Mathematics-II
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Lecture - 10
Evaluation of Real Improper Integrals-1

Welcome to the lecture series on complex analysis for under graduate students. Today's lecture is, on Evaluation of Real Improper Integrals. We are learning about the use of residue theory in evaluation of real integrals. Today, we will learn about the evaluation of improper integrals, using the residue theory. What is improper integral?

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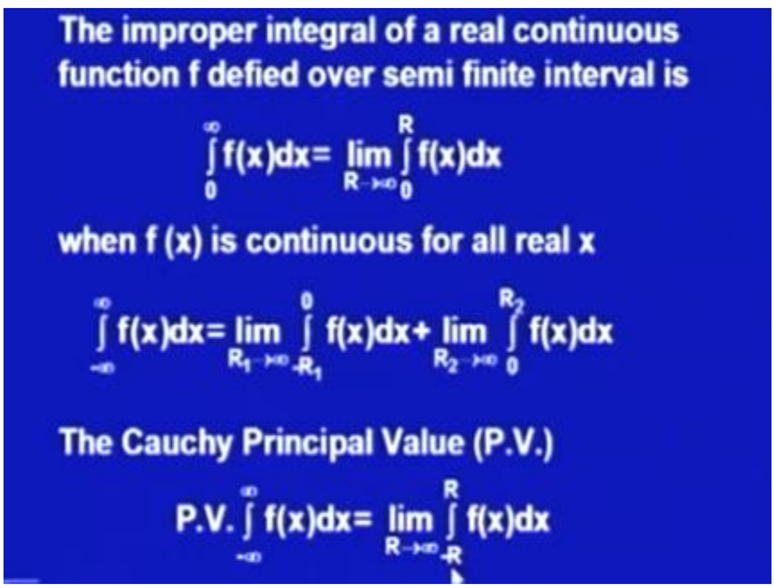

The improper integral of a real continuous function f, define over a semi finite interval. Say for example, for positive real line x greater than or equal to 0 is being defined as integral from 0 to infinite of f x, d x as limit R approaching to infinity. The integral of f x, d x on the range 0 to R. So, if this limit is existing for as R is approaching to infinity. We say, that this improper integral 0 to infinity f x, d x; this is converging. And the value of this integral would be actually the value of the limit on the right side.

Now, if the function is continuous for all x on the whole real line. Then, also, we define the improper integral of the type minus infinity to plus infinity of f x, d x as the sum of 2 limit is. First as limit, R 1 is approaching to infinity, minus R 1 to 0, f x, d x plus limit as the R 2 is approaching to infinity, integral from 0 to R 2 of f x, d x. Here, both R 1 and R 2; we are taking as positive and both are approaching to infinity.

Now, we say, that this improper integral would be convergent. If both these limit is, that is the first 1 and second 1, both these limits are existing. And the value of this improper integral would be sum of these limit is. There is, one more way in which this improper integral is being defined. And that value assign to this improper integral is known the Cauchy principle value.

What is it? It is being defined as the Cauchy principle value is being define of this integral. Minus infinity into plus infinity f x, d x is as limit R is approaching to infinity, integral from minus R to plus R, f x, d x. And we say if the converge, the integral minus infinity to plus infinity f x, d x. There is, if this convergent integral, that is, if both these limits are existing.

Then, the Cauchy principle value would be equal to the value of the integral. Or, in other words, we are saying is, if this integral is convergent. Then, the value of this integral is nothing but, the Cauchy principle value. What it is says is, rather than calculating these 2 limit is, we would calculate only this single limit, where the integral, we are calculating from minus R to plus R, f x, d x.

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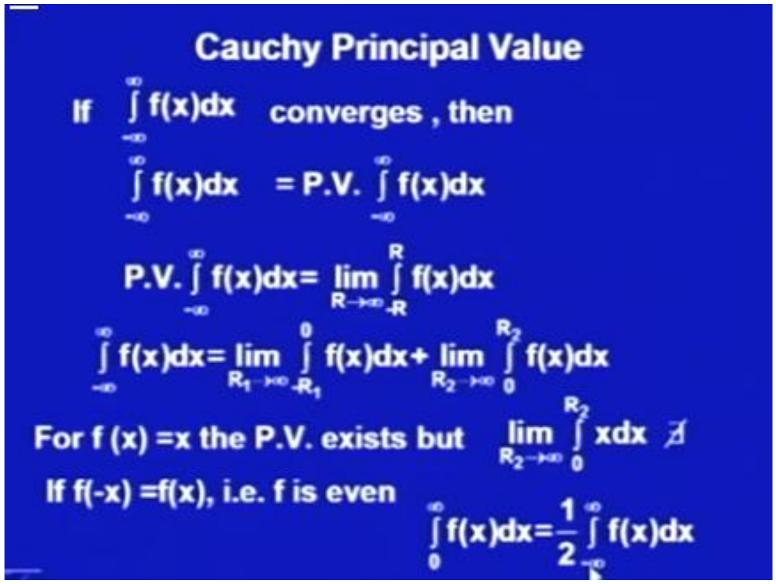

Now, this what, we are saying is that, if integral minus infinity to f x, d x converges. Then, integral minus infinity to plus infinity f x, d x would be same as principle value of the integral minus infinity to plus infinity f x, d x. What is the principle value? Principle value, we have just now seen is that, this is limit as R approaching to infinity minus R to plus R f x, d x.

What it says is that, let us just take very simple example here of the integral, we have defined as limit R 1 approaching to infinity minus R 1 to 0. And limit as R 2 is approaching to infinity, 0 to R 2 of the function f x. This is, what we have defined this integral. We are saying is, if this integral converges, that means, both these limits are existing.

Then, we say that, the sum these limit is would be actually this particular limit. Limit R is approaching to infinity, minus R to plus R, f x, d x. Now, let us say for one example, if I take simply the function x. That is f x is equal to x. Then, from here minus R to R, x d x, we do know, this is the x is an odd function. So, it would be 0 for all R. That is the limit would be 0. So, the Cauchy principle value for the function x would be 0.

