Mathematics - II
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Lecture - 12
Evaluation of Real Improper Integrals – 3

Welcome to the lecture series on complex analysis for undergraduate students. Today's lecture is Evaluation of Real Improper Integrals. We are continuing with the evaluation of real improper integrals. We are learning that is how the residue theory can be used to evaluate real integrals. We had seen, that the residue theory can be apply to evaluate the improper integrals of the form of minus infinity to plus infinity f x cos s x d x are of the form minus infinity to plus infinity f x sin s x d x.

Certainly here we had assume in the last lectures, that f x is a rational function. That is, it is of the form p x upon q x, where both p and q are the polynomials. And we had assume that q does not have any real 0. Moreover we had assumed that the degree of the q, that has the denominator must be at least 2 degrees higher than the degree of the numerator polynomial p.

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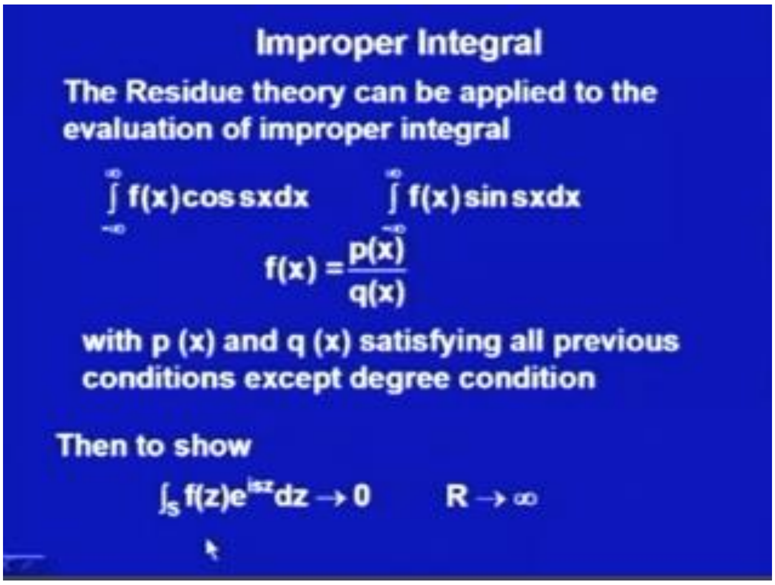

We were requiring that degree condition to show that the integral of this f z e to the power is z d z goes to 0. And a semicircle s as limit s as r is approaching to infinity. Now, assume that this f x is having all the conditions as such that is my denominator does not have a real 0. And moreover this p x and q x they are polynomials, it is a rational function none of them have a common factor.

And but the only condition is that this p x and q x they are not satisfying this degree condition. That is the difference between the degree of denominator and numerator is not at least 2. And suppose here, if the difference is one. Then, we are certainly we are going to face the problem in showing this limit as R is approaching to infinity showing the integral f z into e to the power minus is z d z goes to 0. For that sometimes we require one particular kind of inequality, which helps us in showing this convergence. That is called Jordan's inequality.

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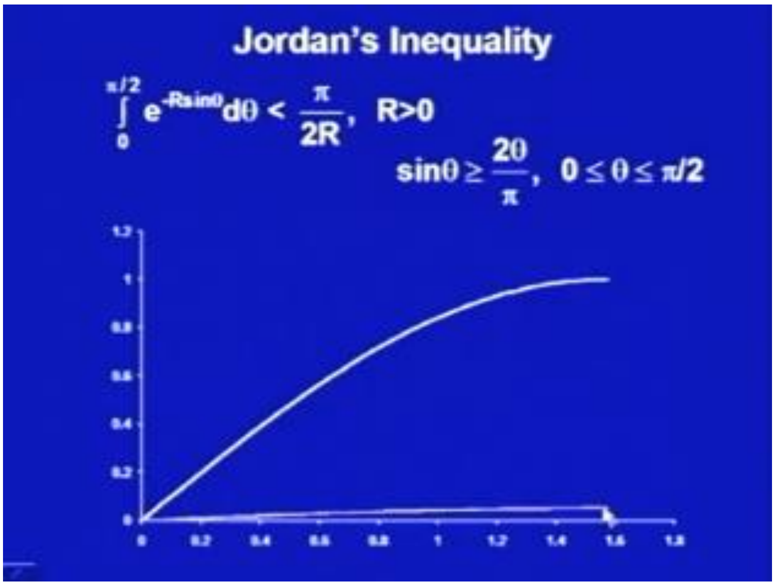

What is this? This is actually integral 0 to pi by 2 e to the power minus R sin theta d theta is less than pi upon 2 R for R positive. What we are having is that the integral of e to the power minus R sin theta with respect to theta, on the range 0 to pi by 2 this is bounded by pi upon 2 R, when R is positive.

To understand that, what is this inequality? And how we are going to use it in our evaluation of integral. Let us first understand this one or rather first we would show that this is true. For showing that this is true, we would like to compare two functions, one is R sin theta another is theta upon 2 pi. So, if we see sin theta, the curve for sin theta between 0 to pi by 2 this is the curve for sin theta.

And the curve for y is equal to 2 theta upon pi, this is the curve for y is equal to 2 theta upon pi from 0 to pi by 2. Now, we see from here that for complete range from 0 to pi by 2, this function 2 theta upon pi, this is lying below than this curve sin theta. What we are concluding from here, we are concluding that sin theta would always by greater than or equal to 2 theta upon pi in the range 0 to pi by 2, because at 0 both of them of are equal.

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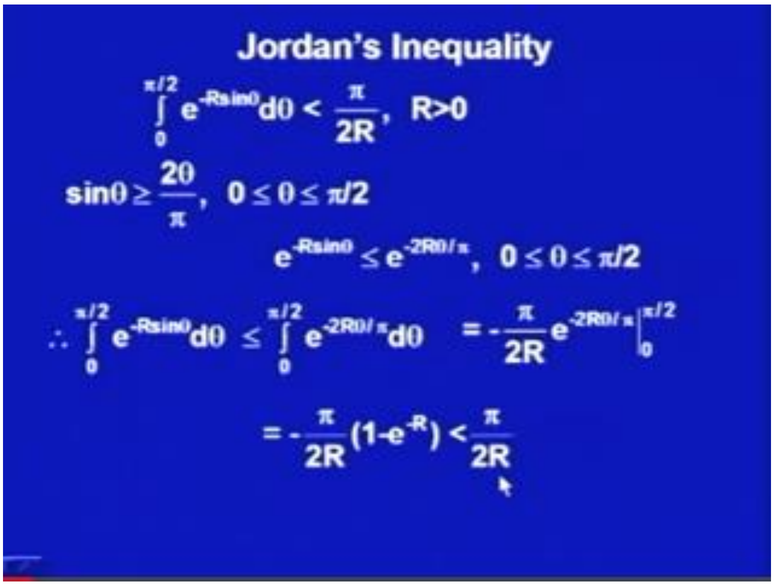

Now, we will use this to show or to prove this Jordan's inequality. We want to prove 0 to pi by 2 e to the power minus R sin theta d theta is less than pi upon 2 R for R positive. We are just shown that R sin theta, that sin theta is greater than 2 theta upon pi. So, we would use that sin theta is greater than 2 theta upon pi. So, we would write e to the power minus R sin theta. It would be actually one upon e to the power R sin theta.

