Mathematics - II
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Lecture - 11
Evaluation of Real Improper Integrals – 2

Welcome to the lecture series on complex analysis for undergraduate students. Today's lecture is, on evaluation of Real Improper Integrals, part 2. We are learning to evaluate the improper integrals, using the Residue theory. Today, we will learn another method for another kind of improper integrals.

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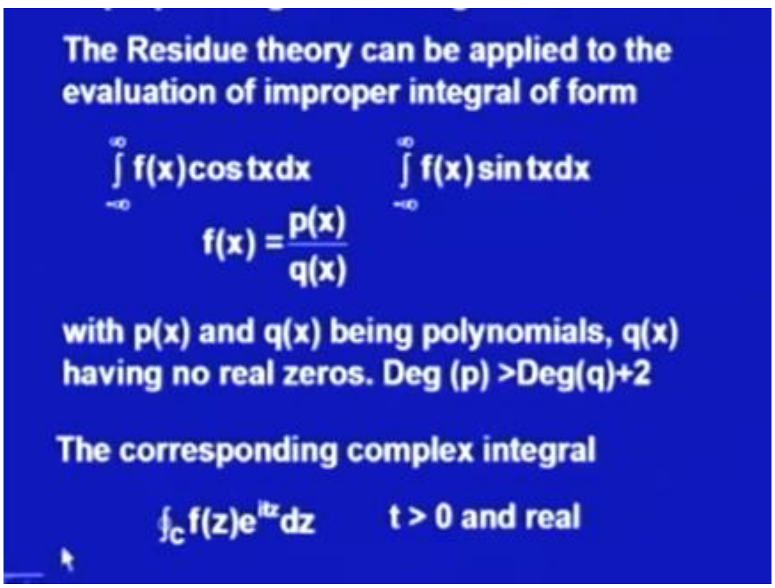

So, today, we will learn about the improper integrals, which are involving sin and cosine functions. The residue theory, can also be applied to the evaluation of the integrals of the form, minus infinity to plus infinity, integral of f x cos t x, d x or the integral of the form integral, from minus infinity to plus infinity f x sin t x, d x. These integrals are very useful, when we are evaluating the Fourier integrals or we are actually calculating the Fourier series of evaluating. Or finding out the coefficients of the Fourier sin and cosine series, this kind of integrals appear.

Here we would be first assuming that f x is satisfying all those conditions, which we have learnt in the last lecture. That f x is again a rational function, which is can be ratio of two polynomials, p x and q x. And this p and q are both satisfying the condition, that the factors to them are not common to each other. And the degree of the q; that is the denominator is at least 2 degrees higher than the degrees of the numerator. So, we are having and moreover this, q x will not have any real 0.

So, on f x, we have used that is all the conditions, which in the previous method are being imposed. All those conditions are also valid over here are being imposed over here. Moreover, now we do have integral of the form f x into cos t x or f x into sin t x kind of function. Actually, these integrals or these functions, we can use, if I do use, only one corresponding complex integral of the form f z e to the power i t z.

And what we would do is, we would like to change this integral to a contour integral of the function f z, e to the power i t z over some contour c. And this contour, we would chose is closed contour. Here, when, I am choosing this one complex function, where f z into e to the power i t z. We do know by all Euler's formula, that e to the power i t z, we could write it has cos t z plus i sin t z.

Say, if I am writing over here, this contour integral could be written as in the form of two contour integrals. One would be 1 f z times cos t z and plus i times contour integral of f z sin t z. Thus, what we would get, we get that, whatever answers to this contour integral, we are getting. We just here, it is also complex value and whatever the answer, we are getting is that, would be a complex value. So, we just get it.

Compare the real and imaginary part and accordingly, we would be able to get the integrals along f z cos t z and f z sin t z. And again, as in the previous theory, we had known that, what we have done is, we had used this Cauchy principle value and tried to show that, this would be actually ranging to the integral minus infinity to plus infinity f x, e to the power i t x, d x. So, let us see is, how we are going to do, first we will take this, t to be positive and real.

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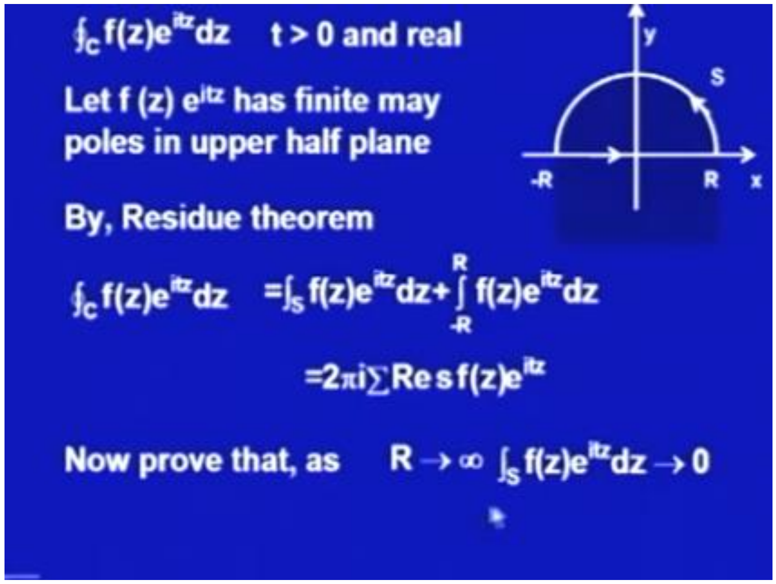

So, this contour integral f z e to the power i t z, d z. This again, as in the previous method, we would use this contour c as the semicircle bounded by the real line. So, f is, it has and what we are assuming is that, since f z is of the form p z upon q z, where p and q, both are polynomials. And q is not having any real 0's. So, what we are having is, q z would have 0's, which are complex numbers.

