Engineering Physics 1
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Module-04
Lecture-04
Diffraction by a Circular Aperture

Diffraction by a Circular Aperture by M K Srivastav, Department of Physics, Indian Institute of Technology, Roorkee, Uttarkhand. In the last three lecture on Diffraction, we have studied the meaning of Diffraction.

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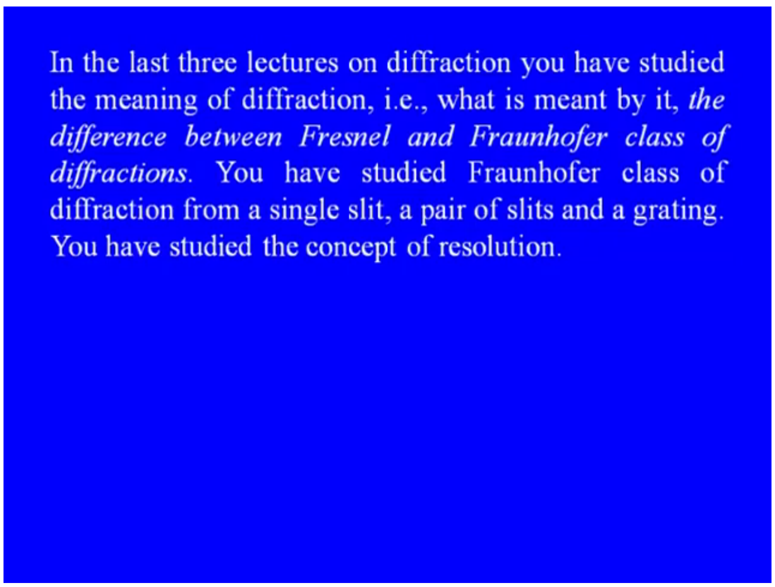

What is meant by it? The difference between Fresnel and Fraunhofer class of diffractions, you have studied Fraunhofer class of diffraction from a single slit, a pair of slits and a grating. You have studies the concept of resolution.

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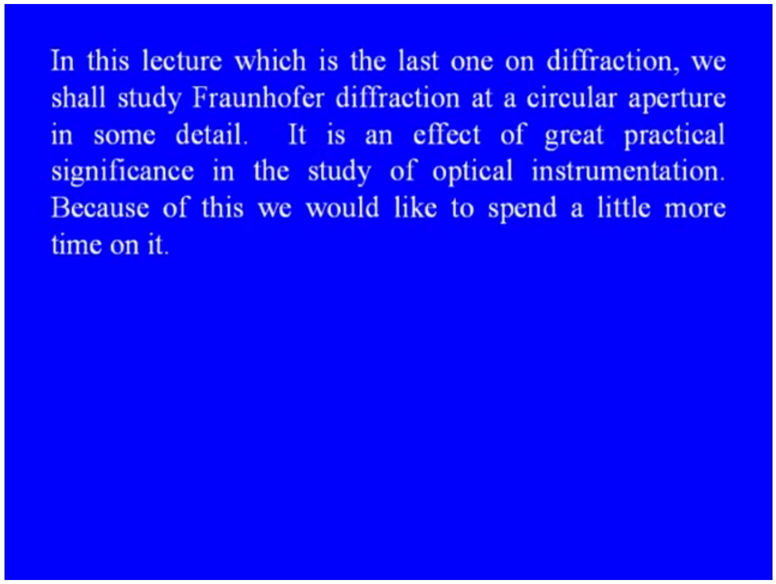

In this lecture which is the last one on diffraction, we shall study Fraunhofer diffraction at a circular aperture in some detail. See, it is an effect of great practical significance in the study of optical instrumentation. Because of this we would like to spend a little more time on it.

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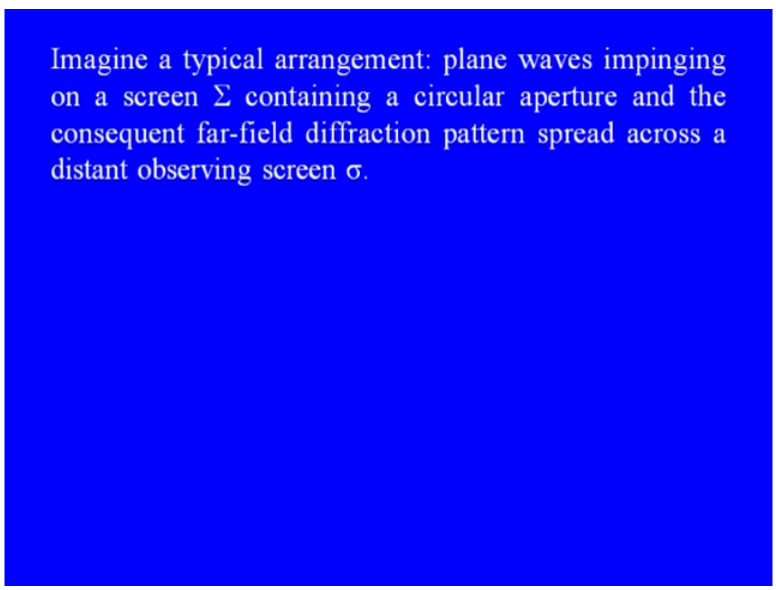

Imagine a typical arrangement plane waves impinging on a screen capital sigma containing a circular aperture and the consequent far field diffraction pattern spread across a distant observing screen sigma.

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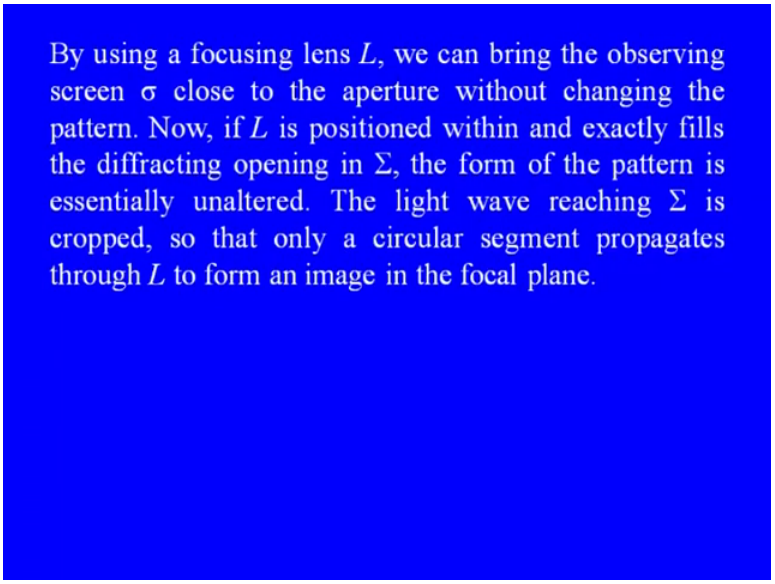

By using a focusing lens we can bring the observing screen sigma close to the aperture without changing the pattern. You see, the screen is virtually infinity. Not really infinity but practically very large distance away from the aperture. Now, if L is positioned within and exactly fills the diffracting opening, that is, completely fills the aperture, the form of the pattern is essentially unaltered by the lens. It is really controlled by the aperture.

The light wave reaching this screen is cropped means it is obstructed so that only a circular segment propagates through the lens that is through the aperture to form an image in the focal plane of the lens.