Since, sin theta is greater than 2 theta upon pi, R sin theta would be greater than 2 R theta upon pi. And that would be in the denominator. So, what would get actual e to the power minus R sin theta is less than or equal to e to the power minus 2 R theta upon pi in the range 0 to pi by 2. We are interested only in the range 0 to pi by 2, because this integral is on the range 0 to by 2.

If this is happening that says that this integrant is bounded by this function for whole the range. The what it says is that from here, that integral 0 to pi by 2 e 2 the power minus R sin theta d theta, this would be less than or equal to integral 0 to pi e to the power minus 2 R theta upon pi d theta. Now, integrated this function, this is most simple function, this is just exponential function, it is integration would be minus pi upon 2 R e to the power minus 2 are theta.

So, minus pi upon 2 R e to the power minus 2 R theta upon pi evaluate it from 0 to pi by 2. Evaluation from 0 to pi by 2 at 0, because it is a minus sign. So, first let say a have evaluation at 0, at 0 this theta is equal to 0, this would give me 1. At theta is equal to pi by 2 this would give me e to the power minus R. So, what I would get it as minus pi upon 2 R 1 minus e to the power minus R, this minus sign not behave.

So, what it says is this is pi upon R 1 minus e to the power minus R. Whatever would be this 1 upon e to the power minus R, that is a number which is a smaller than 1. So, what we would be getting that, this number would be always smaller than that is your subtracting. That is you do have some positive number minus some positive number. So, certain it would be less than pi upon 2 R.

So, we had shown this Jordan's inequality that integral 0 to pi by 2 e to the power minus R sin theta d theta is less that pi by R. Let us see, that is how we are going to use this in evaluation of the integral, we will my f x is failing our a degree condition. So, for the let us see one example.

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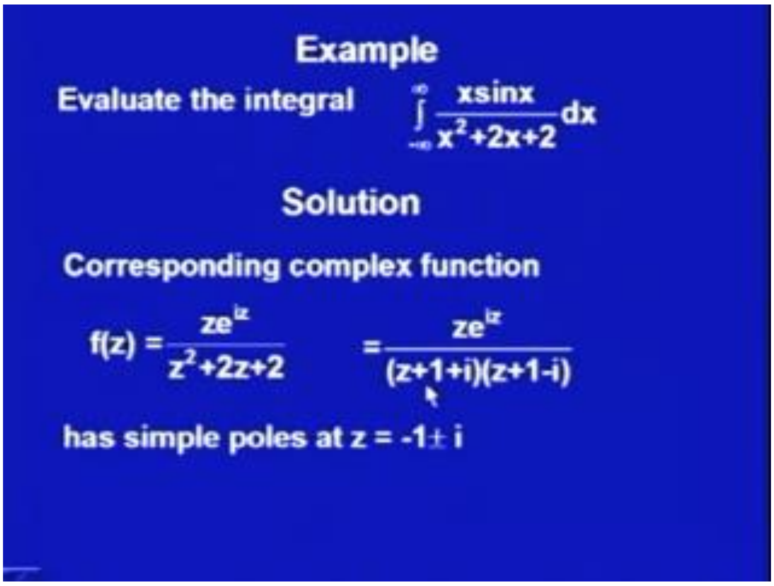

Evaluate the integral minus infinity to plus infinity x sin x upon x square plus 2 x plus 2 d x. Let us see this integral, this integral is of the form of x upon x square plus x plus 2 this is my f x into sin x. So, we are having a f x sin x d x kind of integral. Moreover here, my function f x ((Refer Time: 08:25)) it is x upon x square plus 2 x plus 2. So, it is again of the form of the polynomial.

So, what let us try to solve it, we are having our f x as x upon x square plus 2 x plus 2, we do have that p x would be x and q x would be x square plus 2 x plus 2. P x is equal to x is a polynomial, q x is equal to x square plus 2 x plus 2 this is also a polynomial. If I find out it is roots are it is are 0s, they would be actually x plus 1 plus i and x plus 1 minus i though would be factors of this denominator. So, I would not get the real 0s for this one.

What it says is that, I am having all this conditions satisfied. That is, it is a rational function both are polynomials, the denominator does not have a real 0. But, the condition one more condition about the degree. Degree of p x is 1, p x is equal to x the degree of x is 1 and degree of q x is 2. So, the difference is only one it is not greater than or equal to 2, so that condition is not be satisfied.

Let us see go ahead with the as usual method, that evaluation of integral of minus infinity to plus infinity f x sin x d x. So, for that we always use to define the corresponding complex function as f x e to the power i z. So, here we would take the f x, so f z z upon z is square plus 2 z plus 2 into e to the power i z. So, this we would take our corresponding complex function.

Let us a make this factors over here. So, get z e to the power i z upon z plus 1 plus i upon z plus 1 minus i. That says, that the 0s of denominator 2 z are your minus 1 minus i and minus 1 plus i. Both have a complex numbers and they are your minus 1 plus i this would be lying in the upper half plane and minus 1 minus i this would be lying on the lower half plane.

So, we do have one pole or one isolated similarity in the upper half plane and that is also a simple pole. And z equal to this p z is not 0 at that at not point. So, it says that we would to go with our usual method. Let us start it, that says that I would like to use our residue method, it said is that is if the function is ((Refer Time: 11:23)) using and which has a number of finite poles on a inside a simple close contour we could.

Go ahead with that the integral value would be the some of the residue at 2 pi i times some of the residues at the simple poles. So, let us try to apply it let see.