But, that is not the value of the integral, because, this integral is not existing. Why, if I take the integral from say, 0 to R 2 of f x, d x. That would be your R 2 square by 2. As R

2 is approaching to infinity, it would approach to infinite. Similarly, this one is also going to be approach to infinite. So, what we have, just now said is, that is f x is equal to x. Then, Cauchy principle value exist. But, none of the integral, neither this limit is existing nor this limit is existing.

So, existence of, we said is that, if the result, which I had coated here is, if this integral is convergent. Then, the integral would be equal to the Cauchy principle value. So, what we are saying is that, from this way, it is true. But, existence of Cauchy principle value does not say, this integral is convergent, because just now, we have got this counter example. Now, can we say in with certain condition, that will, it happen that, if Cauchy principle value is existing. We could say is that this integral is convergent.

And that is value of, of course, if it is convergent. Then, the value of this integral would be Cauchy principle value. So, we just want to find out, under what condition, we could say that, existence of Cauchy principle value. That is existence of this limit, says is existence of this integral. That is existence of this limit is, says, existence of both of these limit is.

Let see, if my function is even. That is, f of minus x is f of x, for all x on the whole real line. Then, this integral, actually then the Cauchy principle value, if the Cauchy principle value is existence, it would convey that, this integral is existing. Why? You see, this integral is convergent. If f of minus x is, f of x for all x. Let see this and if, one of this right hand side limit is existing.

Say, suppose this limit is existing for R 2 approaching to infinity 0 to R 2, f x, d x. If this limit is existing, if I change x to minus x, I would reach to this interval, this integral. So, if this limit is existing, this limit would also exist. That says is, only and from here, what we are getting is, we are saying is Cauchy principle value exist. If this is even and Cauchy principle value exist. Cauchy principle value is limit, R is approaching to infinity minus R to R, f x, d x.

Now, f x is even. We do know that, if function is even, from the properties of definite integral, where this R is a finite value. That interval, integral from minus a to plus a, f x, d x, for f being an even function, we could write it as 2 times integral 0 to a f x, d x. That says is, I could write it as, 2 times integral 0 to R, f x, d x. And then, as R is approaching to infinity, what we are saying is, integral as R is approaching to infinity. Limit as R is approaching to infinity, integral from 0 to R, f x, d x. That is similar to the second integral is existing.

So, Cauchy principle value is existing for an even function. Simply says is, that in this definition of improper integral from minus infinity plus infinity. This second limit, this is existing. Since, this is existing and the function is even, we could say that, this limit is also existing. Since, I just change x 2 minus x and this integral would be converted to this integral. And this limit would be this whole limit of this integral would be converted the limit of this integral.

So, if it is existing, this would also be existing. What it says is, if Cauchy principle value for an even function is existing, both these limit is would be exiting. That says is, that my improper integral minus infinity to plus infinity f x, d x would be convergent. And the value of this integral should be the sum of these 2 limit is. What is these 2 limit is? This limit, if I to see, this is nothing but, half of this limit. R is approaching to infinity are that is, we could says is, half of the Cauchy principle value and that is same over here.

So, sum of them is again the Cauchy principle value. So, now, what we have got, that the result, which we have got. The one result, we have got. When, we are defining the improper integral on the whole real line from minus infinity to plus infinity. Then, if the integral of a function is convergent, it would be equal to the Cauchy principle value. Where, the Cauchy principle value, we are defining in this manner.

But, Cauchy principle, existence of Cauchy principle value does not guaranty the convergence of this improper integral. The other thing, which the result, which we have got is, if my function f is and even function for all x. Then, the existence of Cauchy principle value, says convergence of the improper integral. So, we have got the result. That if, f is even and Cauchy principle value exist.

Then, the improper integral minus infinity to plus infinity f x, d x is convergent. And the integral 0 to infinity f x, d x is 1 by 2 or half of integral minus infinity plus infinity f x, d x. So, you see is, that is, we are using this kind of result many times in our calculation of integrals. But, that is, it is not true for minus for this improper integral. That result was, there are property of the definite integral, says is, here the limit is has to be finite. That is minus a 2 plus a 1 only, then it is 2 times 0 to a, f x, d x.

Now, what we are saying is, if the improper integral is convergent. Then, also, then for the improper integral also the same thing would hold true. What it says is, now we would go for evaluation of this improper integral, either this kind of improper integral or this kind of improper integral. If the function is even. Let us just try when the function is a rational function. That is, it is ratio of two functions and those two functions are also a special kinds.

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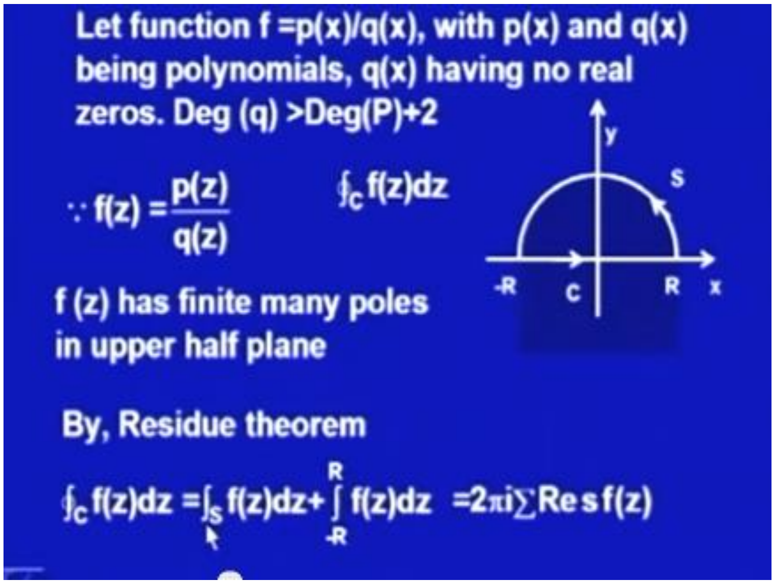

So, let us see, we want to go for the method of evaluating improper integral of rational functions, using our Residue theory. So, let the function f x is of the form p x upon q x with p x and q x are both polynomials. And q x has no real 0. That is what are the 0's of this they are all imaginary. And moreover, we are having one more condition. That p x and q x are such that, no factors are common also. That is both the function, both the polynomials p x and q x, have uncommon factors.