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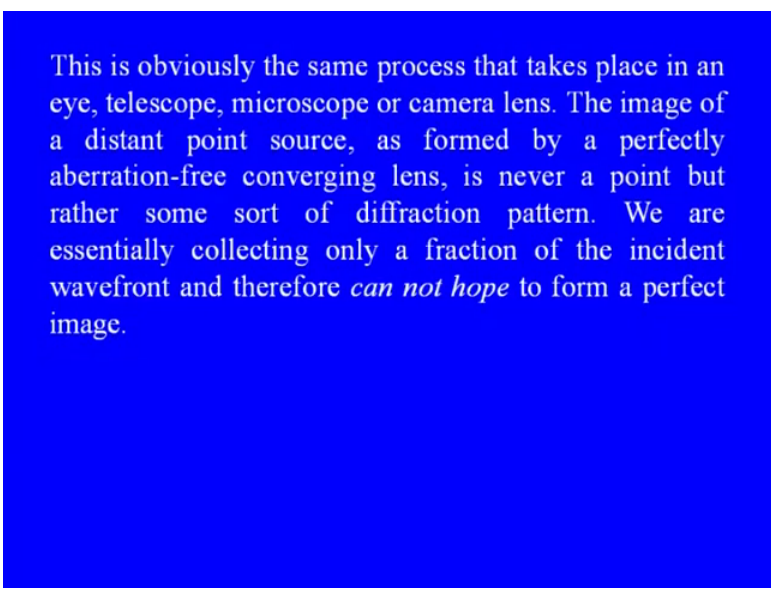

This is obviously the same process that takes place in an eye, a telescope, microscope or camera lens particularly or typically in any optical instrument. The image of a distant point source as formed by perfectly aberration free converging lens is never a point but rather some sort of a diffraction pattern. We are essentially collecting only a fraction of the incident wave front. One that passes through the apertures, most of it is obstructed and therefore we cannot hope to form a perfect image.

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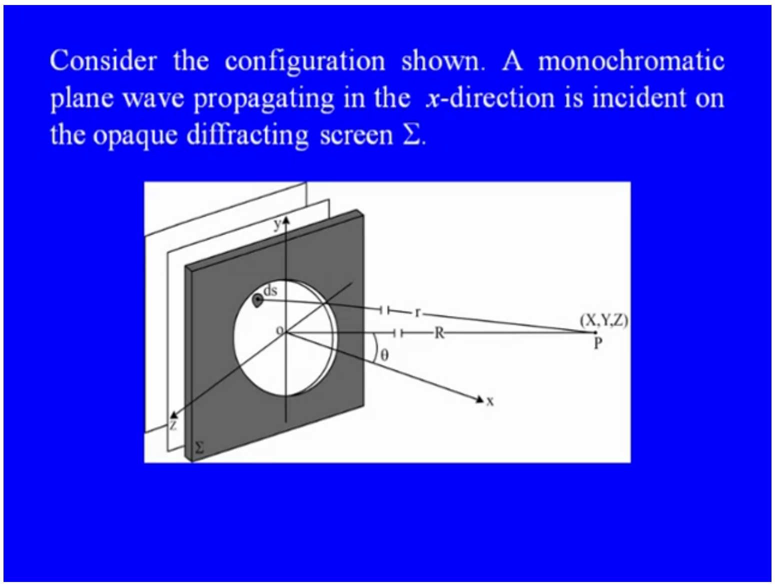

Consider the configuration shown. A monochromatic plane wave propagating in the x direction is incident on the opaque deflecting screen capital Sigma. Remember, this is a plane wave, we are considering. We wish to find the consequent false field flux density distribution in a space are equivalently at some point P. Let us consider a few things here. The opaque screen is in the yz plane, x axis is perpendicular to the plane, O is the origin of the coordinates; they are finding out the intense distribution say, find out density at some point P.

The point P has coordinates capital X, capital Y, Capital Z. Capital R is the distance of this point from the origin of coordinates. The small x for the aperture is 0, it has only dimensions along Y and Z.

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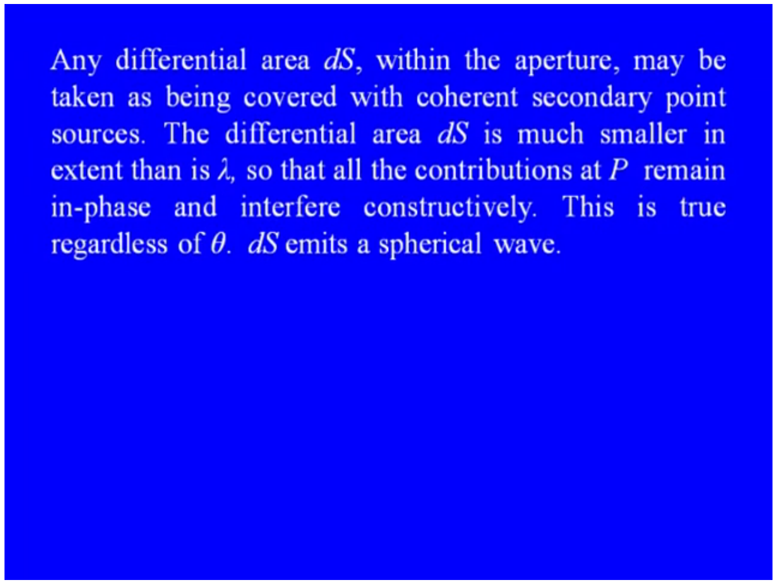

Any differential area dS, within the aperture may be taken as being covered with coherent secondary point sources. Individual sources socialize really with the differential area dS, is much smaller an extent than is lambda so that all the contributions at P, remain in phase and therefore interfere constructively. This is true regardless of the angle theta, dS emits a spherical wave.

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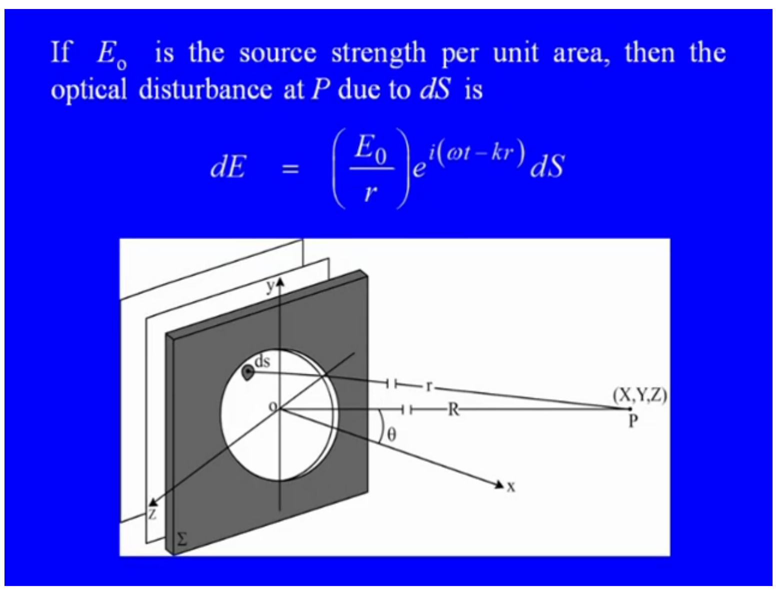

Now if E0 is the source strength per unit area, then, the optical disturbance at point P due to a small elementary area d, is in the aperture is given by E naught upon r. This r in the denominator is because of that it is a spherical wave and E raised to the power I Omega t - Kr. That is the usual wave face vector times the area dS.

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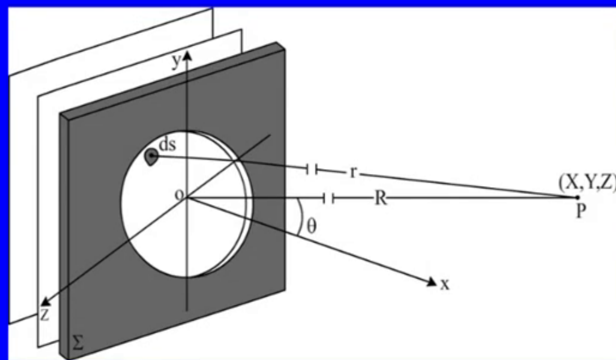
The source strength is assumed to be constant over the entire aperture.