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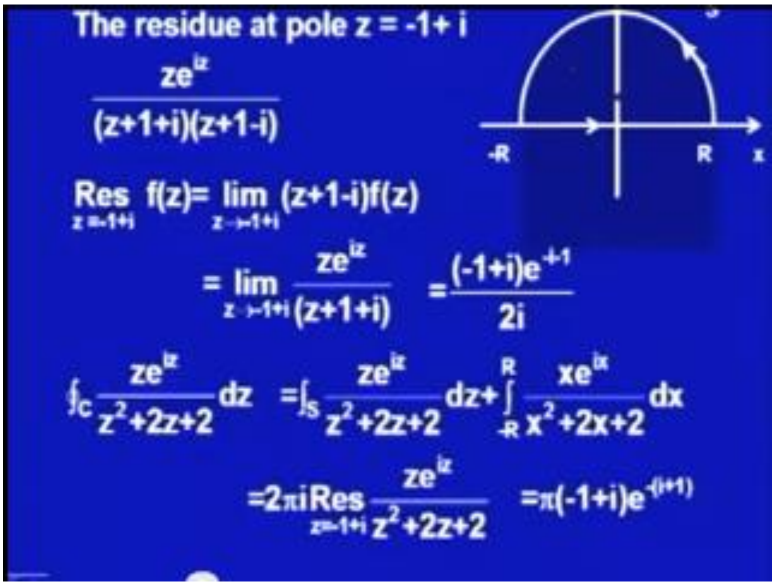

This is what is my curve, which I am taking for which I would like to change my integral on this close contour. The close contour which I would be taking is consisting of the point, your this S and from minus R to plus R on the real lines. So, we are taking the semicircle bounded by the real line from minus R to plus R. My pole in the upper half plane is only this minus 1 plus i, this is your plane, this is your isolate similarity this is only in the upper half plane we want...

So, by a residue theorem we do have that integral along this one should be equal to the 2 pi i times residue at this one. So, let us see what is my function. My function is z e to the power i z upon z plus 1 plus i into z plus 1 minus i. First we are going to calculate the residue at the pole minus 1 plus i of we would use our first definition or first method of calculation of residue, which says that the residue of a function f z at z is equal to z naught when z naught is a simple pole.

It is limit as z is approaching to z naught z minus z naught f z. So, here your z naught is minus 1 plus i. So, we are using this definition residue at z is equal to minus 1 plus i of f z would be limit as z is approaching to minus 1 plus i into z plus 1 minus i into f z. Now, if I multiplying my f z with z plus 1 minus i. We see here, that in the denominator we are having z plus 1 plus i and z plus 1 minus i.

So, what I would get actually, I would get limit z is approaching to minus 1 plus i. Z e to the power i z upon z plus 1 plus i. So, this limit says is just evaluate it at z is equal to minus 1 plus i. If I am evaluating it at this 1 z would be minus 1 plus i e to the power i into minus 1 plus i that would give me minus 1 minus i. And z plus 1 would be i. So, the denominator i would get 2 i.

So, I am getting it minus 1 plus i into e to the power minus i minus 1 upon 2 i. So, using over residue theorem, what I do get integral along this close contour c. Close contour c consist of the semicircle S and the real line from minus R to plus R. So, this close contour we are having is we can subdivide it into integral from S on the semicircle S of the function z e to the power i z upon z is square plus 2 z plus 2. And plus integral from minus R to plus R of the function f z.

Now, minus R 2 plus R this is a real line, this is only the real one. So, I could make this parameterization I could say, this is x e to the power i x x square plus 2 x plus 2. That is, z is equal to x the parameterization we are taking. Now, this integral by residue theorem should be equal to 2 pi i residue at z is equal to minus 1 plus i of the function z e to the power i z upon z square plus 2 z upon 2.

Residue just now we had calculated as minus 1 plus i e to the power minus i plus 1 upon 2 i multiplied with the 2 pi i. So, 2 i 2 i would get cancel it out I would get it minus 1 plus i times pi into e to the power minus i plus 1. Now, the thing which is remaining is just to show that this integral of z e to the power i z upon z square plus 2, along the semicircle s this approaches to 0, as R approaches to infinity, then the second integral minus R to plus R x e to the power i x upon square plus 2 x plus 2.

This will approach to the integral minus infinity to plus infinity or it will actually give the cos c principle value. And then we could find it out this integral. So, let us try to see, this integral on semicircle of this function f z.

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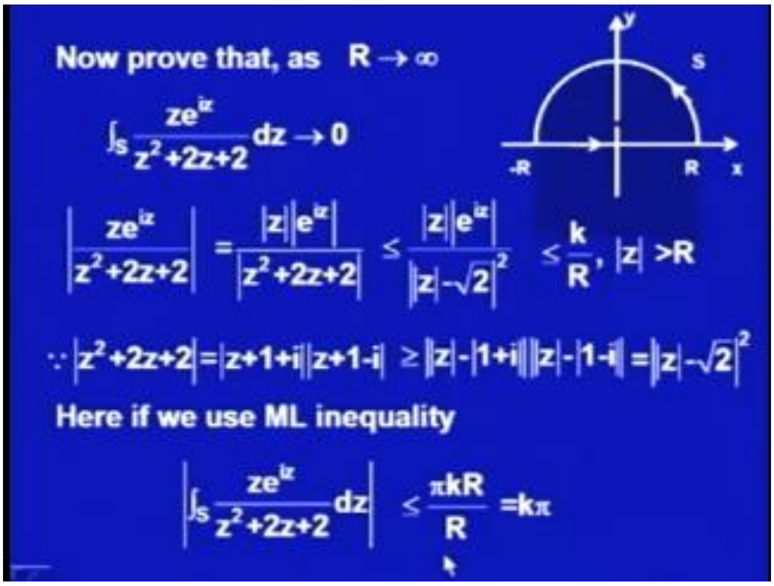

We want to prove that as R approaches to infinity, this integral of z e to the power i z upon z square plus 2 z plus to d z on the semicircle s approaches to 0. See, if I have to take this function z e to the power i z upon z square 2 z plus 2. We could write it as mode of z into e to the power mode of e to power i z upon z square plus 2 z plus 2 absolute values of this R.

This again we could write it as the lower one z square plus 2 z plus i we could say, that is it is z plus 1 plus i and z plus 1 minus i kind of thing. And from their you see, we could get it here let see, your mod z is here, here from here if I take this mod z common. I would get mod z whole square outside and then 1 minus root 2 upon z.

When, mod z is greater than R root 2 upon mode z that could be less than 1. That says is 1 minus this thing, this would be a fix constant, this is smaller constant. And mod z is get and cancel it out over here. And here I would be getting only mod z. Since, mod z is greater than R 1 upon mod z would be less than 1 upon R. And what's remaining e to the power i z, we do know the absolute value of e to the power i z is 1.

So, what thing remaining is constant upon R. So, we have got that this is bounded by k upon R, then your z is greater than R. But, this is let see that is denominator how we have got it out, this is the explanation for this. Z square plus 2 z plus 2 is equal is we could write as z plus 1 plus i into z plus 1 minus i. So, I am writing that absolute value of R the mod of x into y as mod of x into mod of y.

Now, here I would be using is that z plus 1 plus i I would take z as x and this 1 plus i as the y. Then, we do know mod of x plus pi we could write it out as this is a greater than or equal to mod of mod of x minus mod of pi. So, by that we would be getting it as mod of z minus mode of 1 plus i. Then, mode of z minus mod of 1 minus i mod of 1 plus i we do know it is square root 2 mod of 1 minus i is also square root 2.

