Engineering Physics 1
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Module-07
Lecture-06
Consequences of Special Relativity - V

Hello, everybody, so, today we will be talking of another consequence of special relativity.

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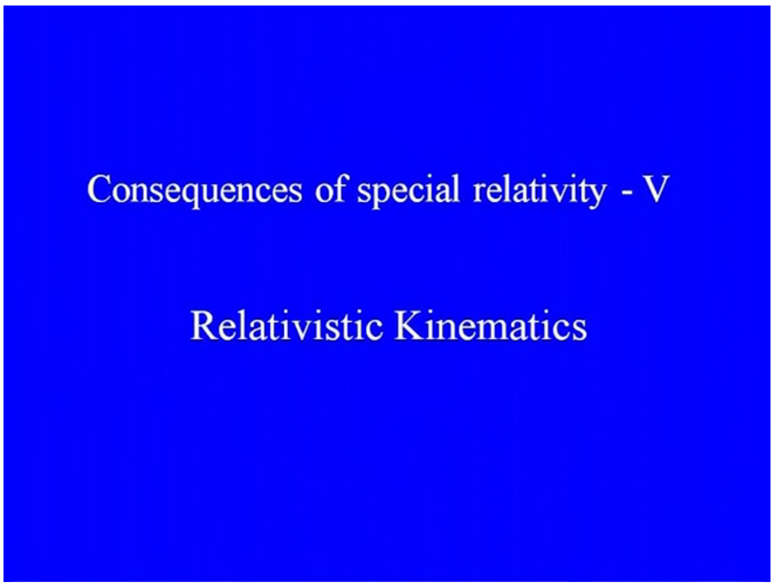

So, we will do some topics in relativistic kinematics and if one recalls we had to redefine mass. We had this famous mass energy equivalence, special relativity gives us that. And we also have to redefine mass so as to preserve the conservation of momentum. Conservation of momentum in relativity and talking of both energy and momentum how are they related?

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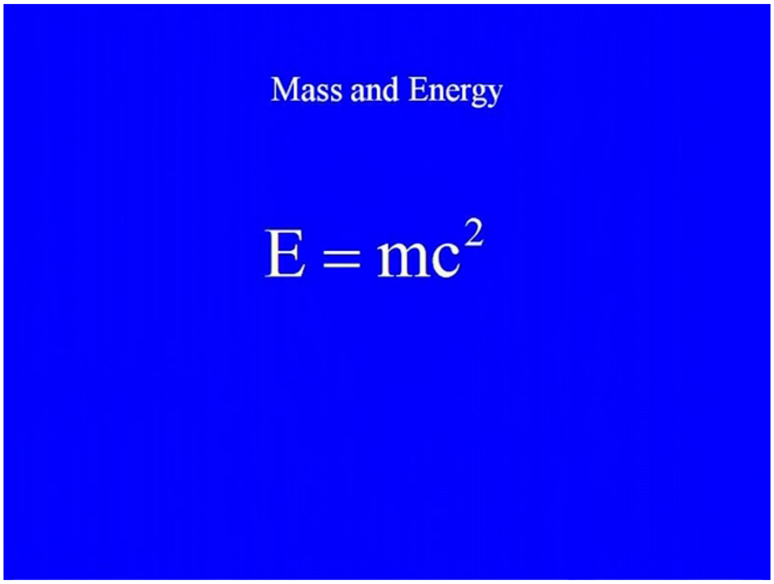

We know that E = mc square and then this m or the total mass of body that is; it is actually related with speed at which this body is moving there ok.

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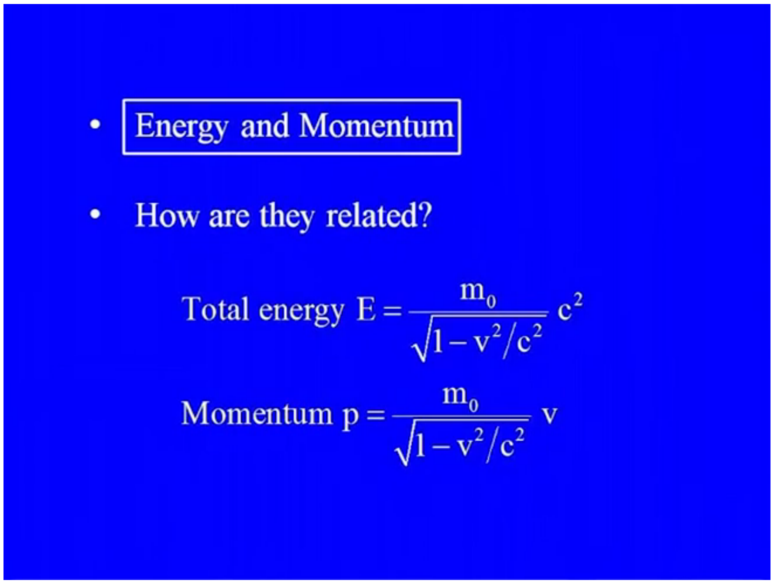

So, if m0 which you see here is the rest mass of a body then if it is moving at a certain velocity v its mass becomes m divided by root over of 1 - v square by c square ok fine. And consequently we also have the momentum defined as the mass multiplied with velocity. But then the mass again here is velocity dependent okay. So, how about the relation between them? Why do not we square them up?

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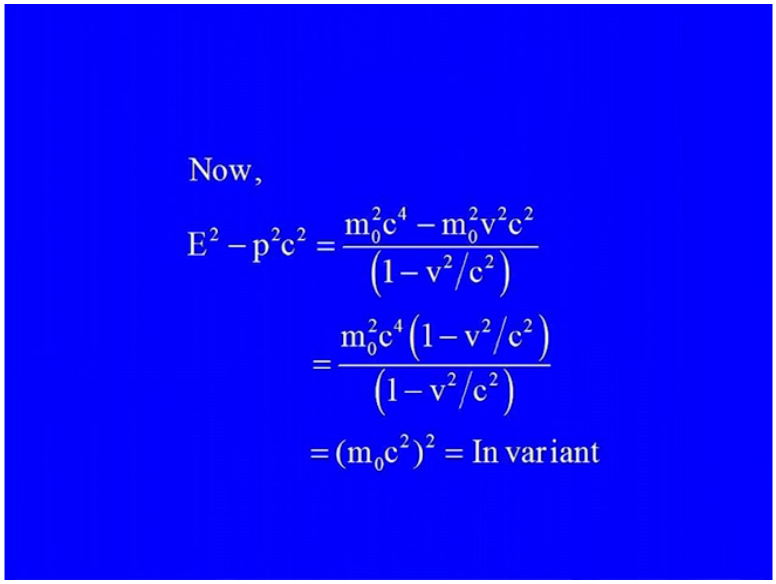

And you know if we square the energy and then multiply a c square to the momentum, do a bit of algebra. It turns out and then we and then we subtract p square c square from the square of the energy it turns out that we get something which is Lorentz invariant or in other words it does not change when you change frames or it does not change in frames which are moving with a certain constant velocity with respect to each other.

So, you actually get m0 c square on the m0 being the response of a body and then c square being the velocity or the c being the velocity of light okay. That is pretty interesting, so, if we now write it in a certain way actually if we write all these things in the dimension of momentum square.

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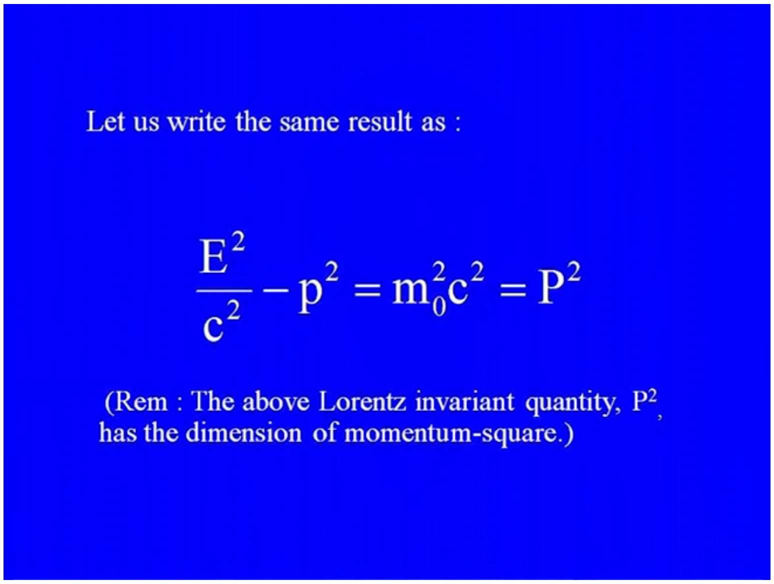

We get a isolation like you get an e square that is the total energy square divided by the velocity of light square that gets the and that is get the dimension of that gets the dimension of momentum square that is and that is the linear momentum square that we are talking of here. And then you subtract the momentum that which I denote by small p okay, you get something which is invariant okay.