Again, what would consider as in the previous method; that the 0's of q z, because it is polynomial. So, it will have isolated 0's only. So, 0's of the polynomial q z, would be actually the poles of this function f z e to the power i t z. And thus, we would go ahead with the residue of the poles, which we are having in the upper half plane of the function, f z e to the power i t z. And using that, we would use this Cauchy principle value.

So, we are having the poles in upper half plane. So, by Residue theorem, we could say is, that if I do take this integral in the closed and we would choose this circle, such that. The semicircle, such that, all those number of poles are inside this circle. So, inside the interior to this closed contour, see, what it says is that, by Residue theorem. We could say that, integral along this closed contour c, would be nothing but, the 2 pi i times, summation of residues of f z, e to the power i t z. Where, summation would be all at all the poles, which are interior to this contour c.

And this contour c is, actually consisting of two paths. One is, the semicircle, another is the real line or that the real axis. So, we are using this semicircle of the radius R. R is being, so is chosen such a large, such that, all the poles of the function f z, e to the power i t z are interior to this semicircle. And we are the choosing this other boundary as, real line from minus R 2 plus R. So, as in the previous method, we are writing this by Residue theorem, this would happen.

Now, as in the previous method, we had gone. That is, if I treat this integral minus R to plus R, f z, e to the power i t z, d z. We see this is on the real line. So, this contour we could say is, we could write it as minus R to plus R, f x, e to the power i t x, d x.

Now, this integral we have to evaluate. This integral is equal to this integral minus, that is the integral on the close contour c of f z, e to the power i t z minus the integral on the path s, which is semicircle of f z, e to the power i t z, d z. Now, what as in the previous method, here also, we will show that, when f z is of the form p z upon q z. And the degree of q z is at least 2 degrees higher than the degree of p z. This integral along this semicircle would approach to 0, as R is approaching to infinity.

And thus, we would get that, the value of this improper integral from minus infinity to plus infinity f z, e to the power i t x, d x. Could be as 2 pi i summation over the residues of f z, e to the power i t z, where the summation would be varying on the all poles in the upper half plane. So, what now we require is, that as R is approaching to infinity integral of f z, e to the power i t z, d z on the contour s, must approach to 0. So, let us first see this portion here.

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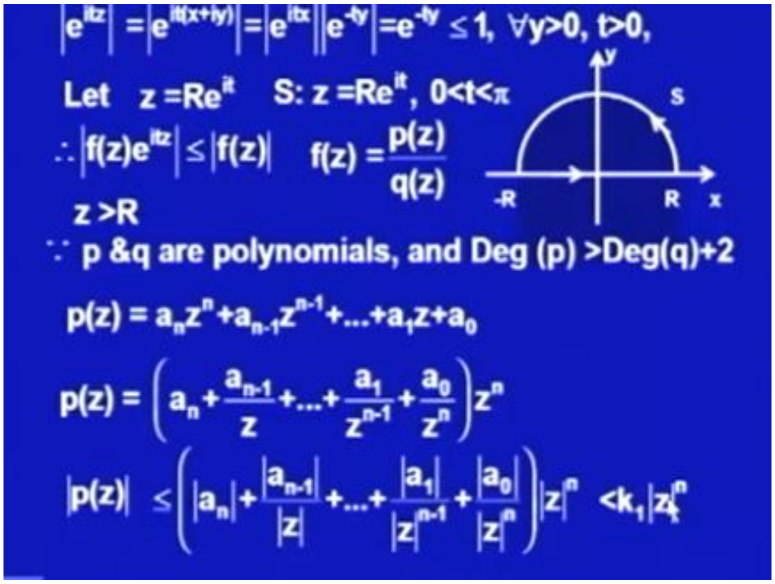

Let us take now, my integrant is f z, e to the power i t z. Let us first see, what is e to the power i t z? E to the power i t z, z is a complex number, we could write it our x plus i y. So, just substituting z as x plus i y, we would get e to the power i t times x plus i y. It would give the first thing is, that e to the power i t x plus e to the power i, t into i y, that is e to the power minute t y. So, we would get this is equal to absolute value of e to the power i t x into absolute value of e to the power t y.

Since, we do know that, further complex numbers x and y, absolute of value of x into y is equal to the absolute value of x into absolute value y. Now, absolute value of e to the power i t x, we do know e to the power i t x by Euler's formula. That it is, cos t x plus i sin t x and it is absolute value is, always 1. Because, cos is square i t x plus sin is square i t x would be 1 for all t n for l x. So, this would be e to the power, the absolute value of e to the power minus t y, e to the power minus t y is a real number. So, it should be same as e to the power minus t y.

Moreover, whatever be this y, y is, we are taking is on the semicircle, which is the upper half plane. That is, y is always positive. And t, we have taken a positive real number. So, both t into y, this is positive. So, e to the power minus t y would be a number, which is more than 1. Because, e to the power x, for x positive is always greater than 1. That is says e to the power minus t y, this should always be less than or equal to 1, for all y positive and for all t positive.

So, what we have got actually from here, that the absolute value of i t z is always bounded above by 1. What it says is, now if I take the integrant, so now, we have to show that, this integral of the contour s of f z, e to the power i t z is approaching to 0 as R is approach into infinity. And this s, this semicircle, if we are going with the parametric 1 or you could say is a polar 1. Then, z is R e to the power i t on this 1.

