was rational function. But, q is having no real 0s.

Now, we are talking about if q has real 0. Now, we are adding up one conditional first, that it is simple real 0. That is its not order higher than one. Then, what we want, we want to change our improper integral to the contour integration of f z d z on some contour. I want to use our, because now f z if I do say is that is of this f z should be of the form p z upon q z.

Since, q z is having a real 0, if I take the contour as usual as we are used to take. That is the semicircle bounded by the real line. Then, what we are having is here, because our q x is having the real 0 say let say it is having at a point a. Then, now what we would have? We would have in this contour c, which is consisting of our semicircle and this portion of real line from minus R to plus R, we would be having a 0 or a pole at on the contour.

So, now the residue theory as such it cannot be applied. Because, in the residue theory we say is that is ours. Therefore, all my poles are lying inside the close contour. But, now here what we would be having, when we do have a simple real 0. We would be having our pole on the contour. Then of course, directly we cannot apply the residue theorem or residue method. What we could do is... So, f z has one simple pole on real line. And now let us see that is how we are going to do it, before doing it let see how we are going to evaluate this integral before that one result, which we would like to have.

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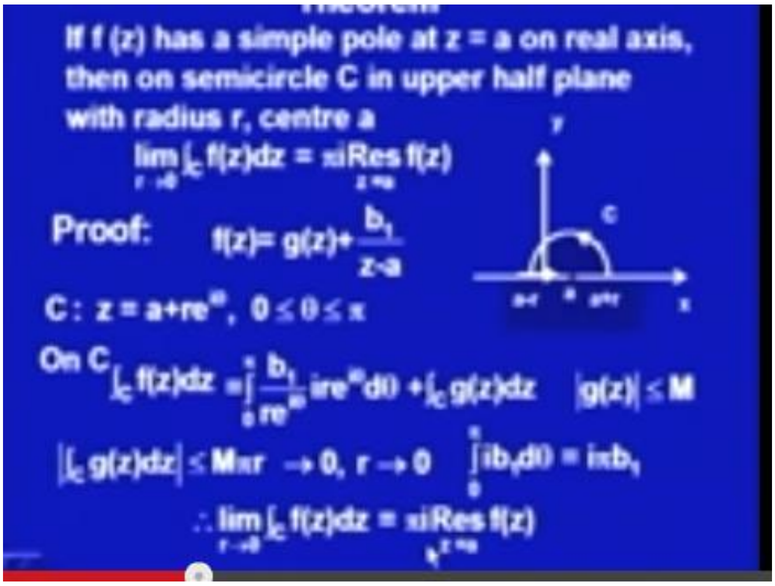

The theorem says, if f z has a simple pole at z is equal to a on real axis, then on semicircle C in the upper half plane with radius r and center a. The limit as of r as approaching to 0 of this integral c f z d z is pi i times residue of f z at z is equal to a. C is the semicircle C is the semicircle in upper half plane. What we are saying is that, if my function f z has a simple pole at z is equal to a, that is on the real line.

Then, if I take this semicircle C around this point a, where we are having the pole of the radius small r, r is the small enough. Then as r is approaching towards 0, the integral on this contour C, that is the semicircle only, not including this one, this would be pi i times residue of f z at z is equal to a. Let us show that is how this would happen.

We are been given, that f z has only a simple pole at z is equal to a. That say is my f z can be written in the Laurent series around z is equal to a. We could written it as f z as g z plus b 1 upon z minus a, because we are having only single simple pole at z is equal to a. So, it should be of this form and g z is one analytic function, which is analytic completely at a also.

Now, b 1 is the residue of f z at z is equal to a. We want this integral on this semicircle C, what is this C? C is a semicircle centered at a with the radius as r. So, parametric representation we could write z on the c as z is equal to a plus r e to the power i theta for theta ranging from 0 to pi. Now, let us write this integral f z d z on c, write it using the simple contour integration, write this parametric once.

So, it should be integral 0 to pi f z we are writing as g z plus b 1 upon z minus a. So, first thing is we are writing f z as g z plus b 1 upon z minus a. So, we are writing two contour integration, one is integration on the g z, another is contour integration of b 1 upon z minus a. Let us first take this one, for b 1 upon z minus a the contour integration, if by definition we do write it goes from 0 to pi b 1 upon r e to the power i theta. Because, z minus a is r e to power i theta.

And z dash theta, that is i r e to the power i theta d theta plus the contour integration on the c of g z d z. Now, this g z this is analytic at a now. So, we are having is that is, if I do take it is analytic. So, by definition it should be continuous and by the definition of continuity, we could find out a bound or a constant capital M. Such that, mod of g z is less than M on this line are on this contour C.

So, what we are getting is that is mod of g z is less than or equal to M, because g z is analytic hence continuous and at a. So, by the definition what are we have taken is a small neighborhood from the point a. So, on this contour C we would get that mod of g z would be less than capital M. For on this we could use this our ML inequality. Since, mod g z is less than M.

So, using ML inequality on this contour, we would get that it is less than or equal to M times the length of contour. Length of contour, that this is semicircle with radius r. So, it should be pi r, so this is pi r. Now, as r is approaching what we have to evaluate this integral as r is approaching to 0. So, if r is approaching to 0, certainly M pi into r this would also approach to 0, so this part is over.