Now this I denote by something like the capital P squared okay and just make a comment that still this is momentum square that is all. Let us see is there something behind all these things okay. Now to check that what is there behind, we if you recall what was a 3 momentum squared okay. So, if you know a 3 momentum square think of the thing in the Cartesian system or in any is the Cartesian system.

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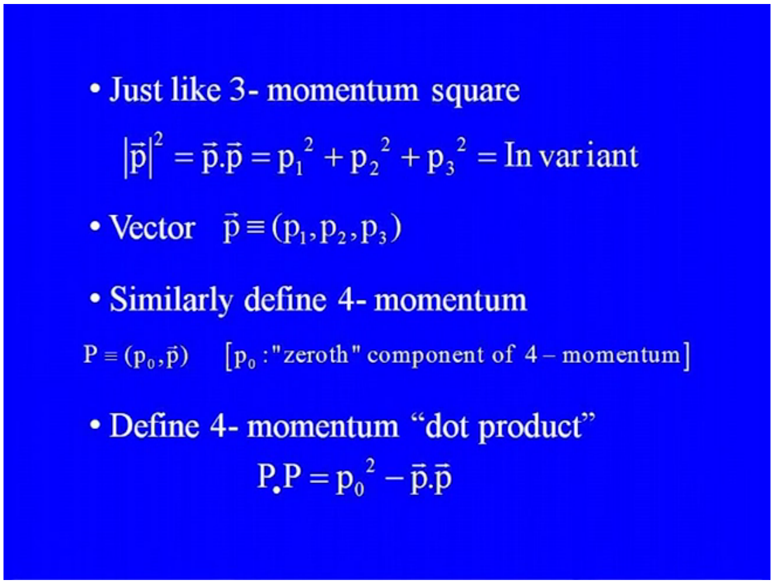

So, what you see that 3 momentum squared you get the x component square the y component Square and the z component square or component 1, 2, 3 squared each, we add them up. Wherein invariant quantity and you all know that this momentum is a vector it has 3 components. Similarly the thing that we talk of when we talk of a 4 momentum we have we can also talk of something called a 4 momentum squared.

And then what do we get and if you define the dot product of; we define a dot product for the case of 4 momentum as you know the initial the 0th component. We call one of the complaint the first component was 0th component and the other 3 components which are equivalent to the usual the vector in the 3 dimension that we all know of and then we define the 4 momentum, the square of the 4 momentum with the help of the special dot product.

That is why when I say a dot product, I do not put a simple dot but a rather a bigger dot okay. So, that is equal to actually p0 square - and the 3 dot product of the momentum, the linear momentum that is here, what we have here okay. Now why do we call it as the 4 momentum more of a little later? But here what we see is that if we identify this p0 of the 0th component of the 4 momentum as E by c.

And then the other 3 components as the linear momentum itself then what we get is the dot product of this 4 momentum dot product and shown by P, capital P dot, capital P is still an invariant quantity ok. Now the notation that I will be using is for a small p it will always be the 3 momentum if I put a vector you also know that it is a this momentum vector. But for a 4 momentum vector not put any vectors on top of it and not only that, I will always use capital letters for it.

So, so capital P is a for momentum and in the small p is the 3 momentum and that we talk of here. Where it is also interesting that we know that the 3 momentum square that is an invariant quantity okay, you know the dot product or when you take the square of a vector. You get the length of a vector which is a scalar. So, that is an invariant quantity. Similarly what we see here is that when we take the dot product of the 4 momentum itself we also get an invariant quantity okay that is something to think about okay.

Now let us let us recall a few more things about the 4 momentum which we may have covered earlier. But given today's topic of relativistic kinematics I thought it will be useful for us to do a bit of the 4 momentum formalism again, a bit ok.

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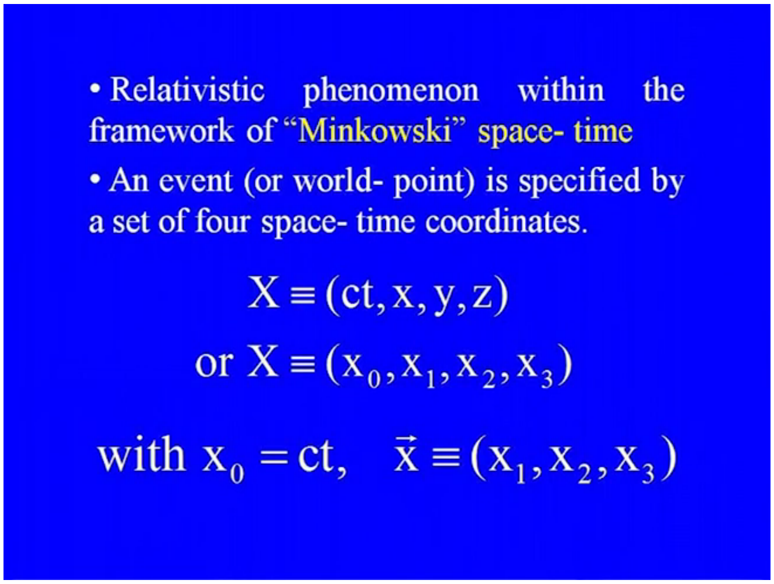

So, just to recall we can also think of the relativistic phenomenon in Minkowski space time ok where an event is given by an event or an world point is specified by 4 quantities 3 space quantities one time coordinate okay. Of course you can multiply the constant velocity of light to time. So, that you have the dimension of length for all the coordinates okay. And then that is what we do we can simply multiply c to the time coordinate and then we have x, y, z.

Let us say so the 4 space 3 space coordinates and then we have the same dimension for all and then we can simply write x0 to be t0 at component the other 3 components for the other 3 the two 3 vector here okay. Now it is also interesting when I take the squared norm or then the 4 dot product of x with itself.

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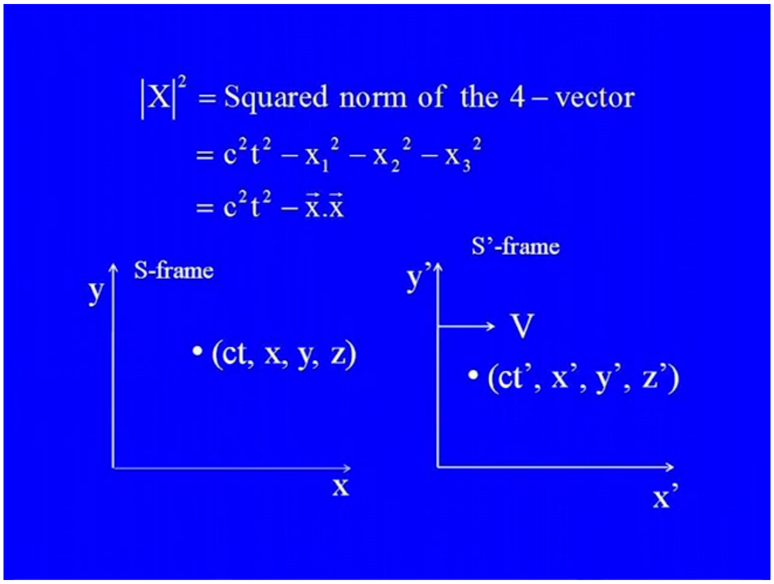

I will get c square t square - X dot X like the 4 dot product we had done earlier fundamental and how does the components change when we go from one system to another. Let us say we go from the S frame where you have the coordinate's ct at a certain time and then x, y, z. And then the same event if it is measured with the help of from S prime frame and then you think of that frame as and you measure it at a certain time t prime and x prime y prime z prime.

Of course you now know that these are related by Lorentz transformations okay. So, the component of this 4 vector is actually related by Lorentz transformations which we have covered earlier and it is act okay. Just to recall let us let us read the first line so it is the how is the X component in the S prime frame related with the X component with the S frame.

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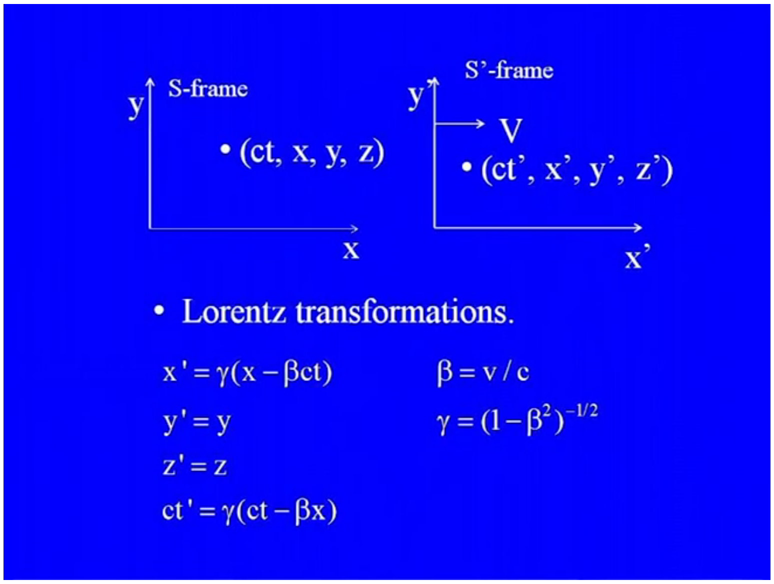

You see that let us express x prime is actually = gamma times x - beta ct ok. So, where beta is the velocity with which the S prime frame is moving divided by the speed of light okay. And then the gamma what you see here is nothing but the 1 by the square root of 1 - beta squared okay. So, what is also interesting is that if one takes the full dot product ok the 4 dot product in the S frame and the S prime frame can easily check that it is norm is equal okay.