You see, the sources strength are assumed to be constant for the entire aperture. The reason for this is because the incident wave front is that of a plane wave front which leads to a uniform intensity pattern on the surface of the aperture.

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The distance from dS to P is

$$r = \left[X^2 + (Y - y)^2 + (Z - z)^2\right]^{1/2}$$

The Fraunhofer diffraction condition occurs when this distance approaches infinity.

Now the distance to dS from dS to point P naturally is given by this expression, small y is not appearing, small x is not appearing here because that is 0, for the plane of aperture. So this is the distance capital X square + Capital Y - y square + Capital Z - z square whole raised to the power half. And Fraunhofer diffraction condition occurs, we know this is, that this distance must approach infinity, must be very large, in the present case would be very large compared to the size of the aperture.

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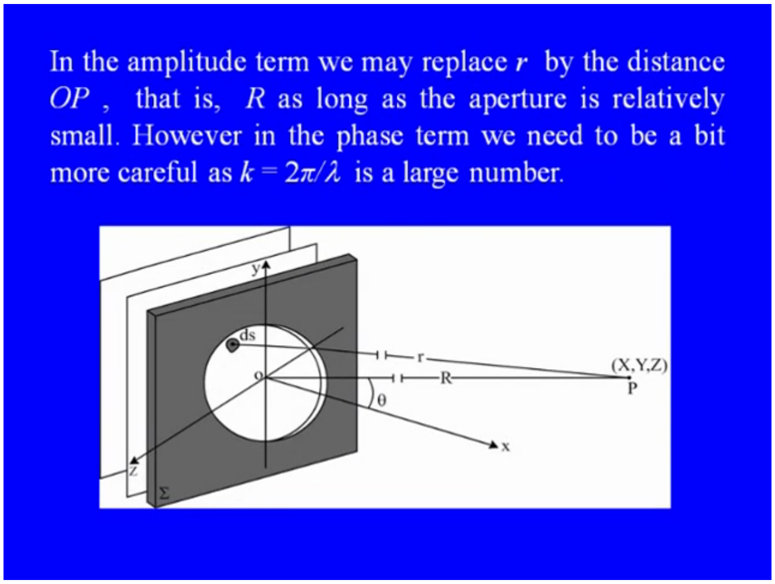

Now in the amplitude term we may replace small R by the distance OP, distance will be the distance of the point P from the center of the aperture. So, we can replace a small r by capital R as long as the aperture is relatively small and it is really small compared to a small r or capital R. So, this we can approximation can be made in the amplitude term. However, in the phase term we need to be a bit more careful. The reason is you are keen to small rk are appearing there and k is pretty large it is 2 Pi by lambda, lambda is a small number.

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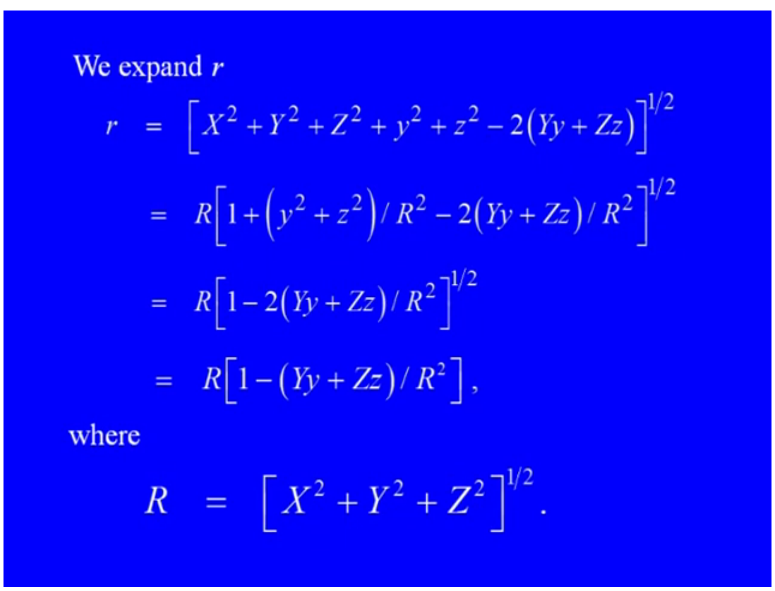

So, for the phase term we expand small r, okay. Write it as capital X square, capital Y square, capital Z square. Then, we have + Y square + Z square and - twice Capital Y small y + capital Z small Z whole square root. Capital X square + Capital Y square + capital Z square gives me capital R square. So, we take R common outside within the square brackets then, we have 1 + Y square Z square upon R square and the last term which is - twice a Yy + Zz upon R square whole square root.

We neglect cropped the second term which is y square + X square, these are very small numbers. They are controlled by the size of the aperture and we know that the size of aperture is very small, compared to capital R. So, we drop this term. The result is small r is now = capital R and then 1 - twice Yy + Zz upon R square, whole square roots. Now expanded to binomial expansion so that whole factor drops done and now we have finally capital R multiplied by 1 - this term Yy + Zz upon R square.

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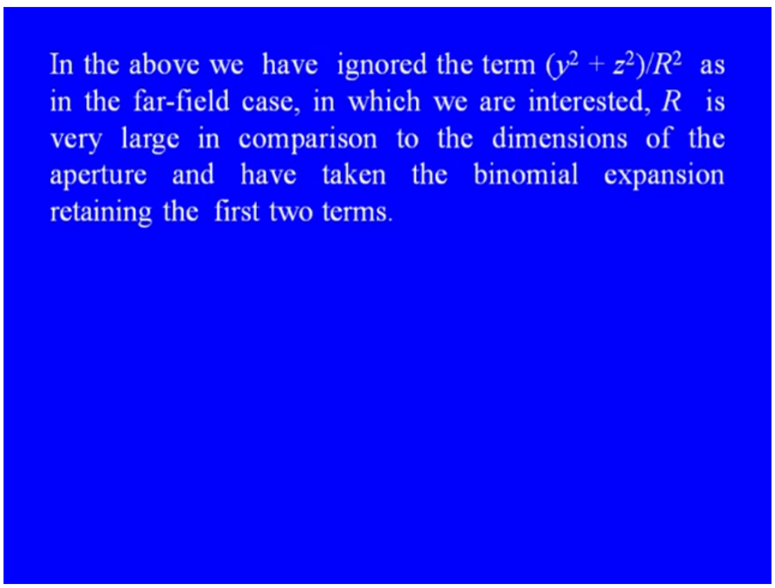

Now in the above, as I pointed out earlier, we have ignored the term y square + z square upon R square as in the far field case, in which we are interested, R is very large in comparison to the dimensions of the aperture and have taken the binomial expansion retaining only the first two terms.