Now, what is this first integral, first integral this r into e to the power i theta is in the numerator, as well as in the denominator r into e to the power i theta. So, what I would get this first integral is 0 to pi i r b 1 d theta i r b 1 all they are constant with respect to theta. So, what we are getting is the integral would be only pi. So, we are getting this i pi b 1. Now, i pi b 1 b 1 is nothing but the from here if we see b 1 is nothing but the residue of f z at z is equal to a.

So, now what we have got as limit as r is approaching to 0 of c f z. Then, this second integral is approaching to 0 and the first integral is pi i residue of f z at z is equal to a. Now, you see is that we can use this result in evaluating our integrals, our improper integral where the we are having a pole on the contours. So, let us see that is how you are going to apply this theorem.

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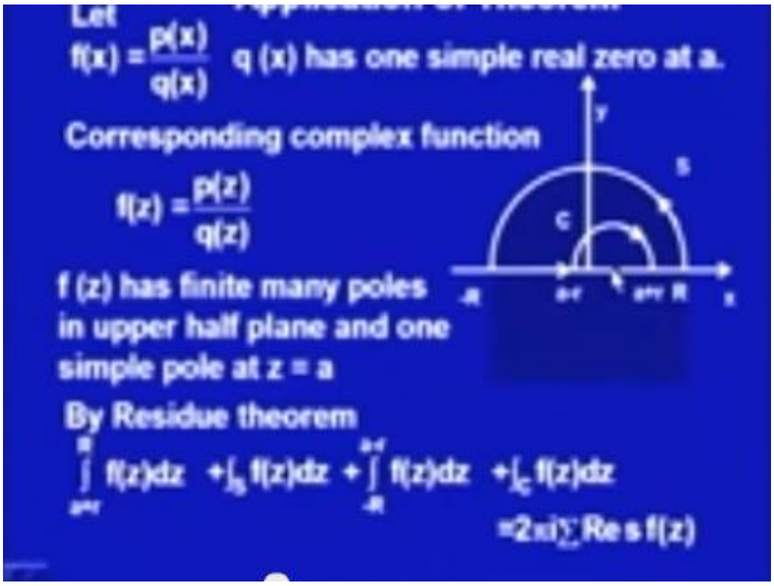

So, application of theorem let my f x is of the form p x upon q x. And p x and q x both are polynomials q x has one simple real 0, we are just talking about that one simple real 0 only. And this condition of course, it can have other 0s, but there has to be a non real 0s. And the condition of the degree is also being satisfied, the degree of q has to be at least 2 degrees higher than degree of p.

Then, corresponding complex function we used to take as f z as p z upon q z. So, in the similar manner if see then f z has finite many poles in upper half plane. And one simple pole at z is equal to z, because this is a polynomial. So, it would have actually maximum, the whatever be the degree let us say the degree is k, then it would have a maximum k 0s. So, there would be finite once and in upper half plane of course, there would be finite one.

Moreover what we are assuming is that q x is having one simple real 0. That says is we would have one simple real pole at the real line that z is equal to a. That is, if I just go as usual we use to go, that is take the semicircle as the contour of integration, the semicircle from the range minus with the radius as capital R. We are having that at z is equal to a we are having one pole.

Now, because the problem is here is that residue theorem cannot be applied. Because, pole we are having on the contour. So, now we are changing our contour. How we are changing our contour? We are saying is that, let us take this contour from this point a plus r point to this positive direction to R, then the semicircle in the counter clockwise, and then from minus r to a plus r. And then this inner semicircle, in the clockwise manner, that is in the negative sense. Now, what we are having, we are having a close contour from a plus r to like this one. And the poles which we are having is they are all inside this close contour. Now, this pole which were we are having on the real line, that portion we have just escaped. So, now we are not having any pole on the contour.

Now, this residue theory can be applied till now we do know that if my function f is of the rational form of the p z upon q z form. And now this q z in this region is not having any real 0s. So, we would have only a finite many your poles in the region. And then using this residue one we could find it out, that integral from a plus r to R plus integral on the semicircle s plus this integral from minus R to a minus r plus this integral along this inner semicircle c.

This must be equal to 2 pi i times summation of the residues of f z, where the residues we would be taking all the residues, all the simple poles in the upper half plane of f z. Now, so see what this r I have to take for bifurcating, this pole at real line. This r has to be small enough, such that it is not having any other pole in inside this semicircle.

So, this r has to be taken very small enough, that says is we are taking is r is approaching to 0 like kind of thing. So, now we do know from our theorem, that if I take r is approaching to 0. Then, we would be we just bifurcating this pole only. And in that case just now we had proved, that if I have to find out this contour integral of on the smaller semicircle. Then, as r is approaching to 0, this is given as the pi i times the residue of f z at z is equal to a. Since, this orientation of the curve is now reversed one, there we have used in the theorem. That is the contour clockwise, here we are taking the clockwise. So, it should be minus pi i this one.