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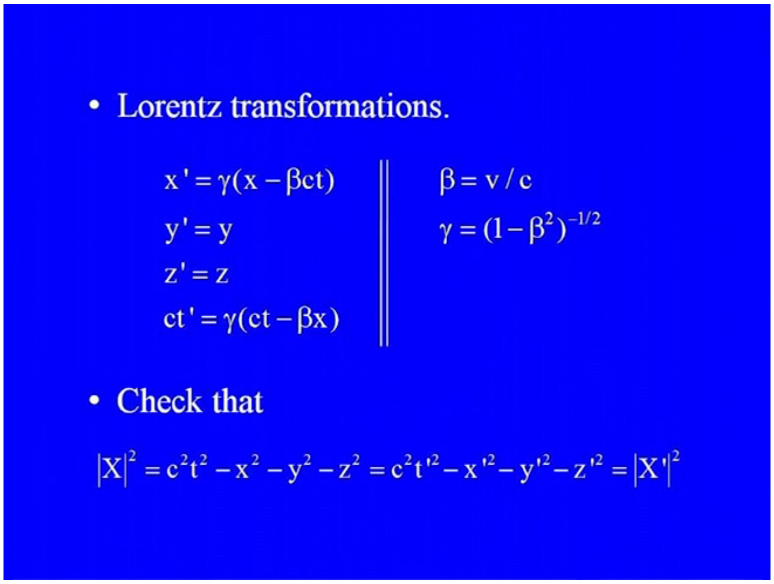

That is again something we keep in mind it is that the norm of a full vector is is becoming you know what we check is or for this in is becoming invariant when is actually invariant when you go from one frame to another ok. Now just as an aside we can also recall that the norm of a 3 vector is also invariant when you change frames okay.

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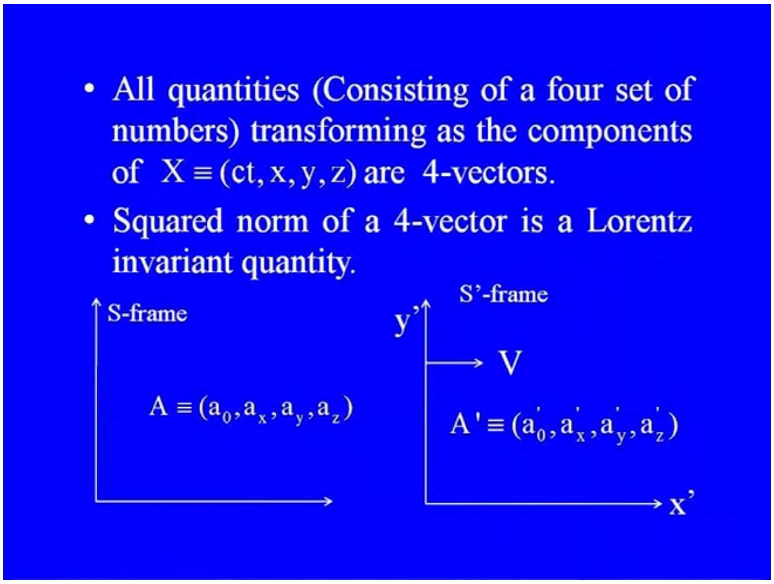

So, to make things a little bit more general we call all quantities or other all quantities which has 4 sets of sets of numbers here. Now and if they transform and if they transform in the same way as Laden's Lorentz transformations and not only that that square norm of such a quantity A as becoming a same whether you are in S frame or the S prime frame. Then such a thing are actually 4 vectors okay.

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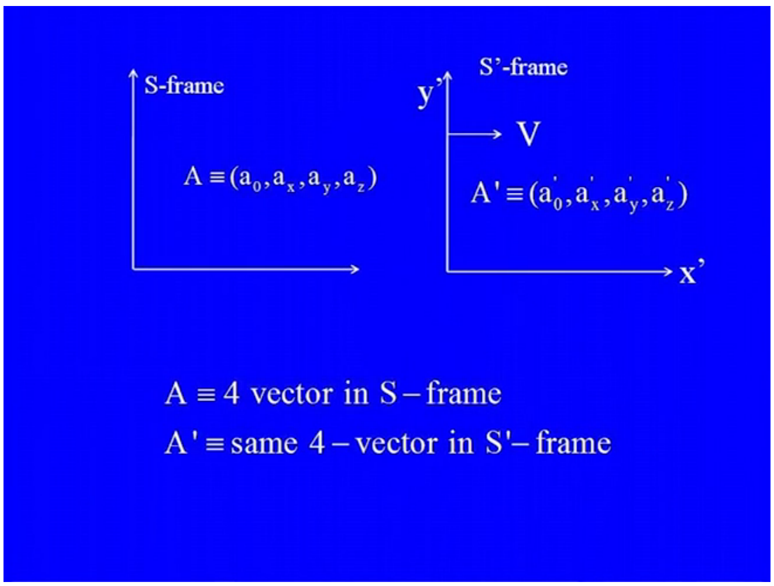

And to summarize what we have is that we have a 4 vector in s frame then in the S prime frame the same 4 vector will have of course the same norm but its components will change okay. But liquid components will change in accordance with what? With accordance with deal a Lorentz transformations okay. So, that is what we have the components of a 4 vector will change. You know the components are related by the Lorentz transformations and then the norm is invariant fine.

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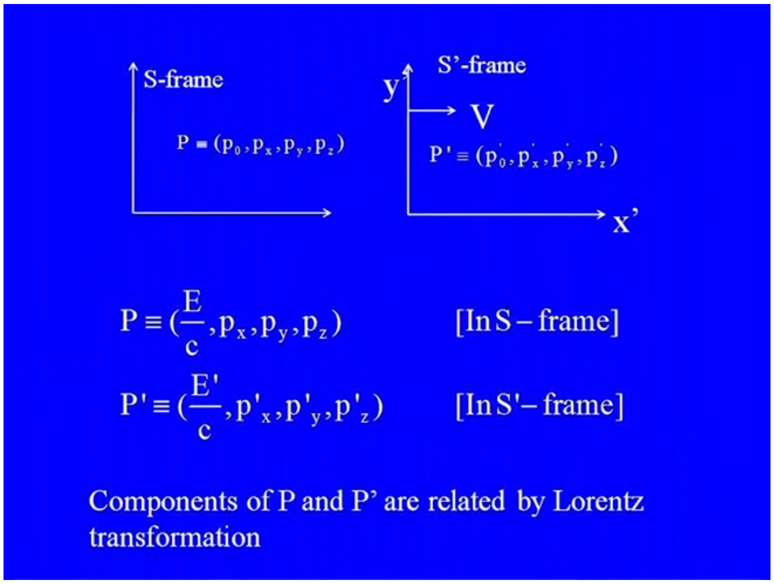

So, just to put things in perspective again if we talk of a 4 vector the momentum 4 vector here okay. So, the 0th component is E by c that is the energy by c and then px, py, pz are the other 3 components and in the S prime frame which is moving with a certain constant velocity V with respect to the S frame. The components will be different okay; of course it will be different. But the norms of the 4 vectors would be the same okay.

So, now you know just as an aside why do we call it as a 4 vector? Okay. Recall what we have in a 3 dimensional space when we have a vector. So, what is a vector? So, if we recall what a vector is? It is actually that mathematical physical entity which has 3 components here as a 3 vector that is why it is called a 3 vector. Now a vector what is the special property of it, it is that if one changes moves from one coordinate system to another.

Now the transformation matrix which one has to use to go from one frame to another okay that same transformation matrix is going to change the same vector which is in the old system to the new system.