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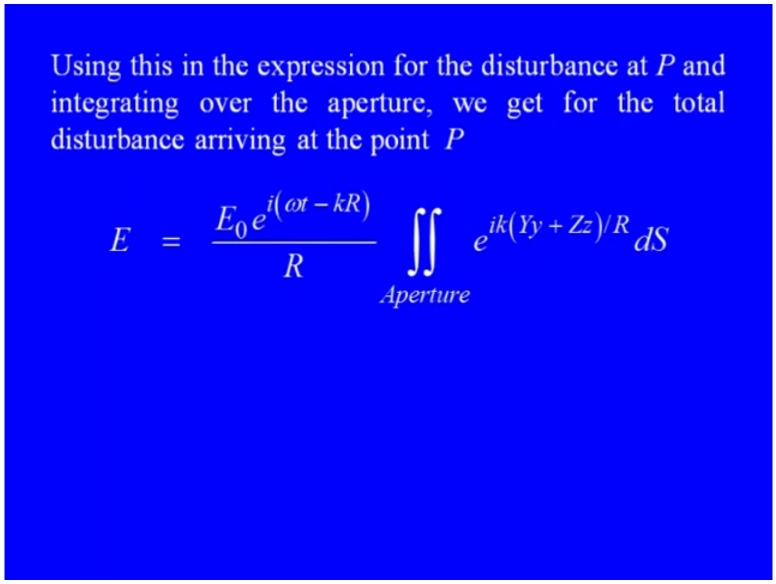

Now, using this expression for the disturbance at PE and then integrating it over the aperture to get the full contribution. So, we get for the total disturbance arriving at the point P, E is = E naught into the usual wave face vector e raised to the power i Omega t - k capital R now here divided by R and an integration over the aperture which is a two dimensional integration. The integrand is e raised to the power i k, Yy + Zz divided by R to be integrated over the whole area of the aperture.

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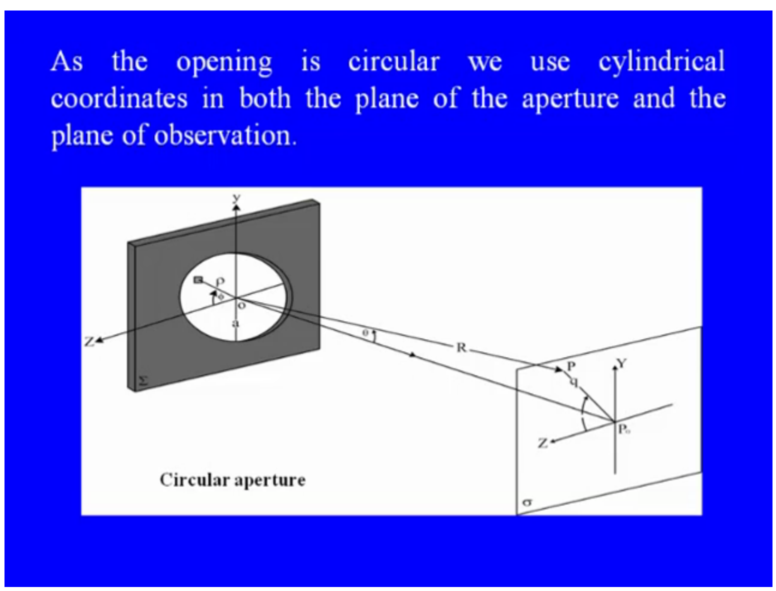

Now as the opening here is circular we use cylindrical coordinates in both the plane of the aperture and the plane of observation for a small y and small z in the aperture, but should we use

Rho and Phi. And here for the point P, where the intensities to be calculated for capital Y and capital Z, we use Q and capital Phi.

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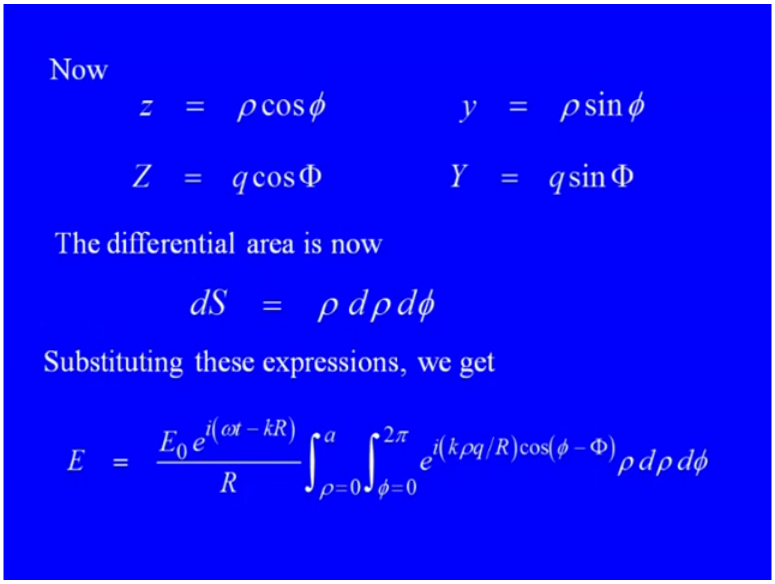

So now, this small z is = Rho cos Phi a small y is Rho sine Phi, Capital Z and capital Y these are for the point P Capital Z is = q cos Phi, Capital Y is = q times sine of capital Phi the differential area dS, which was dy dz now is 0 and Rho d Phi, so, Rho d Rho d Phi. Substituting these expressions in there, in the expression for the electric field, for the intense, for the strength of the disturbance, E is now = E naught times the phase factor as before e raised to the power I Omega t - K capital R divided by R.

Remember again, the division by RL because the disturbance is a spherical wave. To be integrated now, over Rho, the range is 0 to a, a is the radius of the aperture and over Phi making one complete round so Phi varies from 0 to 2pi. The integrand is e raised to the power I, k Rho q divided by R and whole of the sector times cos of Phi - capital Phi Rho d Rho phi. This is the expression we have. This is to be evaluated. This is in the cylindrical coordinates which are naturally appropriate for this problem.

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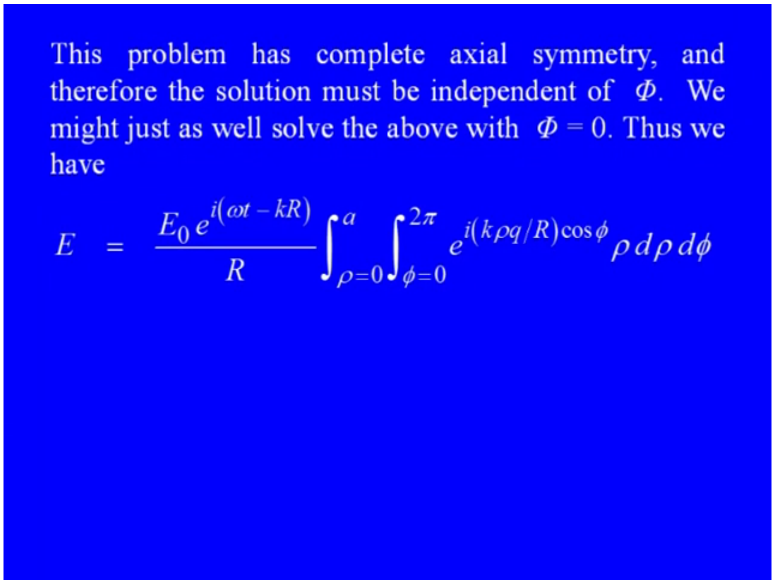

Now this problem has complete axial symmetry. I mean symmetry above the x axis which is perpendicular to the aperture and therefore the final solution must be independent of Phi. The capital Phi does not mean anything, cannot influence, effect the final solution because of the symmetry about the x axis. We may choose it conveniently might just as well solve the above expression taking capital Phi = 0.

We do that and thus the expression now becomes E is = the E naught into the phase vector is the same thing as before divided by R, all no change integration over a from 0 to a integration over Phi 0 to 2 pi as before. The only change here is that e raised to the power I k Rho q divided by R times cos Phi in place of cos of Phi - capital Phi Rho d Rho d Phi.

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The portion of the double integral associated with the variable φ

$$\int_0^{2\pi} e^{i(k\rho q/R)\cos\phi}\, d\phi$$

is the defining relation for the Bessel function (of the first kind) of order zero. It is equal to

$$2\pi J_o(k\rho q/R)$$

Now let us consider, we see, the double integral over Rho and Phi. Let us consider the integration Rho over PI, 0 to 2pi one complete round, e raised to the power i k Rho q upon R the factor times cos Phi d Phi. Now, this double integral which appears in the expression for e, this is the defining relation for the Bessel function, Bessel functions of the first kind and of 0 order.

