Quantum Physics

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Lecture No # 05

Eigenvalues and Eigenvectors Part-02

I started mentioning what the adjoint of an operator was. We still have a little bit of mathematical preliminaries to do but then we need to also get started with the quantum mechanics. So 1 of things we do need however is the idea of linear vector space which is infinite dimensional.

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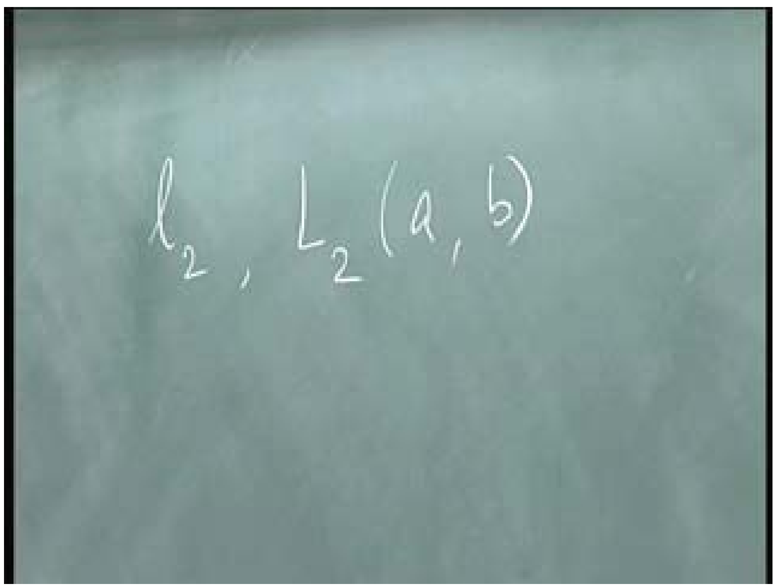

And I mentioned two notable examples of infinite dimensional vector spaces 1 of which was l_2 and the other was L_2 in some specified interval. So, we talked about the space of square summable sequences and the space of square integrable functions defined for some real variable x running between a and b. Now these 2 spaces have a special role to play in quantum mechanics because you will see as we go along that the states of a system would be described by vectors in a linear vector space and the linear vector space would generally belong to 1 of these 2 spaces. So this is the physical reason why we are actually interested in these 2 spaces in particular.

Now when we talk about infinite dimensional spaces there is also the question of convergence. I didn't pay much attention do it so let me spend a few minutes and talk about convergence of states in vector spaces.

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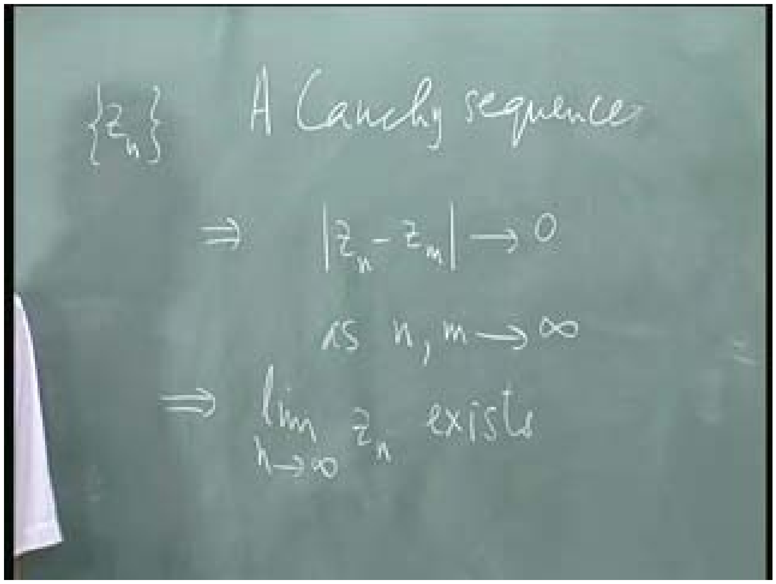

Notice that if I give you a sequence of complex numbers Z_n, Z_1 Z_2 Z_3 etc formed according to some rule. The natural question that arises is does this sequence converge to a limit or not. As n tends to infinity, does it converge to a limit or not. If it does, then that would be the limit point of the sequence and then 1 could ask what kind of sequences would converge and these are called Cauchy sequences. And a Cauchy sequence implies that the modulus of Z_n - Z_m tends to 0 as n, m tends to infinity. In other words if the sequence is such that the difference between Zn and Zm modulus goes to 0 as both n and m tend to infinity, then I call it a Cauchy sequence. Such sequences converge to limits definite limit points. This immediately implies that limit n tends to infinity, Z_n exists. It implies that this sequence has a limit point which is some complex number. This concept is carried over to state vectors. The idea of Cauchy sequence is what we need. Not all sequences are Cauchy sequences. You can easily construct sequences for example, if Zn is -1 to the power n, then of course even ones are +1, odds ones are -1 and this sequence doesn't converge to any limit at all.

On the other hand if this is true then this suffices to show that the limit of this sequence exits. Now as you know, a set of points which includes all its limit points is called a close set. So if you take the sequence Zn and you include the limit as n tends to infinity Z_n you include that limit point then that set is a close set or a close sequence.

