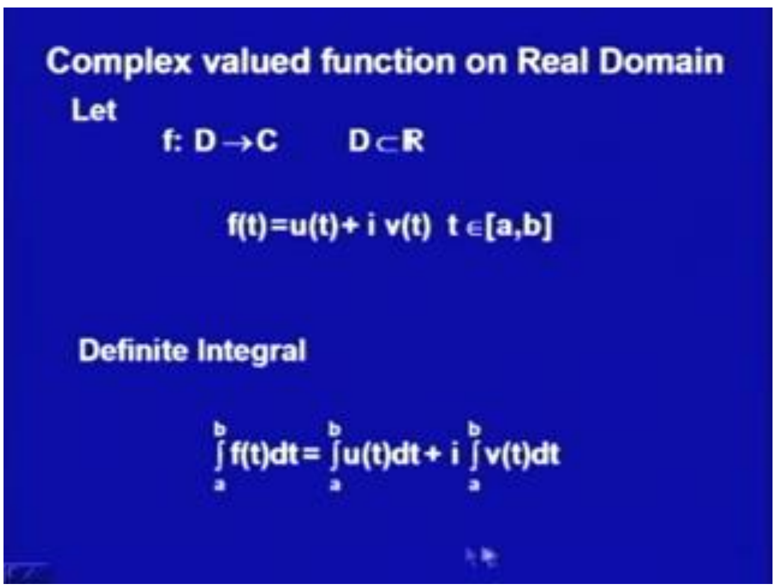
Mathematics-II
Prof. Tanuja Srivastava
Department of Mathematics
Indian Institute of Technology, Roorkee

Module - 1
Lecture - 1
Complex Integration

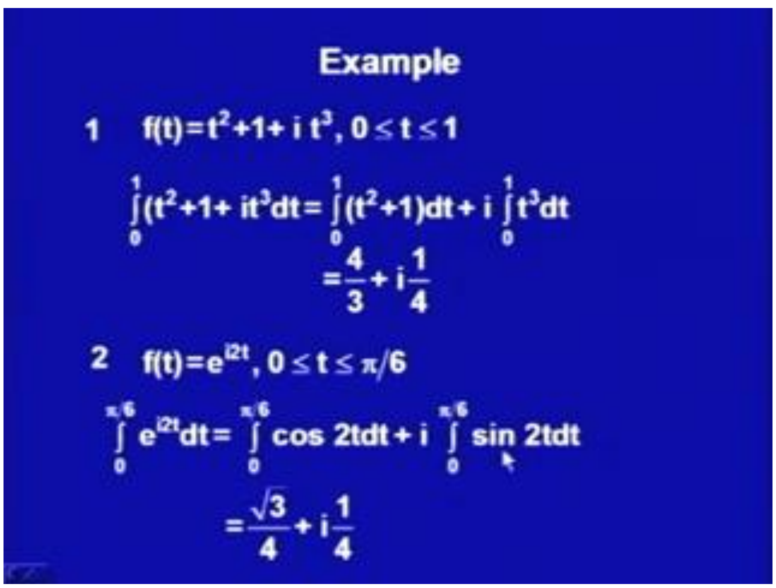
Welcome to the lecture series on complex analysis. Today's topic is complex integration. We had learnt integration in real analysis; we do know that integration by integration, we mean it is the inverse process of differentiation. We have learnt definite integral also in the real one. Let us just go with the complex variables and complex functions. What do we mean by complex integration?

(Refer Slide Time: 1:04)



So, let us first consider the complex valued function on real domain, that is, let us say, f is a function from D to C, where D is subset of R; more precisely, your D would be an interval from a to b, and C is the complex plane. Then, we can say, because it is a complex valued function that we can say, f of t is u of t plus i of times v of t, for t belonging to a to b, interval ab. What does it mean? It simply says that your function f t - it is complex value, that is, we could write it as in the two parts: one is the real part, another is the imaginary part. So, we do have two functions: one is u t - this is a real function, v t - this is also a real function. So we are and t is the real number, that is, it is on the real domain. Now, and this, if I try to define the complex integral just like as the differentiation in the for complex valued functions, we have define we can define it the integral of separate parts of the real parts and then combine it. So the definite integral, we could define as integral a to b f t dt, as the integral a to b u t dt plus i times a to b vt dt. That is, what we are having is, that is if my functions ut and vt are such that these are the real valued functions on the real domain, so, if these functions are such that these definite integrals are existing, that simply says is that these functions have to be continuous on this range t belonging to the interval a to b; then this integral does exist. Since this integral exists and this integral exists, this is the complex integral. So, what we are having is, this is nothing but we are defining same as in the terms of real definite integrals.

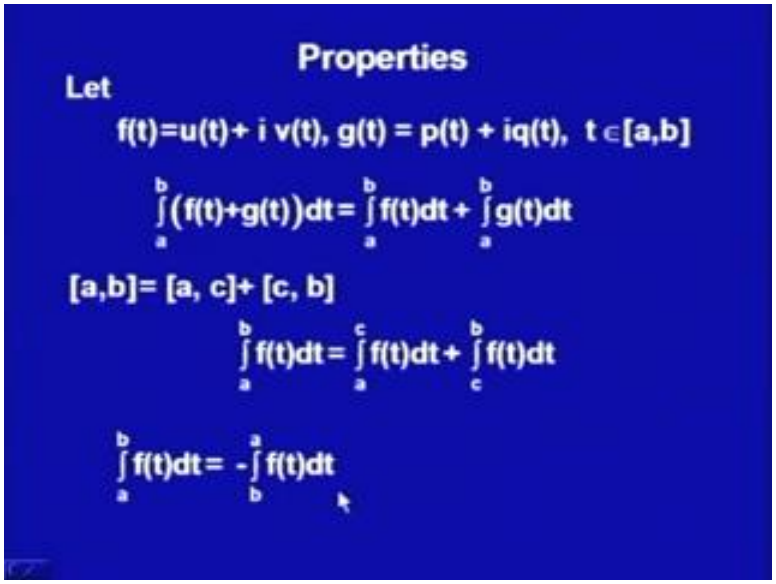
(Refer Slide Time: 03:17)



Let us just do one example over here. Suppose my ft is t square plus 1 plus i times t cube in the range 0 to 1; I have to integrate this on the whole range. So what we do have - this function does has two parts: one is the real part, another is the imaginary part; or rather we could say is, that is, we are having two functions: ut is t square plus 1 and vt is t cube, in the interval defined 0 to 1. So the interval 0 to 1 ft dt - ft is t square plus 1 plus i t cube dt. So now, we could write as the two integrals 0 to 1 t square plus 1 dt and i times 0 to 1 t cube dt. This integral we do know is nothing but t cube by 3, so integral root over here, 1 by 3 plus 1; that is integral is t, so it would be 1. Similarly t cube - its integral 0 to 1 is t to the power 4 divided by 4, and evaluated from 0 to 1 should give me 1 by 4. So what we are getting is, from here 1 by 3 plus 1- that is 4 by 3 - plus i times 1 by 4; that is, the integral of this complex valued function is again a complex number.