This integral is = 2 pi times J0. J0 mean of the order 0 they and the argument of the Bessel function is k Rho q divided by R.

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The expression for the total disturbance reaching the point P now becomes

$$E = \frac{E_0 e^{i(\omega t - kR)}}{R} 2\pi \int_0^a J_0(k\rho q / R)\rho\, d\rho$$

So, we substituted this back in the expression for E, so the total disturbance reaching the point P, now becomes Phi integration has already been done, so, now the expression is E is = E naught the single phase factor divided by R, 2 pi an integral from over Rho from 0 to a. Integral is now over J0 argument k Rho q upon R Rho d Rho. So now, the integral over Rho is to be performed.

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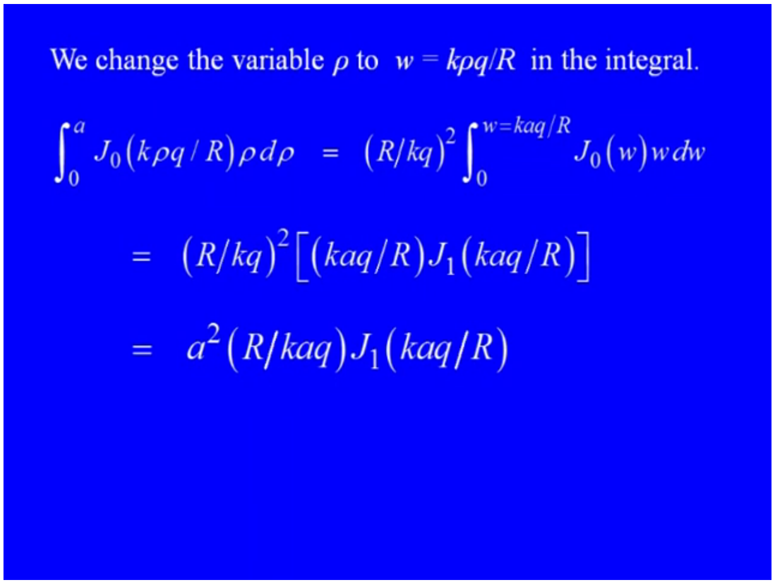

To do that we change the variable Rho to say w = k Rho q upon R in the integral so, this integral from 0 to a J0 k Rho q upon R Rho d Rho now becomes R upon kq squared integral over w now 0 to w = k a q upon capital R the integrand is J0w w dw. This again can be carried out. The result of this integration is there in the second line. So, we get R upon kq squared and the result of integration is ka q upon R times J1 of k q upon R. J1 is again Bessel function of the first kind of order 1. Now, we rearrange the variables outside, outside the Bessel function.

So it can be written as a square remember, a is the radius of the circular aperture so a square times R upon ka q times J1 of ka q upon R.

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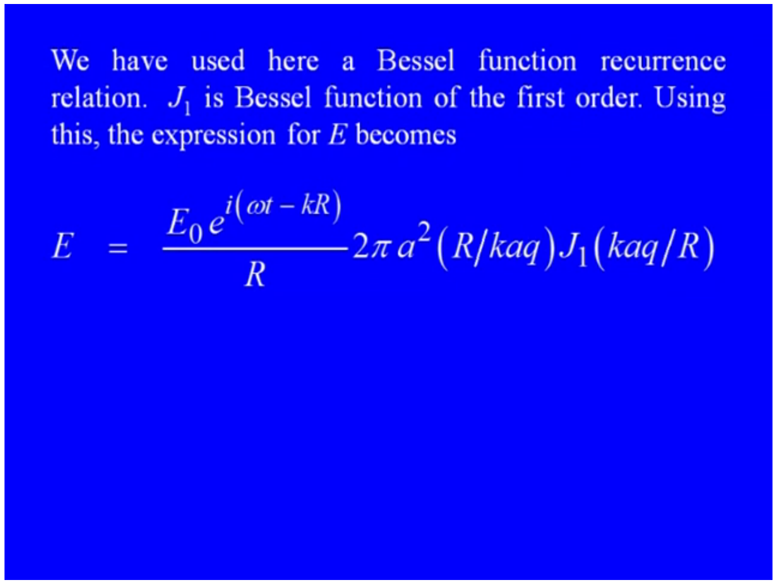

You see, in this derivation integrating over J0, we have used a recurrence relation of a Bessel function we have invoked the properties of the Bessel function. We shall not go into really into the details of these recurrence relations, All right. Using this expression for E now, we have E is = E naught E of i Omega t - k R as before, divided by R, multiplied by 2 Pi a naught a square into R upon ka q + and J1 of k a q upon R. Remember, 2 pi a square is the area of the aperture.

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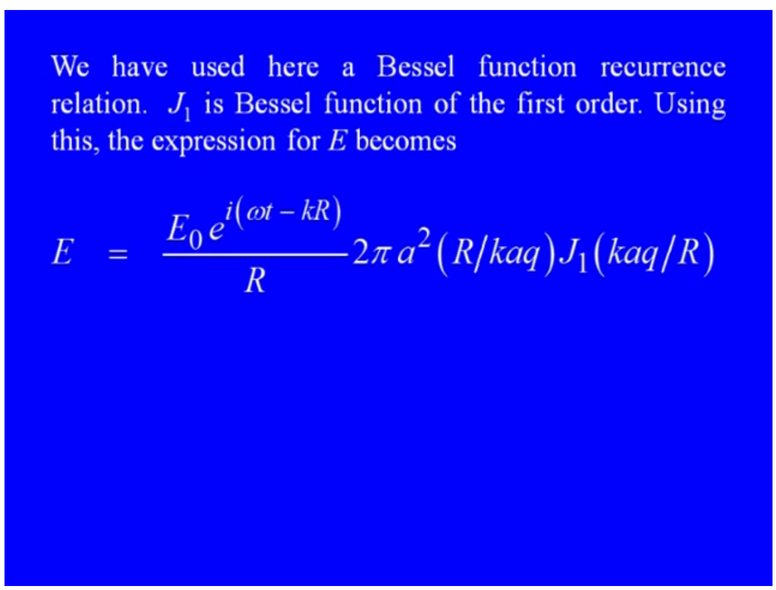

Now, let us spend some time and Bessel function. We shall not go into the real details of these functions. It is enough if we just appreciate that they are just like sine and cosine functions. The differences is they are slowly decreasing like then oscillation functions.

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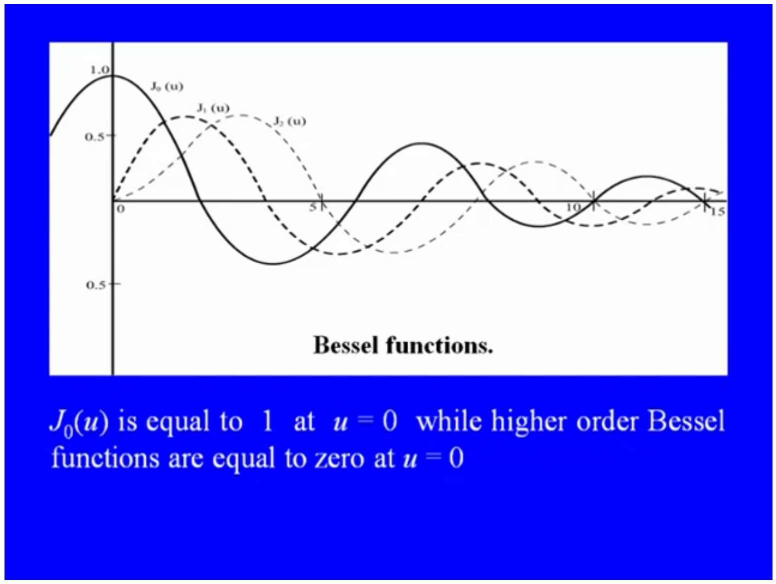

This shows a graphical representation of the various special functions. J 0 of u, u is the argument as a function of u, J1 of u, J2 of u. The main feature, this should be noted is, that these are oscillatory functions, them the oscillate, the amplitude goes on decreasing and J0 is = 1 at 0 argument at u is = 0 while higher order ones they all vanish at u = 0.