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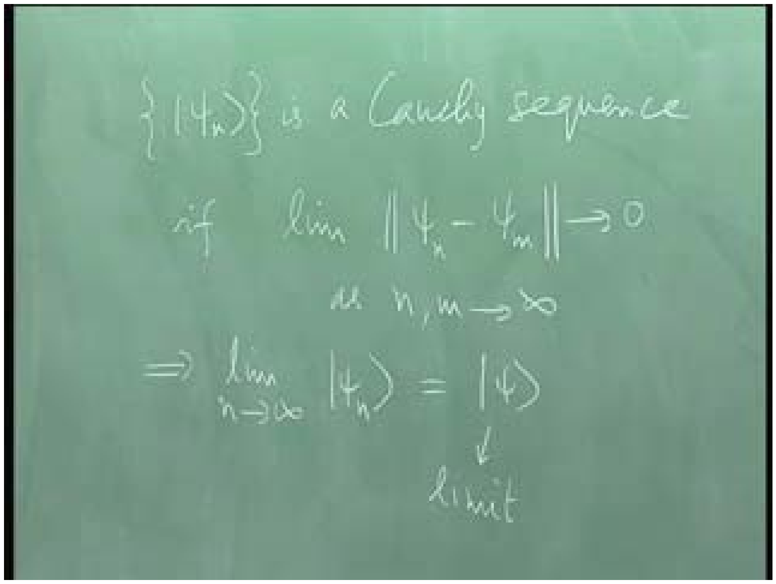

Now taking this over to the idea of vectors in a linear vector space, this sequence in a linear vector space psi $_n$ is a Cauchy sequence if limit of the norm of psi $_n$ - psi $_m$ tends to 0 as n, m tends to infinity. So this vector difference, psi $_n$ vector - psi $_m$ vector must have zero length in the limit as n, m tend to infinity. That sequence is called a Cauchy sequence. Now a linear vector space has a large number of vectors and there is no guarantee that every Cauchy sequence has a limit point which exists in the same space. If we are guaranteed that Cauchy sequences tend to limit points, this implies that the limit as n tends to infinity psi $_n$ = psi. This limit is the limiting vector. So every Cauchy sequence converges to some vector. But there is no guarantee that that vector is in the linear space. If it is then this space is called a complete linear vector space.

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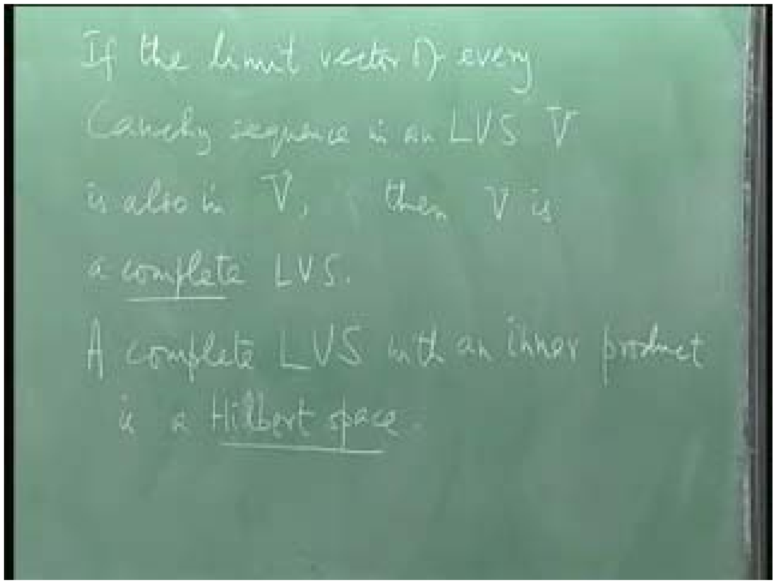

If the limit vector of every Cauchy sequence in a linear vector space V is also in V, then V is a complete LVS. So we will see why we need these ideas. For finite spaces this doesn't matter at all. It will turn out that every finite dimensional vector space is in fact a complete vector space because you form all sorts of linear combinations. You add them and so on nothing will happen. It will never go out of this space. On the other hand, in the case of infinite dimensional spaces you have to specifically make this requirement that the limit point of every Cauchy sequence remain in the linear vector space because we are going to serve operations with infinite dimensional spaces.

If this (Refer Slide Time: 08:36) is so, then the linear space is said to be a complete linear vector space. In quantum mechanics, we are going to use complete linear vector spaces all the time. In fact we also need the idea of an inner product. We have already talked about the scalar product. So a complete linear vector space equipped with an inner product is the space that we use in quantum mechanics and it's called a Hilbert space. It will turn out that these are the natural spaces in which the state vectors of systems exist. The reason you need these mathematical niceties, none of which I am proving, is to make sure that operations with infinite sequences and infinite dimensional spaces don't lead to errors. Now all finite dimensional linear vector spaces are complete and once you define an inner product on them, they actually are Hilbert spaces. It's only the nontrivial cases that are the infinite dimensional spaces.

Now even given a Hilbert space, we should now ask what about basis vectors in such a space. As you have seen, a basis set in a linear vector space is a set of linearly independent vectors which span the linear vector space completely. Once you are given a basis you can make it orthonormal and so on. Now given an arbitrary infinite dimensional space, there is no guarantee that you can find the basis set which is actually denumerable, namely you can count vectors in direction 1, direction 2, direction 3 all the way to infinity. If you can then it's a countable or denumerable basis. So 1 more a little piece of technicality is needed.

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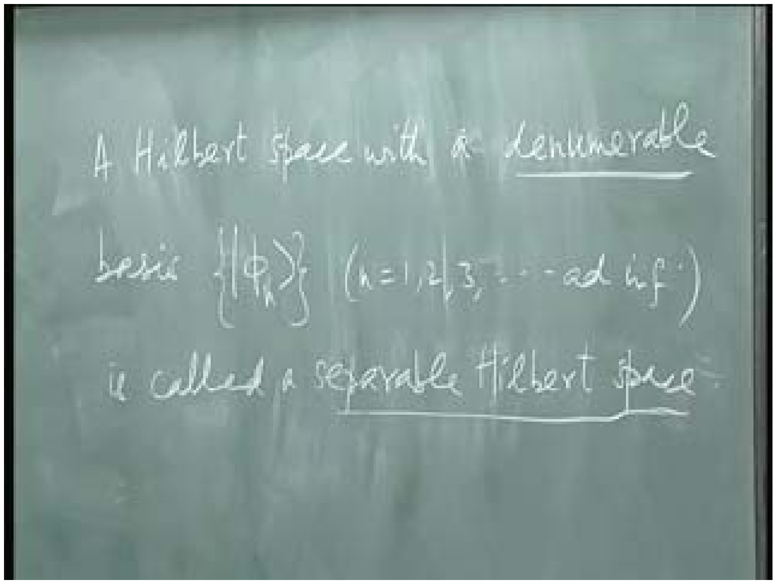

A Hilbert space with a denumerable basis, let's call it phi $_n$, n = 1, 2 till infinity, is called a separable. We are going to work specifically with separable Hilbert spaces.