And what is my, this parametric equation of this semicircle s? That is, z is equal to R e to the power i t for t ranging from 0 to pi. Now, f z e to the power i t z, this absolute value would be actually absolute value of f z or you could say modulus of f z into modulus of e to the power i t z. Modulus of e to the power i t z is, less than 1. Just now, we have proved, that says is, this should always be smaller than or equal to absolute value of f z.

Now, f z we had all ready assumed that, it is of the form p z upon q z, where p and q, both are polynomials the degree of, q is at least 2 degrees higher than the degree of p. So, again as in the previous method for z greater R, this p and q, we would start. Let us say, p is a polynomial of say degree n. Then, p z, we could write as a n, z to the power n plus a n minus 1, z to the power n minus 1 and so on, plus a 1 z plus a naught.

Now, using the, take this z to the power n common from here, because we are taking this z is greater than R. We could write it as, a n plus a n minus 1 upon z and so on, plus a 1 upon z to the power n minus 1 plus a naught upon z to the power n into z to the power n. Now, take the absolute value of this 1. So, absolute value of or modulus of p z would be now, we are going to use the result. That is, mod of x plus y is less than R equal to mod of x plus mod of y.

So, we are getting is, less than R equal to mod of a n plus mod of a n minus upon z. And that we are using is, mod of x into y is mod of x into mod of y. So, this is mod of a n minus upon mod z plus so on, mod of a 1 upon mod of z to the power n minus 1 plus a naught upon mod z to the power n into mod z to the power n. Now, since mod z, z is greater than R, so mod z could be greater than R, that says is, 1 upon mod z would be less than R, 1 upon R.

What it says is that, this would be less than mod of a n minus 1 upon mod R, upon R and so on. We are getting is that, it is this last value should be less than mod of a naught upon R to the power n. R is all ready a positive number, large number. So, 1 upon over R as R to the n is increasing the power of 1 upon R to the power n; that would be decreasing. So, this would be bounded by or this would be actually, we could find out, that is all these things would be limit. Or, that is, this would be bounded by some constant. Let us say k 1 times mod z to the power n.

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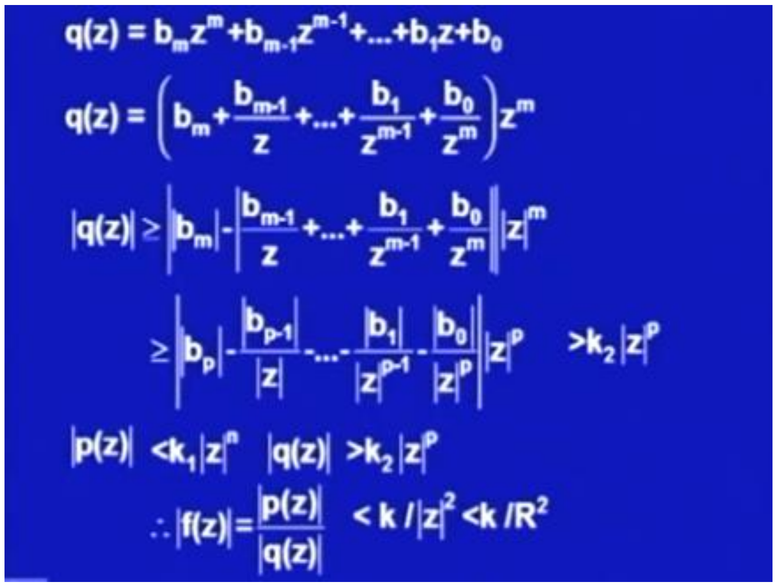

Similarly, assume that q is a polynomial of degree m. So, let say that q is of the form b m, z to the power m plus b m minus 1, z to the power m minus 1 so on, plus b 1 z plus b naught. Again, in the similar manner, we will take z to the power m, common. So, what we could write it as, b m plus b m minus 1 upon z and so on, plus 1 b 1 upon z to the power m minus 1 plus b naught upon z to the power m into z to the power m.

Now, again take the absolute values, so absolute value that is mod of q z. Now, here, what we are using, we are using the inequality of the mod, that mod of x plus y is always greater than or equal to mod of mod of x minus mod of y or absolute of mod of x minus mod of y. So, first, we are taking is b m and all these terms is the second term. So, it should be mod of b m minus mod of b m minus 1, upon z and so on, plus b naught upon z to the power m.

Absolute value of all these into z to the power, absolute value of z to the power m, which says is that, this is greater than or equal to b m. And here, what we would be getting is, that is, again I am using. Now, the inequality, that mod of x plus mod of y is less than or equal to mod of x plus mod of y. So, since, it is in the negative sign. It would be greater than and we are getting here, all that terms b m minus 1 and upon z, mod z, b 1 upon mod z to the power m minus 1 and so on, b naught mod z to the power m into mod z to the power m.

Now, since, mod z is greater than R, 1 upon mod z would be less than 1 upon R. So, minus 1 upon mod z would be greater than 1 upon minus 1 upon R and like that, we just go on. That says is, this value in the first absolute 1's first modulus sign, this should be bounded above by some constant k 2. So, what we get, this is greater than or equal to k 2 times mod z to the power m. Now, my p z is less or equal to k 1 z to the power n, q z is greater than or equal to k 2, mod z to the power m and my f z is mod p z upon q z.