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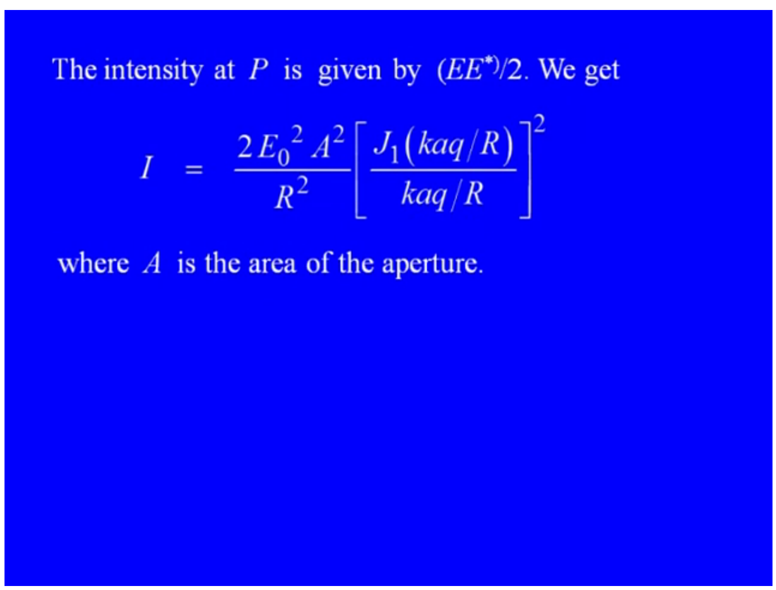

So now, using all this let us calculate the intensity at the point P,naturally, the intensity is given by E into complex conjugate of E divided by 2, that is a very standard expression. So, we get for the intensity at the point P is 2E naught square capital A square. Remember, capital A is the area divided by R square times the square of this combination J1 of k a q upon R divided by k a q upon R.

The expression for R is the intensity. One thing we should note here, is see this is the cylindrical symmetry. So, the position of the point P on the screen is really defined by q. q is sort of a radial distance on the screen. Everything else is fixed; R is taken care by the distance of the screen, A takes care of the size, k is 2 pi by lambda determined by the wavelength capital A is the area and E naught square depends on the intensity of the incident wave.

This is really a function of the radial distance on the screen of the point P determined by q.

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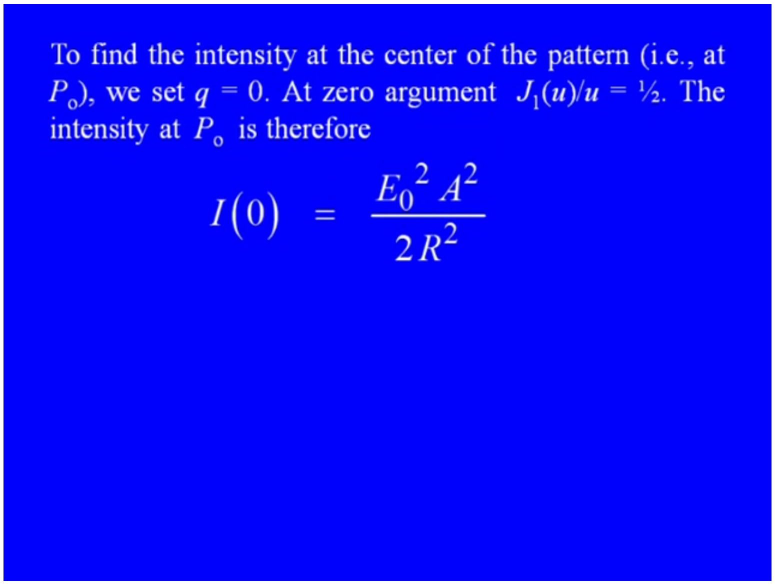

Now to find the intensity at the center of the pattern that is corresponding to q is = 0 which means, the argument of Bessel function will be 0. We again use the property of Bessel function, you know, J1of u goes to 0 but J1 of u upon u goes to half. So, we can use this to obtain the intensity at the point P0 that is in the exactly in the forward direction, that intensity is given by E naught square capital A square capital A then, that is the area divided by 2 R Square. So, let us substitute this expression.

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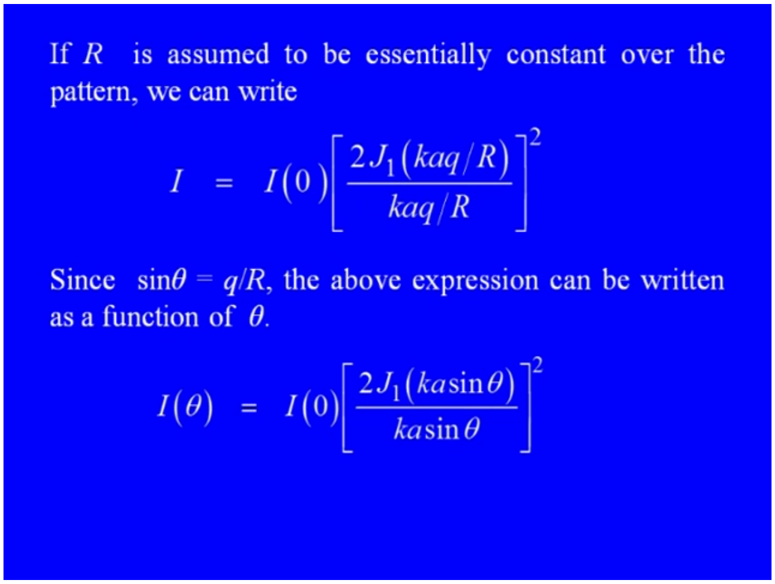

Finally the intensity now appears in this form I is = I naught, twice J1 of k a q R divided by the factor k a q upon R and holding the squared. Now, sine theta, remember that, so that is q over R so, we can also write it in terms of sine theta. So the above expression can be written in this form, as a function of theta is = I0 and the square of this quantity twice of J1 K a sine theta upon k a sine theta.

If you remember this expression is very close to what one has in a rectangular single slit. There, the expression comes out to be in the form of some sine beta upon beta. Beta is some function like this. In place of sine beta over beta, we have J1 of beta over beta. I mean, that is the qualitative change in the expression, when the rectangular slit, thin slit basically changes into a circular aperture.