Similarly, let us take another example. f t is e to the power i2t in the interval 0 to pi by 6. t is real value -real numbers in the interval 0 to pi by 6; function is e to the power i2t. Now either I write it out as i2t is, I could write it as cos 2 t plus i sin 2 t; that is now, integrals, we are breaking into two parts - ut is my cos 2 t and vt is my sin 2 t. Both sin 2 t and cos 2 t - they are real functions. So I will integrate it over 0 to pi by 6. Integral of cos 2 t dt, we do know is would be 1 by 2 times sin 2 t and the integral of sin 2 t is minus 1 by 2 time cos 2 t - evaluated from 0 to pi by 6. Here what we have got: 1 by 2 times sin 2 t; at t is equal to 0 sin is 0, at t is equal to pi by 6, cos sin pi by 3 is root 3 by 2 and 1 by 2 was there. Similarly, here when we get half cosine 2 t - at t is equal to 0, we would get it 1 so it is 1 by 4, and at pi by 6, that is cosine pi by 3, that is 1 by 2. So we are getting this as simply 1 by 4. So we have got is, integral as a square root 3 by 4 plus i times 1 by 4 - again a complex number. Now what we have defined the complex definite integral here on the real domain: that is, my function is complex valued but it is defined on the real domain. So, since it is defined on the real domain, I can break it into two integrals on the real analysis or in the real domain on the real line, and so all the properties which we do know about definite integrals, they should satisfy over here precisely.

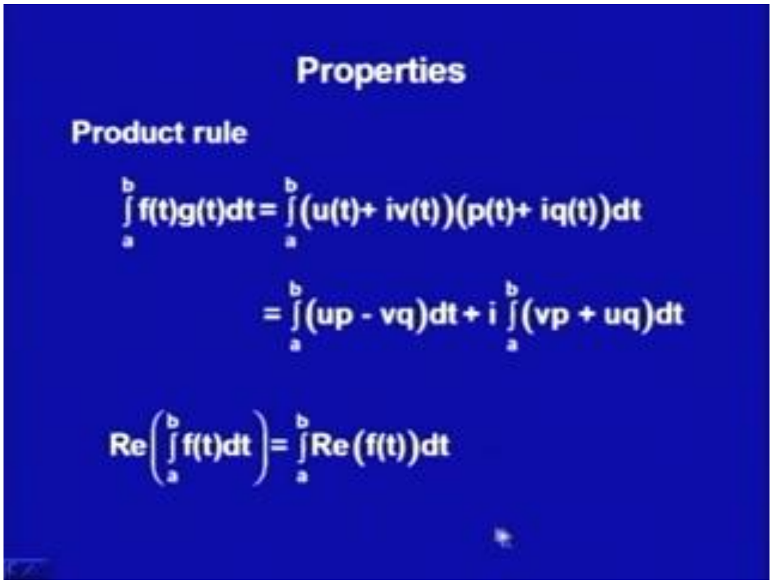
(Refer Slide Time: 06:59)



If f is breaking as ut plus i vt, and gt is as pt plus iqt, for any way interval t belonging to a to b, then we do know that the properties of sum of a function, that is, a to b ft plus gt dt should nothing be the sum of the integrals, that is integral a to b ft dt plus integral a to b gt dt. Moreover if this interval a to b - if I can divide the interval this interval at the point c, such that my interval ab can be written as a to c and then c to b, my integral can be divided into two, that is, a to b ft dt can be given as a to b a to c ft dt plus c to b ft dt. Now all these properties you can check: you have to put you have to do is, you have to prove you can prove it by putting your f as u plus iv and g as p plus iq. So if I am keeping here f as u plus iv and g as p plus iq, we do know the sum of complex numbers. What we are having is for a fixed t, ut and vt and u plus i ut plus ivt is a complex number. So we can use the all the properties or all the sum formulas for the complex numbers. So what we would get: we get here, u plus p plus i time v plus q; so what we would get is that is we would get 1 integral, 2 integrals : 1 integral for for u plus p and another integral for v plus q. u plus v ,u plus p - this integral- using the real integral, the real definite integrals property, we can write it as integral of ut plus integral of pt. Similarly, integral of v plus q

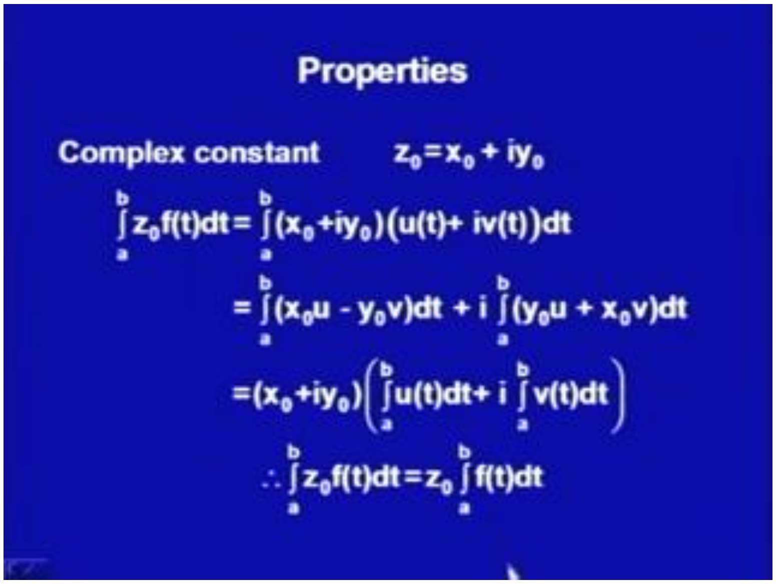
- we could write as integral of vt plus integral of qt - and that would give me, that is again, a summarize a again, collecting the term we could write them as this one. Similarly, here, if you do break, we have to write it u plus iv and then each one because u is real function, v is real function; again, using the properties of definite integrals and the real functions, we could just get this solution as well. So we can prove these properties very easily. Then, the says that is if the interval from a to b, I change it to the interval b to a, then, as there in the real one, we are having that, integral a to b ft dt should be same as minus of b to a ft dt; again we can show it using just u plus v.