And the difference between this n and in the degrees of q and p is at least 2. So, let us assume that, it is equal to 2, actually. Then, what we would have this is, less than or equal to mod k upon z square, because k 2 upon k 1, we are taking it as, that is a another constant k. Since, mod z is greater than R. This should be less than or equal to k upon R square. So, as in the previous case, here also we had obtained that, absolute of f z is bounded above by k upon R square. And the integrant f z e to the power i t z, that is completely bounded by the absolutely value of f z and which is bounded by k by R square.

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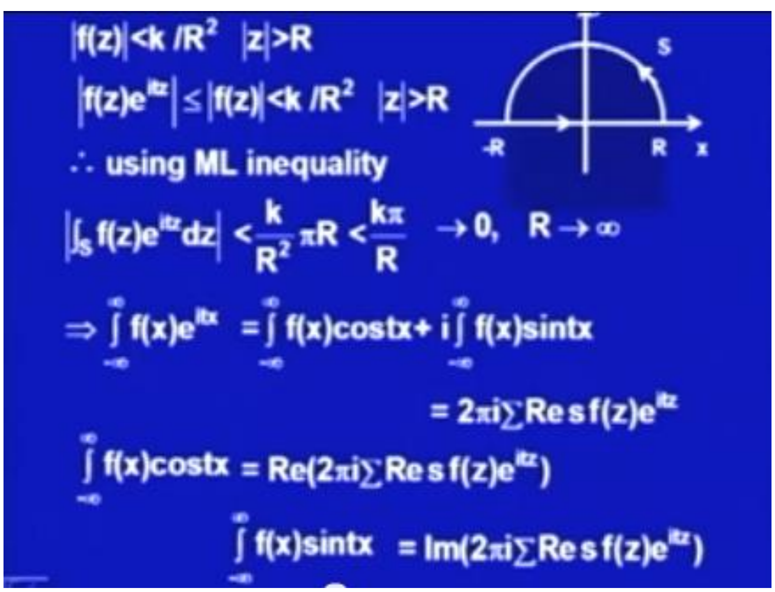

So, now what we have got? We have got this result that, this is bounded by this 1. And absolute value of f z, e to the power i t z, this is less than or equal to absolute value of f z, which is bounded above by k by R square. For mod z, greater than R, greater than or equal to R, you could say. So, using the ML inequality on this integral of f z, e to the power i t z on this semicircle, that is in on this path.

Using ML inequality, this since the integrant is bounded by this value k upon R square. This would be less than or equal to k upon R square into the length of the contour s. The contour s is the semicircle. It is parameter would be pi R. So, we are getting a pi R. That is, it is actually k pi upon R. So, what we have got that this, absolute value of the integral f z, e to the power i t z on this semicircle s is bounded by k pi by R.

Now, this value is positive as R is becoming large and large, this value would approach to 0. So, what we are getting is that, as R is approaching to infinity. The integral of f z, e to the power i t z with respect to z, on the contour s, this would be approaching to 0. So, what it says is that, now integral minus infinity to plus infinity f x, e to the power i t x. This would be same as the contour integral the integral of f z, e to the power i t z.

And the whole contour closed contour c, which is equal to the 2 pi i times some of the residues at the poles, which are interior to this. See, that is, the poles, which are in the upper half plane. So, we have got that formula now. Actually, what we are evaluating? We were evaluating the function, the integral of the form minus infinity to plus infinity f x cos t x and f x sin t x. So, let us using the Euler’s formula, write this integral as f x cos t x plus i times f x sin t x.

Now, what we are getting is, this should be equal to now 2 pi i summation of residues of f z, e to the power i t z, where summation is over all the poles on the upper half plane. This is also a complex number. So, comparing the real and imaginary part from here, we would get actually the 2 formulas. One is, integral from minus infinity to plus infinity f x cos t x is the real part of 2 pi i summation of residue of f z e to the power i t z. Where, the summation is over all the poles in the upper half plane.

And when we compare the imaginary parts, we do just get is that, integral of f x sin t x, d x should be the imaginary part of 2 pi i, summation of the residues of f z, e to the power i t z.

Where, summation is, running over all the poles in the upper half plane. So, we have got these two formulae's. You could say our rather than a method, where we are reaching, that is, we could obtain these integrals, improper integrals. Using the residues of the function f z, e to the power i t z in the upper half plane.

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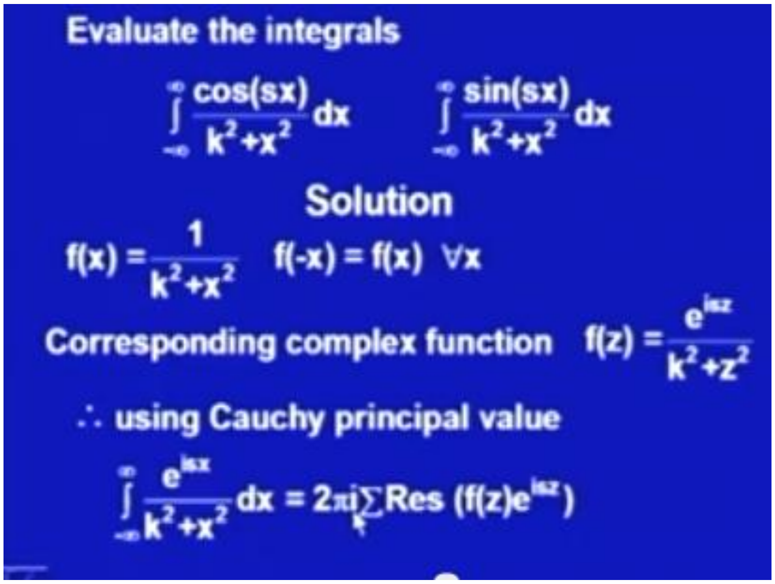

Let us, apply this formulae or this method to some of the examples. Evaluate the integral minus infinity to plus infinity, cos s x upon k square plus x square d x and integral, minus infinity to plus infinity, sin s x upon k square plus x square d x. You see, here in this example, we are going to evaluate two integrals. And both the integrals, we find out certain similarities. The similarity is the denominator for both this is same k square plus x square.