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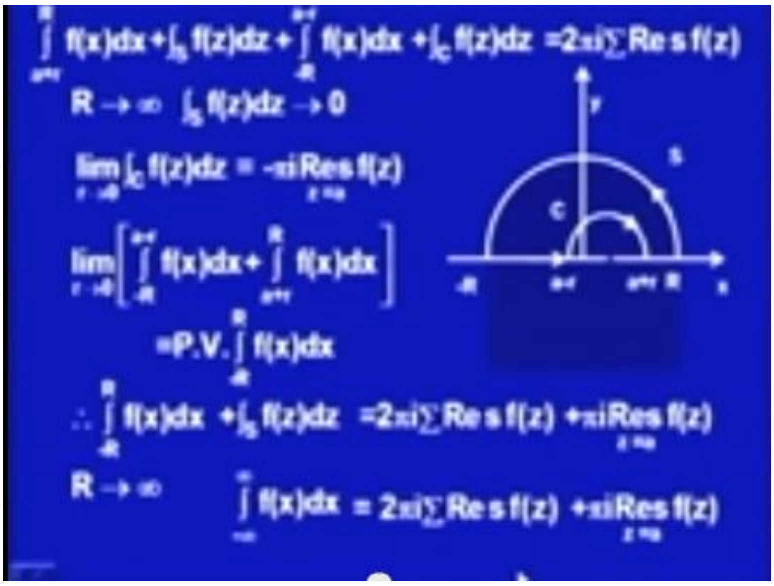

So, what we do know, now let us consider this result we do have. Now, as R is approaching to infinity, we do know that integral along this bigger semicircle capital S. Because, that were all the conditions are being satisfied, this would be approaching to 0. Because, the degree conditions are satisfied and showing that, that we had all ready shown.

And as a small r is approaching to 0, this integral is minus pi i residue of f z at z is equal to a. So, let us first take the limit as r is approaching to 0 on this expression. What it says is limit r approaches to 0, the first integral a plus r to R f x d x and then this third integral minus R to a minus r f x d x. If I we can rewrite it as integral minus R to a minus r plus integral a plus r to R, as r is approaching to 0.

So, now this r would be same as the Epsilon we do say. So, what we could have is what is this? This is nothing but the and since my f x is having actually a real 0. It says is that is q x is having a real 0 it simply says is f x would be approaching to infinity at x is equal to a. So, this is actually our principle value of the integral minus R to plus R f x d x by definition.

So, now or now this take this limit as a small r is approaching to 0 on this expression what we do get is minus R to plus R f x d x plus integral of f z on s semicircle, that bigger semicircle s is equal to 2 pi i summation of residues of f z plus pi i times residue of f z at z is equal to a. Now, you see we are having is that this summation would be on all those residues, which are in the upper half plane. Upper half plane means, we are excluding the real line.

So, this summation would be on the all poles in the upper half plane. And then plus pi i times the residue for the pole at real line. Now, here if I take the limit as this capital R is approaching to infinity. Then, this integral would be actually minus infinity to plus infinity f x d x plus this would be approaching to 0. Because, we do know that as capital R is approaching to infinity, integral along this semicircle s of f z d z approaches to 0 I would get as 2 pi i summation residue of f z plus pi i times residue of f z at z is equal to a.

Where the summation would be varying on the or summation is over all the poles, in the upper half plane of the function f z. So, we have got now, one you could say is here we have got is as principle value and that I have written is here as integral from minus R to plus R. So, moreover you could say is we have got, if my function f x is such that, in the interval of integration it becomes infinite.

Now, if that interval of integration is itself finite or infinite, that we do not have here, now here what we have find it out. If my limits are also infinite and the function also becomes infinite, somewhere on the real line. Then, this improper integral, the Cauchy principle value of this improper integral can be find out using this residue, which says is it should be 2 pi i times summation of residues of f z, where the summation should be on all the poles, which are in the upper half plane of the f z plus pi i times residue of f z at pole at z is equal to a on the real line. We have talked about one simple pole. Let us do one example of this kind, where this kind of function is having and evaluate the integral, then we will move to extension.

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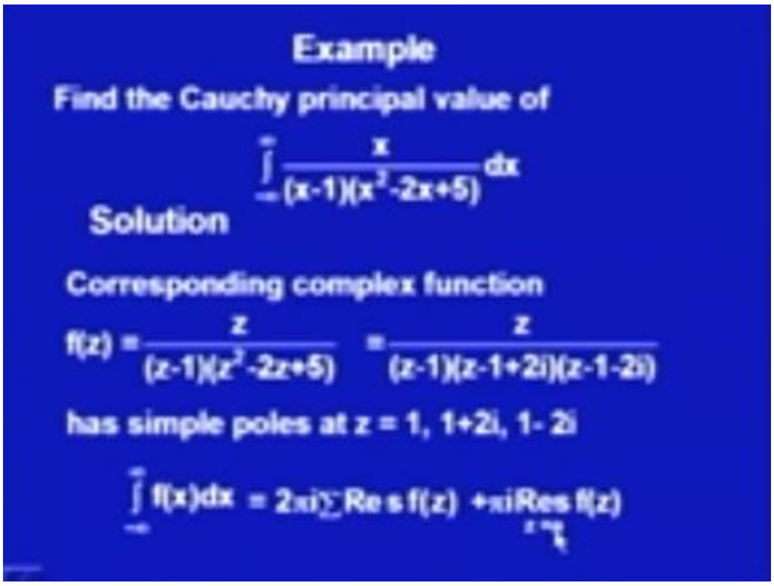

Find the Cauchy principle value of integral minus infinity to plus infinity x upon x minus 1 into x square minus 2 x plus 5 d x. Let us see, we do have my function f x as rational function p x is x, q x is x minus 1 into x square minus 2 x plus 1 of the 0 of p is 0 and the 0 of q are 1. And from here we would get it as 1 plus 2 i and 1 minus 2 i. So, none of the factors are common.