And moreover, one more assumption on the degree; that is degree of the denominator, q is at least 2 degrees higher than the degree of the numerator. So, with these conditions, let us just try to evaluate the improper integral minus infinity to plus infinity f x, d x. What we will do, to apply the Residue theory, the improper integral minus infinity to plus infinity f x, d x. We would change to contour integral f z, d z, where, f z, we would take the corresponding complex function.

As the function f x is here, p x upon q x will replace this x by z, that with the complex variable. And we would take this complex integration. That is the contour integration. With contour, actually being the semicircle in upper half plane. The contour, we would take this c is that the semicircle in the upper half plane plus the line x is equal to 0. That is the real access.

So, we will take this semicircle for as the contour c close contour c. And then, we would apply our Residue theory in this one. You see, if it is, we are taking our here. Since, we had assumed that, my f from p x upon q x. And q x is having no real 0. That says is, I am replacing my or making the corresponding complex function f x as f z. Then, f z would be of the form p z upon q z. Since, q x is not having any real 0. That says is q z would have the 0 all the complex 1's.

Complex 0's means is, they would be isolated similarities of the function f z would be either on the, that none of the similarity would be on the real line. They would be either in the upper 1 or either over on the upper plane or on the lower plane. Now, since, we had assumed, that this f z and q z both have polynomials. So, whatever the 0's of this polynomials, they would be all this singularities would be isolated singularities.

And if any function is having isolated singularities. Then, certainly they would have the finite number of singularities only. So, if I am taking on the upper half plane, they would be sum finite number of isolated singularities only. So, let us take this semicircle of the radius R. R, we are taking larger enough. Such that, it is covering all the isolated singularities of this function f z.

That is, all the isolated singularities of this function f z in the upper half plane. They are interior to this semicircle, which is bounded, which we are bounding by the line x is equal to 0 or the x axis. So, what we would get is, now if I transform my integral minus infinity to plus infinity f x, d x. So, now if I see is, this f z, which is having isolated singularities, inside this close contour. Then, we do know that, for finding out the integral along this contour, we can use the Residue theorem.

If it is having more than one isolated singularities are 1's isolated singularity. We could give the integral along this whole contour as sum of 2 pi i time, sum of the residues at all those isolated singularities. So, what we have doing it, since, it is having 0's, isolated singularities would be 0's of the denominator. So, we do know with our theorem, which say, we relates with the poles with the 0's.

That, if denominator q z is having 0's, then any function of the form p z upon q z, were at all those, because we have assumed here, that both this p x and q x are such that. The none of the factors are common. That says is, none of the 0's of q would be the 0's of p. So, at none of the 0's of q, p z naught will not be 0. That says is the conditions for thus, that theorem of relation between poles and 0 is satisfy. Hence, f z would have finite many poles in the upper half plane only.

And by the Residue theorem, we could say, that the integral along this close contour C, which is consisting of this half circle. And the line x is equal to 0 between minus R to plus R. So, this, we are saying is, now I could write it as dividing into 2 contour integrals. One is, only half circle, which I am writing it by S in the contour clock manner, S, f z, d z and plus the sum minus R to plus R of f z, d z. We see, when we are talking about the integral along this real line minus R to plus R, f z, d z.

We do know that, this is simply x. So, we do say that, this integral would be actually minus R plus R, f x, d x. By Residue theorem, it should be same as 2 pi i summation of all the residues, summation of residues of f z, where summation is over all poles in the interior to this upper half plane. Now, what we have to get this integral from minus infinity to plus infinity.

The only thing we have to show that, limit as R, it is approaching to infinity. First thing is that, this integral will approach to improper integral minus infinity to plus infinity f x, d x. Since, here what we are having is, we are talking about the residues, residues of certain finite number and the sum of these things. So, the right hand side would be the finite one. So, when I am putting the limit over here, as R is approaching to infinity on both this side. This side is being a constant not depending on R. It would be same as this one.

Here, this would change to the improper integral minus infinity to plus infinity f x, d x. And now, what we have to remaining to show is that, this improper integral is, this minus integral along this contours S of f z, q z. With the S is nothing but, the half circle or the semicircle. What we have to show, that as R approaching to infinity rather the plan which we are going to do is, that is, we will show that as R is approaching to infinity.

The integral of this function f z along this contour or this semicircle s is approaching to 0. So, let us see, that is, now what we are, then what we would get is that, we could get to improper integral using the Residue theory.

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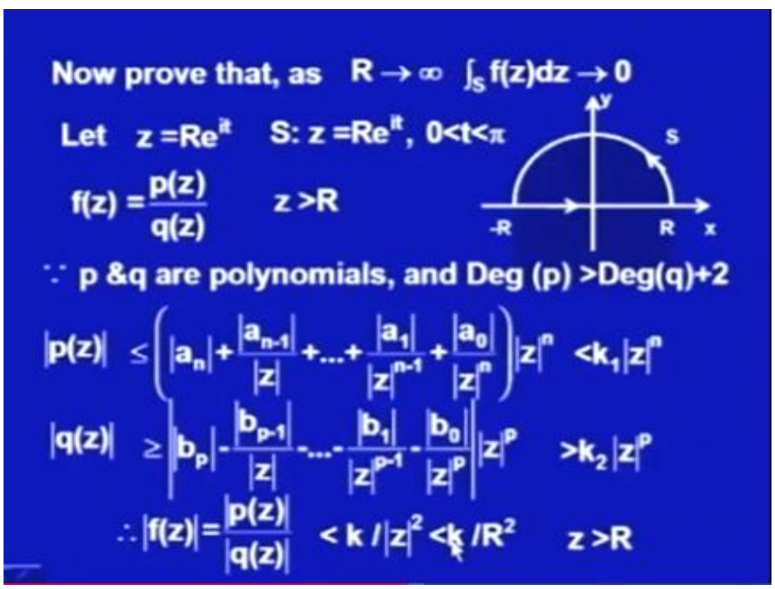

So, let us see, what now we want? We want to show that as R is approaching to infinity. The integral along the contour S, f z, d z is approaching to 0. S is my semicircle on the upper half plane. Let us write, see that is, what this is, now, since we are talking about this semi circle. Why we want that is, because we have taken it, we want minus infinity to plus infinity. So, let say this semicircle, we could parametric representation would be R e to the power i t, were t is ranging from 0 to pi.