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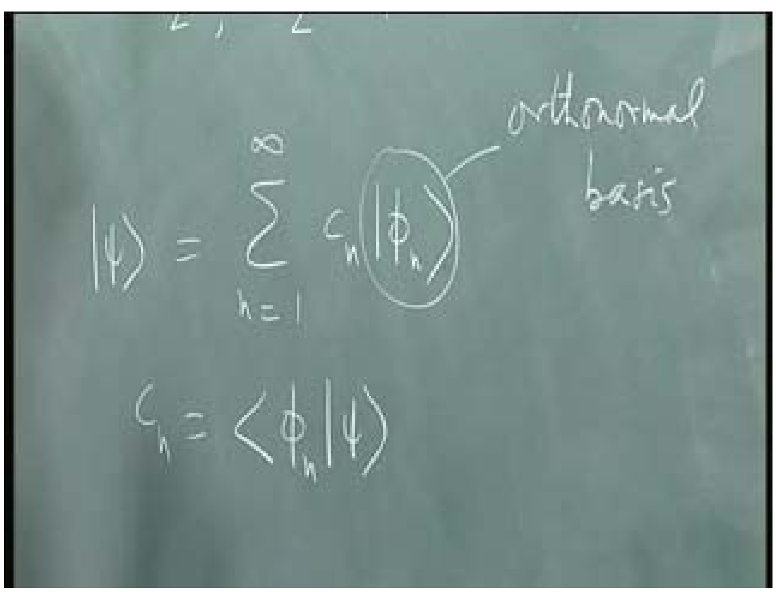

Now once you are given the fact that you are going to look at a separable Hilbert space of this kind, it means that any vector psi in the space can be written down uniquely in an orthonormal basis. If the basis is orthonormal it can be written down in the form n = 1 to infinity, c_n phi $_n$. so the c_n's are unique quantities. Given psi, you can find c_n uniquely and of course the inversion formula c_n is just phi $_n$ psi because this is an orthonormal basis. I will assume that whatever basis you give me, it's denumerable and I will make it orthonormal by the Gram Schmidt procedure. Then this (Refer Slide Time: 13:26) is the expansion and this is the inversion formula. So the set of number c_n suffices actually to specify this vector uniquely and moreover, the fact that we want these vectors to have finite norm.

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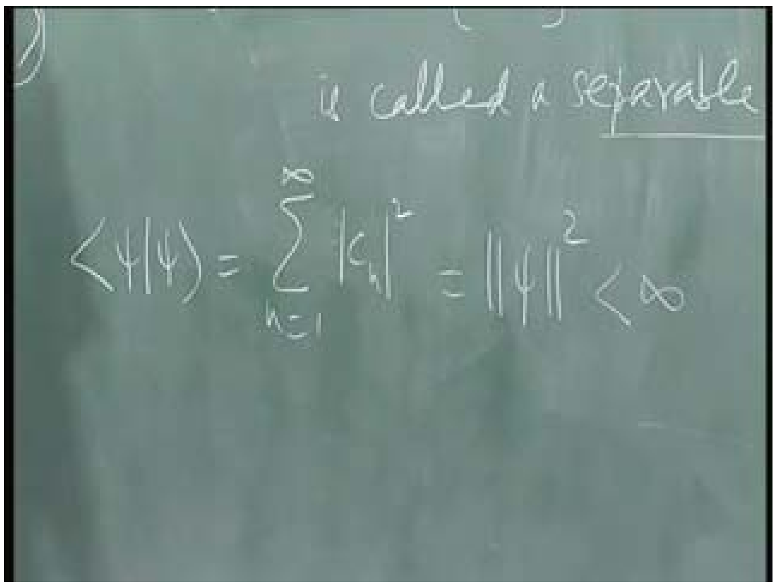

Which means psi psi is nothing but n = 1 to infinity mod c_n squared. This quantity here (Refer Slide Time: 14:07) is the norm of the vector. So by definition this is a square of a norm and it means this is finite. What do you call sequences in which this quantity is finite? They are square summable sequences, l_2. So you see how square summable sequences are going to play a very fundamental role in everything.

We start by saying our system is going to be described by a state vector in some separable Hilbert space equipped with some orthonomal basis. Since I am going to look at these vectors which are going to have a physical meaning and we are going to insist that they have finite norm, this quantity should be less than infinity, so our knowledge of the state vector can actually be summarized by our knowledge of these (Refer Slide Time: 14:48) coefficients here in this basis and these coefficients would belong to l_2. So that's how l_2 is going to play a very basic role in everything. Of course we also know how to go from 1 basis to another. Basis sets are not unique. We saw that the formula for translating from 1 to another is very straight forward. All you have to do is to use the completeness of these basis vectors and insert the identity operation at all times.