(Refer Slide Time: 09:56)



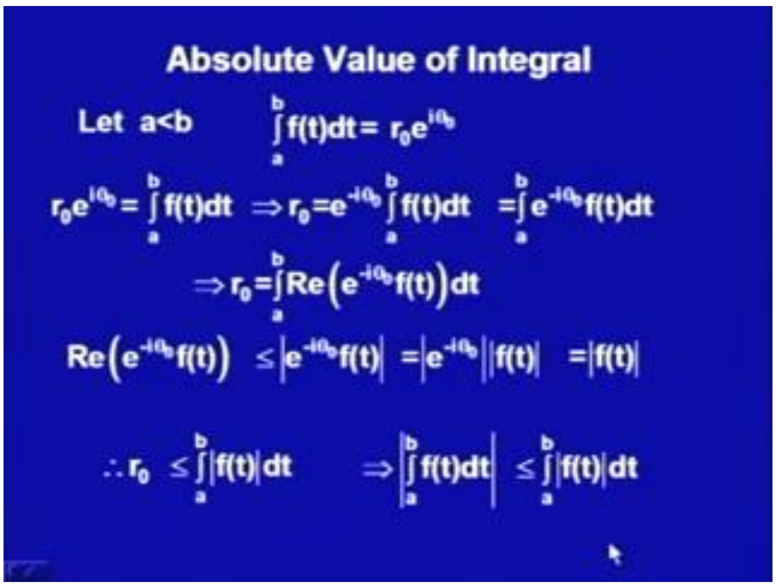
Moreover, the product rule; now if my f and g as I have defined earlier, that f is u plus iv and g is p plus i cube, then what should be this integral of ft gt dt, that is, the product one? Of course how could we find it out? We just substitute the value of these functions f and g - u plus iv and p plus iq. Now, use the product of the complex numbers, what we would get: u into p minus v into q. So, we would get u into p minus v into q and i times vp and uq - this is with the plus sign; so now what we are getting is, actually we are getting this multiplication as up minus vq plus i times vp plus uq. So, we can we are getting these two functions again - because u is real function, p is real function, b is real function, q is real function - so all these are real functions; similarly all these are real functions; so, I can write it as the integral, by the definition integral a to b up minus pq dt plus i times a to b vp plus uq dt. Now you see, as that is I am just leaving this part, that is, function evaluated at the point t, just for the writing space. This is actually the function of t: up, vq - all are the functions of t; that is why it is we integrated with the respect to t. This is standard type of writing. So what we are getting is, this is again a complex number, because this integral - this is integral of real functions - and here, we can use product rule; of again we can use it as integral a to b up dt minus integral a to b vq dt. And then for the product rule up dt, we do know that is how to evaluate the product on the real once. Moreover, we do have one more property which says the real part of a to b ft dt is same as integral a to b of real part of ft dt. How? If we do remember our definition of definite integral in the complex valued function, we have defined integral a to b ft dt as integral of ut with the respect to t, in the range a to b, plus i times integral of v with respect to t, in the range a to b. You could see here also, that if I replace this by u and this replace by v, then what is the real part of this one? We do get is the real part in the examples we have got, that is, all the time we have got the integrals as a complex number - and you can check in those all those examples or you can do some more examples as well to find it out that the real part of this one. So, by definitions, it is very clear from that, that is, real part of integral a to b ft dt would be nothing, but the integral of ut with respect to t from in the range a to b and what is ut? ut, if I see by the definition - f is u plus iv - so u is nothing but real part of ft; that is what we are seeing. So this is very simple just from the definition we are getting and you can verify with the examples.

(Refer Slide Time: 13:29)



Then we do have one more property which is coming from the complex; involving something is complex number. If I do have a complex constant z naught, as x naught plus i 1 y naught; that is, x naught and y naught - all are constant. Then, integral a to b z naught ft dt - we could write it at z naught as X plus x naught plus iy naught and f as u plus iv. Now, again use the multiplication of the complex numbers; at fx point t, we get x naught q minus y naught v plus i times y naught u plus x naught v. So, that is what we have got: integral a to b x naught u minus y naught v dt, plus i times integral a to b y naught u plus x naught v dt. Now, if I take this x naught - here, is the first term - x naught u dt integral a to b and from here, if I take the second term integral a to b x naught v dt - this is with i time. Now, from these two if I take, the these two integrals, if I take and I treat this imaginary number i - is square root of minus 1 - as a constant. Then what do I get? Using the properties of the real definite integrals, we do get it that it would be nothing, but x naught times integral a to b u dt plus i times integral a to b v dt. And similarly, if I take the second term over here minus y naught v and first term over here plus i times y naught u - now y naught, I am taking common; moreover I am taking i also outside. If I am taking i outside, that says is, here, I have to multiply with the plus i, because it is a minus 1. So what I would get it - i times y naught i am taking out - and I would get from here u plus i; so again, what we are getting is that, again, if you just simplify the terms, you do get it should be x naught plus iy naught times integral a to b u t dt plus i times a to b vt dt. That is, says this is x plus iy naught this is nothing, but the z naught; and this is nothing, but the integral of ft in the range a to b with respect to t. So what we have got: that if there is a complex constant in the integral, we can take that complex constant outside the integral sign; as we are taking the real constant outside the integral sign. In the similar manner, the complex constant can be taken outside the integral sign.