While, the numerator is being changed only to the extent that, the numerator of the first integral is cos s x and the numerator of the 2nd integral is sin s x. So, what we could say is, we are going to evaluate the integrals of the form f x, cos s x and f x, sin s x, where of course, f x is same as 1 upon k square plus x square. That says is, we could apply the method, just now, we had learnt. And both these integrals, we can solve using only one complex function, where we would use e to the power i s x.

So, f x is, 1 upon k square plus x square. The method, which we had learnt just now, that required certain conditions for f x to gets satisfied. The conditions for the f x, were that f x should be a rational function, which we could say is, that of the form the ratio of two polynomials. Here, it is, of course, the rational function the numerator is 1 the constant, a denominator is k square plus x square.

Certainly, a constant can always be treated as a polynomial of degree 0. And the denominator is x square plus k square. So, this is polynomial of degree 2. One more condition, we required on the numerator and denominator, that none of the factors should be common. Of course, here we are not having any factors common. Moreover, this denominator x square plus k square does not have or does not have any real 0's.

Moreover, the degree of the numerator is 0, because it is a constant. And the degree of the denominator is 2. So, the degree condition is also satisfying, that says is that all the conditions of the method, just now we have discussed, they are satisfying, so we could. And moreover, here we are finding out that, f of minus x is same as f of x also. This condition actually, when we are in evaluating this integral from minus infinity to plus infinity is not really very much required.

But, if we have to evaluate the improper integral of from 0 to infinity, then this condition or this even condition of that function is even, would be helpful, when we are evaluating the integral from 0 to infinity. Then, we could say is that, the integral 0 to infinity would be half of integral minus infinity to plus infinity of f x, cos s x, d x. So, let us say, apply our method. That says is, the function, we would require as e to the power i s z upon k square plus z square.

And we would require to see, the poles of this function in the upper half plane. And then calculate the residues. And use the result, with just now the things, which we had obtained. That integral minus infinity to plus infinity f x, e to the power i t x, d x would be 2 pi i times, summation of the residues of this function. Where, the summation would be on all the poles in the upper half plane.

So, just using this Cauchy principle value, we would get this result. That is, integral minus infinity to plus infinity e to the power i s x upon k square plus x square d x, should be 2 pi i times summation of residues of f z, e to the power i s z. Where, the summation would run over all poles on the upper half plane. So, let us see the residues.

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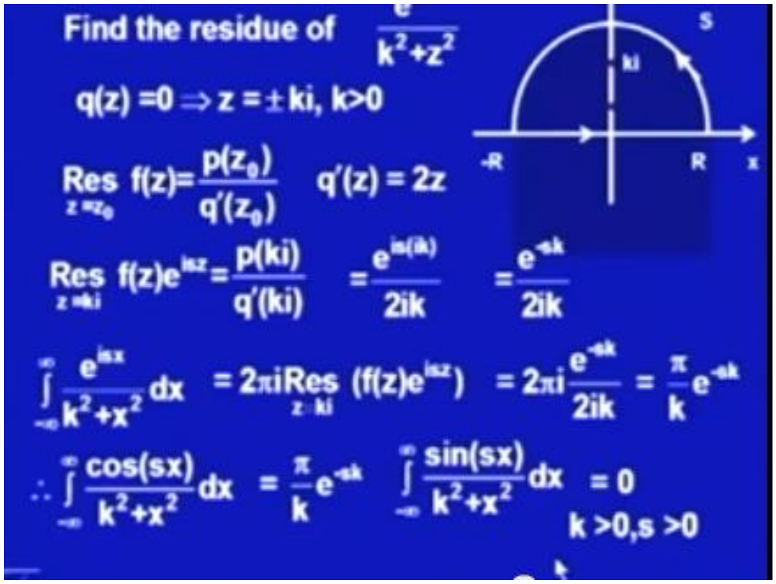

So, now we have to find out the residue of the function, e to the power i s z upon k square plus z square. This function, now we will treat as p z upon q z. And here q z is, the 0's of this q z implies. That is, z is equal to plus minus k I. When, k is, now I am making one assume, because here it was k square. So, whether k is positive or negative, it was not making any effect. Now, let us assume that, k is positive.

Of course, why I am assuming k is positive, that also you have would have got it. If k is positive, then plus k i would be in the upper half plane. And minus k i would be in the lower half plane. If k is negative, then actually minus k i would be in the upper half plane. So, without loss of generality, if k is negative, I can say is k is some minus n, where n is positive. And thus, we can always assume that, k 2 be positive.

So, that is, we are assume that, let us say k is positive. So, this 0, we would have at plus and minus k. That says is, that the function e to the power i s z upon k square plus z square, this will have pole at only one point, k i. That is, on the imaginary axis at k, in the upper half plane. So, we have to calculate the residue at that point. Calculating the residue at that point, says is, since the function is of the form p z upon q z. We would use the formula of reside at z naught of any function f z as p z naught upon q dash z naught.