This factor x square minus 2 x plus 5, this does not have any real 0. But, we do have real 0, that is at x is equal to 1. Now, the condition which we were having for the evaluation of the improper integrals for the rational functions. The condition that q x does not have a real 0, that has been failed. We are having a 0 at on the real line at x is equal to 1. Or you could see that, this function f x is a nonexistent at x is equal to 1 or it is approaching to infinite as it is x is equal to 1.

So, we would, but that is only single point. That is we are having only single point and this 0 is ordinary 0, that is simple 0. So, we could apply the method just now we had obtained. So, we do take our corresponding complex function as z upon z minus 1 z square minus 2 z plus 5 or write out all the factors z minus 1 z minus 1 plus 2 i z minus 1 minus 2 i.

So, 0s of the q z would be 1 1 minus 2 i and 1 plus 2 i. That says is this on all these 0s are simple 0. So, we would have that my function has simple poles at z is equal to 1 1 plus 2 i and 1 minus 2 i. This 1 minus 2 i would be in the lower plane. So, in the upper plane we would be having upper half plane, we would be having only 1 plus 2 i and on the real line we would have 1.

So, we do have two poles, one in the upper half plane, one at the real line which we would be mattering. So, now if we do apply our the results, just now we had obtained which said is that integral minus infinity to plus infinity of a f x d x. When, f x is having one simple real pole can be given as 2 pi i residue of f z summation of residues of f z plus pi i times residue of f z at z is equal to a. Where z is equal to a is the pole, on the real line. And here the summation is for all the poles which are in the upper half plane. So, what is required is that for this function, we have to calculate the residues. Since, both these poles are simple poles.

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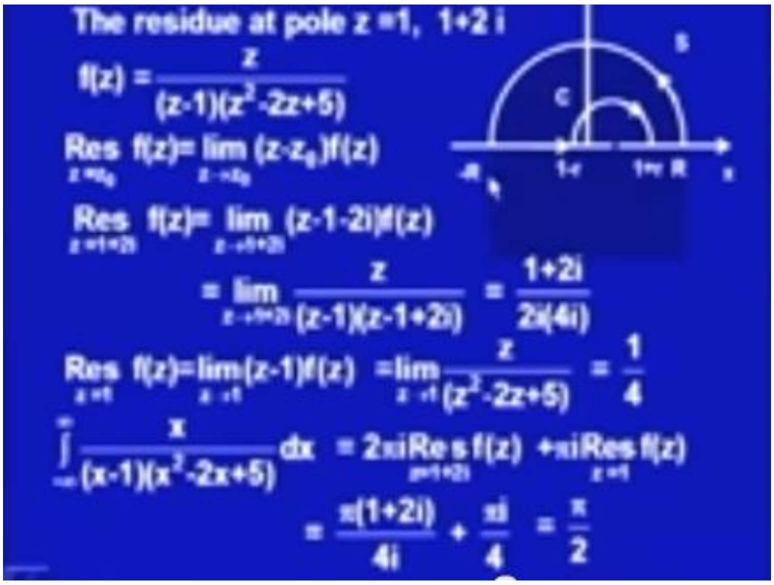

So, we could go for finding out these residues. Simple poles we will use of our first formula for the simple pole and the function is also simple. So, using that limit formula as residue at z is equal to z naught of f z is limit as z is approaching to z naught z minus z naught f z, that would be easy to apply. Since, the function we could factorize it easily. So, it would be much easy to use this limit formula.

So, first in the upper half plane at 1 plus 2 i. So, residue at 1 plus 2 i of f z is limit as z is approaching to 1 plus 2 i f z minus 1 plus 2 i into f z. This factors we do know z square minus 2 z plus 5 are z square or that z minus 1 plus 2 i and z minus 1 minus 2 i. So, z minus 1 plus 2 i would cancel it out, we would get the function as z upon z minus 1 into z minus 1 plus 2 i.

And what we it should be z minus 1 1 minus 2 i. So, it would z minus 1 plus 2 i is here. So, limit as z is approaching to 1 plus 2 i, we would be getting is z minus 1 is approaching to 2 i. This is approaching to 2 i into 2 i, that is 4 i and it is 1 plus 2 i. So, evaluation we get 1 plus 2 i upon 2 i into 4 i we are not simplifying it further. Because, we require to multiply 2 pi r.

Then, residue at z is equal to 1 the pole at z is equal to 1. Again by the same definition, it should be the limit as z is approaching to 1 z minus 1 into f z. The function would be z upon z square minus 2 z plus 5 evaluated as z is approaching to 1. That is evaluate at z is equal to 1. Z is equal to 1, the numerator would be 1, denominator would be 1 minus 2 plus 5. So, we would be getting it as 1 by 4.