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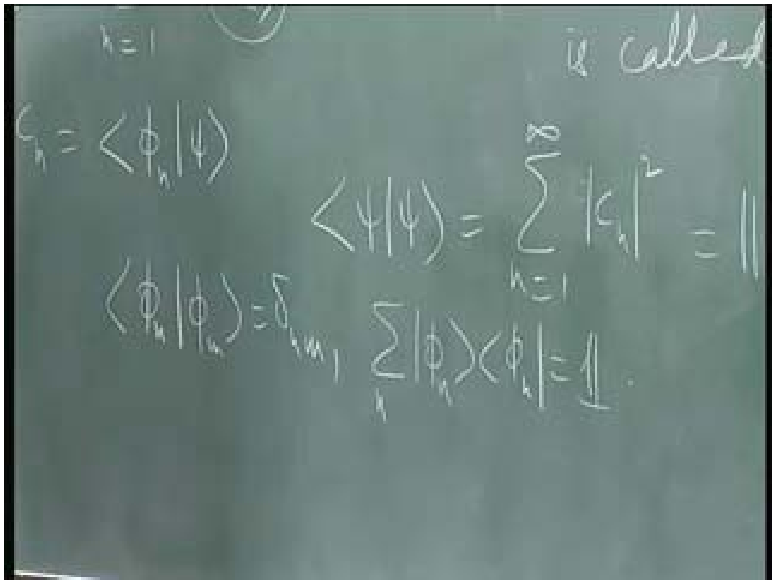

So in other words phi $_n$ phi $_m$ = delta $_{nm}$ and summation over n phi $_n$ phi $_n$ = identity operator. these are this is orthonormality and this is completeness (Refer Slide time: 15:30) and to go from one basis to another etc it becomes very trivial, as long as you go on inserting the identity operator in between.

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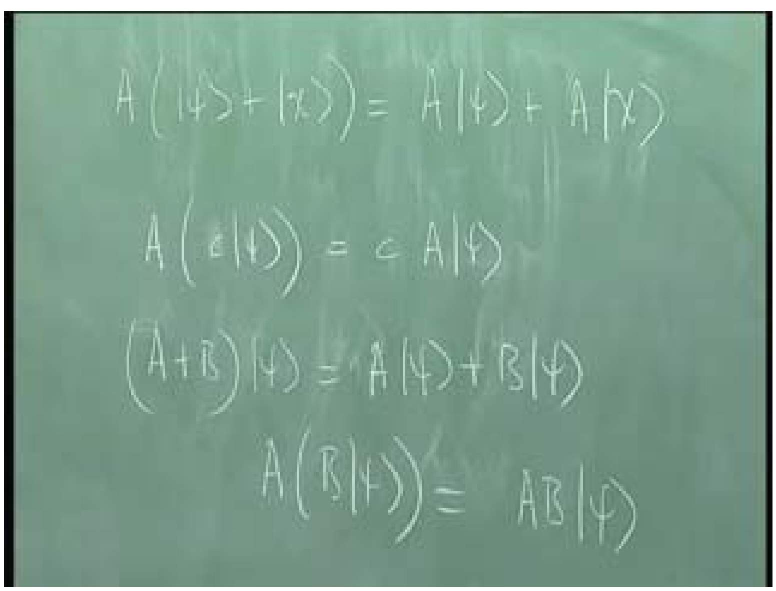

Now coming to operators, operators act on vectors in a linear space and linear operators are those which act in a linear manner. In other words A is a linear operator. If A acting on psi + chi = A acting on psi + A acting on chi. and A acting on c times psi, where c is some scalar, which multiplies this vector of a linear vector space, this is the same as c A psi. Moreover (A + B) acting on psi is the same as A on psi + B on psi, if A and B linear operators and A acting on B psi is the same as AB acting on psi by definition. The product of 2 operators is successive but not commutative.

Now let's put in a little bit of physics into this and I am going to do this by defining a whole set of postulates. These postulates have come based on our experience with quantum mechanics, ultimately based on experiments. But at the moment we can write down a fairly axiomatic formulation of quantum mechanics. Then of course the test is finally to ask if this way of calculation are compatible with experiment or not and to our best of our knowledge, over the last hundred years quantum mechanics has stood the test of time.

There are still many unresolved questions in it. There are very delicate questions of interpretation which are not been taken care of fully. But as a calculation tool, it is in fact quite robust and has been tested over and over again. As far as we know there are no violations at all.

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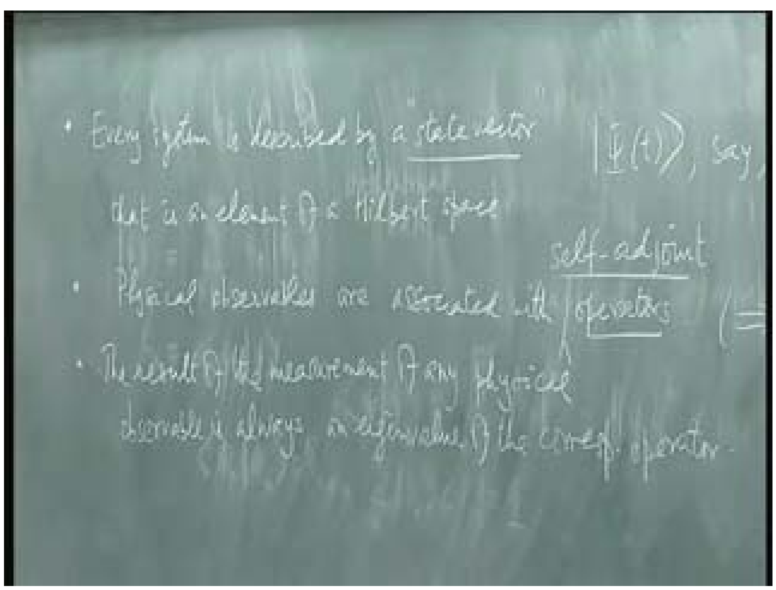

So whenever I make a postulate let me put a little dot here to show this is 1 of the axioms. I start with a very general thing and then of course we will make it very much more specific. This is not the most general way of describing systems in quantum mechanics, but for the moment we start with this.

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Every dynamical system is described by a state vector, say psi for the notation that is an element of a Hilbert space. Of course different systems would have different Hilbert spaces. Now what do I mean by a system? It could be one particle, ten particles, a collection of particles, a human being, all the people in this room, this earth, this galaxy etc. all these are examples of physical systems. And the statement is that every one of these systems is described by its own state vector.