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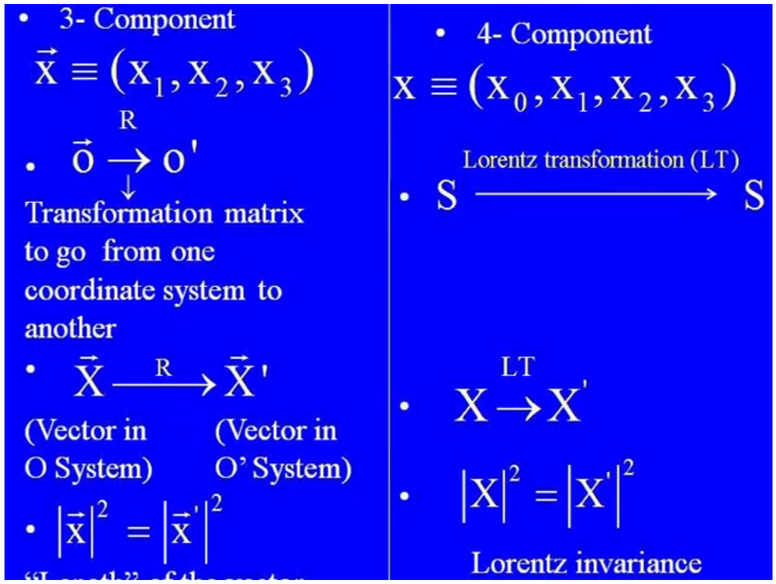

For example here in this example if X is a vector in O system and X prime is a vector in the O prime system see the transformation matrix which is transforming O to O prime will be the same which transforms X to X Prime and not only that the length of this vector is actually the same as a scalar okay. So, that does not change when you change your coordinate systems. And if you recall what we have just talked of this quote unquote 4 vector. See some amazing properties which are quite similar with the 3 vector first of all it is got 4 components.

And then the the components of the you know when you go from one system to another the what is the transformation matrix here it is the Lorentz transformations. And it is the same Lorentz transformations which is transforming the components of this quantity X of this 4 vector from one system from one frame to another and not only that it is norm is fixed okay. So, that of course justifies the word the vector here what you have used and also it has 4 component that is how you call it a 4 vector okay.

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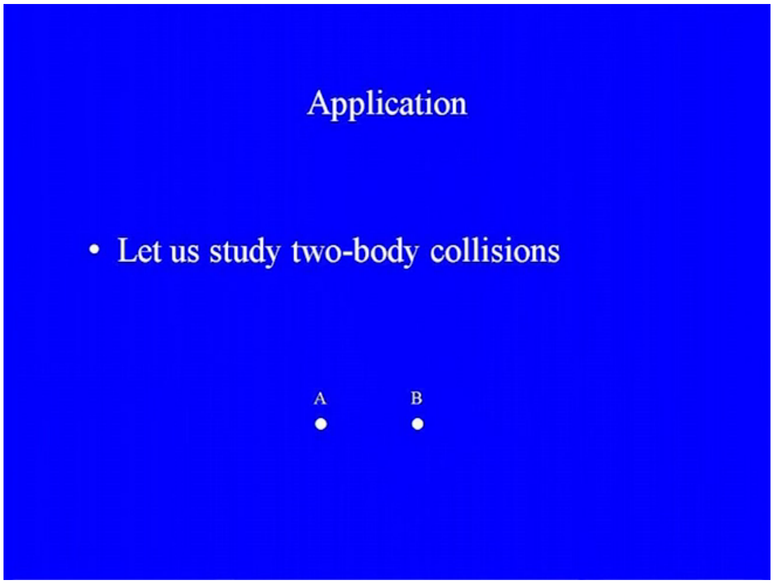

So, with this, let us with this background now I think we can go over to an application of relativistic kinematics okay. Where let us take a simple example let us study two body collisions okay.

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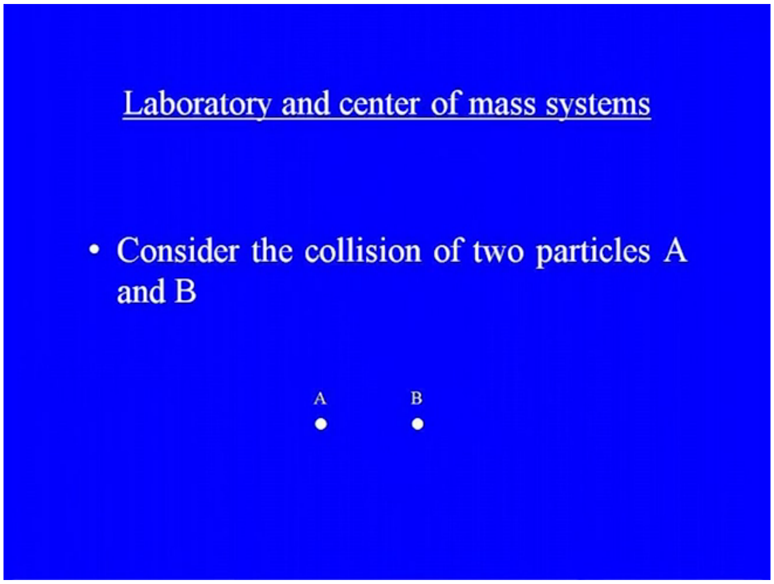

Now as in classical mechanics we can study this in the lab system and also in the center of mass system okay. So, what is a lab system when we consider the collision of two bodies okay? So, the lab system is one in which one of the bodies it may be in rest and the other comes and hits it okay. To keep things simple I mean we just take the direction of motion of one of these bodies of one which is moving to be along the x axis okay and it is on a plane let us say okay anyway.

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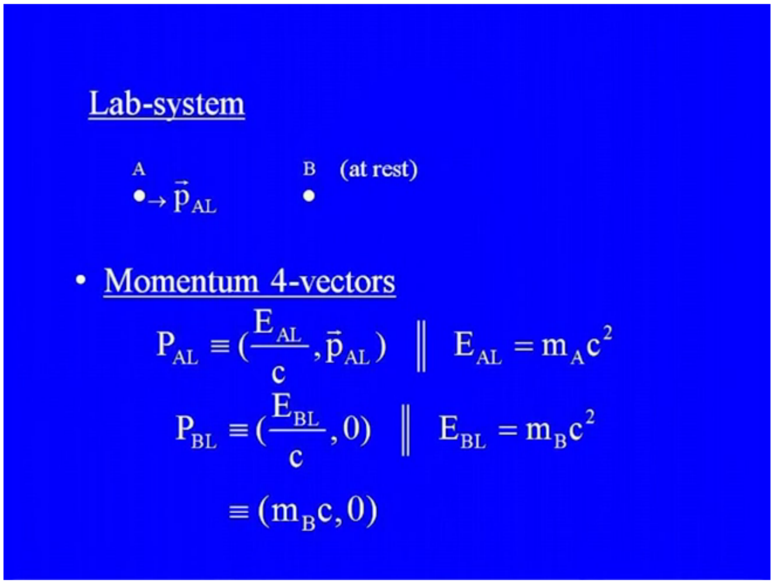

We also take one of the bodies at rest, so, let us just take the body B at rest and the body A which is coming with a certain linear momentum PA and if you see we have also put a subscript L and also a vector. So, that is that is an indication PA L. So, that is the indication the P has a vector so that is A 3 vector and it is related with the first body A and then it is in the lab system. So, that is the quantity L that is that is the reason I put this quantity here.

Now what would be the 4 vector for such a quant for these bodies okay? If recall that the 0th component of the 4 vector relates with the energy okay. So, we have the energy of particle A in all the body A in lab system divided by c that is the speed of light. And then the other 3 components are the 3 vectors okay or the 3 components of a 3 vector okay. And what is EAL, so that is nothing but mA into C square okay that is the mass of E.

And then what about PBL that is the 4 vector PB L. So, recall that again it sits at rest. So, the 3 momentum part of it is 0 okay but it is interesting that it is energy is not 0, why? Because of course it has a rest mass and then you have rest mass energy, so you have mB into c square that will be the rest mass energy okay. Now if you take mBc square divided by c, you will get mBc and so you get the 4 vector PBL that is capital PBL to be mBc.

And then the since the 3 vector part is 0, so you have it is 0 okay. So, there is of course another system in which we go over to the center of mass of the system. So, all the measurements are being made from the center of mass of the system here.

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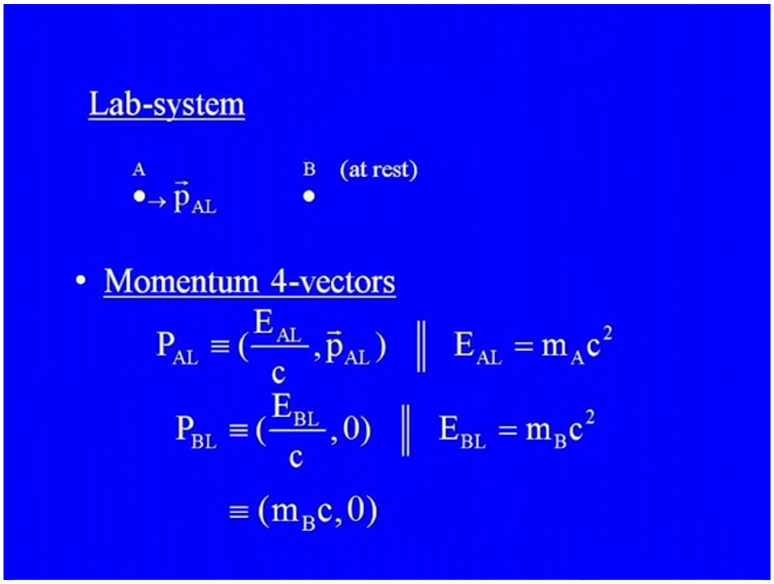

And in this particular case we have the case in which the bodies are moving in a set such a way that the total linear momentum turns out to be 0 here okay. So, the same collision process you are looking at the frame you are looking in a way in which the both are moving towards each other. In such a way that the sum of the linear momentum are c okay and it is easy how to write the momentum for vectors here for PA.