What is q dash z here? Q dash z here is, 2 z only. So, we would get the residue at k i f, f z, e to the power i s z as e to the power, that is, p k i and upon q dash at k i. That is, e to the power i s z, is i k upon 2 z, z is i k, that is, at z is equal to k i or i k. What it says is, in the numerator, we would be getting is, e to the power minus s k upon 2 i k. This is the residue. So, now, let us apply our Cauchy principal value. We got it, that is integral minus infinity to plus infinity, e to the power i s x upon k square plus x square d x. This would be 2 pi i residue times residue at k i of f z e to the power i s z.

Now, residue of this one is, just now here, which we had calculated. This is, e to the power minus s k upon 2 i k. So, we just multiply, that is 2 pi i into e to the power minus s k upon 2 i k. so 2 i and 2 i got canceled it out, we would get it as pi by k e to the power minus s k. So, what we have got, we have got integral of minus infinity to plus infinity e to the power i s x, k square plus x square d x is equal to pi upon k e to the power minus s k.

What it says is, this e to the power i s k, we could write as cos s x plus i sin s x. So, we would get the two integrals. One is, cos s x upon k square plus x square d x. Another would be the integral minus infinity to plus infinity sin s x upon k square plus x d x and with plus i. That is, we would be getting the real and imaginary part. So, compare this real and imaginary part of this. So, this is here, we are having two parts. One is, real part and another is imaginary part.

Real part would be integral from minus infinity to plus infinity cos s x upon k square plus x square. And imaginary part would be sin s x upon k square plus x square. It is integral on the range minus infinity to plus infinity. The evaluation, we have got only the real number. So, it has only real part, it is imaginary part is 0. So, what we had got finally, when we are comparing this, that integral minus infinity to plus infinity cos s x upon k square plus x square d x should be pi upon k e to the power minus s k.

And the integral of sin s x upon k square plus x square plus with respect to x on the range minus infinity to plus infinity should be 0. So, we had evaluated it, using this residue theory. We had shown that, this would be the integral. Now, let us do one more example and this formulae's would be valid. When s is positive and k is positive.

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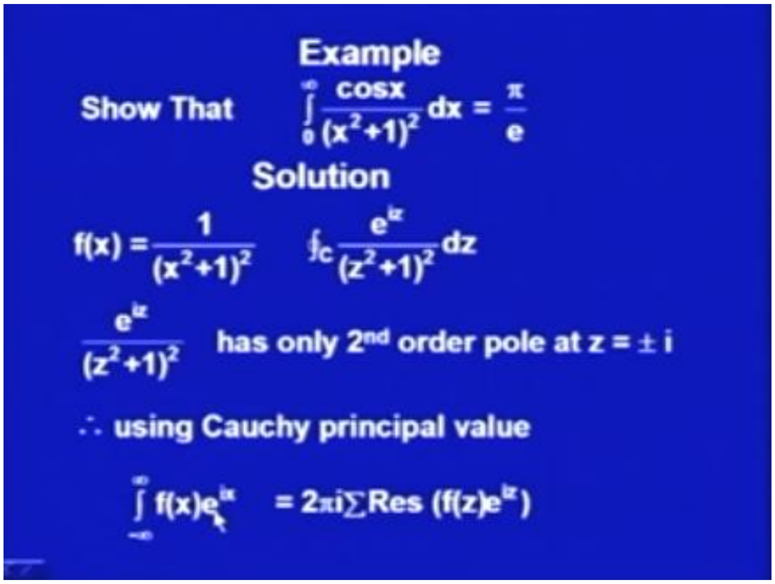

So, let us do one more example of this kind. Show that, integral 0 to infinity of cos x upon x square plus 1 whole x square d x is pi upon e. So, now, we have to prove this formula. Now, we see, that the integral is now, from 0 to infinity. The integrant is having one term, cos s x and another term is, 1 upon x square plus 1, whole square. So, we see that, we could write this as the form of f x cos s x kind of integral.

So, we have to check about the conditions for the f x. If it is satisfying all those conditions, we can move to the method, just now we had learnt. So, let us see, what is my f x? My f x is 1 upon x square plus 1, whole square. Of course, we can say, this is a rational function of, from p x upon q x, where p x is the constant. So, it is a polynomial of degree 0. The denominator q x is x square plus 1 whole square. Certainly, it is also a polynomial, whose degree is 4.

So, we are having is, this is f x is the ratio of two polynomials, none of the polynomials have any factors in common. And the degree of the denominator is 4; degree of the numerator is 0. So, the condition is also satisfied. One more important condition, we require that the denominator should not have 0 on the real line. That is, it should have no real 0's. That root should not be a real.

Certainly, x square plus 1 has the roots as plus or minus i. So, we would have the double roots plus i and minus i. So, this is not having any real roots.

So, all the conditions are being satisfied. Moreover, since, I have to evaluate this integral from 0 to infinity. Now, Residue theorem, we are applying in the integral from minus infinity to plus infinity. So, for that, we have to check one more condition of the even function. If x, we are replacing with minus x or this x square, will remain as same as x square. So, we are getting is, f of x is same is f of minus x for this function or if I treat this complete integrant, cos x upon x square plus 1, whole square.

So, if I change x to the minus x, I would get, because cos of minus x is same as cos of x. And this is the x square plus 1 whole square. That is, this whole function is an even function. That says is now, we could use the condition that, integral 0 to infinity of f x, d x, where f x is the even function would be half of 0 to infinity integral f x, d x. So, we are going to use that formula. Now, we want to change this integral to the contour integral.