And the presumption is that all the information you can get about the system is buried in the state vector. And the subsequent rules of quantum mechanics tell you how to extract physical information given the state vector. Since it's a dynamical system we are talking about, this state vector psi is a function of time. Right now we are talking about non-relativistic quantum mechanics. So I am assuming what you would assume in Newtonian mechanics namely, there is some frame of reference, there is an observer in the frame of reference etc.

And time is a variable in which dynamics occurs and the state vector is a function of time. And the next thing of course would be to say it changes with time. As the state evolves, it changes with time and the rule we need is to say how it changes with time. The analog of what you would say in Lagrangian or Hamilton mechanics where you would give rules or equations saying how the q's and p's of a system change with time. This is what we will do next. But we need a few more preliminaries. Every system is described by a sate vector. Then the question is what physical observables are in the system. What are the physical properties of the system?

For this purpose it's convenient to think of this system as a particle. That's the way you do mechanics too by starting with 1 particle and then you assign coordinates and momenta to it or coordinates and velocities. In a same way I think of a particle and it's in interaction with the rest of the universe and I assign a state vector to it. Then it has physical attributes. This particle has a momentum, velocity, position, angular momentum, energy and so on. In quantum mechanics, physical observables also called measurables are associated with operators. So for every physical observable, there is an operator for the position, the velocity, and the momentum and so on.

These operators need not necessarily have to be independent because if you look at the kinetic energy which is 1/2 mv squared for a particle, once you give me the velocity operator, I know the kinetic energy operator. But with every observable I associate an operator. Then the question is what values these operators can have.

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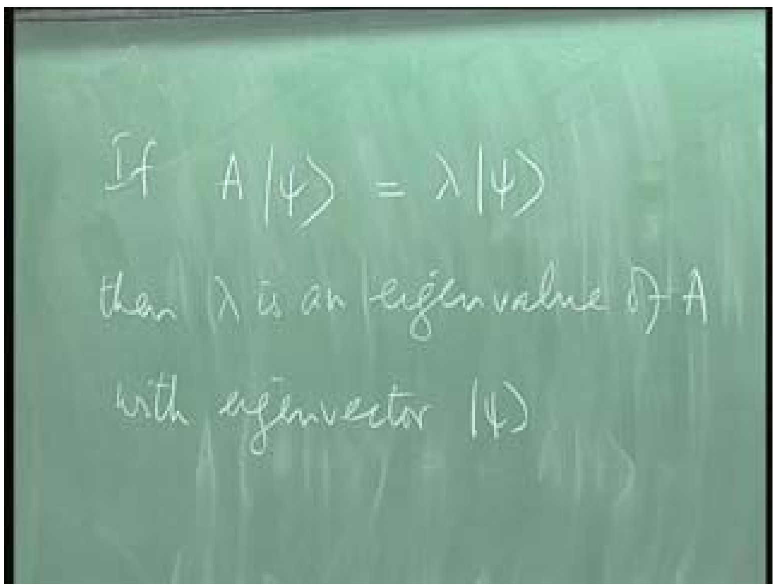

The natural way to do this is to talk about the Eigen values of this operator. so given a linear operator A in a linear vector space acting on a state vector psi and this is equal to lambda times psi for some special value of lambda and a special value of psi, then lambda is an Eigen value of A with Eigen vector psi. So this is going to be our basic equation.

Of course a given operator in an abstract sense we have lots of Eigen values think of a matrix think of a finite dimensional vector space then psi could be written as a column vector in n rows and 1 column, then every operator A can be written as n by n matrix. And when it acts on this column vector, it produces another column vector in general. But there are special column vectors such that when this matrix acts on those column vectors, it reproduces the same vector multiplied at best by a scalar. And those scalars are called the Eigen values and these are the eigenvectors, exactly as you have in matrices. Now of course for a given Eigen value lambda, you may have more than 1 eigenvector then you would say this Eigen value is a repeated Eigen value. If you have 2 or more linearly independent Eigen vectors corresponding to a given Eigen value, then the Eigen value is a repeated Eigen value or a degenerate Eigen value. It's also possible that the number of Eigen values is infinite if the vector space is infinite dimensional.

If it's an n by n matrix then you can at best have n Eigen values which could be n distinct complex numbers or maybe degenerate partially but in an infinite dimensional space, you may have an infinite number of Eigen values. So this is an Eigen value equation and the postulate we need is to make sure that the Eigen values which in some sense will be the only measurable values of an operator corresponding to a physical observable should be real. If the observable you are talking about is a real physical observable then the Eigen values should be real. And there is a theorem which guarantees the kind of operators which have real Eigen values and those operators are self-adjoint operators. Square matrices have real Eigen values. Let's talk about n by n matrices. So an arbitrary n by n matrix in general has n complex Eigen values. If the entries are real and the matrix is symmetric, you are guaranteed that the matrices have real Eigen values. But if the entries are complex, then symmetry is not enough. If a matrix is equal to its Hermitian conjugate then, the Eigen values are real.