Thus we have got that this would be larger than or equal to absolute value of z minus square root 2 whole square. This is what we have written over here. So, thus we have got it what we have got now. So, from here if I use the ML inequality and this integral. This is function is bounded by k by R. And this is on this semicircle with the radius R.

So, here using the ML inequality, we could say this integral absolute value of this integral would be less than or equal to m. That is the bound, that is k upon R into the length of the arc. The length of arc here would be pi into R. So, what we are getting is this is bounded by a constant. Now, this constant is independent of R. So, now if I put limit as R is approaching to infinity, this constant is not going to change.

So, what we have got we are we do not know, how could we say that this is, because till now what how we had shown it. We have shown that is whatever this bound was coming that was depending on R and thus we are saying is as R is approaching to infinity. That bound is becoming a smaller and smaller and since it is positive quantity. So, that has to be equal to 0. But, here we are not able to find out the bound, that it is also decreasing as R is increasing.

So, that we cannot do, does it says is that this integral does not approaches to 0. No, it is not the only thing is that we had use this method. That is bound which we had find it out this is little bit more upper bound it may happen, that we may find out some more nice bound, which is depending on R and which is approaching to 0 can we use. Here comes, that is how we are going to use a Jordan's inequality let see.

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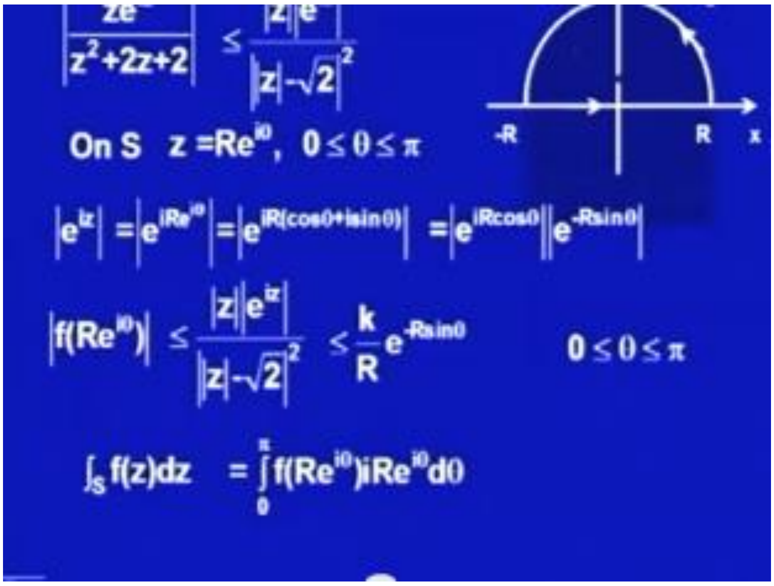

Again write this function z e to the power i z upon z square plus 2 z plus 2. This again we would be writing is absolute value of mode z into mod of e to the power i z upon a mod of z minus root 2 whole square, this is what is we have already obtained. Now, we want this integral on this semicircle S. And this semicircle S, this semicircle S we can use this parametric representation.

Then, on this S z i could write as R into e to the power i theta for this fixed R. And theta is ranging from 0 to pi, this is the parametric representation of by semicircle. Now, that says is we want a this bound on this semicircle only. So, let see on the semicircle if I write it out what my function will happen. E to the power i z on the semicircle, if I try to find out it is bound z I am replacing as R e to the power i theta.

Now, e to the power i theta by Euler's formula, I would be writing it as cos theta plus i sin theta. That says I would get e to the power i R cost theta plus i sin theta. Now, let see here I would get e to the power i R cos theta and here I would get e to the power minus R sin theta. So, what we are getting is absolute value of e to the power i R cos theta into absolute value of e to the power minus R sin theta.

Now, absolute value of e to the power i R cos theta whatever be this e to the power i it is modulus is always one. Because, it is cos x by the Euler's formula, it is cos x plus i sin x. And so sin x square x cos x square x plus cos x square x is always one. So, whatever be this i times this one. So, this is one and R we are taking as a positive value. Sin theta we are saying is the theta is laying between 0 to pi, we do know in the upper half that is from 0 to pi sin theta is always positive.

So, we are getting is that this quantity is positive, this would be some quantity. And e to the power any quantity, that is always going to be the positive. So, the absolute value would be e to the power minus R theta. Now, write this function f z as now R z I am representing in this parametric representation. So, I would get the integral on S f z d z as integral from 0 to pi your f R i theta and your d z would be getting from here R i R time e to the power i theta d theta. So, that we are going to do it.

So, let us see f z this is the s one. So, I am writing it as f R e to the power i theta. From here if I do write I would be getting this mod z e to the power i z upon z minus square root 2 whole square. This we could says is, because the first term we had already shown here, what I would get it R e to the power i theta mod of R e to the power i theta that would be simply R.

And here I would be getting is mod of R. Rather you could says that is in the before and we had obtain that this whole quantity was less than k by R. So, let us use it from there that is k by R and e to the power i z is equal to mod of e to the power i z is equal to e to the power minus R sin theta. So, now you see we had obtained a modified or final bound for our f z. Previously you had find out the bound for f z, that was k by R only.

Now, I had find out my bound for f z as k upon R e to the power minus R sin theta, where my theta is actually on this parametric representation of the semicircle S my z can be written as R e to the power i theta. So, we have got this a final bound. Let us use this final bound. So, now we are writing by integral s f z d z, if I just represent it in the parametric form, we could write it as a line integral 0 to pi f of R into e to the power i theta into your...

If you do remember that is f z is equal to your some z t, then you are writing it as f of z of t and then z dash t d t. So, that is what we are written here z dash theta d theta. So, it is i R e to the power i theta d theta. Now, use this ML inequality now.

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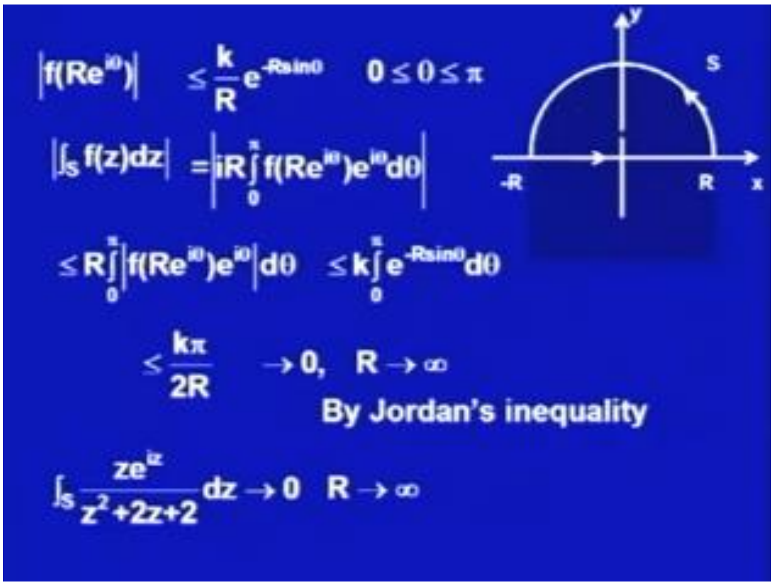

So, we have got f R e i to power theta is less than or the bound as k upon R e to the power minus R sin theta. And the integral s f z d z now on this one, we could write as integral of 0 to pi f R e i to the power theta e i theta d theta. And this R is constant with respect to theta. So, I can take this i in the outside. Now, take this absolute value, this I could say is this is x and this is y.