So, we got the residue at both the poles at z is equal to 1 as 1 by 4 and z is equal to 1 plus 2 i as 1 plus 2 i upon 2 i into 4 i. Now, use the result integral minus infinity to plus infinity x upon x minus 1 into x square minus 2 x plus 5 or the principle value of this would be equal to the 2 pi i residue at z is equal to 1 plus 2 i of f z plus pi i times, the residue of f z at z is equal to 1. Since, in the upper half plane we do have only single pole.

So, the summation is reduced to only one residue. And similarly here we are having only one pole. So, this is also this one. So, now the residue at this is 1 plus 2 i upon 2 i into 4 i. So, 2 i and 2 i would get cancel it out, we would get pi times 1 plus 2 i upon 4 i and here it is pi i upon 4. So, now if I multiply this with minus i on the upper one and then solve it. So, what I would get is minus i means i would be getting minus pi i plus 2 pi.

So, minus pi i and plus pi i that would cancel it out. And we would be getting is 2 pi upon 4 i pi upon 2. So, we have got the Cauchy principle value for this integral as pi by 2. Here, what we had used, now our method when we are having a pole on the real line. Then, we have used this formulation. Now, let us extend this result if I do have more than one simple pole on the real line.

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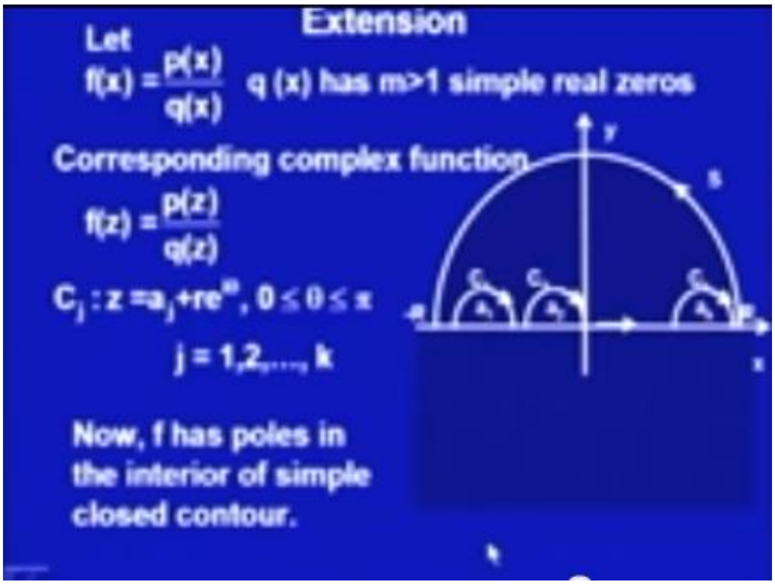

So, see the extension, let my function f x is of the form p x upon q x. And q x has m simple real 0s, where m is greater than one. All other conditions are satisfying, that is p x and q x both are polynomials, the degree of p is 2 at least 2 degrees lower than the degree of the denominator q. So, the corresponding complex function, if we do write we would have p z upon q z and since both p z and q z are polynomials.

We would have that the function f z would have a finite number of isolated singularities, that says in upper half plane, we would have finite number of singularities. And since, we are having is that, it is having q x is having m simple real 0s. It says is that, we would have simple poles on the real line m simple poles on the real line.

So, let us say that is those poles are at the points a 1, a 2 and say they are k poles. So, let us say that jth pole I am calling is a j. Then, I would like to choose the same kind of contour. That is a big that semicircle from minus r to plus r. But, now since we are having at many points on the real line as a j as the poles. So, around each pole a j we would take one semicircle c j with the radius r.

This r is a small enough. So, we are taking this semicircles z as c j as a j plus r e to the power i theta for theta ranging between 0 to pi, this is for all k poles 1 to k. How this we are choosing, let see in the figure. We have chosen this our contour s as usual, that semicircle with the radius as capital R. What we are having is that in usual method I am having here, that the some simple poles at the real line or on the contour.

So, we have to forgo about or we have to exclude those simple poles or those on the contour. So, we just take it you see is that each a, suppose this is a 1 is at the first pole on the real line. I will take one semicircle around this a 1 with the radius r. This we are taking as such is r. And similarly around a 2 again we are taking this r, this r we are taking small enough. Such that it is not containing any other singularity of the function f z.

So, that inside this contour we do have only one isolated singularity. So, we have this r small enough, such that each this is small semicircle contains only this pole or this isolated singularity at a j and no other one. Now, we will take this close contour ranging from a k plus r small r to capital R. Then, this semicircle and then this region of real line from minus R to a 1 minus R, then this a small semicircle C 1. Then the portion of real line from a 1 plus r 2 a 2 minus r, then this semicircle and so on. So, now this is our close contour and we do have the poles only interior to this close contour and there is no pole inside the on the contour. So, f has poles in the interior of simple close contour. Now, in this case we could apply our reside theorem.