Notice that I have not used any subscript L here so except maybe when I only the subscript for the bodies or the particles have been used here. So, the other subscript the cm is not required when superfluous thing. So, if I do not write it means that it is the center of mass system. Otherwise we assume that I which in the lab system when I write it by this letter L.

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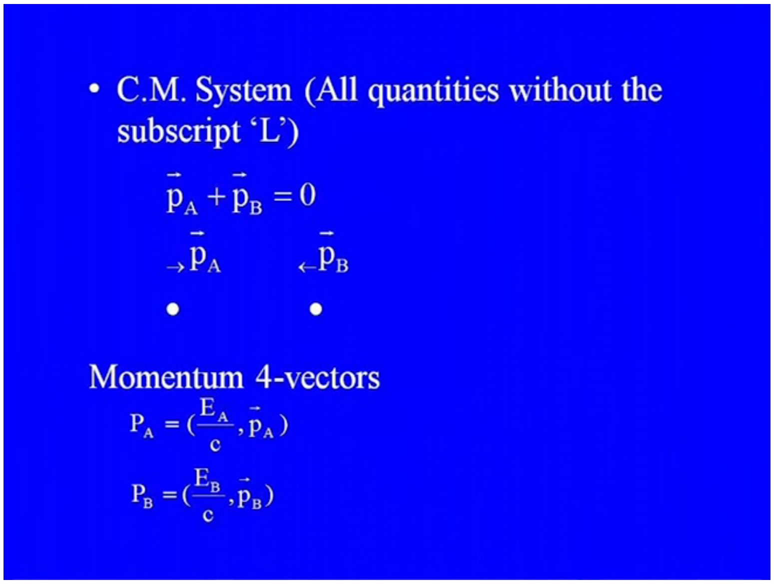

So, the momentum PA that is the 0th component we know that EA by c again all quantities in the center of mass system and then the 3 vector that is the PA and then for the capital PB that is the 4 momentum for the particle B that is EB by c and then it is got a 3 momentum PB okay.

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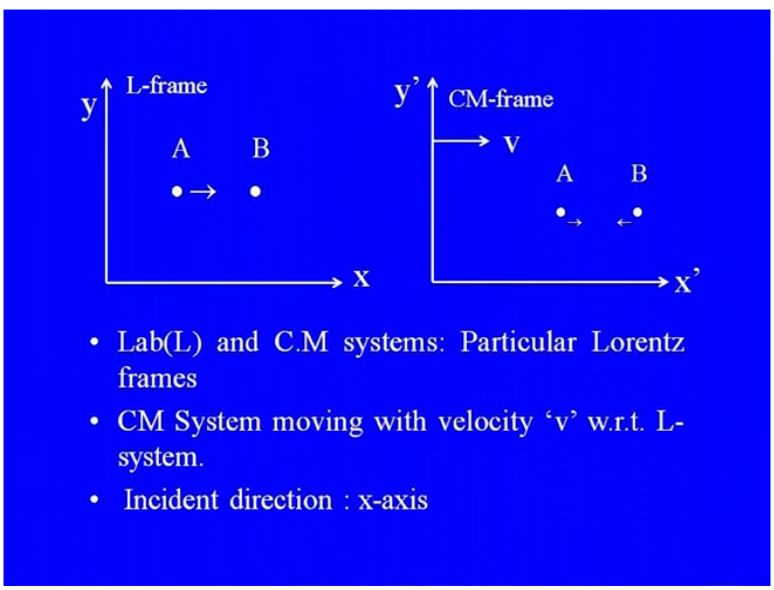

So, how are they related with each other member, so, the center of mass system can be thought to be moving with a certain constant velocity V with respect to the lab system. And then if our incident direction of collision is the x axis you know so we just have we just written the S frame on the S prime frame we just rename them as L and L frame that is the left frame and the center of mass frame respectively here.

So, in the lab frame notice that the particle B or the body B is at rest and the particle A is coming in coming towards it with a certain momentum velocity and certain energy here okay kinetic energy element. Of course for particle B here in the lab system although it does not you might recall you might recall that it has some energy which is associated with the rest mass energy. And then in the center of mass frame what we have?

We have the particles moving with mentor towards each other equal momentum so that they are opposite oppositely directed and when you add them up they cancel out okay. So, how are the things related with each other and recall that the coordinates in the lab and the center of mass frame they are they are Lorentz frames here. So, they are they are related by the Lorentz transformations okay.

And what about the components of the 4 vector of the 4 for momentum, so since it is also a 4 vector momentum we are talking of the components of momentum 4 vector the components will transform according to Lorentz transformation itself okay. So, it is interesting that it is very easy you can you can find out if the momentum of any particle is you know one of them in the left frame and in the other frame.

And that is in the center of mass frame which is moving with a constant relative velocity with respect to the lab frame. You can simply find it out by the help of Lorentz transformation that is the relation between the lab and the center of mass moment and of course the energy the energy being the 0th component here okay.

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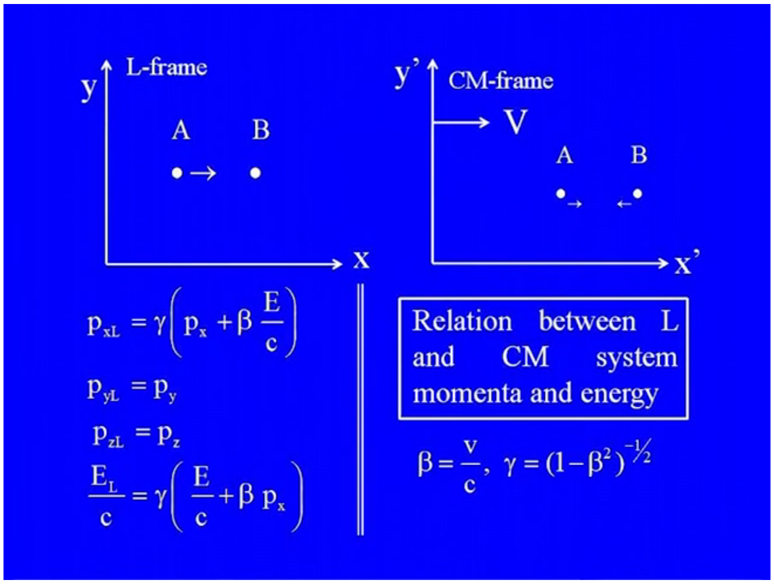

It is simple here for example the and if px is the momentum in the small px that is the linear momentum along the x axis in the center of mass system. And then E is just the energy in the center of mass system then the px L that is the x component of the momentum in the lab system is related with the help of Lorentz transformation that is gamma times px + beta E by c ok. So, it is just the, and then since it is we have taken the motion to be along x axis.

So, the x and the y and the z components are the same here ok. And what about the energy part remember this is transforms like the time like components or the ct component when we are doing the position 4 vector okay. So, what is the energy in the lab system so that is simply related with the energy in the center of mass system by if EL is the energy in the lab system EL by c that is actually equal to gamma times E by c.

So, E being the energy of the center of mass system + beta times px, px being the x component of the 3 momentum in the center of mass system ok fine.

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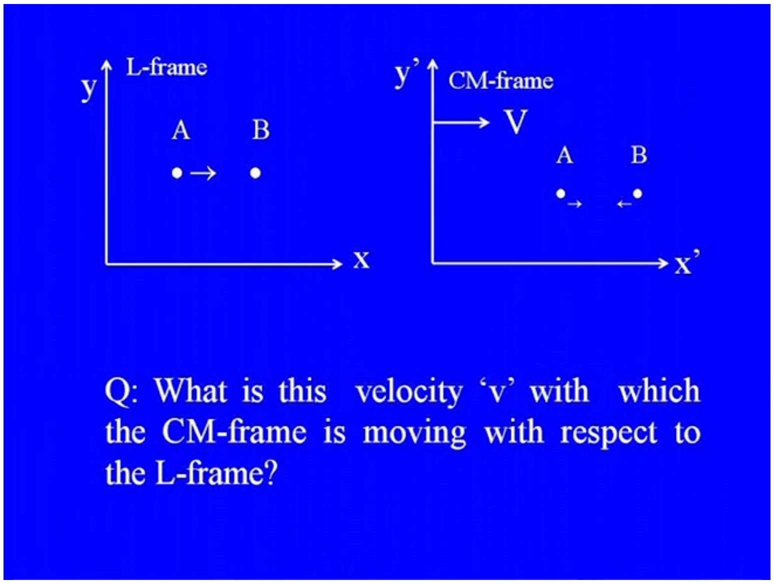

So, what is this velocity then with which the center of mass is moving with respect to the lab ok. What is this velocity? How can we estimate it with the help of known components with the help of known quantities? Like the mass of the particles in the lab system and in the linear momentum of the particles in the lab system ok. So, Let us find it out. And to do that, what we can do is that we can simply apply the conservation of linear momentum and the conservation of momentum and also conservation of energy.