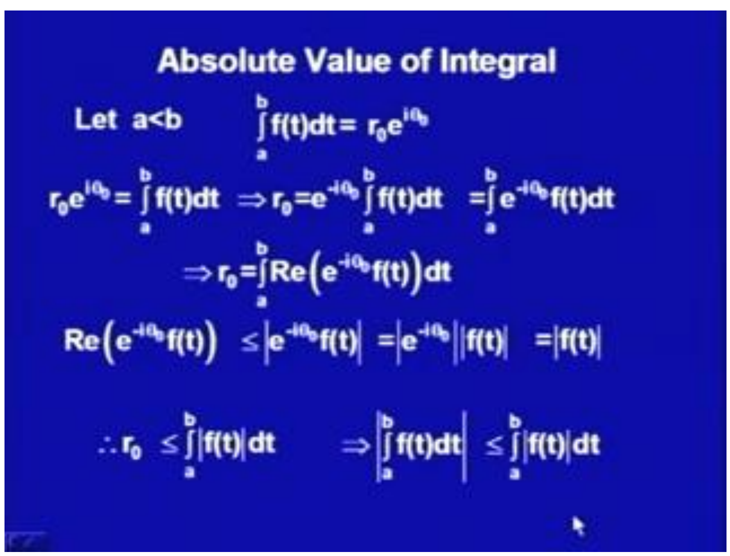
(Refer Slide Time: 16:36)



Then there is one basic property of the integral, that is, what we are calling is absolute value of the integral. Now, let us suppose my a is less than b - the interval. So, integral a to b ft dt - we do know this is a complex valued function. So, this integral comes as a complex number. Now, let us write that complex number in this form: r naught e to the power i theta naught, where r naught is your modulus and theta naught is the argument. So if I do write this one - now from here, if I see rewrite it - I am writing r naught e to the power i theta is not is equal to a to b ft dt. Now, from here, I want what is the modulus; because absolute value of the integral - we are saying is the absolute value, when we are talking about on the real valued functions are real numbers, then absolute value simply says is that without the sign , are that removing the sign of the value. But in the complex number, we do know that the absolute value means is modulus; so here, the absolute value simply says is the modulus. So, in that form, the modulus is r naught; we want to find out what is this r naught. e to the power i theta naught - this is constant - so I can rewrite it as e to the power minus i theta naught times a to b integral ft dt. Just now, we have done the property which says is that the constant can be taken outside the integral; so, this constant can be taken inside this integral. So, if I am taking like this one, I would get e to the this as a to b e to the power minus i theta naught ft dt. Now, my f is u plus ivt. What we are having from here: we have got that r naught is equal to a to b e to the power minus i theta naught ft dt. Now r naught - r naught is the modulus of the complex number r naught e to power i theta naught - that says, r naught is real. Now, if this is real, what we are getting that this integral is actually real number. So, now if I am taking both the sides - the real part of this one - we have done one more property which said is real part of integral ft dt is same as integral of real part of ft dt. So taking the real part on both the sides, what do I get? From the left hand side, r naught is real itself; so real part of r naught is r naught itself. From the right hand side, the real part of integral a to b - this function e to the power minus i theta naught ft dt- that, with the property just now we have done, we could write it as, it is integral a to b real part of e to the power minus i theta naught ft dt. That says is, now find out what is the real part of e to the power minus i theta naught ft - ft is u plus iv. So, if I am writing it out - my e to the power minus i theta naught - this should always be - whatever be the real part of any complex number - that should always be less than of or equal to the modulus of this one. Real part of this means is - if I just give you one reminder - that is, suppose my number is - complex number is - x plus iy. Then the real part of x plus iy - z is equal to x plus iy - would be x. And what will be the modulus of x plus iy? That would be square root of x square plus y square. We do know that whatever be this y, the minimum value could take a 0, because y square: we are having in the modulus (square root of x square plus y square). So that will always be smaller than square root of x square plus y square. So, we are just using this property; what it says is, see from here, now in the multiplication - we do know that modulus of z 1 z 2 is equal to modulus of z 1 into modulus of z 2. So, we are just using that property - it's same as modulus of e to the power minus i theta naught into modulus of ft.

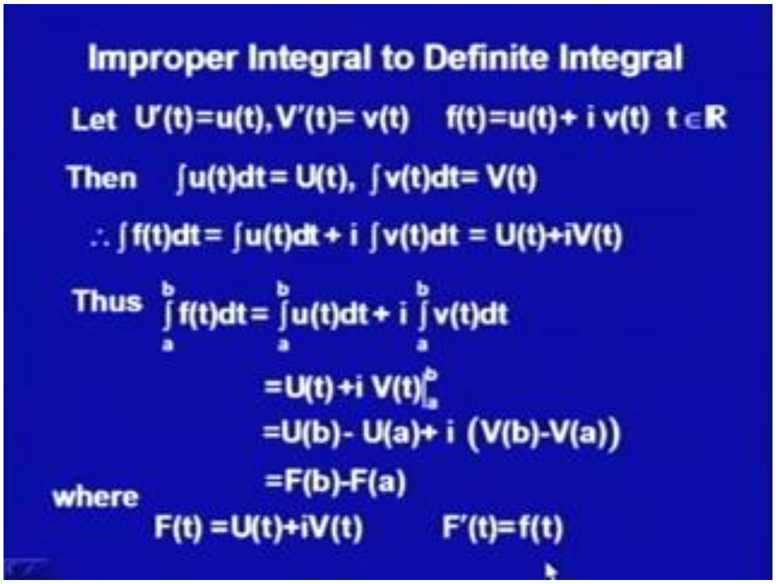
Now, if I just simplify, the that is, if we just substitute for this one as this so what I would get r naught? And e to the power the modulus of e to the power minus i theta naught - that would be 1, because i theta naught e to the power minus i naught is nothing but cos theta naught minus i sin theta naught and its modulus is always 1 - because the modulus part, the r naught part we have taken already out - so that way, also you can understand that it is 1. So what I have got now from here: that the real part this is less than or equal to modulus of ft. Now, substitute it over here that's says: r naught which is equal to this is less than or equal to integral a to b mod of ft dt; since this part is less than or equal to ft dt, so we do get this integral should also be the same way. Now what is r naught? r naught is nothing but modulus of this integral. So what have we got: that modulus of integral a to b ft dt is always less than or equal to the integral of the modulus of ft dt. This is what is called the absolute value of the integral or this is what you are having; this in equal to, this is the basic property of integral.