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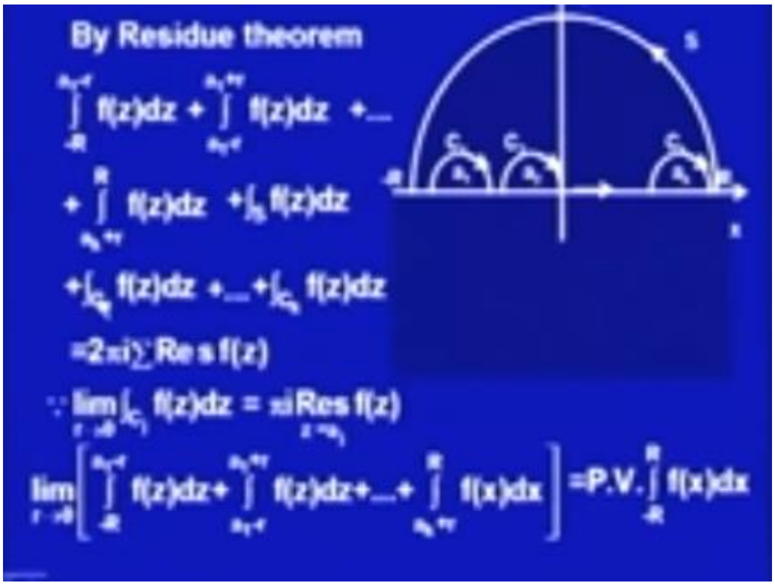

And that says is by residue theorem integral from minus. Now, I am writing it no I am not writing from this side of course, I had started my contour from this side to move like this one. But, now what I would do is that is I will write the things in little bit different manner. So, that this is an order rather than going into it is usual manner should be that is I should have a started a k plus r 2 r.

And then so first I am writing all the intervals of the real line. So, first I would take this minus R to a 1 minus R f z d z, then I would take a 1 plus this a 1 minus R to a 1 plus R. Then, we do have this a 1 plus R to a 2 minus R and so on we would have a k plus R to R, then this integral along this bigger semicircle s, s f z d z. Then, I would be writing again this in this form that c 1, c 2 and so on and this one.

So, adding it up integral of c 1 along the path c 1 f z d z plus so on the integral along the path c k f z d z. Since, it is having a finite many poles interior to this one by residue theorem, it should be equal to 2 pi i summation of residues of f z, where summation would be ranging on the all the poles, which are interior to this circle. So, now as limit R is approaching to 0, because we have to take this as small enough.

So, if I am taking this limit r is approaching to 0 for any of this one. We do have that integral by the theorem, which we have shown that, if I do take any one particular one semicircle a smaller semicircle around to this a 2 let say. Then, as r is approaching to 0, we do know the integral along this path would be minus pi i times residue at z is equal to a 2.

So, like that if I take this the jth the semicircle a smaller semicircle. Then, we do know that limit r is approaching to 0 integral along the c j of f z d z would be minus pi i times residue z is equal to a j of f z. And if I take this integrals along this real lines, what we do have minus R to a 1 minus r a 1 plus r to a 2 minus r and so on. As my R is approaching in toward 0, this would give by definition the Cauchy principle value of integral minus R to R f x d x.

Because, what we have defined if it is having only single point, at which the function is going to infinite. We are saying is our Cauchy principle values is says is minus R to a minus R a minus Epsilon f x d x plus integral a plus Epsilon to r f x d x. Now, if I am having more than one points, where it is happening is we would take we will break our interval, in that many portions.

And then this principle value would be defined as limit as the R is the approaching to 0. So, R is here is your Epsilon. So, we get that this is principle value. So, the this first integral which is along this real line, that is the principle value of the real integral. And all this integrals which are on the a smaller semicircle c 1, c 2, c j they would be approaching to pi i residue z is equal to a j f z. That says is summation over all summation of residues of f z for all the summation would vary for all aj's that is for all the poles on the real line.

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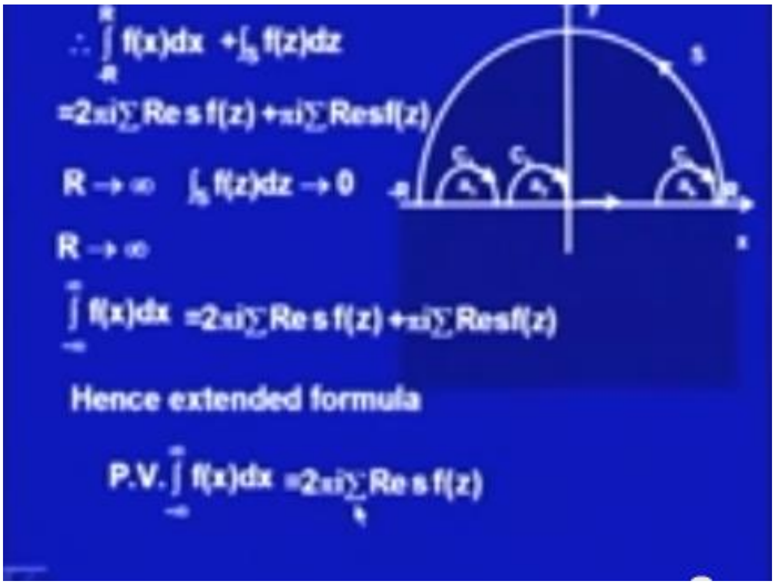

Moreover, now if I take... So, what we have got that integral minus R to plus R f x d x plus integral of f z along this contour s d z. That is 2 pi i summation residue of f z by this residue theorem plus pi i times summation of residues of f z, where this summation would now vary on or this summation is on all the poles which are on the real line.