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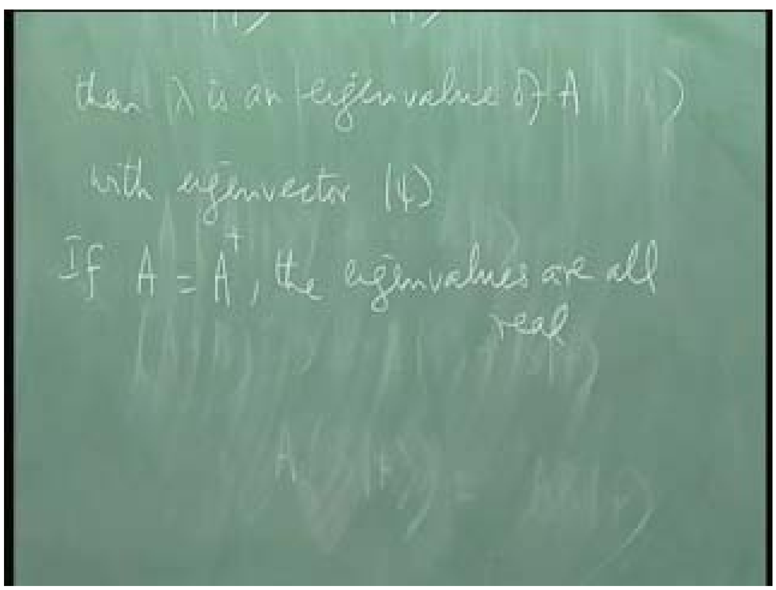

Now of course for matrices it's quite clear. If you give me a matrix I can inspect and find out if it's Hermitian or not. If it's Hermitian, the Eigen values are guaranteed to be real if not that's not true. But for arbitrary operators, maybe differential or integral operators and other complicated operators, you need a generalization of the idea for Hermitian conjugate and I am going to call them self-adjoint.

If you notice on an L_2 space, I defined the operator d over dx, the derivative operator. And then I said lets look at those vectors which remain in the space when (d / dx) acts on it. It then turned out that the adjoint of (d / dx) was – (d / dx). So that operator is not self adjoint and there is no guarantee the Eigen values are real. Now that's immediately obvious because if I put e to the power ix and I do (d / dx), I get i e to the power ix. So there is an Eigen value which is i.

On the other hand you are absolutely guaranteed that if the operator is self adjoint then the Eigen values are real. So, physical observables are associated with self adjoint operators. A little later when we look at problems involving 1 dimensional motion and so on then I will point out carefully the difference between a Hermitian operator and self adjoint operator.

Actually there is a certain difference and it's not necessary that a Hermitian operator be a self adjoint operator although in elementary treatment on quantum mechanics, these 2 words are used interchangeably because in the case of matrices, they mean exactly the same thing but we got to be a little more careful.

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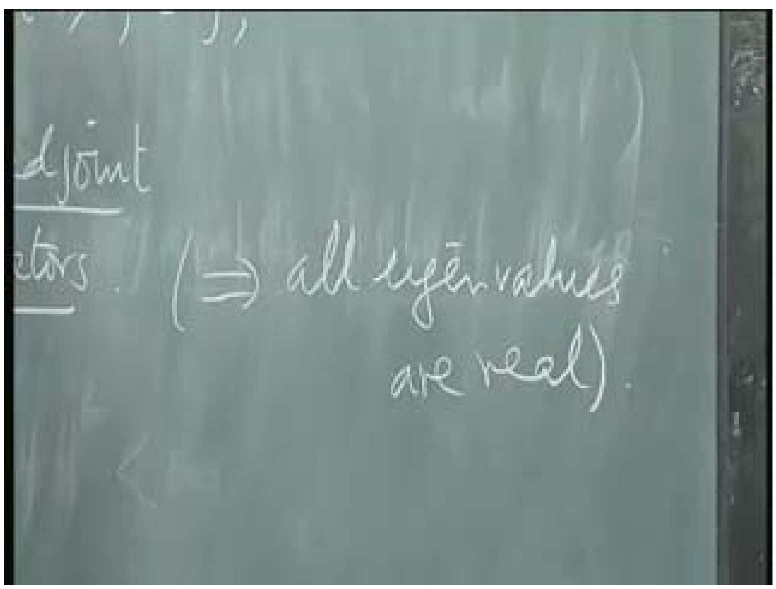

At the moment let me be rigorous and correct and say that the operators have to be self adjoint operators. This would imply all Eigen values are real. Also the number of Eigen values of an operator maybe infinite. It's also possible that the Eigen values are not even countable. For instance if I look at the position of a particle moving on a continuous line - infinity to infinity, this position can take continuous infinite number of values .Therefore we must allow for the fact that the set of Eigen values of an operator may actually be a continuous set. The Eigen values are not even countable. That's a very important possibility and we need to take care of it. So right away it says that matrices may not be enough to do this whole thing because there would be more general operators which would have continuous sets of Eigen values. On the other hand I look at the matrix I know its Eigen values are all countable. Even if it's infinite in number, it's still countable. So we must allow for this possibility. At moment I simply say all the Eigen values real. The physical quantities are associated with self adjoint operators. These operators will act on the state vector and possibly produce other state vectors and so on.

Then the result of the measurement of any physical observable is always an eigenvalue of the corresponding operator. So you make a measurement on a system described by some state vector the measurement has to be designed by you cleverly to measure some physical quantity and the result that you record will be a real number and it's guaranteed to be 1 of the Eigen values of the operator associated with this observable.

What is not clear and what is not definite about quantum mechanics is that you cannot say which eigenvalue is going to emerge. This is where quantum uncertainty comes in. it will be 1 of the eigenvalues but we can't say which 1 before the measurement. On the other hand you can repeatedly make lots of measurements, take the average over all the eigenvalues and you would call that the mean value of this observable.

So you can do this by preparing identical imaginary copies of a system, making a measurement on copy 1, copy 2, copy 3, etc all the way, take arithmetic average of the results. That would be a distribution or an ensemble and the result would be the mean value or the expectation value of this operator. What quantum mechanics does is to give you a rule for calculating the expectation value.

So let me repeat this again because it's a very basic point. It is imagined that every system is described by a state vector. I would like to take this system and measure, let's say its angular momentum. In classical physics, once I identify the generalized coordinates, generalized momentum and so on I have a formula for the angular momentum in terms of the q's and p's. And in principle there is nothing to stop me, given initial conditions, finding what these q's and p's are, at a later instant of time. I can calculate its angular momentum. You can then measure it and test it against the experiment. If the system is a quantum system, then if it's in some state, we don't know what state exactly it is in. I measure by an apparatus designed to measure angular momentum. The answer I get is going to depend on the state in which the system is but the answer I get even for a given state of the system in general cannot be predicted before the measurement. Even with the knowledge of that state, you cannot predict the answer.