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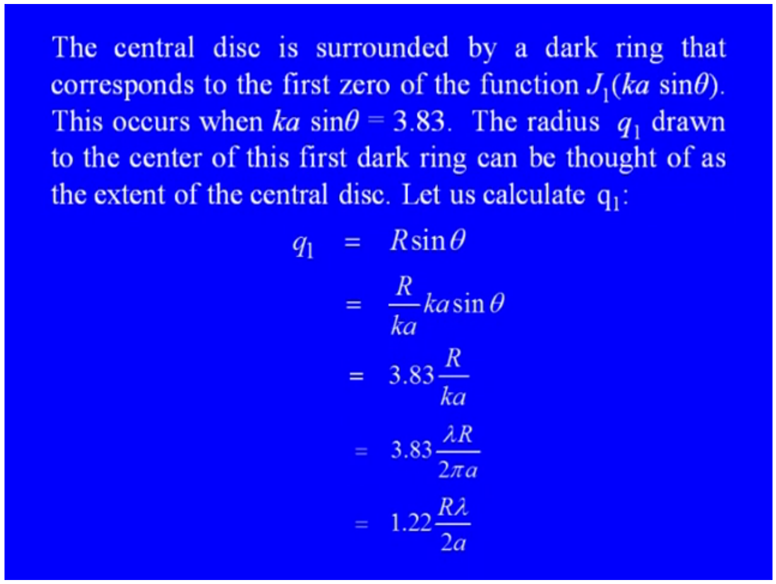

Now let us study this expression the intensity is maximum as theta = 0. Now the central disc, this is surrounded by why I am calling it a disc, because of the symmetry about the x axis, the central disc is surrounded by a dark ring that corresponds to the first 0 of the Bessel function J1, J1 k sine theta. And this 0 occurs, when the argument k sine theta is = 3.832. The radius q1 drawn to the center of this first dark ring, we can, can be thought of as the radius of the central, the size of the central disc, the extent of the central disc.

Let us calculate this q1. The q1 is = R sine theta. I divided it, multiplied by k a. ka sine theta, I put the value 3.832 for the first 0 of J1 multiplied by R upon ka k is 2 pi by lambda. So, I have that expression now 3.83 divided by PI that gives me 1.22, so, 1.22 Rlambda by 2a. 2a is the diameter of the aperture.

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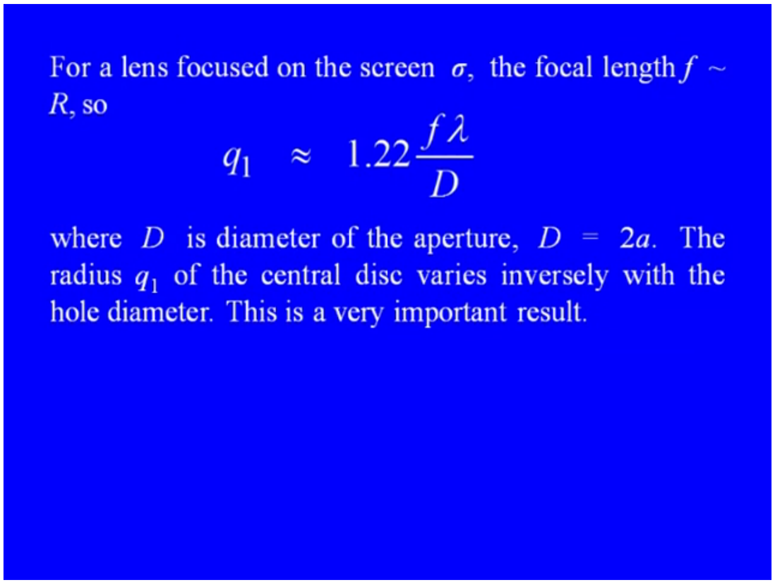

For a lens focused on the screen, the focal length will replace R. So, if the lens has been used then in that case q1 will be given by 1.22 f lambda by D as I said D is the diameter of the aperture. Now, the important result here is the radius q1 of the central disc which is the most important, varies inversely with the whole diameter. That is the important result.

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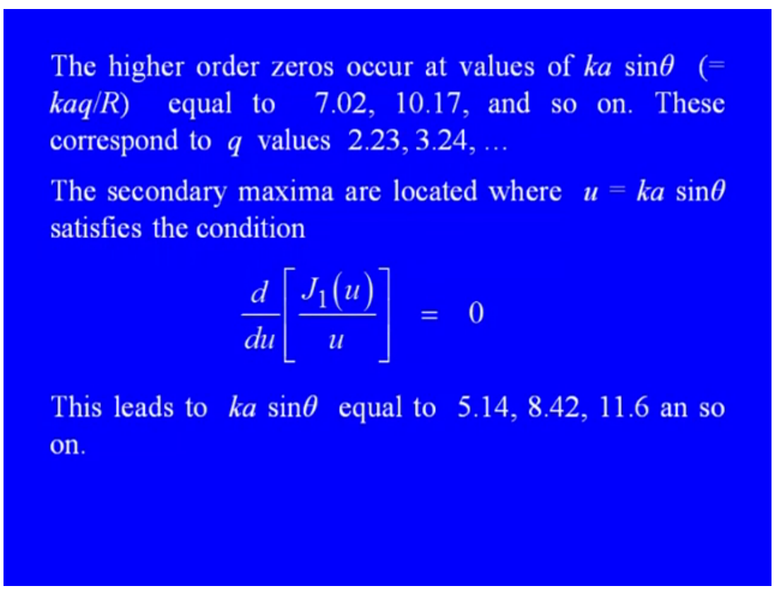

Now the higher order 0es of the Bessel function which will, they will lead to rings around the central disc they are given by when k sine theta which is = ka q upon R. Other values are 7.02. You see, first 0 was at 3.382 to that at present that we had just now. But then, the other next 0 is at 7.02,10.17 and like this if we divide these by pi then they correspond to the values 2.23, 3,24.

Remember, for in the first case, it was 1.22. Now, when these minima are there corresponding to 0s of the Bessel functions in between there will be secondary maxima. Naturally they are located, where the argument k sine theta satisfy this condition you see, the intensity depends on J1 u upon you so it is derivative, put it = 0, get the value of u. And when this calculation is done the k sine theta comes out to be = 5.14.

So, remember you had the first 0 then at maximum at 5.14 minimum at 7.02, again a maximum at 8.42 minimum at 10.17, a maximum at 11.6 and so on.

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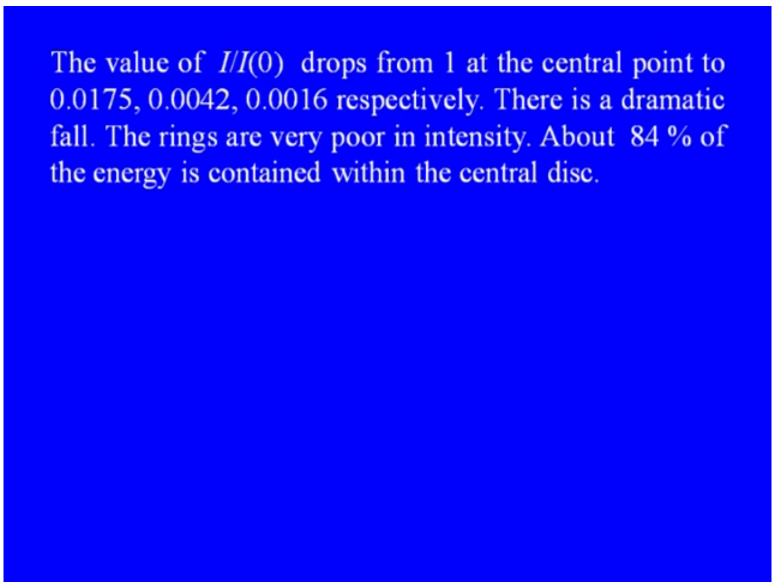

Now the value of I upon I 0 that is the interesting thing, it drops from 1 at the central point that is the important thing. The central maximum if we take its intensity as 1, then, relative to that the intensity of the first of the first amidst the secondary maxima away from the center 1 that is the intensity in the first ring, is 0.0175 much less in the second ring 0.0042 is still smaller in the third ring when 0.0017 is very, very small compared to centre.