So, if I take z in the, so first thing is, we are making thus z in the polar coordinates. And on this semi circle, we are having is, that z is R times e to the power i t, what 0 to t is ranging from 0 to pi, R is fixed one. So, on this semi circle on this contour, my z is R e to the power i t with R as fixed one. Now, we want to show. So, let us write, what is that, my this contour integral. Moreover, my f z is the of the form p z upon q z.

Let us just try to see is, that this integral of f z along this contour S is approaching to the, what the method, we would gives is, we would use our ML inequality. So, we find it out, that this ML inequality, which we had shown that, if my f z could be bounded by some constant. Then, it is that m, then the absolute value of the integral is always is smaller than are equal to m time length of contour. You see that is, how important that is, we are using it in so many places to find out, the solution of many problems.

So, here again what we would find out, first will find out, what is the bound for this f z. That bound we do not have to find out inside this one. Because, what we have to assume that, as R is approaching to infinity. So, we would say that, when R is large. That is, when R is greater than this fixed value R. So, for all those z for which z at greater than R. That is, absolute value of z is greater than R.

Then, we want that is, we have to find out it is bound. We had assume that, p and q both are polynomial. So, let us see, for z greater than R. Since, p and q are polynomials and degree of q is at least greater than 2 degrees higher than the degree of p. So, let us say, p z is suppose of the degree n. And let say, this the polynomial of the form, a n z plus a n minus 1 z to the power n minus 1 and so on.

So, we could write, p z as we have done in one of earlier lectures. We could take the z to the power in as common. And we could write our p z as a n plus a n minus 1 upon z plus a n minus 2 upon z square plus a naught upon z to the power n into z to the power n. So, if I take the modulus and both the size. We do know that, mod of p z should be less than are equal to, we do know that, my mod of x plus y is always less than are equal to mod of x plus mod of y.

So, we had applied this one. So, each term, we had written here separately. Also, we do know mod of x in to y is always equal to mod of x into mod of pi. Using that, one we could again separated. So, finally what we are getting, that mod of p z would be less than or equal to mod of a n plus mod a n minus 1 upon mod z plus so on. Mod a 1 upon mod of z to the power n minus 1 plus a naught upon mod of z to the power n in to mod of z to the power n.

Now, so from here, what we have got, if z is greater than R. If z is greater than R, we do get 1 upon mod z would be less than 1 upon R. Since, this R, I am taking this fixed value, some fixed value R naught. So, 1 upon mod z would be less than 1 upon R, 1 upon mod z to the power n minus 1 would be less than 1 upon R to be power n minus 1 and so on. So, what we are getting is, each of these terms are bounded by some constant.

And what that constant is, that R is very large enough. So, 1 by R would be e, whatever your 1 by R, that would always be larger than 1 upon R to the power n. So, we are getting is, that we are being bounded with the functions which are lesser, lesser and lesser. And so, what we could say is, that all these terms, which are inside this bracket, they are your bounded by some constant say k 1. So, we could write this is bounded by k 1 times mod z to the power n.

Now, let us see the denominator q z. Now, q z again we are saying is that, this is the polynomial of degree. Let say p, which is at least 2 degrees higher than degree of p z. So, that says is, that I should have at least of degree n plus 2. So, let us see, then, let us call it polynomial as say b p plus 1, b p minus 1, z to the power p minus 1 and so on, b naught, z to the power p, b naught.

So, again we would be taking in the similar manner, this z to the power p common. What we would be writing is, rather than using the inequality, that mod of x plus y is less than or equal to mod of x plus mod of pi. We would use the inequality, that mod of x plus y is greater than or equal to mod of mod of x minus mod of y. So, we would write it as, b p, this q z would be greater than equal to mod of b p minus mod of b p minus 1 upon mod z minus, minus. Mod of b 1 upon mod z to the power p minus 1 minus b naught z upon mod z to the power p in to the z to the power p.

You do find out, that is, this kind of things we have a d1 in 1 of earlier lectures. You can just refer it to over there and find out that, details over here. I am explaining here, that is, first I have breaked it, b p minus all those terms. And then, we had use the x plus y less than or equal to x plus mod of x plus mod of pi. That is why; we are getting like this one. So, here again, when z is greater than R 1 upon z would be less than 1 upon R. That says is, minus 1 upon mod z would be actually greater than minus 1 upon R.

So, each of these terms are bounded above by some constant. This is bounded by minus mod of b p minus 1 upon R. This is bounded by minus mod of b 1 upon R to the power p minus 1 and so on. And this term is some constant and what we are having R is some finite number and large number. So, all these terms, which are these bounds, they would certainly be some small number. So, we are subtracting and we are taking, whatever b small or large, we are taking it is absolute value.

So, that say is, this complete absolute value should be bounded above by some constant k 2. So, we do write it as, this is greater than sum k 2 times mod of z to the power p. Now, what we have got? We have got that, when z is greater than R, my numerator p z is bounded above by k 1 times mod z to the power n. And the denominator is bounded below by k 2 times mod z to the power p.

That says is, that I could write f z is less than k 1 times mod z to the power n upon k 2. Because, 1 upon q z is bounded above. So, 1 upon q z would be less than or equal to k 2 times, 1 upon k 2 times z to the power mod z to the power p. So, what I am writing, k 1 upon k 2, let us say some other constant k upon mod z square. Because, at least, the difference between these p and n is 2. So, it has to be greater than or equal 2. So, at least to be 2.

Now, when z is greater than R, we do know 1 upon mod z to the square would be less than 1 upon R square. So, we say is, this less than or equal to k upon R square. Now, we do know as R is approaching to infinity, this k upon R square would approach to 0. Once, we have find out, that is mod of f z. so, we had find out actually, that when z is greater than R, function f z is bounded by a constant. That is, we have find out and upper bound for this function f z for the reason, when R is higher.