(Refer Slide Time::22:22)



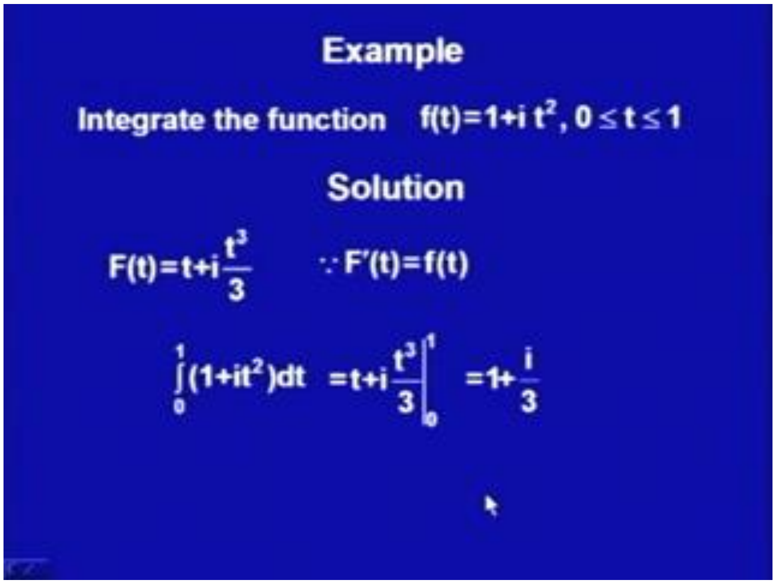
Now, till now, we have done this complex valued function defined on a real interval and the definite integral. In the similar manner, we can define the improper integrals as well, that is, where we are not having the completely; that is, D is subset of - it's in the not necessarily going to be an interval of just on a of finite length, it may be from a to infinity or it may be from minus infinity to plus infinity or we do not have any limits - that is, we are talking about improper integral. Again f: we are saying is that is the complex valued function; it is varying to the C. So we do have is: f as u plus iv - t for t belonging to R. So, we can define this improper integral, that, is just integral ft dt as integral ut dt plus i times integral vt dt. Can we define? The question comes here is: can we define? If we see this: we are saying is, this integral equal to this integral plus i times this integral. Now this would be existing or this would be there, if right hand side integrals are there. So what we say: if I do have a complex valued function such that, if I can break it into real and imaginary parts - and that you have already learnt that we can always break any complex valued function into the real part and imaginary part, where we do say is imaginary part is i times a real part; so if my these two functions - these two real functions u and v - are such that these two improper integrals does exist, then this integral will always exist. Now if we do remember from the real analysis, what are the conditions for existing existence of these improper integrals, that simply says is that these functions must be continuous. Moreover if we do remember your integral calculus from the real analysis part, you do know that if you do have antiderivatives of these functions u and v, then this integral does exist. So, the what does it simply says that if you do have any functions, say U and V such that the difference derivative of capital U is a small u and capital V is small v. Then, we can say that these integrals can be written as these integrals do exist and in that case this will also exist. what it says So, we are just giving here as definition, that this integral does exist provided the right hand side integrals exist - and that we do have come across that this my ut and vt have to be continuous. All the properties of real integrals hold true here also, that is, in the improper integrals whatever the properties in the real integral we do have - all those properties should hold true over here, even the basic property of your absolute value. That should also hold true over here - even if I do not have this a to b, rather than if I do have a to infinity, this property also holds true. Now, I am leaving this as an exercise for you to verify all these properties and find out that is how much significance you do have.

(Refer Slide Time: 26:05)



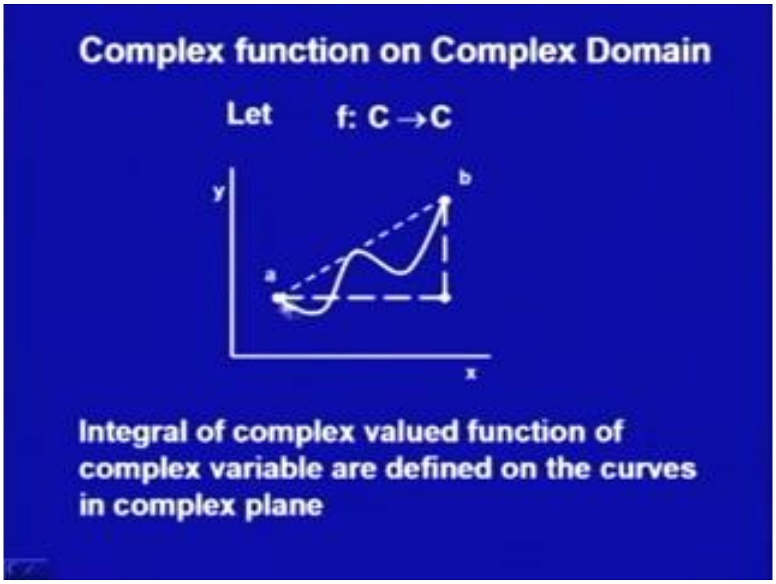
So what we have done till now: we have done that is from the improper integral, that is what is I said is that if antiderivative derivative or that is whatever you have learnt the first integral calculus. Now from improper integral, can we evaluate the definite integral? If you do remember, we have done in the real analysis this part, that is, how did we learn the definite integrals? First we had learnt that, definite integral; that, if I do have any function f whose antiderivative is F, then we simply said is integral of ft dt's capital Ft and if I could put the limits on t, that is from a to b or so, we said is that is we evaluate the function capital Ft from a to b that way once we have learnt the definite integrals, that is, using the limits. Now, let's see, that is, what we are saying is can be moved from here also, in the improper integral to definite integral. So let us say we do have for both the functions ut and vt the antiderivative, so that is, they are some functions - U and V; such that, the derivative of U is a u and derivative of V is v. What are these u and v? They are actually my the real and imaginary parts of the function ft. So where function ft is ut plus ivt for t belong into R Now Till now, you have learnt this complex differentiation; from there, you can learn that if ut plus ivt is this function, we do know that Ut plus i times Vt- if I do write that function's derivative would be ut plus i vt: that you have already learnt. Now, we are going to use this one. If this is happening, then we do know from the real analysis that, integral of ut dt should be capital Ut and integral of vt dt should be capital Vt. Hence, integral of ft dt, which is integral of ut dt plus i times vt dt - just now, we have defined - should be Ut plus Vt i times -that is, the imaginary part. Now So, if I try to use this definite integral a to b ft dt, I should write it as integral a to b ut dt plus i times integral a to b vt dt. Since integral ut dt is capital Ut, so I just write it as capital Ut and this integral of small vt - that is capital Vt - so we just get it capital Vt evaluated from a to b. What do we get? This Ub minus Ua plus i times Vb minus Va. Let me write it out as Ub plus i f i Vb - let me write it out as Fb- and remaining part is capital Fa.What we are saying is now Ft - the function i am defining - is capital Ut plus i Vt. We do know very well from the differentiation of the complex one, that, actually the derivative of Ft is ft. So what we are saying is if antiderivative of the real and imaginary part - that is, both are the real functions; so, if the antiderivatives for them exist, we can evaluate these integrals using these antiderivatives.