And we will simply take them out from there okay by adding the corresponding quantities for these two particles okay. So, first Let us do the conservation momentum conservation first for the momentum for these particles okay. And what we find is that if we add these two quantities in the lab system and then of course we relate them to what these quantities are in the center of mass system.

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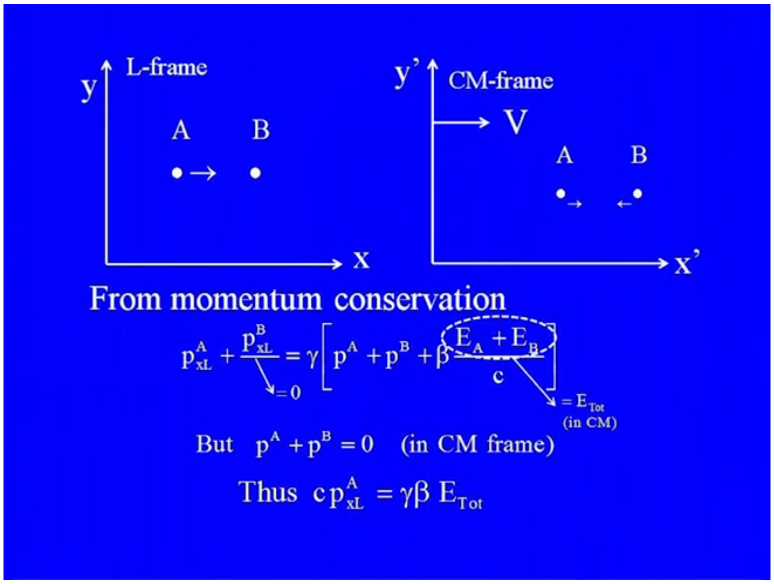

So, what we see is that we add pAx that is the x component in the lab system for particle A and pB x L that is the that is the x component of the of t of the second particle B in the lab system. By the way remember that it is actually 0 that the x component that the curve that the momentum of particle B is 0. So, pB x L that is actually 0 okay, so, that is x = gamma times pA + pB + beta times EA + EB by c.

Now pA + pB so that is 0 in the center of mass frame and then but the energy is EA + EB how are they related? Let us simply call them as a total energy in the center of mass frame okay. In that case we get a rather simple looking relation which relates the component of the component the momentum of A that is pA xL in the lab system with the total energy in the center of mass system.

And remember what is this beta, beta is nothing but velocity with which the center of mass frame is moving divided by the velocity of light and then gamma is another kind of mathematical quantity related with beta. And then the c here is the velocity of light okay or the speed of light if you wish.

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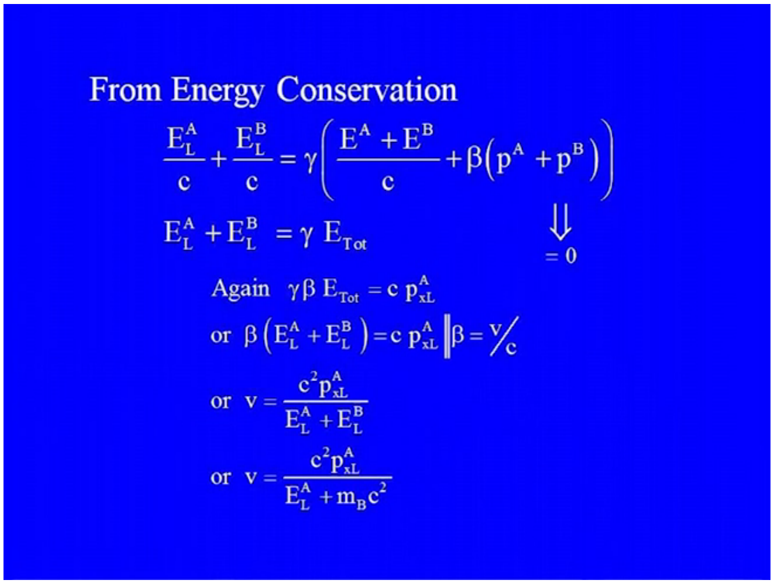

Again from energy conservation we simply add up the energies in the lab system and then corresponding and then you equate that with the corresponding quantities with the corresponding patience in the center of mass system. And we simply also use the quantity that the total momentum the linear momentum in the center of mass system is 0 then we also get another interesting quantity okay.

What is it? That the total energy in the center of mass system and the total energy in the lab system how are they related you simply take it out from this solution. But again from the conservation of momentum we have seen that the total energy in the center of mass system can be related with the total momentum in the lab system. Now we can use that relation again to drive out a little some other quantities here.

And then from these two equations the one which relates the total center of mass and the lab energies and then the one which relates the total center of mass energies with the lab momentum of particle A, we can find out what the velocity with which the center of mass frame is moving with respect to the lab system you can simply work out that it is nothing but 3 squared times I mean times the momentum of the particle A divided by the total energy with which the particle A is moving.

Plus the rest mass less mass energy of particle B okay. So, you have everything with respect so we in terms of everything that you started with in terms of all lab system quantities okay. So, what about the collision process we have not used the 4 vectors as yet okay. We have just used the conservation of linear momentum and the conservation of energy separately and we found out this velocity.

Now what if we use the conservation of the 4 momentum in the lab and the center of mass system, so, what we get from there okay.

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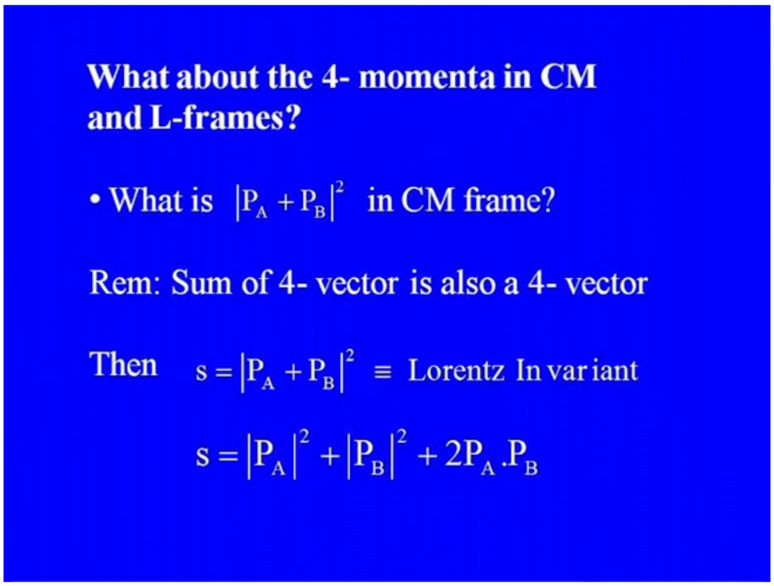

So, for that Let us simply add up the 4 momentum of particle A and particle B and take its norm in the center of mass system. And remember that the sum of this 4 vector is also 4 vectors then you could take a norm of that it is going to be that is going to be invariant whether you are in the lab system in center of mass system. Let us simply call this norm as small s and given that it is still a 4 vector you add two 4 vectors you get a greater 4 vector, I think it is norm.

It is Lorentz invariant, so, S here is a Lorentz invariant quantity then what is it? It is the norm of PA square it is it is the norm of PA and norm of PB these are the 4 vector norms plus the 4 vector dot product twice the 4 vector dot product of PA and PB. Now we all know how to do these 4 vector dot products and calculate these norms.

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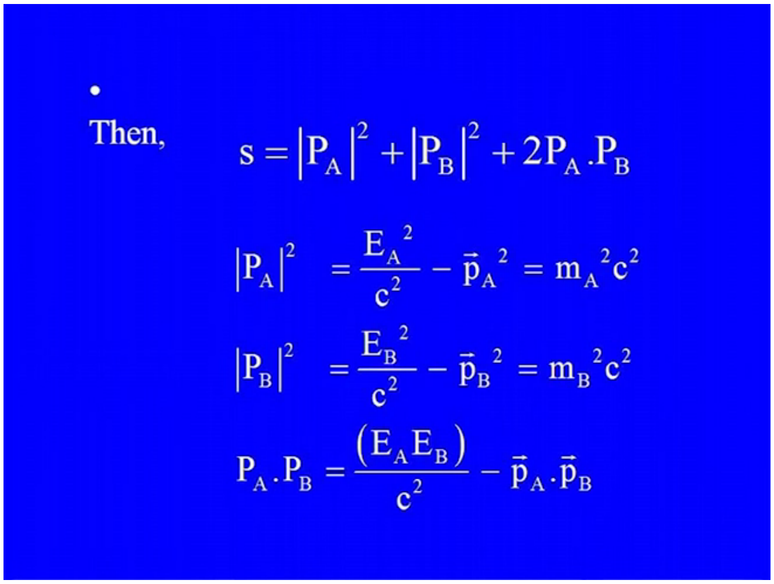

And for example what is PA squared that is the 4 vector of a square and center of mass system we know that it is also it is EA square by c square that is the energy of particle A in the center of mass system minus the 3 vector norm of particle A. And then the norm of the 4 vector is nothing but mA square into c square that is the rest mass energy square of that that is the rest mass square of particle E multiplied by c square.