That is, for all these z, above this one or what you says is, that is, we have taken this semicircle and we are trying to take then limit as, R is approaching to infinity. So, whenever this, we are saying is, some fixed R, we are approaching towards, we are going to the higher values.

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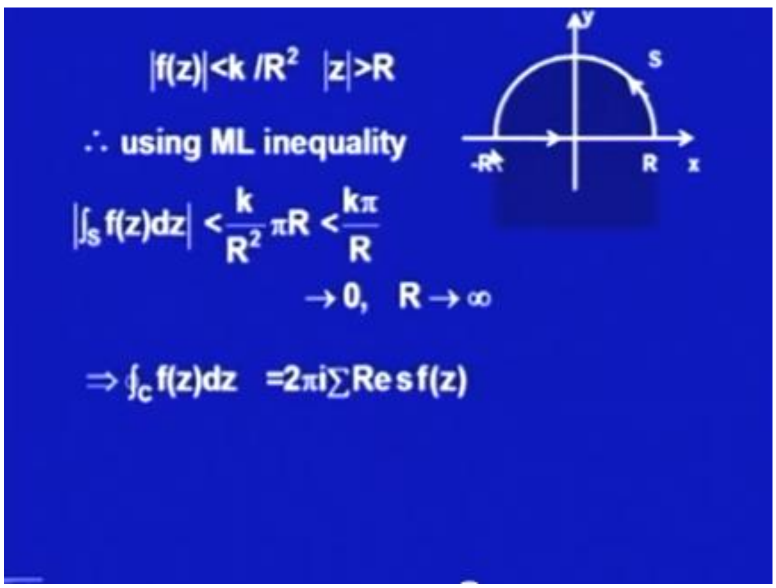

Now, what it is say is, it says is that, so what, now we do, have we had find out, that f z, absolute value of f z is bounded by k upon R square, when mod z is greater than R. And so, use that, ML inequality for the integral S, integral of f z, d z on the contour S. Using the ML inequality, this is absolute value is, when it is on this arc, we are saying is that, sum R naught, we have taken and it is greater than. So, it is k upon R square.

And into the length of contour L, what is, this is the semicircle. So, the length of this contour, that is the your parameter of this semi circle is, we do know pi times R. So, it is into pi R. So, what we have got, that this integral is less than or equal to k times pi upon R. Now, whatever this absolute value, this is always going to be positive. On the right hand side, we have got that this absolute value is less than a constant upon R.

What is this R, so if I take this S is, such that, the limit, R is approaching to infinity. That says is, I am increasing my radius of this semicircle. So, as z is becoming greater than R. That is, we are reaching out and out. My semicircle is increasing, what I would be getting is that, since, this is bounded by this positive quantity. And this positive quantity would go on decreasing. That says is, the value of this integral must be 0.

So, what it says is, it could approach to 0 as R approaches to infinity. This says is, now that this integral S, f z, d z is 0. So, now, what we have got, we have got that along this close contour, the integral of f z, d z is. By the Residue theorem, it is 2 pi i times summation of the all residues of f z, interior to this, same is this contour C. And this, what is this integral over here, that, we had break in to two parts. One is that, contour integral on the semi circle S. And another as the contour integral on the line, are the real access minus R to plus R.

Now, we are taking this as the limit is approaching. So, what we have got that, actually integral minus infinity to plus infinity f x, d x would be equal to 2 pi times, summation of residue of f z. Where, the all the residues, we would take at all the isolated similarities inside this semicircle. So, what the method, we had actually find it out by this result, which we had shown. Actually, we are saying is, that if we have to evaluate an improper integral minus infinity to plus infinity of f x, d x; where my f x is a rational function.

That it is, it is of the form p x upon q x, where both p and q are polynomials. And the polynomials are such that, we do not have, we are have to make those polynomials should not have any factors common. That is, we are making them as, we could says, residues polynomials, which cannot be reduce more further. Moreover, we had assumed one more thing, that the degree of the denominator polynomial must be higher than the degree of the numerator polynomial. And the difference has to be at least 2.

Then, we are saying is that, this improper integral, we could evaluate using the Residue theory. Where, what we are taking is that, all the simple poles, we would take a circle such that, all the simple poles are lying inside on the semicircle. And we would talk about only the positive. That is, on the y positive, that is in the upper half plane only. And they we are calculating that residues, all the residues in the upper half plane only.

And we would, all the poles in the upper half plane and will calculate the residue of the function f z in the upper half plane, like that we could do. So, let see is, that is, how we would going to apply with the help of one example.

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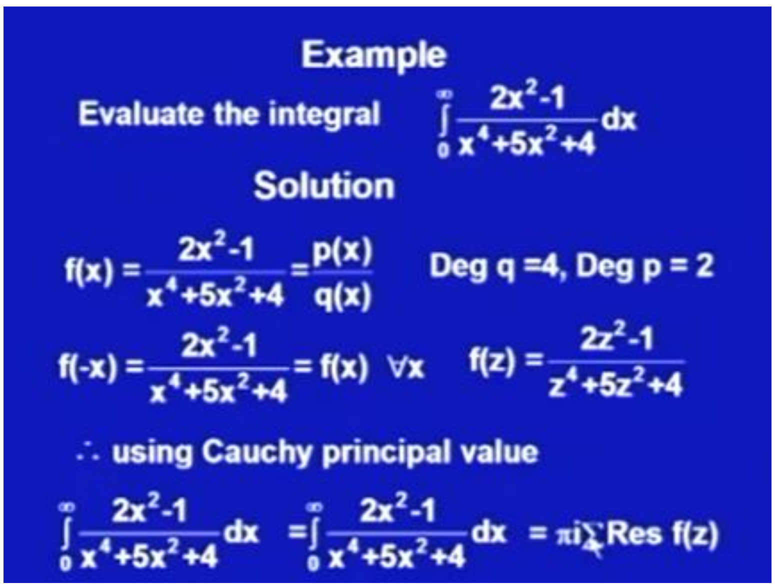

Say, evaluate this integral 0 to infinity, 2 x square minus 1 upon x to the power 4 plus 5 x square plus 4 d x. See, we are having my function f x as 2 x square minus 1 upon x to the power 4 plus 5 x square plus 4. This is of the form p x upon q x, p x. This is 2 x square minus 1. This is a polynomial of degree 2. The q x is of the power x to the power 4 plus 5 x square plus 4. This is also a polynomial of degree 4.