While this summation is on all the poles, which are interior to or which are in the upper half plane of the complex plane, other than the real line. So, now this formula we have got now here, now we take limit as R is approaching to capital R is approaching towards infinity. We do know that my function f is satisfying those conditions, that it is polynomial with degree condition. That degree of numerator is at least 2 degrees smaller than the degree of denominator.

That says this integral would be approaching to 0. That is R is approaching to infinity integral s f z d z would be approaching to 0. Thus, if I take this limit as R is approaching to infinity on this one, this integral will approach to our improper integral minus infinity to plus infinity f x d x. Or you could say is Cauchy principle value of this minus infinity to plus infinity of f x d x that would be equal to 2 pi i summation of residues of f z plus pi i times summation residues of f z.

This summation is on all the poles, which are on the upper half plane. And this summation is on all the poles, which are on the real line. So, thus we have got the formula, if we do have one thing is clear, that is we have assumed that all these poles are simple poles. Because, we had used that result which said is that integral along this smaller circle would be pi i times residue of f z at z is equal to a, where z is equal to a was a simple pole.

So, we have got this result or this extension and extended formula. Now, we could use this extended formula for one more example. Let us see extended formula, that is principle value of integral a in minus integral plus infinity f x d x is 2 pi i summation on the residues of f z, where summation is over all the poles in the upper half plane plus pi i times summation of residues of f z, where this summation is on the all the poles on the real line.

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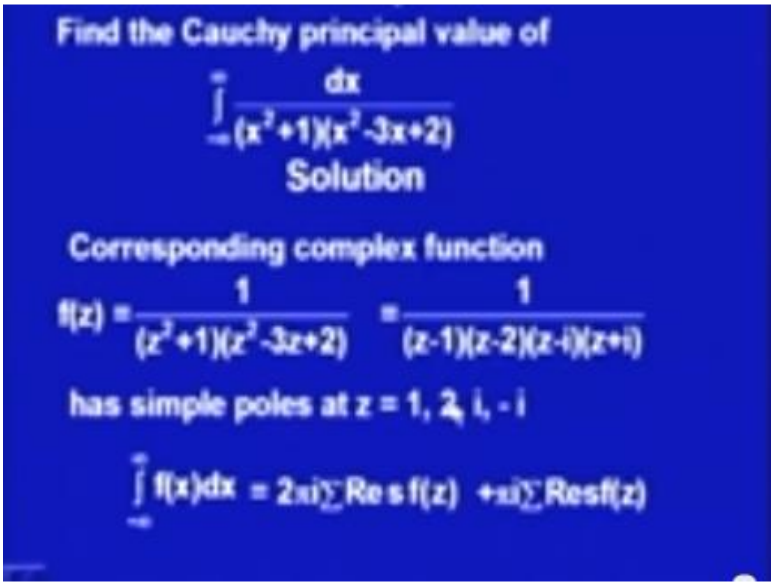

Let us see this example, find the Cauchy principle value of the integral minus infinity to plus infinity d x upon x square plus 1 into x square minus 3 x plus 2. Let us see, this function, we do have this function is of the form 1 upon q x kind of thing. So, p x is your constant. So, it is of degree 0 and the q x is the a polynomial of degree 4. The factors of q x are here x square plus 1 and x square minus 3 x plus 2.

If we do go for the factors of this is it would be x plus i and x minus i. That is it is having only non real 0s. But, here we do have that factors would be x minus 2 and x minus 1, that says is we would have one and two that is two poles on the real line. So, now we would use our extended formula. So, for using this extended formula, we require this corresponding complex function, this complex function is 1 upon z square plus 1 z square minus 3 z plus 2.

Let us write it is all the factors 1 upon z minus 1 z minus 2 z minus i into z plus i. What it says is all my 0s are 1, 2 i and minus i all these are simple 0s. So, it has simple poles at 1, 2 i and minus i. Out of which this minus i would be in the lower half plane only one single pole is in the upper half plane. But, we do have two poles on the real line. So, we would use this formula, that is integral minus infinity to plus infinity of f x d x as 2 pi i summation residue of f z plus pi i times summation residues of f z.

The first summation is on the all the poles, which are in the upper half plane. And the second summation is on all the poles, which are on the real line. So, here in this example again in the upper half plane, we do have only one pole and that is also simple pole. So, this summation would reduce only to the single residue. While is on the real line we do have 2 poles. So, here this we would have this summation would be running on the 2 pole. So, the job remaining is now to find out these residues.