Similarly you can find the PB square that is the norm of the vector the 4 vector for particle B and then you can also find the 4 vector dot product of PA and PB that is nothing but the product of the energies minus the 3 vector dot product of PA and PB that is a bit of an algebra. Now when you arrange all these things put it plug it back into the expression for S okay.

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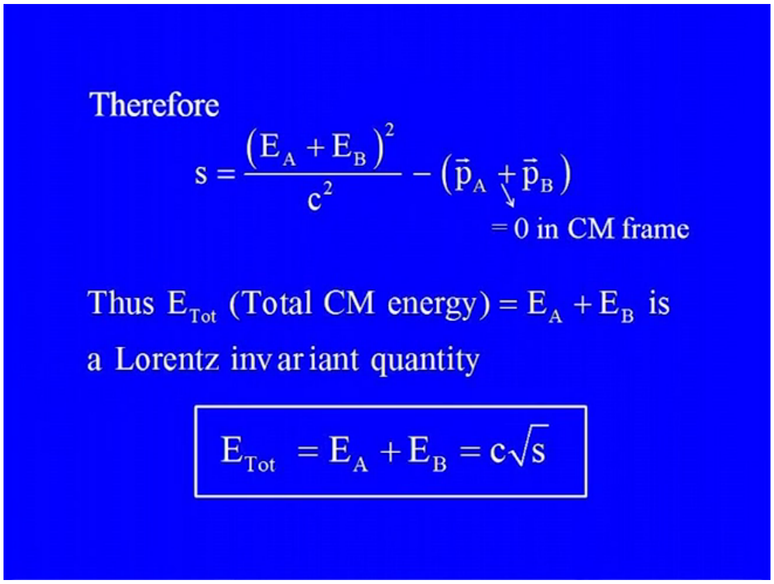

What you get is that the sum of the total center of mass energy squared divided by c square minus the minus the PA + PB whole squared okay. Now if you recall that the total momentum at the 3 momentum when you add these two up in the in the center of mass system is 0 okay. So, the total energy in the center of mass system which is EA + EB is a Lorentz invariant quantity here.

You can it is and if you put it in you know remove the square here and you see that it is actually equal to c times root S okay. So, S being a Lorentz invariant quantity okay, so, that is a rather important relation we have got here it is that the total center of mass energy for this two body collision that we see here is actually a Lorentz invariant quantity okay fine. Now S again is a Lorentz invariant quantity.

So, we can also find the thing in the lab system okay. So, we do the thing in the lab system we take the sum of the 4 vector in the lab system of particle A and particle B and take its norm okay.

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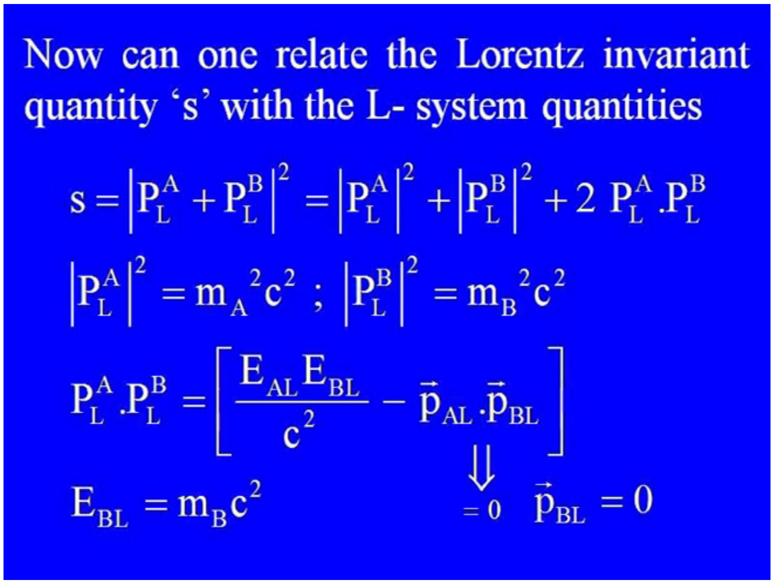

Then do the algebra again so we have PA L stands for the subscript L is telling us that it is a lab system PA L squared + PB L square and then twice PA L dot PB L okay. So, that is the 4 vector dot product. Now if we use the norm of the 4 vectors being invariant in being invariant here, so we know that the norms. And we also use the algebraic expression for the 3 a 4 vector dot products.

Also remember that the any using and in writing the 4 vector dot product we just remember that the of the second particle here B was 0. So, we are not going to have the 3 momentum part here in the 4 momentum dot product. And then the energy of the particle B in the lab system is just the rest mass energy.

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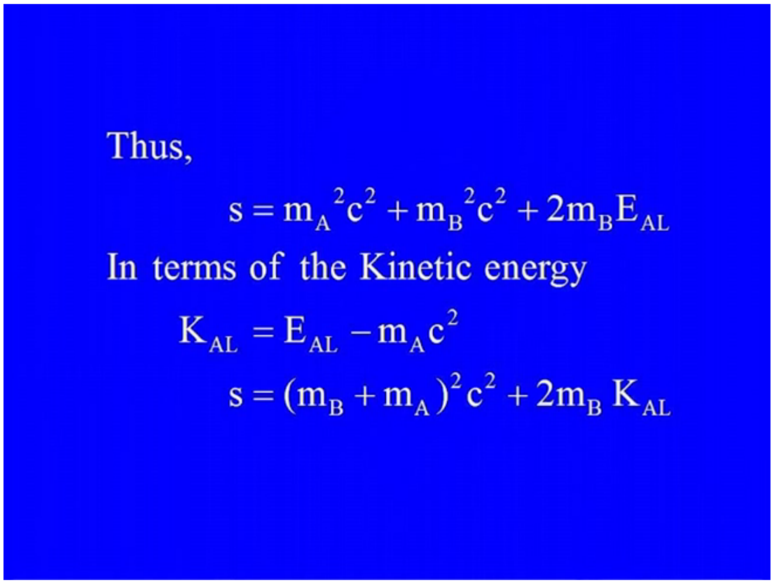

So, we plug in all these things into the expression for the invariant S, what we get is? Is a quantity which is in terms of the masses and the energies of and then the energy the initial energy of particle A in the lab system okay? Just as an aside what is the kinetic energy of particle a in the lab system? It is just you subtract the rest subtract the rest mass energy from EA L that is the total energy. And you can you can write the invariant quantity S in terms of the kinetic energy also okay.

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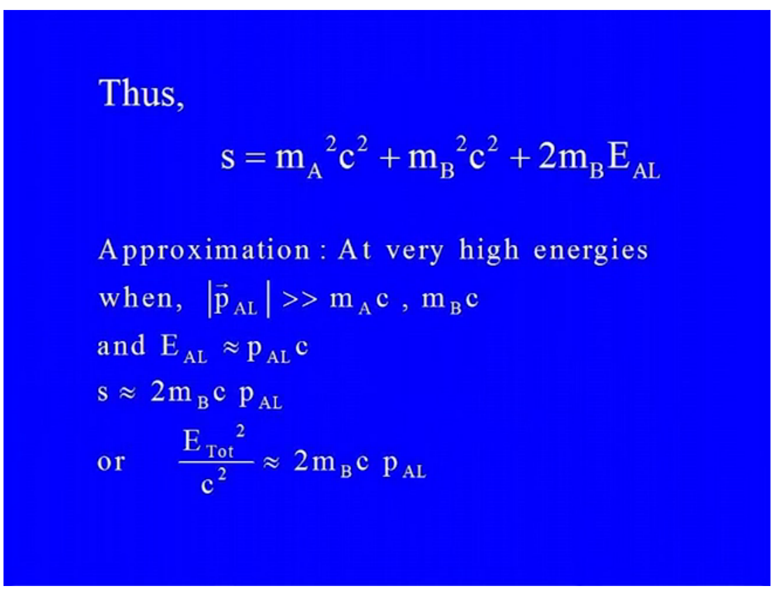

Also we can use a approximation when you are at rather very high energies things called the ultra relativistic cases. When the momentum of is actually much, much greater than mA times c that is the rest mass times the velocity of light okay. If it is rather very big then how does this expression for the invariant S turn out to be? And in that case you know that the energy of the total energy of particle A in the lab system is nothing but P AL that is the momentum in the lab system times t times the velocity of light.