So, the difference between the degree of p and q is degree of q is 4, degree of p is 2. So, the difference is 2. So, we want, that is, at least 2. So, it is 2, it is satisfying the conditions of the method, just now, I had explained. Another condition was that, the denominator q should not have any real 0. From here, we see is that, all the signs are positive. So, certainly, it is not going to have and if I just try to find out the 0's of this, x to the power 4 plus 5 x square plus 4. We will not have any real 0's over here. All the roots are image, the complex numbers over here.

Moreover, f of minus x, since x, it is x square. So, it will remain as x square. So, it is 2 x square minus 1 x to the power 4, for minus x, it will remain as x to the power 4, 5 plus 5 times x square, it will remain as the plus 4. So, it is same as f of minus x is equal to f of x, for all x, that is the function is even also.

So, that says is, we could apply the Cauchy principle value. And the second formula, which says is that, for using this Cauchy principle value. We could say that, integral 0 to infinity 2 x square minus 1 upon x to the power 4 plus 5 x square plus 4 d x, should be half of minus infinity to plus infinity integral of this function. Same function 2 x square minus 1 upon x to power 4 plus 5 x square plus 4, should be equal to the 2 pi i.

So, half would cancel 2 pi i times summation of the residues of f z, where f z is the corresponding complex function. So, what will be my f z, corresponding complex function as 2 z square minus 1 upon z to the power 4 plus 5 z square plus 4. So, what now is remaining to evaluate this integral is, that we have to find out the residue of the function f z So, let us find out the residue of the function f z.

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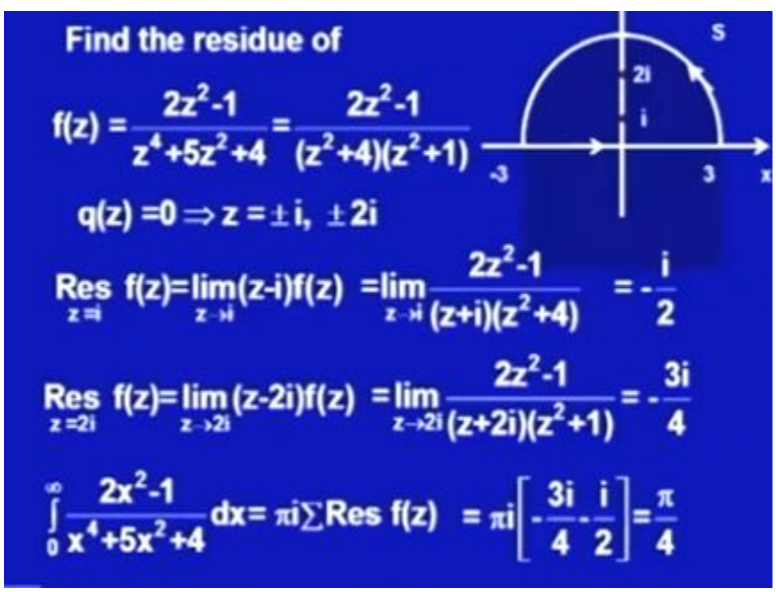

Let see, the function is my 2 z square minus 1 upon z to the power 4 plus 5 z square plus 4. This we could write as, z to the power 4 plus 4 z square plus or that is the z square plus 4 in to z square plus 1. So, the factors, we have got from here, if we do see, the 0's of the denominator would be z square plus 4 is 0. That z is equal to plus minus 2 i and z square plus 1 is equal to 0. That is z is equal to plus minus i.

That says is, this function has four simple poles. So, q z is 0 at plus minus i and plus minus 2 i. Hence, all these 0's are simple 0's. So, f z would have all the simple poles at z is equal to plus minus i and at z is equal to plus minus 2 i. Now, we want, that is our method says is, we want the residues at all those poles, which are only in upper half plane.

So, here if we are having is plus i would be there, minus would be somewhere here, plus 2 i would be here and minus 2 i would be somewhere here. So, we want that in the upper half plane, we do have only two poles. One is at i, another is at 2 i. So, we see here in this particular example, we are having both the poles are on the imaginary axis. So, let us choose my circle of the radius 3. So, it is covering both this poles.

What it says is now, I would have to find out this integral only inside this 1. And then, we would go ahead, because we want the integral from minus infinity to plus infinity. So, we just directly use the formula. This is just, I am showing is that, however, if there are, for the function is of the form p z upon q z. Then, it could have a finite number of isolated similarities.

And since, it is of the from the q and p are the polynomials. It will have all the poles and all those poles, we could find out R. Such that, all the poles would line interior to the semicircle, bounded by the real line. So, let us find out, for using that formula, we have to find the residue at plus i and plus 2 i. So, first residue of i, we would use the simple first formula of finding out the residue, which says is residue at z is equal to z naught of any function f z. Is limit as, z is approaching to z naught z minus z naught the f z.

So, here my point is isolate similarities at i. So, residue at z is equal to i of f z would be limit as z is approaching to z minus i time f z. This we could write as f z 2 z square minus

1 upon z square plus 4 in to z square plus 1. This z square plus 1, we could write as z plus i and z minus i. So, when, we have multiplying it by z minus i, we would get as 2 z square minus 1 upon z plus i into z square plus 4, limit as z is approaching to i.

That says is, evaluate this function at z is equal to i. If I am evaluating at z is equal to i, I would get z square is equal to minus 1. So, it would be minus 2 and minus 1. That is minus 3, my numerator would be minus 3. See, that denominator at z is equal to i, this term would be 2 i. At z is equal to i, z square is minus 1. So, minus 1 plus 4 would be 3. So, we are getting 3 in to 2; that is 6 i. While, in the numerator, we are getting is minus 3.