So, it would be mod of it is absolute value of x into absolute value of phi. Absolute value of i R would be R only. And absolute value of 0 to pi f R time e to the power i theta e to the power i theta d theta. We would be using another inequality, which is on the line integral when we say is that, absolute value of a integral along a line f z d z is always less than or equal to integral along the line of f absolute value of f t d t. This is from the line integral we are getting.

So, if I do use those results I would get that this would be less than or equal to R 0 to pi integral of absolute value of f R e to the power i theta into e to the power i theta d theta. Again this we could treat as two functions. So, absolute value of e to the power i theta, we treat as one. And f R e to the power i theta, that we have just absolute value of this we have find out this bounded by k upon R e to the power minus R sin theta.

So, what we get is that, this would be less than or equal to k upon R e to the power minus R sin theta. Since, this k and R they are constant. So, I could take them out from the integral sign. So, R into k by R I would that as k and then the integral from 0 pi e to the power minus R sign theta d theta. Now, here I can use this is now in the integral, which in the Jordan’s inequality we are finding it out.

So, I could use that Jordan’s inequality and I could write it as less than pi upon 2 R that is this is less than k upon pi 2 R. So, now what I have obtained the upper limit of that absolute value of this integral along the semicircle of the function f z. F z is now normally the function that actual function we have a started at z e to power i z upon z square plus 2 z plus 2. That is bounded by k pi upon 2 R.

Now, you see you have the we had find it out, there are we had find out first one was k upon R. Now, it is that was k pi, now we have find out k pi upon 2 R a more fine of a upper value. And which is depending upon R, that says this as we take R larger and larger, it will approach to 0. Now, since this is a positive value, absolute values is positive value.

So, if it is approaching to 0 as R is approaching to infinity it is says, that value itself must be 0. What it says is that, the integral of f z s integral of f z along this semicircle s must be 0. So, we had shown here, that z e to the power i z upon z square plus 2 z plus 2 it is integral on the semicircle s with respect to z is approaching to 0 as R is approaching to infinity. So, here we had use this one result, which is called as Jordan’s inequality or sometimes that result which we had prove the Jordan’s inequality. That is also called Jordan’s lemma.

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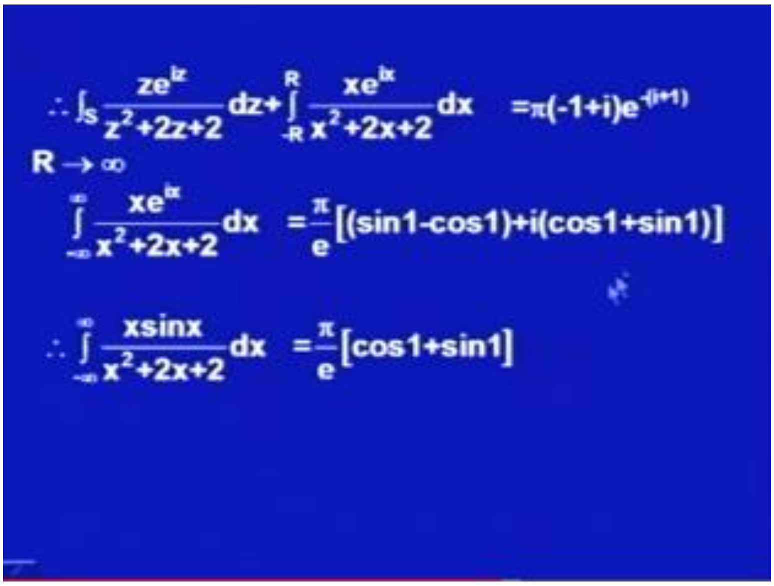

Now, if we use this one, that says this that now we have got the result as. So, now what we have obtain that integral along the semicircle s of the function z e to the power i z upon z square plus 2 z plus 2 plus the integral along the real line minus R to plus R x e to the power i x upon x square plus 2 x plus 2 d x. This is by the first result pi times it minus 1 plus i into e to the power minus 1 plus i.

Now, since R is now take the limit as R is approaching to infinity. Then, just now we had shown, that this integral would approach to 0. So, and this integral would be actually the principle value of this one or rather you could says this integral would approach to minus infinity to plus infinity x e to the power i x upon x square plus 2 x plus 2 d x. This would be this right hand side.

Now, right hand side I am writing in expanded manner, it would be e to the power minus i minus 1. So, e the power minus 1 I am writing as pi upon e, then e to the power minus i we could write as cos 1 minus i sin 1. So, I would get minus 1 plus i into cos 1 minus i sin 1. Write this real and imaginary parts separately after multiplication, it give me sign 1 minus cos 1 plus i times cos 1 plus sin 1.

Now, see in the left hand side, left hand side we do have this integral of x into e to the power i x upon x square plus 2 x plus 2. While I have to evaluate the integral of x into sin x upon x square pus 2 x plus 2. So, e to the power i x here also we could write as cos x plus i sin x. So, this left hand side also we can break it into two integrals as the real and imaginary part.

Now, equate from the both the sides the real and imaginary part. So, we want sin x, sin x would be coming in the imaginary part. That is from both the sides imaginary part we have to equate. So, from the here the imaginary part would be integral minus infinity to plus infinity x sin x upon x square plus 2 x plus 2 d x, which would be equal to pi upon e times cos 1 plus sin 1. So, we have evaluated this integral where of course, my function f x was not satisfying the degree condition. Let us see one more example of this kind.