So, we would change this integral to the contour integral of the function, e to the power i z upon z square plus 1 square, 1 whole square d z. Where, the contour c is a semicircle, bounded by the real line and that semicircle, we would have the radius r. And that r should be such a large number, such that all the poles of this function should be interior to that semicircle. Now, let see this integrant, e to the power i z upon z square plus 1 whole square. The denominator does has 0's at plus minus i and both the 0's are of the order 2.

So, this function will have only 2nd orders pole, 2nd order poles at z is equal to plus and minus i. So, we can apply the Cauchy principal value, because I would have only one pole of the order 2 in the upper half plane. We could use this integral minus infinity to plus infinity f x, e to the power i x, d x should be 2 pi i summation of residues of f z, e to the power i z. Now, the work remains is to find out of residue of this function at all the poles in the upper half plane.

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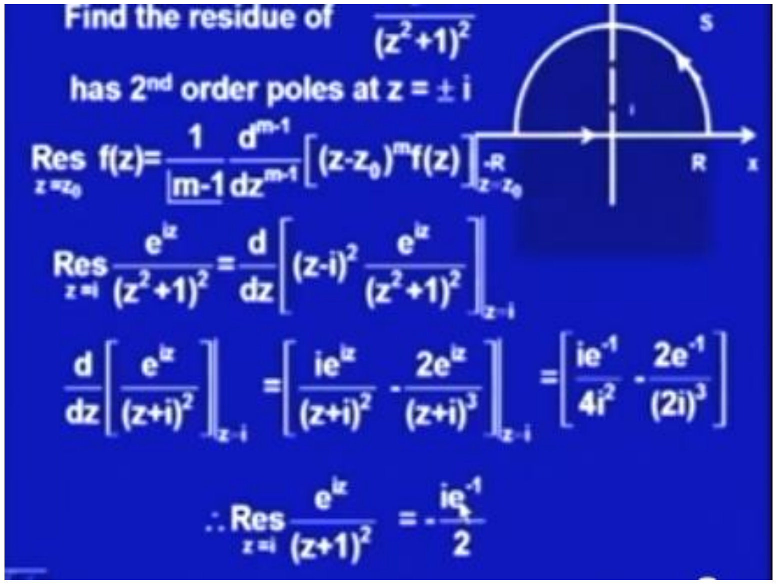

So, let us come to that part, we have to find out the residue of this function e to the power i z upon z square plus 1 whole square. Just now, we had seen that, this function has only two poles of the 2nd order. One is at plus i, another is a minus i. So, on the upper half plane, we would have only one pole, this is of order 2. So, whenever we have to find out the residue of a pole of order m, greater than 1, we would use this formula.

So, we are going to use the formula of finding out the residue, residue at z is equal to z naught of a function f z is nothing but, 1 upon factorial m minus 1. The m minus 1, m th derivative with respective z of z minus z naught to the power m into f z evaluated at z is equal to z naught. Here, my m is 2, so m minus 1 would be 1. So, we will use this formula. And now, we will calculate the residue at z is equal to i.

This is a 2nd order pole. So, I would have m is equal to 2. That says is, I would get, z 0 at z is equal to i of the function, e to the power i z upon z square plus 1 whole square. Since, m is 2, so I would get 1 upon factorial 1; that is 1. And m minus 1 is 1, so I would get the first derivate. That is d by d z of z minus i whole square into the function, e to the power i z upon z square plus 1 whole square.

Now, this should be evaluated at z is equal to i. This z square plus 1, we could write as z plus i into z minus i. So, I would get in the denominator here, z minus i whole square into z plus i whole square. So, z minus i whole square and z minus i whole square, will cancel it out. What we would left is, d by, that is derivative of with respective z of the function e i z upon z plus i whole square. It should be evaluated at z is equal to i.

So, find out it is derivative, not very difficult function. First the derivative of the first function, it is i e to the power i z upon z plus i whole square plus the first function in to the derivative of the 2nd functions. So, e to power i z is as such, the derivative of 1 upon z plus i whole square is minus 2 upon z plus i whole cube. This we have to evaluate at z is equal to i. So, if i keep z is equal to i here, I would get e to the power i into i. That is, e to power i square, so it is e to the power minus 1.

And here, I would give me i plus i, that is 2 i. So, what we have got i e to the power minus 1 minus 2 e to the power minus 1 upon 2 i whole square. That is 4 i square and here 2 i whole cube, i square is minus 1. So, we would get it minus i e to the power minus 1 upon 4. And here, 1, 2 is going to get it cancel it out, i cube is minus i. So, we will get plus e to the power minus 1 upon 4 i, that again would give me minus i upon i times e to the power minus 1 upon 4. So, what we have got, actually the residue at of the function e to the power i z upon z square plus 1 whole square at z is equal to i as minus i e to the power minus 1 upon 2.