What you can do however is to be sure of however is it, when you make this measurement the result that comes out will be 1 of the angular momentum operator. Which Eigen value it would be, you can't say before the measurement. So one imagines making an infinite number of copies of the system, all in the same state and then perform measurements or different people perform measurements on different copies of the system and then you record all the answers.

The first measurement maybe Eigen value lambda 1 emerges, in the second measurement Eigen values lambda 6 emerged and so on. You add up all these Eigen values and take the arithmetic average in the limit in which the number of measurements is infinite. That would be the average value of this physical observable and it's called the expectation value. And quantum mechanics is going to give you a rule for calculating the expectation value. This is what it will do and that's going to be our target. Of course it will also give you a rule for calculating the scatter about the expectation values. It's no good just giving you 1 number. It's like giving an average. It will give you the mean, the mean square, the mean cube and so on. It gives you rules for calculating everyone of the moments.

Now it's conceivable of course that if I measure make a sufficient number of measurements on my physical system and compute a sufficient number of moments of each of the these observables. Then in principle, I know everything I need to know about the system and all information I want about the system is acquirable. So our target would be to find out what kind of measurement should be made on a system in order to find out all possible physical information. Now this is the probabilistic content of quantum mechanics namely; it gives you a rule for calculating average values. So in that sense, it is a probabilistic theory.

Student- Does it mean that quantum mechanics does not provide with time evolution changes? Professor - we are going to come to that The rule that I am going to talk about is going to depend on the state vector. it says in the state, this is the expectation value. Now of course if the state changes from 1 instant of time to another, you need to know what the rule is by which the state vector changes. And that rule will be 1 of the postulates and that rule is precisely the Schrodinger equation.

Schrodinger equation is just the differential equation that tells you how the state vector of a system evolves with time. Therefore you can calculate the state vector at any instant of time given the initial state vector. But even a knowledge of the state vector is not enough to give you a 100% reliable prediction on what the result of a measurement is going to be. That's the way quantum physics is. The point is it possible that the state of the system is such that I can gain complete information about the system?

Go back to classical mechanics. Look at a single particle moving in 1 dimension. It's described by generalized coordinate q and a momentum p. therefore in phase space in the q-p plane; I specify a point that tells me everything I need to know above the system. Those are the only independent attributes of the system and I know it completely. In quantum mechanics, I am saying that the system is described by a state. Now suppose you take the same system, same particle which is quantum mechanical and lets say that the independent dynamical variables are its position and its momentum in 1 dimension.

Then the question is, is there a state of the system such that the position has a sharp value. So I give you the state and I immediately know the position. In other words, the question is, is it a position Eigen state. If it is then of course I apply that position operator on the state vector I get a number. There is no distribution. The answer is yes you have position Eigen states. You also have momentum Eigen states but no state can be simultaneously a position Eigen state and a momentum Eigen state. Such a state doesn't exist. So that's where the uncertainty comes in. so it will turn out that for those variables which are canonically conjugate in the classical sense, namely the Poisson brackets are equal to 1 for those variables in quantum mechanics, it is impossible to find simultaneous Eigen states.

There can be no state of a system of the particle in which both and x and both x and p can be simultaneously specified to infinite precision. The question is, is it possible that I may miss out on some of the Eigen values when I make measurements. The point is, given the states of the system; the idea is that the states will cover all possible values of all possible observables. Only then would it be a complete Hilbert space.

So everything is included. All possible eigenvalues maybe highly improbable. You may need special efforts to produce a state of the system corresponding to a very large eigen value of some observable. But the idea is they are also in the same space and nothing is left out. It's not a question of without doing the experiment. The point is that you want to describe the results. The whole idea of physics is that you want to describe the results of experiment. So you need a formulism which will tell you how to calculate and then you compare with experiment.

So these are 2 different things. The system is a part of nature. It is acting in some fashion quantum mechanics is a formulism which helps you to describe the system in precise terms. So I would always like to have rules for computation of physical quantities which I can compare against the results of measurements. So we won't confuse the formulism with the actual experimental observation or the measurement. These are 2 different things all together. The statement I made was, given an arbitrary state of a system, if I look at some particular physical observable corresponding to the system such as the position of a particle, if I measure the position of the particle the answer I get is guaranteed to be 1 of the eigenvalues of the position operator. But, I cannot tell you a priori which eigenvalue will emerge on the basis of 1 measurement even if I know the state of a system. What I can do is to assign probabilities with which different eigenvalues would emerge given that state of the system.

So suppose the position you take some physical observable corresponding to a system and lets say this observable can take the values 1 2 and three. Those are the only eigen values allowed no matter what state your are in. Now I put the system in an arbitrary state and I make a measurement. I may get the value 2. I make another measurement on an identically prepared copy and may get the value 3 and so on.

Given the state of the system I can't tell you whether I am going to get 1, 2 or three in a given measurement but even before you make the measurement, I can tell you the probability with which you are going to get eigenvalues 1 as p 1 and similarly for p 2 and p 3 based on a knowledge of the state of the system. Therefore it follows that one time p 1 + 2 times p 2 + 3 times p 3 is after all the average value of this variable. If I took an infinite number of copies of the system, identically prepared and made measurements, I would get 1, 2, 3 etc a large number of times.

I draw histograms to denote how many times I get 1, 2, and 3 and so on. Then the relative fraction of times I get p 1 is of course p 1 and similarly p 2 and p3. Now what I can tell you from a knowledge of the state of the system is I can give the numbers p 1, p 2, p 3, etc. And of course once I give you p 1, p 2, p 3, you will say 1 times p 1 + 2 times p 2 + 3 times p 3 is in fact the average value and that's precisely what you will get by making this infinite number of measurements and taking the arithmetic mean. So that's a sense in which quantum mechanics is probabilistic.