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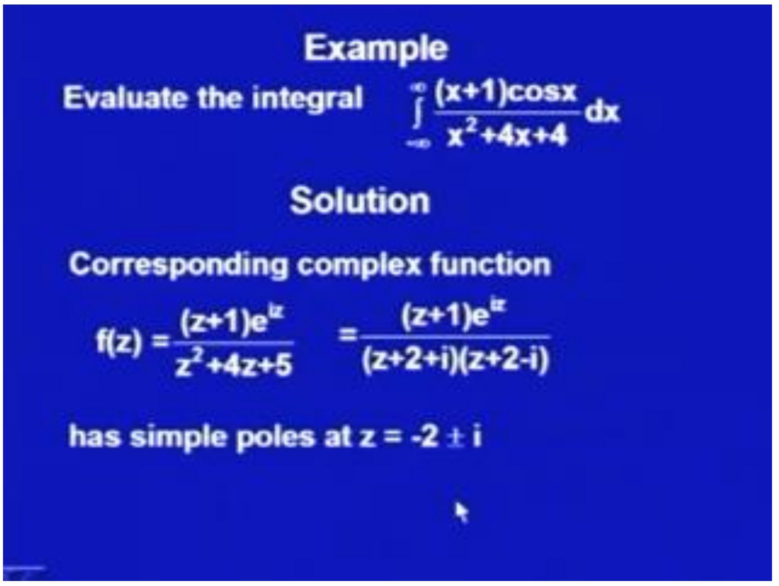

Evaluate this integral minus infinity to plus infinity x plus 1 cos x upon x square plus 4 x plus 4 d x. Now, you see here we are again having the integral of the form minus infinity to plus infinity f x cos s x d x cos x d x. Now, the function f x here is your x plus 1 upon x square plus 4 x plus 4. That is, it is rational function having two function p x upon q x both p x and q x are polynomials. P x is a polynomial of degree 1, q x is a polynomial of degree 2. Q x is not having any your real 0s.

So, what we do get as a solution we would start with, but the condition is that degree condition is not been satisfied. So, we will just find out the corresponding complex function z plus, this function is x square plus 4 x plus 5, otherwise it will have real 0s. So, when it is pi only then it would not have real 0s. So, this question has to be x square plus 4 x plus 5 here.

So, corresponding complex function we would write z plus 1 e to the power i z upon z square plus 4 plus 5. Make this factors, it would be z plus 1 into e to the power i z upon z plus 2 plus i into z plus 2 minus i. The 0s of the denominator are minus 2 minus i minus 2 plus i minus 2 plus i would be in the upper half and minus 2 minus i would be in the lower half. So, we do have that this function is having only two simple poles at minus 2 plus i and minus 2 minus i. So, in the upper half plane there is only one simple pole. Let us use the residue theory over here.

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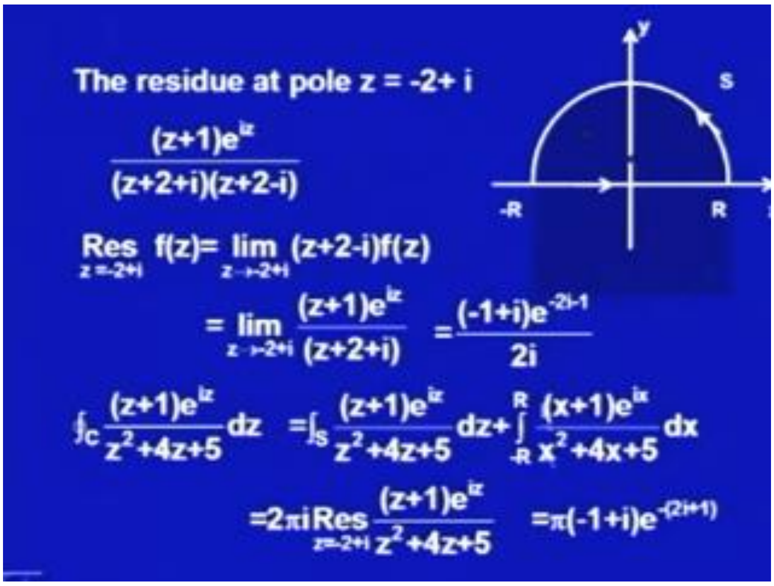

So, we will choose the contour as usual the semicircle, which if you from choosing it of the semicircle of the size of the radius 3, the that would include over minus 2 plus i. So, just one semicircle and then from minus R 2 plus R this real line. This close contour we are taking in this one simple pole in interior to this one. So, using the residue theory we do now this integral along this close contour would be 2 pi i times residue of the function f at the simple pole.

So, first let us use this calculate this residue at a simple pole minus 2 plus i. Again, because it is a simple pole I would use the first formula of this function residue at z is equal to minus 2 plus i into f z as limit is of z is approaching to minus 2 plus i z plus 2 i minus i into f z. If I multiply z plus 2 minus i with this f z I would get z plus 1 e to the power i z upon z plus 2 plus i.

So, I would get limit z is approaching to minus 2 plus i of z plus 1 times e to the power i z upon z plus 2 plus i to evaluate this limit. First we just have to write our value, that z is equal to minus 2 plus i. So, from here what I would get it z minus 2 i. So, z plus 1 would be your 3 plus i. So, what we are getting is minus 1 plus i e to the power minus 2 i minus 1 upon 2 i as usual.

We are just now using the residue theory, along this close contour c. This close contour we are just now again breaking into two parts. One is the semicircle circular one, another is this part of the real line. So, we are writing it as integral of z plus 1 e to the power i z upon z square plus 4 z plus 5 d z along the semicircle s of the radius R. Plus the integral along this real line from minus R to plus R. And that we are writing in the x format.

So, x plus 1 e to the power i x upon x square plus 4 x plus 5 d x. Now, this we have to now show that, integral along this 2 pi i into residue at z minus 2 plus i z plus of 1 of this residue of this function, this residue of this function just now we had calculated this one. So, if I multiply it with 2 pi 2 i and 2 i would cancel it out, I would get again minus pi into 1 plus i into e to the power minus 2 plus 1. Now, the thing remaining is that is now we have to take the limit as R is approaching to infinity. So, I am showing it that, this has to go to 0, then this would go into the desire integral minus infinity to plus infinity.

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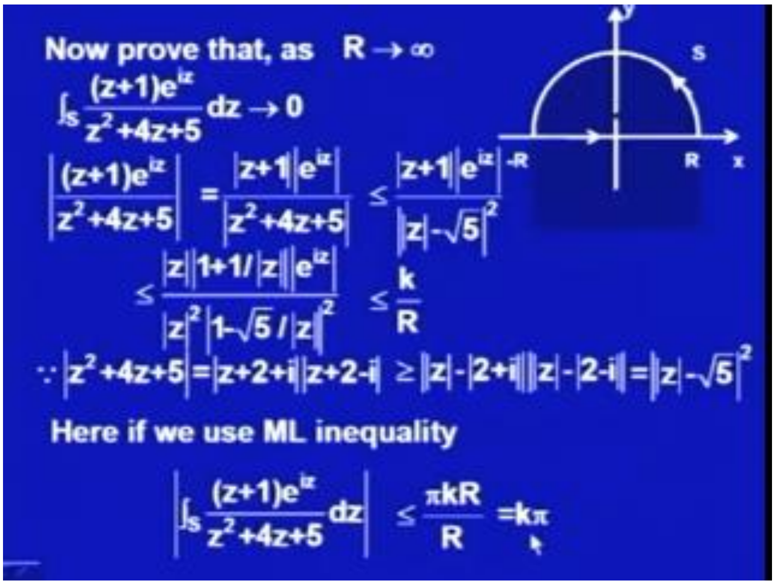

So, let us try to show that the integral of the function f z. That is z plus 1 e to the power i z upon z square plus 4 z plus 5 along this semicircle s of the radius R is approaching to 0 as R is approaching to infinite. Let us go ahead with our usual method, find out the absolute value which is being this bound above of this function. What is the, and then using this ML inequality.