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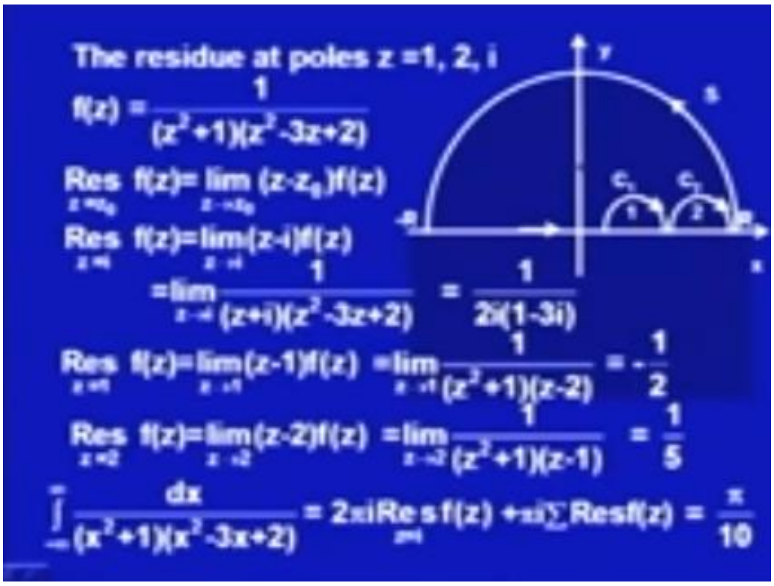

So, if we see in the figure we are having is our function is having the poles at 1 and 2 and we are having it at i minus i would be downwards. So, that is we are not in the lower plane. So, that we are not interested, so we want the residues at the poles 1, 2 and i, where 1 and 2 are on the contour. So, we would calculate these poles for this function 1 upon z square plus 1 z square minus 3 z plus 2. Again all the poles are of the simple form. So, we would simple poles. So, we would use again our simple.

And the function is also of easy one in the form of if I do use the first formula, that residue at z is equal to z naught of f z as limit z is approaching to z naught z minus z naught f z, this would be much easy to apply. So, using it for z is equal to i, the residue would be limit as z is approaching to i z minus i into f z. Z minus i; that means, the first function which we are having here z square plus 1. We would go ahead and we will find it out that z plus i and z minus i.

So, we would get 1 upon z plus i z square minus 3 z plus 2 evaluate it as z is approaching to i, which goes as evaluate it at z is equal to i. Here, I would get 2 i here when z is equal to i this would be minus 1 minus 3 i and plus 2. So, we would be getting is 1 minus 3 i. So, what we would get is 1 upon 2 i into 1 minus 3 i we will keep it as such here.

Then, we want the reside at z is equal to 1 again by definition, it should be the limit z is approaching to 1 z minus 1 into f z. Z minus 1 is the your factor of z square minus 3 z plus 2, that is z minus 1 into z minus 2. So, what we would get it is actually 1 upon z square plus 1 into z minus 2, limit as z is approaching to 1.

So, evaluate it at z is equal to 1, the first term would give 2 and the second term would give 1. So, I would get it as 1 upon 2, second term would give actually 1 minus 2 that is minus 1. Similarly, z is equal to 2 the residue at the pole z is equal to 2 would be limit as z is approaching to 2 z minus 2 f z. And that we would be getting is the factor as 1 upon z square plus 1 into z minus 1.

So, we had got it as evaluate it as z is equal to 2, the first function would give me 5 and the second function would give me 1. So, we would get it as 1 upon 5. Now, use it in the formula our value of the improper integral or that is Cauchy principle value of this improper integral minus infinity to plus infinity d x upon x square plus 1 into x square minus 3 x plus 2 would be 2 pi i residue at z is equal to i f z. Because, we are having only single pole in the upper half plane plus pi i times the residue of f z.

Now, summation here the summation we do require, because we do have two poles at z is equal to 1 and z is equal to 2. So, we just substitute these values, what we do get is first 1 2 pi i the residue is 1 upon 2 i 1 minus 3 i. So, I would get 2 i is get cancel it out pi upon 1 minus 3 i. And we would be getting is here pi i times your minus 1 upon 2 plus 1 upon 5 1 upon 5 minus 1 upon 2 is actually 3 pi by 10.

And here what we are getting is, we are getting is pi upon 1 minus 3 i, make it rational we do get is 1 plus 3 i upon 10. So, we would be getting is minus 3 i plus 3 i and minus 3 pi i that would cancel it out, we would be reaching it as pi upon 10. So, we have got the value of this integral or that Cauchy principle value of this integral as pi upon 10. So, we have learn one more method of finding out the improper integral.

Here, we have talked about the improper integral, when my function was becoming infinite in the range of integration. And more over we have talked about the integral or you could say is that we are talked about the improper integral, where the limits are from minus infinity to plus infinity. And we are having that the poles or that numerator, this denominator q x is having real 0s.

That is we are having our poles on the integration of r on the contour, then how to choose another contour. So, we had judiciously chosen another contour use that one to that contour to apply this residue theory. And then we had actually used that residue theory again and this Cauchy principle for value theorems also to evaluate the integrals along those small contours as well.

And then we had find it out that formula, which is applicable in evaluation of improper integrals, when the function also is vanishing or the denominator is vanishing at certain points on the real line or the function is actually increasing towards infinity at some points in the interval of integration. So, that is all today for this kind improper integrals we had used this residue theory.

Thank you.