(Refer Slide Time: 29:41)



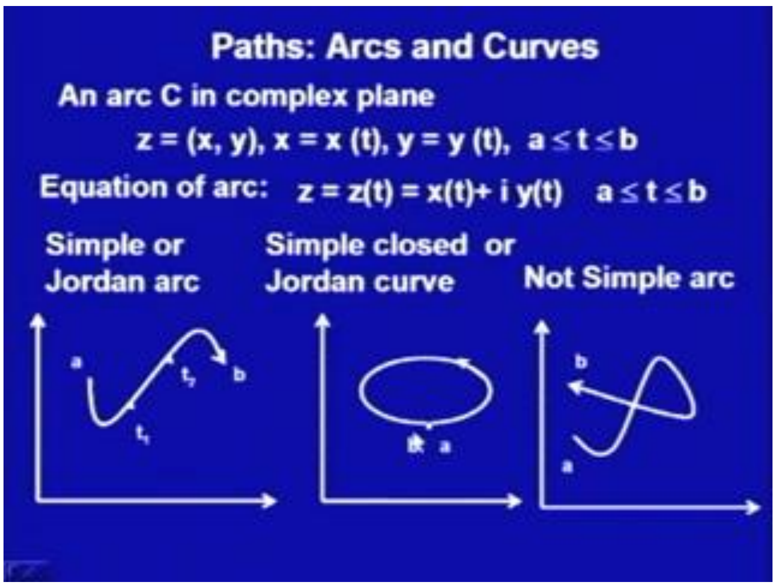
So, like let's it do one example here. Integrate the function 1 plus i t square from 0 to 1. That is, what we have to do is, we have to now we are going to find out this definition; we have to evaluate actually the integral 0 to 1, 1 plus i t square with respect to t. So what we are going to do here, is rather than finding it out separately this one, we just try to find out - is there any antiderivative for this function; does exist because complex differentiation we had already done. Now, let us use this function Ft - capital Ft - as t plus i times t cube by 3. Now, let's see what is the its derivative? Its derivative with respect to t would be 1 plus i times t square because the derivative of t cube is 3 t square. So 3 t square upon 3 - that is t square; so I have got this function - you do know that is how we found it, that is the formula is we know. So, F dash t is actually ft. So what we say is, that is, now evaluation of this integral 0 to 1, 1 plus it square dt, we are not going to do it separately as the real integrals. We will do it directly as the complex integral using this antiderivative; that is the antiderivative, we have find out this one. So, we say we write this antiderivative t plus i times t cube by 3 from 0 to 1; now evaluation from 0 to 1: that is at 1, what I would get - 1 plus i by 3 - and at 0 it is 0. So we have got it, 1 plus i by 3. So this is we are evaluating the definite integral using the antiderivative. So if we know the antiderivatives for certain functions that exist, we can use this also to evaluate the definite integrals.

(Refer Slide Time: 31:37)



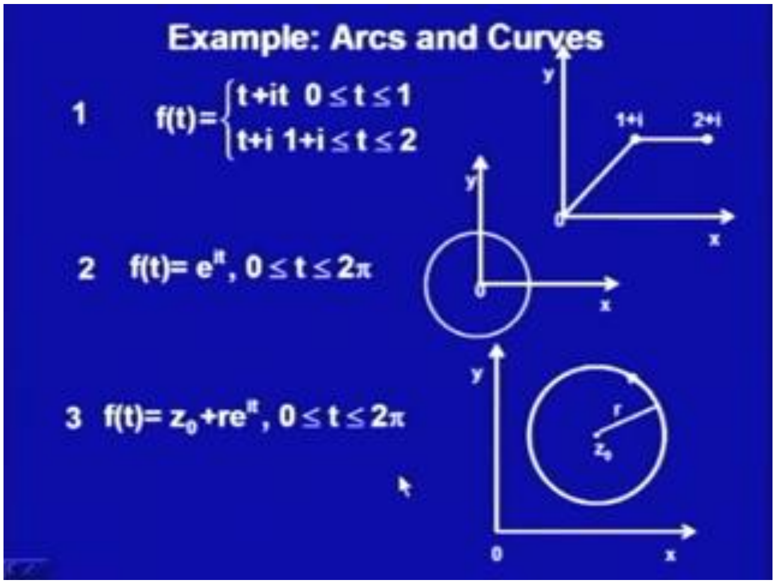
So, till now what we have done is, that is, simple extension of the real integrals or real analysis; we have taken these 2 real functions and induced one imaginary number or we could say just the i and say it is that this is the complex integral - of course there is not much from, it's a simple extension. Now, let us come to the complex function and complex domain. That says is Now, what we are having a function from C to C; so my domain is also C. Domain is complex plane, that is, now we have moved from the line to the plane. Now in this one, it is say is if and the line, if I have to move from one point to the other point, that is, from a to b, there is only single direction - because that was if you see is this is the line - real line. So, if I have to move from a to b, there is only one way in which I have to move from a to b - or I could move to from b to a, that is but that is moving from b to a - from moving from a to b, I have only one way. Now here, if I do have two points a and b - two complex numbers a and b; say a and b. Then, how can I move from a to b? Either, I can move with the straight line that is, from a to b; or I can say is from first I am moving from a to this part - that is, this is the x and this is the y; this x plus iy, that is both the z we are having - and then, from this point to this point. So what we are having is: now these two points, we do have or we can move from a to b in this manner; let us say from here I have gone just like this one, or I can have any path like this one. That says is Moving from a to b in a complex plane, where a and b are two complex numbers, I do have many ways in which I can reach from one point to the other point, because now I am talking about the plane. So finding out in definite integral of a complex valued function on the complex domain - that's says I am interested in finding out the integral a to b fz dz. So, whether I should integrate from here to here, whether I should integrate like this one, or whether I should integrate in this manner? And are these paths which in which I am moving from a to b - do they affect the evaluation of the integral or is it just immaterial; whether I am moving from this path or this path, I should just reach in any manner from a to b. That says is We have to first explore that, or rather first we will try to find out the complex integral of complex function - the complex domain - as according to the path and then, we will check the properties whether it is affecting; the path is affecting the value of the integral for any function are not. So, what we are saying is, the integral of complex valued function of complex variable are defined on the curves in complex plane. So, first thing is, what are these curves? How are these curves going to signify how we are going to evaluate? So, of course, what we would like to do is, we would like to take these curves in the parametric form. That is, we just take one parameter, say for example, if I have to reach from a to b in this manner - that is, this dig dash line. What would I do first? I will move from a to this point – say, say this point c in the direction of x axis - and then we move in the direction of y axis. So, what we would say is, that is, we are reaching - we are first we can make it - because we are, at to one time, we are have only one direction, or if I am moving here, we are taking the direction of this straight line from a to b, and if I am moving in this one, I am taking direction of this path - this is also one line, whatever this curve. So, we want that this parameter for this curve - whether these maybe these straight lines or maybe any curved ones, that we are calling is the parametric one. So, we want to define the curves in the parametric form.