So, we are getting is, minus 1 upon 2 i, that we could write as i by, minus i by 2. So, we have got it minus i by 2. Now, residue at z is equal to 2 i, limit as, again using the same formula, residue at z is equal to 2 i of f z. We could find out as the limit, z is approaching to 2 i, z minus 2 i into f z. So, now we would write this function, the term z square plus 4, we would break as z square plus 2 i into z square, z plus 2 i into z minus 2 i.

So, what we would get the function, remaining function, when we are multiplying it by z minus 2 i, as 2 z square minus 1 upon z plus 2 i into z square plus 1. And find out the limit as z is approaching to 2 i. So, that says is, just we have to evaluate this function at z is equal 2 i. When, z is equal to 2 i, z square would be minus 4. So, we would get here, minus 8 and minus 1. That is minus 9, z is equal 2 i, this term would be 4 i. Here, z square is minus 4 minus 4 plus 3 is minus 3. So, minus 9 minus 3 would get 3 upon 4 i.

So, what we are getting minus 3 i upon 4. Now, let us find out use that Cauchy principle value formula. So, the integral 0 to infinity 2 x square minus upon x to the power 4 plus 5 x square plus 4 d x. Would be given as, the formula say, pi i times the sum of all residues, which are in the upper half plane. We have got the two residues in the upper half; two poles in the upper half plane both are simple poles. We had calculated the residue over there is i by 2 and 3 i by 4.

So, we do get it pi i times minus 3 i by 4 minus i by 2. That is pi by 4. So, this is plus i by 2. So, it is pi by 4. So, we have got evaluated this integral, using the Residue theorem. Let us do one more example.

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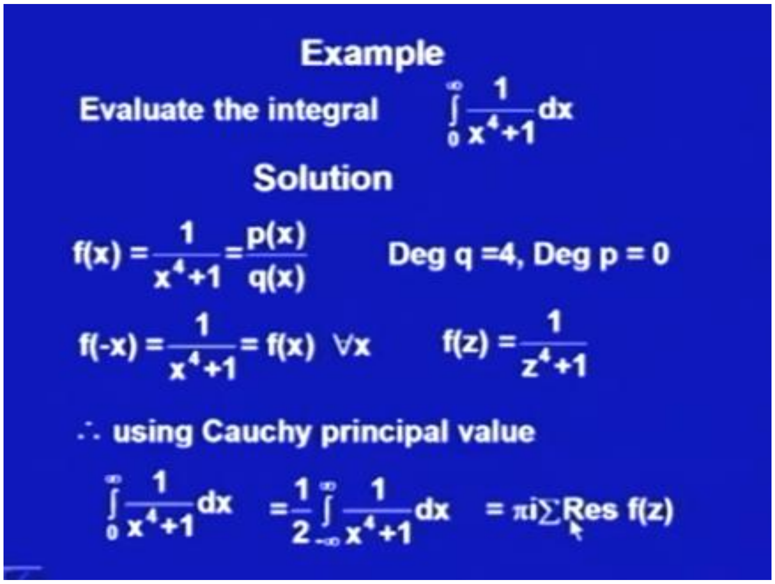

Evaluate the integral 0 to infinity, 1 upon x to the power 4 plus 1. Again, we are going to use the same method. Why? You see the function 1 upon x to the power 4 plus 1, we could write it again as of the p x and q x, where this numerator is constant. Constant, can also be treated as polynomial of degree 0. And denominator is certainly a polynomial x to the power 4 plus 1. Since, it is degree is 0, it is degree is 4. So, we do is, satisfy the condition that the denominator should have degree at least 2 higher than the numerator.

Moreover, there is no term, which common to each other. Moreover, the x to the power 4 plus 1; does not have any real 0's. And we do have that f of minus x is same as f x for all x. So, again the formula for Cauchy principle value, this Cauchy principle value formula would be applicable. And method, just now, we have described, that is applicable. For that, we require a corresponding complex function. So, the corresponding complex function would be 1 upon z to the power 4 plus 1.

Hence, using this Cauchy principle value formula, we do know that, integral 0 to infinity 1 upon x to the power 4 plus 1 d x. Would be half of integral minus infinity to plus infinity 1 upon x to the power 4 plus 1, which is, half of 2 half times 2 pi i into sum of residues of f z. So, that is pi i times sum of residues of f z. So, now the work, which is remaining to do is, to find out the residues of f z.

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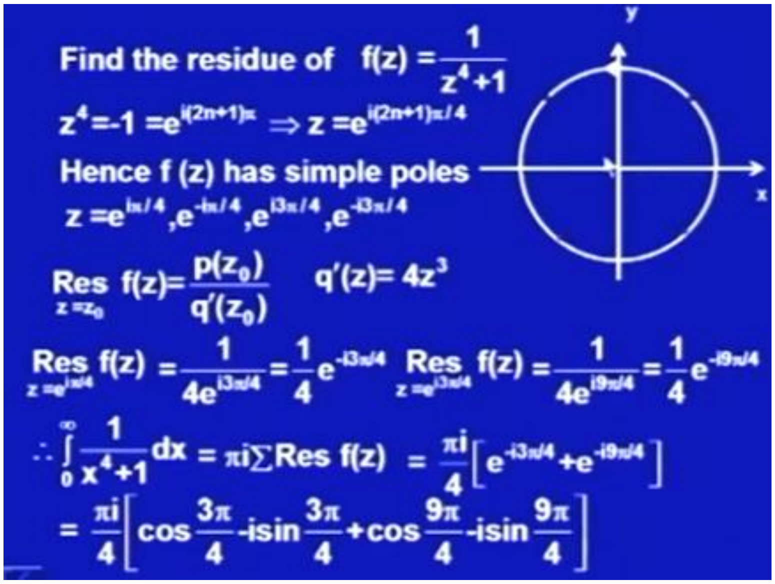

My function is, 1 upon z to the power 4 plus 1. It is singularities would be, where the denominator polynomial z to the power 4 plus 1 has 0's. Now, z to the power 4 plus 1 would be 0. When, z to the power 4 is minus 1, minus 1, we do know, we could write it as e to the power 2 n plus 1 pi i. So, here I am writing it is e to the power i 2 and plus 1 pi. From here, we do get, for all in 0, 1, 2, 1, so on.