So basically in actual practice, what we will observe? We will observe wave but in bright central disc whose radius is determined by the diffraction pattern and that factor comes out 1.22. That is the important factor and then there is a dramatic fall in the intensity. The Rings are very, very poor there but and calculation shows that about 84% of the total energy is contained within the central disc.

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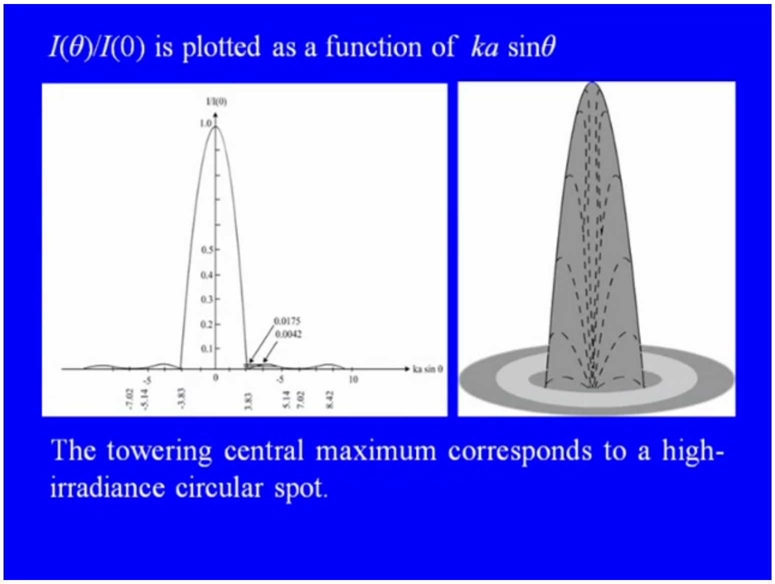

Here we have a plot; you see this on the right side this towering central maximum corresponds to the higher radiance circular spot, carrying an intense total energy content of 84%. Then it shows a ring, second ring. On the left we have a graphical picture, showing the intensity of the central maxima and related to that, we get a minimum at 3.83 then we get another minimum at 7.02 really 7.016 in between there is a maximum.

Similarly a maximum, so this sort of picture there but the essential feature is, we can forget about the rings; the main thing is the central disc carrying most of the energy and the radius of this central disc, which is important.

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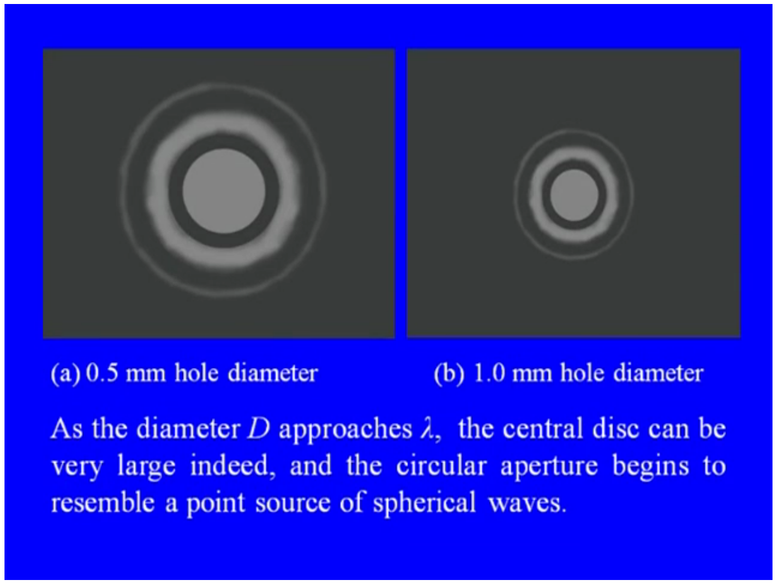

This picture shows for a aperture diameter of point 5 millimeter hole, a pattern will appear like this. For a 1 millimeter hole diameter, the main thing is as it D approaches lambda D becomes very very small the central disc can become very large indeed. And the circular aperture begins to resemble a point source of a spherical wave coming out in all directions, so the size of the disc becomes infinite.

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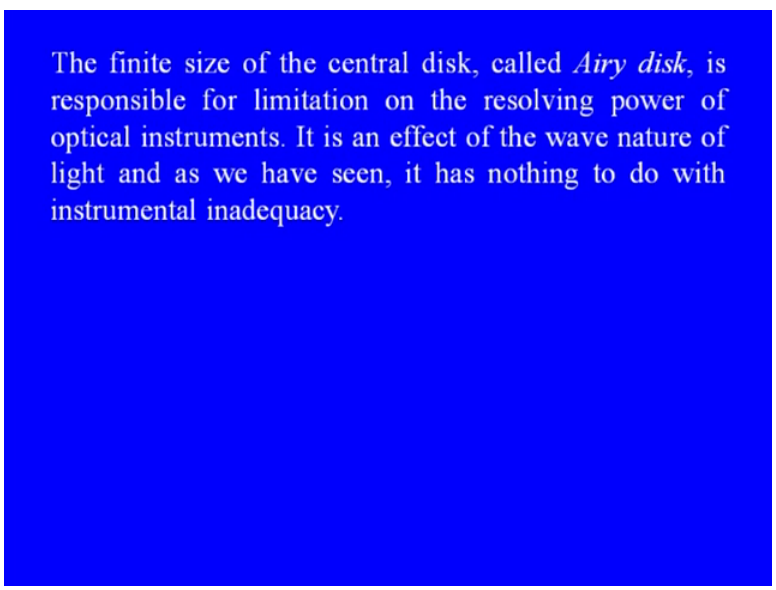

Essentially, to the finite size of the center disc this is called a Airy disk, and is responsible for the limitation on the dissolving power of all the optical instruments. You see, it is an effective wave nature of light and as we have seen it has nothing to do with, with the instrumental inadequacy, even if we have perfect optics, all the lenses completely aberration free, monochromatic aberration as well as chromatic aberration everything.

Even then, the size of this the airy disc may be there and its style, its radius, will depend on, on the aperture of the, I mean, aperture in the clean basically on the size of the lens.

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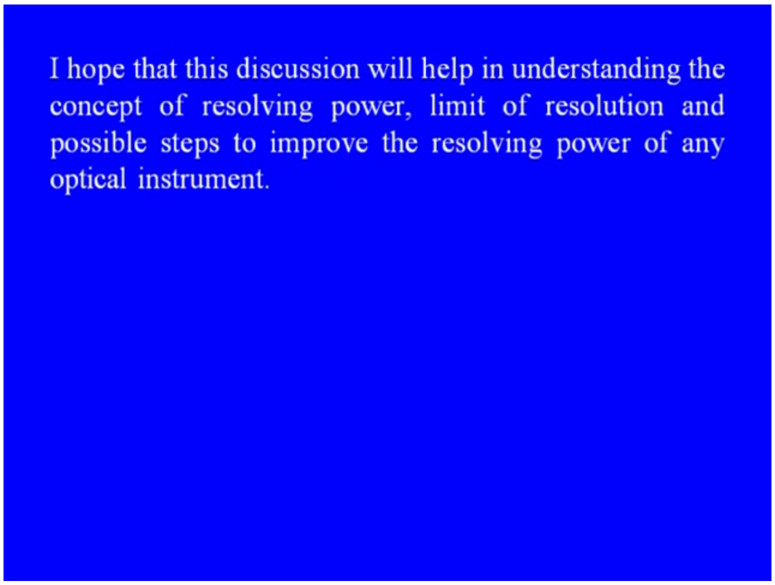

That is the basic result and this is coming out because of the wave nature of light. It is a basic diffraction effect, okay. So, I hope this discussion will help in understanding the concept of resolving power, limit of resolution and possibly steps, to improve the resolving power of any optical instrument. Thank you.