(Refer Slide Time: 36:30)



Let us just call these ways to reach to from a to b as the paths. So if they are paths, are arcs are curves. So an arc C in the complex plane; z is the set of points xy, such that we want to define it in parametric form, that x is x of t and y is y of t for t, lying between a and b on the any real number. The equation of arcs: so, in this manner, what we do get z as z of t and what we do get as z? as Since it is we are seeing is the points xy; that means, each z point is is actually x plus iy in the form of complex plane or, that is how are we are writing it out. So, that say is for a given value of t my zt is should be xt plus i yt. That says is Now, the equation of arc, we are getting in single parameter t, ranging from a to b, and these functions x and y, we are getting according to that arc; that is, whatever this set x and y, that, we want in the form of single parameter t. Let us just talk about the some classification of these arcs; we call an arc to be simple, if it does not cross itself at any point, except, at possibly at the initial point. So let us first see - this is a simple curve, it is thus starting from a, reach into the b; it is at none of the point, time it is not crossing none of the point this arc is crossing itself. Let's say these are the two points: t1 and t2 - and this one. So what do we want? We want this parameter t, so that we do say is this interval -so we could say is that the interval t1 to t2 - we are just trying to say is, that is, from here to here, t is equal to a to b rather than these points a and b, which are the complex numbers here, as I have defined earlier.. Now, I can treat these a and b as the real numbers, that is, these real numbers - I can signify with this x axis or I can signify with the y axis, depending upon what kind of the equation of the curve, or what kind of the curve – and having that we will explain little later. So, this arc which is not crossing itself at any point, and my initial point and the ending points are different - this is called a simple arc or this is also known as Jordan arc. As I said, is except possibly at the initial points a and b, they may be seen that is it may be of this form; that is, starting from a, switching to this one and moving in this manner. It's say is That is, I am having an elliptic kind of curve; which is orientation this orientation has to be defined, because here, if and I am saying is the the same point is both a and b. Then, orientation has to be defined, that is, from which direction we are moving; this oriented this is called closed curve- that is simple closed curve - because it is not crossing itself at any point. This is called a simple close curve or its also other name is Jordan curve. Let us take an example of not simple arc. So, you see, here we are starting from a - and this way we are taking - this is crossing itself at this point. So, this is not a simple curve. Similarly, we could have a if I have could have added it up to here, that is, closed it, this would have not been a simple arc. So, and we do have is that is the classification of the arcs, that is, rather we have defined simple and not simple. And In the simple, we have taken the close curves - closed arcs are also, that we called the curves - so we have arc, we have curve; simple arcs, simple curve. And We are not talking about this kind of or many other kind of arcs or curves.

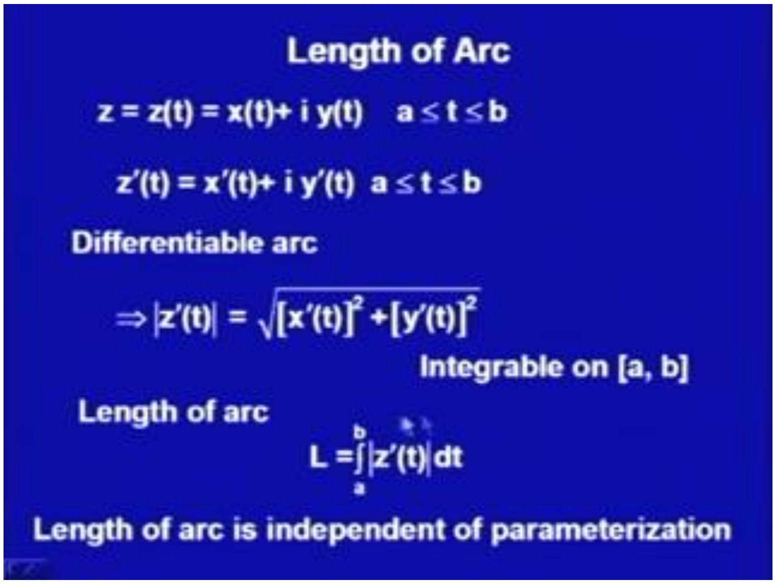
(Refer Slide Time: 41:05)



From here, let us see that is how this parameterization, we do get. So let us see first example. We do have in this arc two lines - one is from 0 to 1 plus I, another is from 1 plus i to 2 plus i. We want the parameterization of this; is we want the equation for this curve - how we can write in the parametric form. We see the first line is from - this is in the complex plane, that is, xy plane - so this point, if I treat it as a 2 dimensional plane, it would be 0 0; this point, we could treat as 1 1. So we are reaching from 0 0 to 1 1, that says is, that this is the straight line with 45 degree inclination; the equation of this line is y is equal to x. So, both the y is equal to x; either I can treat x as parameter. Now let us treat t parameter, then y and x - both are same as the parameter t. and From here to here, 1 plus i to 2 plus i; so in the place 0 to 1 plus I, what have we got? Both x as t and y as t, that is or you could say is - actually t from 0 to 1. Then, from 1 plus i to 2 plus I, if I try to reach from this x of the t parameter, so it is from 1 to 2. In the 1 to 2, this line is a straight forward moving in the same direction as the direction of x or the direction of t. That is the other part, why part has been kept. So, let us see that is how we can write it out. We can write for this part 0 to 1, t plus it, because what we are having is x is t, y is t and the form we can write x plus iy. So x t is t, yt is t; so xt plus it in the range 0 to 1 - t is varying from 0 to 1.Then, when t is varying from 1 to 2 what have we got? We have got only this t variable is varying, but the y is not varying. So what do we get? y is fixed at 1;