So, again if we are going it in the usual manner, we are the finding it out that we could write it as mod of z plus 1 into e to the power i z upon z square plus 4 z plus 5, which we could get it as a should be less than or equal to mod of z plus 1 upon mod z minus square root of 5 whole square. This again you we would be getting in the form the inequality that we would explain.

Now, let us come over here mod z if I am taking common from here, I would get 1 plus 1 upon mod z. And this we are writing as mod z plus 1 is less than or equal to mod z plus mod 1. So, and from here if we are taking common mod z I would get mod z square and 1 minus square root 5 upon mod z whole square now.

Since, mod z is greater than R, if I have to take I would get this is bounded by some constant, this is bounded by some constant. That is, this is bounded by some constant, this is bounded by some constant. And here is a since mod z is greater than R and we are getting as 1 upon mod z. So, this should be less than 1 upon R. And e to the power i z if I treat it as 1 I would get the bound as k upon R. Again we have got the bond as k upon R for the function f z.

So, if now I use then this ML inequality with this bound, what I would get that would be again a constant. Now, let us see this explanation for getting this z square plus 4 z plus 5 as larger than mod z minus square root 5 whole square. This z is square plus 4 z plus 5 with the factors z plus 2 plus i into z plus 2 minus i. Again writing it as greater than or equal to mod z plus minus mod of 2 plus i and mod z minus minus 2 minus i.

The absolute value of 2 plus is 4 plus 1 that is 5 square root of 5 similarly the absolute value of 2 minus i is also square root 5. So, this is simply mod z minus square root 5 whole square, this is the explanation for this denominator. Now, if I use this ML inequality over here, I would get the absolute value of the integral s a integral along this semicircle s of z plus 1 e to the power i z upon z square plus 4 z plus 5 d z.

Because, the absolute value of this is bound is k upon R this M. And L is the length of this semicircle, which is pi arc. So, what we would be getting is less than or equal to k R into pi R oh that is again a constant. So, now this constant bound is not depending upon R or so. This we cannot guarantee, that this would approach to 0. So, again we have to use the method, that e to the power i z we won not take absolute value, the bound for this as one. Rather we would try to write it out as e to the power minus R sin theta. And then the integral along that a semicircle s we found out more final bound.

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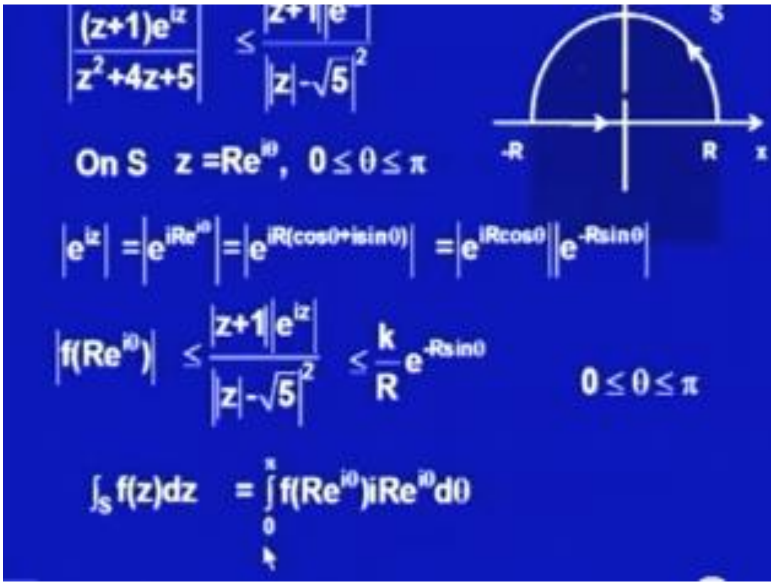

So, again we are taking this function z plus 1 e to the power i z z square plus 4 z plus 5. This we had just shown, that is less than or equal mod z plus 1 into e to power i z upon mod z minus square root 5 whole square. Now, e to the power i z on s we would be writing it as e to the power i R e to the power i theta. Because, the semicircle this s is R e to the power i theta in the parametric representation for that, theta lying between 0 and pi.

E to the power i theta, again in the similar manner we would be writing cos theta plus i sin theta. So, what we would be getting is absolute value of e to the power i R cos theta into e to the power minus R sin theta this is one. So, again we would be getting is e to the power minus R sin theta. In the similar manner... So, this bound for f z, that is now I am writing as f R e power minus theta.

This what we would be getting is that is, in this part again as such this mod z or rather you could say this for this part we had already got. That, this is less than some constant k upon R and e to the power i z is e to the power minus R sin theta. So, we would get it k upon R e to the power minus R sin theta. Now, we use this final bound of this function f for writing this.

This is true for all theta, because this Jordan's inequality is true for this 0 to theta lying between pi. So, we have to found it out that, this on this whole path of integration, this is this bound is true. So, now we get s f z we would be writing again in the form of this parametric representation, we could write this as 0 to pi f R e i to power theta into i R times e to the i theta d theta. This now we would be using these absolute values.

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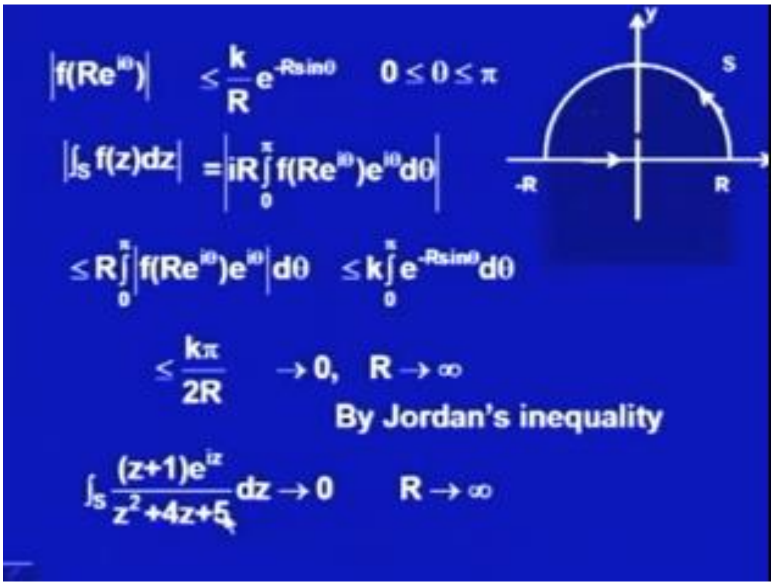

So, this just now and this would be less than or equal R, this you could says i R integral 0 to pi f R e to the power i theta e to the power i theta d theta. Now, again we would be using for the line integral. That is line integral along a path of absolute value of the integral, this is always a smaller than the integral of the absolute value of the function.