So, from here, what we are getting, that z would be e to the power i times 2 n plus 1 pi by 4. Now, let us find out, that is, what are these 0's are, they would be the poles of function f z. When, n is equal to 0. We will have e to the power i pi by 4. Moreover, it is actually, it is plus minus, we could have minus sign also. That is also, even then, it is true. So, with n is equal to 0. We would have the two roots or rather we could say, two poles at i pi by 4 and at minus i pi by 4. At n is equal 1, we would get it 3.

So, what we would get e to the power i 3 pi by 4 and with the minus sign e to the power minus i 3 pi 4. Then, if i take n is equal to 2, I would get pi by 4, pi by 4 means is, I am getting is 2 pi plus pi by 4. And if we do talk about, because here, if I am writing in this manner, that is 1 as e to the power i 2 n plus 1 pi. It is says is, I am using the polar coordinates.

What it is says is, that if I am increasing my n values, I would be actually unless until I am changing my R, I would be on the same path, I am making it. Since, it is of the polynomial of the degree 4, it will have only four 0's. So, those four 0's, we are having e to the power pi by 4, e to the power minus i pi by 4, e to the power 3 pi by 4, e to the power minus 3 pi by 4.

Thus, my f z would have and all is 0's are simple 0. So, my f z would have simple poles at these points. Let see, what are these points? If I make this as unit circle, then the points are, this is pi by 4, e to the power i pi by 4, means is a, this is the point pi by 4. This is the point your 3 pi by 4. This is the point minus pi by 4, e to the power minus i pi by 4. And this is the point, e to the power minus i 3 pi by 4.

So, from here, if we see, we are getting only two poles in the upper half plane. So, we are having for this function f z, we are having two poles in the upper half plane. So, now, when we are going to use this Cauchy principle value formula. We want the residue, only at the two points, two poles, e to the power i pi by 4, e to the power i 3 by 4. Again, we would use the formula, that residue at z is equal to z naught is, p z upon q dash z.

Because, here p z is a constant, so we do not require actually to evaluate p z at different values of z naught. The only thing is that q dash z naught, that we have to find out. Since, we are having a here q z as z to the power 4 plus 1. So, q dash z is 4 z cube. So, actually we had got a simple formula for residue of z is equal to z naught as 1 upon 4 z naught q. So, let us find it out at both the points. That is, e to the power i pi by 4 and e to the power i 3 pi by 4.

So, residue at e to the power i pi by 4 would be 1 upon 4 times e to the power i pi by 4 cube. That is, e to the power i 3 pi by 4. Or, in other words, we could write 1 upon 4 times e to the power minus i 3 pi by 4. Similarly, the residue at z is equal to e to power i 3 pi by 4. From here, it would be 1 upon 4 e to the power i 9 pi by 4 or 1 by 4 times e to the power minus i 9 pi by 4.

So, the residues at both the poles, we had calculated. So, using that our formula, that integral this is equal to pi i times. The sum of residues at although poles on the upper half plane, we have got the two poles. Both are simple poles, we had calculated the residues over there. Add it up pi i upon 4, e to the power minus i 3 pi by 4 plus i, e to the power minus i, 9 pi by 4.

Now, we are going to use our Euler's formula, for e to the power i theta is cos theta plus i sin theta. What it says is, this pi i by 4 is as such, this thing we could write, cos 3 pi by 4 minus i times sin 3 pi by 4 plus cos 9 pi by 4 minus i sin 9 pi by 4. Now, evaluate this, cos 3 pi by 4 means, this array, this angle is your 3 pi by 4. And we do know the cos 3 pi by 4 has value minus 1 by root 2.

While, sin 3 pi by 4 does has value 1 upon square root 2. So, here what we would get, minus 1 by root 2 minus i by root 2, cos 9 pi by 4, 9 pi by 4 means is, 2 pi plus, we are getting it again to pi by 4. That is this angle. So, we do know is that, cos 2 pi plus theta is, cos theta sin 2 pi plus theta is, sin theta. So, we are getting is, it is same as cos pi by 4 and sin pi by 4, cos pi by 4, we do know is, 1 by root 2, sin pi by 4 is also 1 by root 2.

So, here, what we would get here, we have got minus 1 by root 2 minus i by root 2. Here, will get plus 1 by root 2 minus i by root 2, 1 by root 2 plus sign and minus sign. That would cancel it out, I would get minus i by root 2 minus R by root 2. That says is, we would be getting is minus 2 i upon root 2. Just multiply it, i into i, that is, i square is minus 1, we would be getting it as pi upon 2 root 2. So, we have evaluated this integral 0 to infinity, 1 upon x to the power 4 plus 1 d x as pi upon 2 root 2.

We have today learn to evaluate the improper integrals of the form, 0 to infinity of f x, d x or integral of minus infinity to plus infinity of f x, d x. Actually, what we have learn is, that is, f x is a rational function. That is, if it is ratio of two polynomials, that says is and moreover, we have talked about the polynomials. Such that, the degree of denominator is, at least 2 degree is higher than the degree of the numerator.

In that case, we said integral of minus infinity to plus infinity, f x, d x can be given as the 2 pi i times. Sum of all residues, where the summation is at all the poles, which are in the upper half plane of the x y plane or the z plane. We had, we were able to evaluate the integrals of form, 0 to infinity f x, d x, only for the functions, which are even. That is f x is same as f of minus x for all x.

So, we had learned one method of evaluation of improper integral and we had seen that is, how this Residue theory, we can apply over here. So, the theory was little bit involved, were we have to show that, the limit as it is on the semicircle is approaching to 0 as we are making this. R to be large enough and from there, we are obtaining this one. But, the formula, which we have got, that was much simpler or it is.

Actually, making us much simpler to calculate the integrals, just by calculating the residues and then, applying this Residue theory, will learn them, some more improper integrals. So, this is all for today’s lecture.

Thank you.