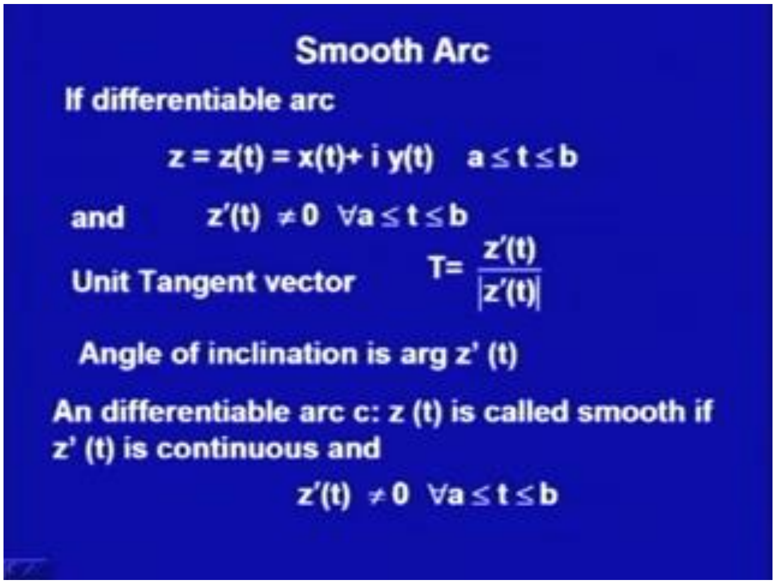
so we have got t plus i for 1 to 2. So, Similarly, let us take another example; this is the unit circle. Now this unit circle, we can write in the parametric form by this equation e to the power it 0 to 2 pi. So what we are having is, unit circle means my radius is 1, and if I take the parameter t as the angle which any point over here - at this point, if I do take any particular point over here - this point, if I draw the line from this point, the angle which this line or this radius would be making with the x axis - that angle i am taking as theta. So, if I take this orientation anticlockwise, if I take the orientation anticlockwise then my t should range from 0 to 2 pi, that is, angle of radius with x axis - that would be ranging from 0 to 2 pi - and this is what is my parametric form of the unit circle oriented anticlockwise. Now, if I generalize it, that is, I take circle with radius r and center z naught. So this center, z naught; this is the radius r; it is oriented anticlockwise what would be this equation of this one -the similar manner here we have got that it is e to the power it. Now, what I am having is: the radius is r here, the radius was 1. So, I just have the radius as the r. So, now now if I do take how we are moving; this point, if I do see if just we just go with the Pollard corners let's say what would be to would be r theta? What is theta? theta is the angle the x axis is making from this point r. So, if I take r theta, we do know the complex number I can write as r e to the power i theta naught - i theta,where theta is the angle which it is making with the x axis and r is the modulus, that is, the direct distance from the point to form origin to that point. Now I have shifted origin from 0 to this point z naught. So what we do get is z naught plus r e to the power it as t - ranging from 0 to 2 pi. So, these are some examples in which we can present these arcs and curves in the parametric form.

(Refer Slide Time: 46:04)



Now, come to another thing; if I do have my this curve or arc zt as xt plus i yt, is such that these function x and y both these functions are continuous and differentiable. That says is, I can get the derivative z dash t as x dash t plus iy dash t, in the range a to b. Then, we call this arc to be differentiable arc, that is, if for any arc, I do have this presentation xt plus iyt, such that both these functions xt and yt - they are continuous and differentiable in the whole range ,for t is a to b, then these arcs are called differentiable arcs. Moreover, if I do have that, this x dash and y dash are also continuous, what it's says is that the modulus of z dash t is nothing but x dash t whole is square plus y dash t whole is square. If they are also continuous, this function is also a continuous function. So, we say is that this is an integrable function on a to b, by the properties of integral, because this is the real number. So, we do have is that is a real function would be integrable; if it is continuous on the real line on the interval, we do call it it it is integrable. If it is integrable, then the length of an arc from a to b, we are defining as integral a to b modulus of z dash t dt. From here, what we are getting is, that is so this is one term which we would be actually using in the next few works. So, I am just introducing here you the length of arc. This length of arc is independent of parameterization. What does it mean? That says is that is Either, I use this parameterization t, or I use any other parameterization tau. If I use any other parameterization tau, that says is my arc which is - we are giving here by the equation z is equal to z t - that could be given as z is equal to say, some pi of tau; and what is that tau? tau - we are saying is that is t is the function of tau, so lets say t is a function of psi of tau. So, if I just substitute all these z dash, and all those things -you can check it by yourself - that finally, your L would come as that phi dash of tau d tau in the interval – say, alpha to beta- because, if we are mapping that ; what I am saying is that, is if rather than having this presentation z is equal to zt in the interval a to b, if you do have any other presentation that z as, say sum phi of tau where tau is ranging from see alpha to beta - again tau is real number, and alpha and beta are real numbers; even then your integ this length would remain same, that says is, it is independent of parameterization. So, - as I have told you - that is it should take let us say t is a psi of tau, and then use this all this differentiation and change of parame change of the variable, and you will find it out that we would be reaching to the same point.

(Refer Slide Time: 49:49)



Now the differentiable arc, let us see. We do have differentiable arcs: differentiable arcs mean I do have an arc z is equal to zt. So, when I am talking about the arc, of course, the curves are included; is xt plus i vt, such that x and y, both, are continuously differentiable functions. Moreover, if the derivative of z is not 0 on whole interval. So, if it is a differentiable arc, that is, my x and y are continuously differentiable functions; moreover none of them are 0 or rather, you could say is zt – total - z dash t is not real for any t in the whole interval a to b, then we call in that case we can get the unit tangent vector of this arc z as z dash t divided by modulus of z dash t. And The direction, or that is, the angle of this tangent would be nothing, but the argument of z dash t. Now, let us define one more classification of the arc; that is called smooth arc. A differential arc zt is called smooth, if zt dash t is continuous and z dash t is not 0 for all t in the interval a to b. So, we are saying is that, in that case, actually what will happen is that the length of the arc, we can define actually. So, we are calling this as smooth arc. So, what we have learnt today: the complex integration - first we tried the complex valued function on the real domain and then we learnt that is, it is similar to whatever we have learnt in the real analysis as the integration. All the properties over there were holding to and we can modify it to also for the complex valued function on the real domain itself. Then the problem - we find it out that if I just try to make it the complex valued function and the complex variable, that is, now we are not on the real line, now we are in the complex plane; there if I try to define the first thing, that is definite integral, we find it out that we can move from one point to another point in many different paths. So, before reaching to the or moving to the complex integral of the complex valued function of the complex variable, first, we have to learn little bit about those basics; that is, how can we evaluate? So, before giving the definition, we have to learn certain terms and those terms we had learnt in the terms of paths. And For the paths, we had classified as arcs - simple arcs and smooth arcs. Next, we would learn little bit more and then we will move to complex valued functions of complex variable. That is all for today’s lecture. Thank you.