It can be approximated by that because the other quantity that is other quantity which depends on the rest mass will be much less compared to this okay. We might do an example for this too to check it out whether that is or not okay. So, in that case we can simply write that the total totally invariant S in terms of just the lab system in the ultra relativistic case okay. We also know that the invariant S that we have defined here that we have got here can be written in terms of the center of mass quantities.

Remember the total energy squared divided by c square that is also an invariant quantity actually that was also S okay. So, in this way we can relate the quantities in the center of mass system the energy the total energy in the center of mass system with the total momentum in the total 3 momentum in the lab system. So, in a way, it is a method for us to check if you are giving this much amount of momentum in the lab system?

How much of it goes to the total energy in the center of mass system okay. So, as I said let us do a small example okay so here.

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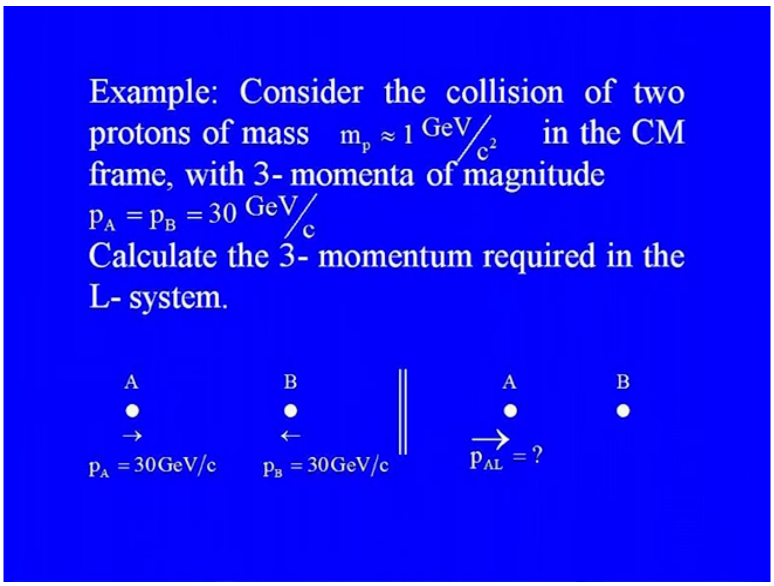

Let us consider the collision of two subatomic particle Let us say we consider the collision of two particles and then rest masses you know the rest mass of a proton is something like 938 940 G eV by c square. But it is approximately let us take it as 1 GeV by c square. Now and then we consider the collision of this in the center of mass system with 3 momentum of magnitude 30 GeV by c okay.

This actually rather very picky okay so here you can look at this figure for exam we have these two protons A and B they are coming towards each other with moment at 330 G V by c. Now the question is what is the what is the linear momentum what is the momentum in the lab system which will be necessary so that you have this much amount of momentum in the center of mass system okay.

So, we simply, let us find the center of mass energy of the protons first okay. Now it will be of course the same with the center of mass energies of the move both these protons can be found.

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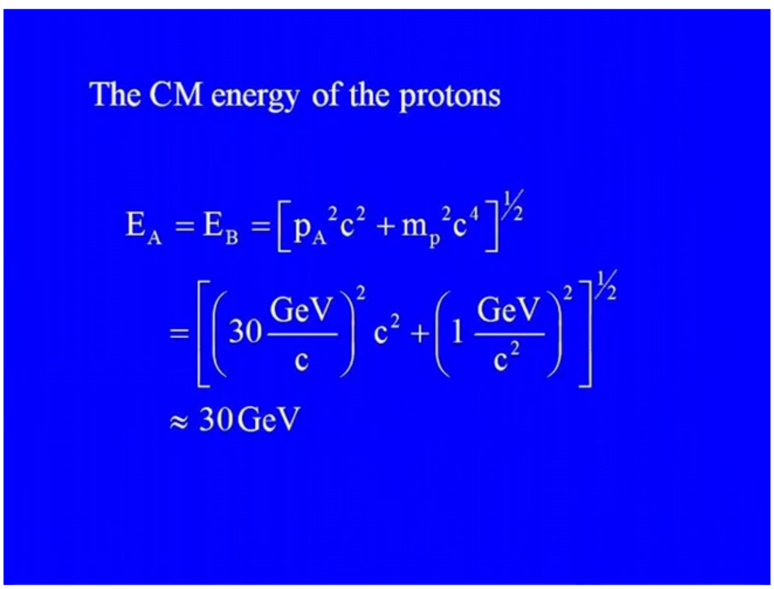

By simply you know be a square and c square plus mP square c power 4 and then you take every take the whole thing to the power half okay. So, what is that so PA square c square what is that? That is that is 930 squared okay. Now to 900 if you add 1 that is the rest mass rest mass energy is squared okay. It is nothing 900 becomes simply 901 so if you take the square root of 900 or 901 it is approximately 30 okay. So, it is actually in the ultra relativistic keys that we are talking here.

So, we can neglect the rest mass compared to the momentum here okay or rather rest mass times t the velocity of light compared to the moment here okay. Now we also know we need to calculate what is the mentor in the in the lab system okay.

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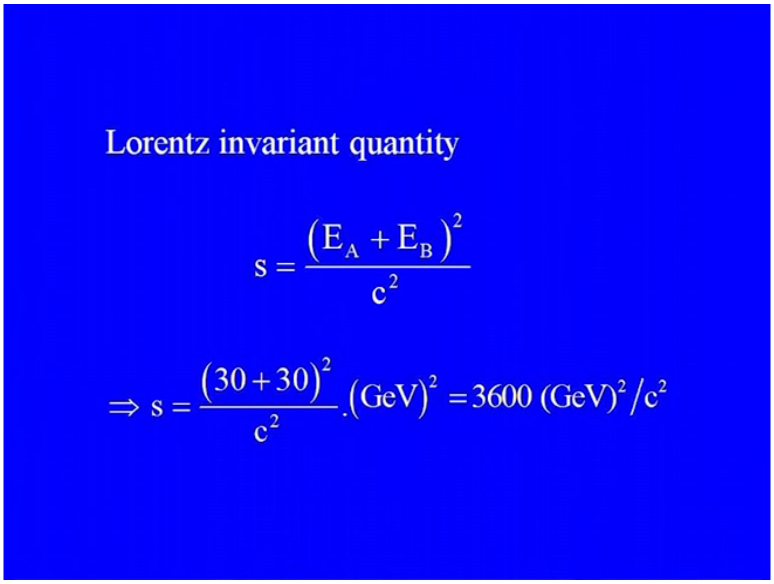

So, for that let us calculate the invariant Lorentz invariant quantity S, we know that that is related with the total center of mass energy squared right. S, S will be 30 + 30 here because each of these protons have total energy is 30 in the center of mass system. So, S is nothing but 3600 GeV square by c square fine. We can also relate this with we will we have seen how it is related with the lab system went up okay.

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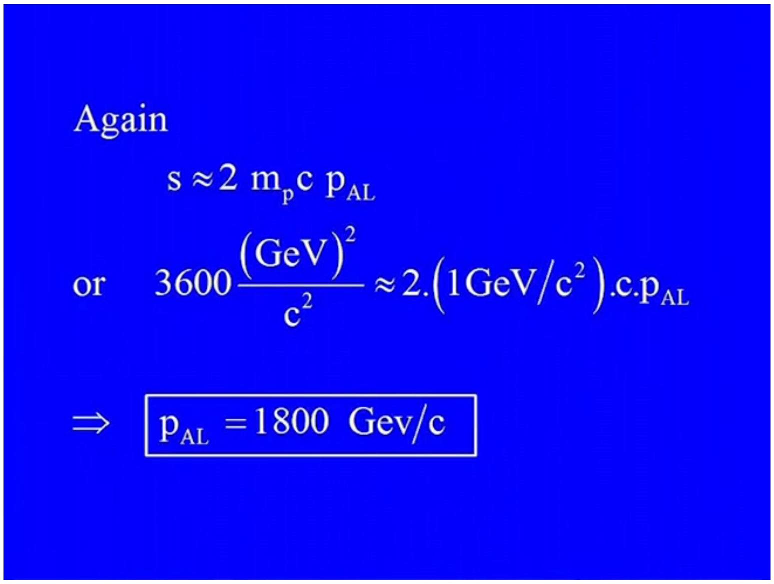

That is actually equal to twice the mass of the protons times the velocity of light times the linear momentum okay. Now if you plug in all these quantities you will get the linear momentum of in the lab system to be right 1800 GeV by c which is huge okay which is rather huge. So, you see to get a energy of 30 GV Per c you need such a huge momentum in the lab system okay and this also and then this number actually justifies the approximations that we have used that of our ultra relativistic case.

In which the momentum was considered to be much, much larger than the rest mass times the velocity of light here okay. So, I hope I have convinced I have, you know shown you some examples of relativistic kinematics more specifically relativistic collisions and the use of the 4 momentum of the power of the use of 4 momentum in relativistic kinematics thank you very much.