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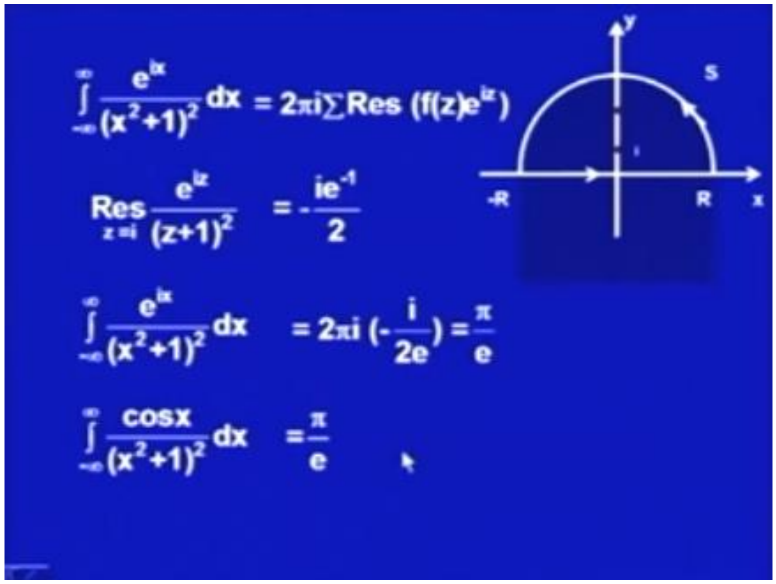

Now, use this in our formulation. So, we have got the integral, we would get from minus infinity to plus infinity, e to the power ix upon x square plus 1 whole square d x as 2 pi i summation of residues of f z e to the power i z. Where, the summation should run over all the poles in the upper half plane. In the upper half plane, in this particular example, we were having only single pole at i.

So, we would just have, here the summation will replace, only the single value that is, residue at z is equal to i of e to the power i z upon z square plus 1 whole square. And that we had just now calculated as minus i, e to the power minus 1 upon 2. So, my integral minus infinity to plus infinity e to the power i x upon x square plus 1 whole square d x has come up as 2 pi i into minus i upon 2 e, i into i is i square, that is minus 1. And 2 is 2 is getting cancel it out, we are getting pi upon e.

So, again the evaluation of this integral, we have got that value is, only the real number. And this integral is actually consisting of two integrals, which is one integral, we could say as the real part of this whole integral. And another integral, we could say as the imaginary part of this integral. The real part of this integral is cos x upon x square plus 1 whole square, d x integral from minus infinity to plus infinity.

And the imaginary part of this integral is, integral from minus infinity to plus infinity, sin x upon x square plus 1 whole square, integrated with respect to x. So, compare both the parts, we would get the real part is equal to the real part, gives me integral minus infinity to plus infinity cos x upon x square plus 1, whole square into d x is of equal to pi upon e. So, we were able to prove this formula.

So, we have proved the formula minus infinity to plus infinity integral cos x, x upon x square plus 1 whole square d x is equal to pi upon e. Actually, you can generalize this formula also, you can use here cos a x and here x square plus b whole square, where b is any number. Then, you have to find it out that, your b, whatever you are choosing, because your pole would be at plus minus b i.

So, again b you have to take positive, without loss of generality. Because, if b is negative, you have to use this positive or negative parts in the different manner, that is all. And so you would be getting here in the formula, that whatever you would be achieve in the right hand side, the difference of positive or negative only. So, you can generalize this formula using here as the cos a x and here as x square plus b whole square.

And then do it by yourself. It would be a nice exercise, where you could find it out, that what the new formula you are achieving. And for that, of course, a also you would require to be positive while. We have done is, that is, whenever we are taking this function, e to the power i s z, we have taken s to be positive and real. So, we do take a positive and b, we would take positive. So, that, we are getting is that, our pole b i is coming in the upper half plane. So, for that b, we would take as the positive.

So, wherever it is coming, we take the b positive, so that it would be coming in the upper half plane. So, today we had learn that, if our improper integrals are of the form, where the integrant we could write as f x into cos s x or f x into sin s x. And if our integral is, from minus infinity to plus infinity, that is, it is again an improper integral. Then, we can use this Residue theory again. Of course, we required the method, which we had learned today, that effects should satisfy certain conditions.

Those conditions are, that my f x should be a rational function or more precisely, my effects should be the ratio of two polynomials, p x and q x, where p x and q x should not have any factors common. That is, it is in the most reduced form. And the 0’s of q x should not be real. So, that, we do have poles only. So, that, we do not have any isolated singularity for the corresponding complex function e to the power i s z, f z on the real line.

And then we could use this Residue theory and we could find out the integrals. Again, one more condition, which was required for us to show that the integral around the semicircle is going to the 0, was that. We require the difference between the degrees of the numerator and denominator must be at least 2. That is the degree of the denominator must be higher than the degree of the numerator and that difference has to be at least 2.

So, we had learnt one more method to evaluate the improper integral of the form, integral minus infinity to plus infinity cos s x, sin d x or integral minus infinity to plus infinity sin s x, d x. We do know that, these kinds of integrals are occurring, when we are finding out the Fourier integrals or the Fourier series. You do remember that, many times, we do require to evaluate, those integrals from 0 to infinity.

But, if you do remember the Fourier series, if you have done that, the integrals 0 to infinity, were occurring, when the function was, f is either even or odd kind of things. That is, we do have f function, if the function f is even. That is, we do have f of minus x is equal to f of x, only then we would be having the integral 0 to infinity f x cos s x, d x we can evaluate. When, we are not having this even or we are not satisfying the conditions and the degree or anything any other conditions, which we had imposed on a f x, they are failing. Then, this method would not apply.

Next, we would learn, when this condition of degree is failing about a special kind of functions. That functions, we are having is rational functions, but they are of the form this p x upon q x. But, your condition and the degree, that is the difference between the degree of the numerator and denominator remains at least 2. That is not being satisfied. If it is fails, one special kind of integrals, we could solve that, we would see in the next class. So, for today, it is all for this, one more method of evaluation of improper integrals.

Thank you.