So, absolute value of i R is R only and this is less than 0 to pi integral along 0 to pi of the absolute values of f R e to the power i theta into e to the power i theta d theta. Now, absolute values of e to the power i theta is one and absolute that we will take f 1. And the bound for f of R e power i theta, we take as k upon R e to the power minus R sin theta.

So, since this is less than or equal to... So, the integral would also be less than or equal to R times integral 0 to pi k upon R e to the power minus R sin theta d theta. And this k upon R is a constant. So, that we would take out. So, we will get less than k time 0 to pi e to the power minus R sin theta d theta. Now, for this integral we would use, because this is from 0 to pi we would use the our Jordan's inequality. And that says is at a k times pi upon 2 R.

Now, we have got for this integral the absolute value final bound as in the previous method. And this final bound is depending upon R. So, as we are increasing R, this bound would be decreasing towards 0. If bound is decreasing towards 0 what it says that this integral must be 0 for that large R or that we could says is that, this integral s along this f z this approaches to 0 as R is approaching to infinity.

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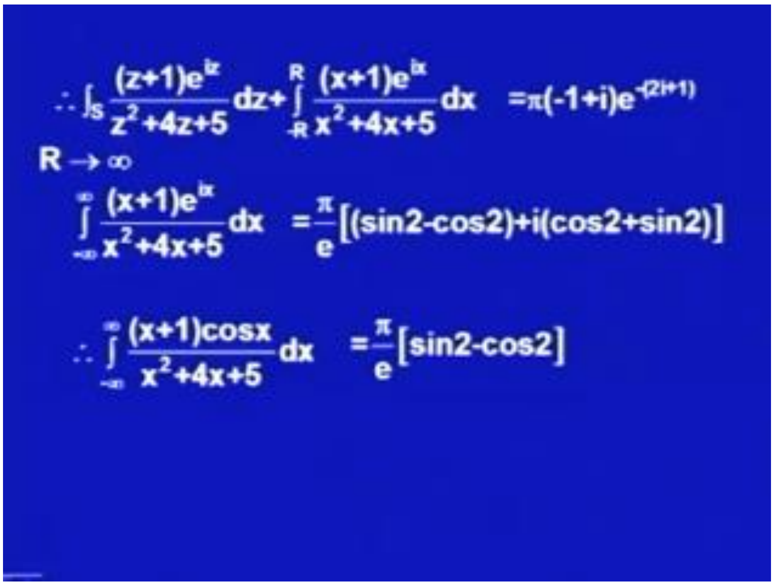

Now, so what we have got is? We have got that integral along this s z plus 1 e to the power i z upon z square plus 4 z plus 5, and integral along this real line minus R to plus R x plus 1 e to power i x upon x square plus 4 x plus 5 d x. This we had already obtained is equal to pi times minus 1 plus i e to the power minus 2 i plus 1. Now, take the limit as R approaches to infinity.

Then, the second integral would be actually we could write as integral minus infinity to plus infinity x plus 1 e to the power i x upon x square plus 4 x 5 d x. And this integral would be approaching to 0 just now we have shown, as R is approaching to infinity. So, they should be equal to this part. Now, this I am writing as real and imaginary part separately as in the previous one.

So, e to the power minus 1 that I am writing as pi upon e, e to the power minus 2 i that we could write as cos 2 minus sin 2. So, we are having minus 1 plus i into cos 2 minus pi sin 2 multiply it. And write the real and imaginary parts separately, we get sin 2 minus cos 2 plus i cos 2 plus sin 2.

Now, here the left hand side e to the power i x, again we can using the Euler's formula. And we can write it as cos x plus i sin x, that says this left hand side I could I have got at least two improper integrals. One is integral 0 to infinity x plus 1 upon x square plus 4 x plus 5 cos x d x plus i times minus infinity to plus infinity x plus 1 times sin x upon x square plus 4 x plus 5 d x.

Now, the integral which we have to evaluate that was integral of minus infinity to plus infinity of x plus 1 cos x upon x square plus 4 x plus 5 d x. Now, this is the real part of this integral. So, equated with the real part, real part says pi upon e sin 2 minus cos 2. If I have to find out the imaginary, that is if it would have been sin x I would have use this imaginary part.

So, thus today what we have learn, that even if our function f is not satisfying the degree condition. There is one more term or one more method we had learn one more result we had learnt. That is called the Jordan’s lemma, which said is integral 0 to pi e to the power minus R sin theta d theta is bounded by pi upon 2 R, and that result has helped us to evaluate some more integral of the form minus infinity to plus infinity f x cos x R f x sin x d x.

Now, you see all these proofs when we were doing, we had find it out that in the examples. We were find out that is a first you we use to take the degree condition, that that the difference between the numerator and denominator degree must be at least 2. So, that the convergence condition are that integral along the semi circle of the function, should approach to 0 as R is approaching to infinity.

When we had find out the difference is only one, we were reaching towards a k that is says is we are not actually able to find out. That how it is reaching to 0 or how it is should says that it is equated to 0. So, that we could the residue theory directly. Then, we had use this Jordan’s inequality. And find it out it still if the function is of the form f z e to the power i z kind of thing.

We could still write it out in the form on the semicircle, we could write it out as R into e to the power i theta. And from there we had got the clue, that is we could use this Jordan’s inequality. And again we could show, that the integral along this path, that is the semicircle as R is approaching to infinity is and get into 0. And then again we could use this residue theorem or residue theory.

So, the examples we have done only one simple pole. If we do have a more than one simple pole, certainly we can use the residue theorem. And then it could be 2 pi items summation of a residues at all those simple poles of the function f z. All this poles what we have taken, they over in upper half plane. And we had although and we have taken that my f has to be rational function. So, that we could show a integral along this semicircle approaches to a 0.

So, we have got that is either it is greater than 2 or we had reach, that if it is greater than or equal to 1 a still it is happening. So, we are not talking about the constant functions. Now, if my poles are or if the singularity of the function is on the real line. All the times we have put condition that q x should not have a real 0. That is we should not have the function to be singular on the real line.

Now, if the function is singular on the real line, what will happen? That we would see in the next lecture. That is, if this condition that q x has no real 0, if this has been failed is still can we still we do apply this residue theory. The answer we would get in the next lecture class.

So, today we had got that we can solve the functions, the find out the integral of the functions of the form f x cos x or f x sin s x a form of the integrals from minus into plus infinity, where the f x R failing with degree condition on the numerator and denominator of greater than 2. Or let see that is we have got the results, that these kind of integrals have been evaluated. We can use the degree condition that, it is greater than or equal to 1. So, that is all for today's lecture.

Thank you.