Student- Will the average value be one of the Eigen values? Professor- It could be but the average value of the height of the students in this room need not be one of the heights of the students. The average value of a set of integers need not be an integer. It depends on the distribution. *(A question by a student which the professor repeats)*- The question is, is this a short coming of quantum mechanics or is nature inherently random. Our belief is that nature is this way. Firstly, it is not a failure of experiment. So the uncertainty in quantum mechanics is not due to the fact that you don't have sufficiently precise measuring instruments. Even if you had infinitely precise measuring instruments, this would still be the case.

Second, it is not an artifact of our formulism of quantum mechanics. We believe it's because there exists a fundamental constant called Planck's constant which is non-zero. So the fact its numerical value is irrelevant that is 10 to the power – 34, its numerical value is linked with the fact that objects as big as us don't have many quantum corrections. I mean we know we have essentially classical objects like electrons and the quantum corrections would be very significant. That's what the numerical value of Planck's constant is it determines.

Had Planck's constant been 34 orders of magnitude greater, the quantum uncertainties would be so wild that we wouldn't have the classical world we have but the fact that its non-zero is what's leading to quantum mechanics and we think that's what it is. i don't know if we can ever know the why of things at an ultimate level always because this ignorance will simply be pushed back one step further to whatever. if in future some other theories subsumes quantum mechanics and it is superior theory which actually includes quantum mechanics as a special case, even then the question would persist why that.

Student- What do you actually mean by copies of a system? Well, suppose I asked you to measure the resistance of one meter of copper wire. There are 2 ways in which you do this. You take a long spool of copper wire, a kilometer long, cut it into a 1000 meter lengths and have thousand different people make measurements 1 each. That would be an ensemble average. The assumption is that the copper wire is homogeneous and therefore whatever is 1 meter here is the same as 1 meter there and properties haven't changed. Then that gives you the arithmetic average. So that's the sense in which I shall make copies of the system and in principle you need an infinite number before you get the true arithmetic average.

Of course in dynamics, remember that we talked about ergodic systems. There we said that the time average is equal to the ensemble average. So the idea was you don't have a 1000 pieces of copper wire. You have just one piece and you repeatedly go on making measurements on this one piece. And the presumption is that all the values or realizations that these 1000 pieces of wire would have given of this random process would already be realized as time goes on in the single piece of wire and that's what we called ergodicity where I said the ensemble average is equal to the time average here. So that's the specific assumption in dynamical systems. We are not making any such assumption here at the momentum.

Now all we have said is physical observables which are real observables should really have real values when you make measurements and therefore we looked for all those operators which have real eigenvalues and they are self adjoint operators. Now, the spectrum of an operator maybe continuous or discrete. It could be integer valued, non-integer valued, continuous, partially discrete, and partially continuous and so on.

Let me give you an example. The energy of an electron in a hydrogen atom in quantum mechanics as you know is -1 over n squared in Rydberg units for the bound states. But then those are negative energy states and they are all discrete - 1 over n squared where n runs 1, 2, 3, and 4. As n becomes larger, these states come closer to 0 and then all possible positive values are also allowed.

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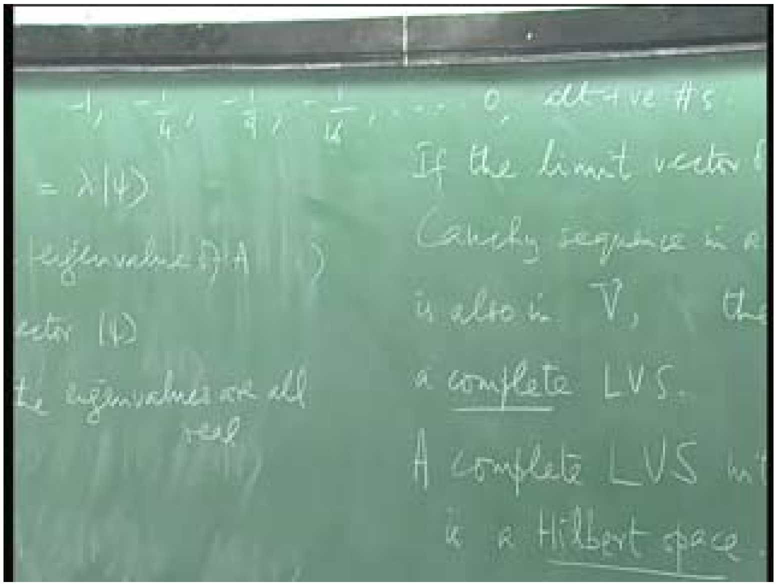

So that spectrum consists of the set of – 1, - 1 fourth, -1 over 9, -1 over 16 till 0 and then all positive numbers. That's the total spectrum of an electron interacting with the proton by Coulomb interaction in quantum mechanics. So the negative numbers are the bound states of the hydrogen atom and the positive numbers correspond to a free electron scattering of a proton. They are still part of the spectrum. What will become significant is that the state of the electron in this part of a spectrum is very different from the state of system there. And that's what will distinguish the scattering states from the bound states.

Student- Is it possible to make a measurement without changing the state of the system?
Professor- This is again a very deep question does it change the state of a system when you make a measurement. This is very deep question and this is where the problem of measurement in quantum mechanics arises. The answer of course is, in general, when you make a measurement and produce an eigen value as the answer, it implies that the measurement has somehow converted your state into an eigen state.

If I make a measurement on this and I get lambda 1 as the eigen value and psi 1 as the eigen vector corresponding to the eigen value lambda 1, it implies that this operation of measurements has somehow affected the system and instantaneously changed its state from whatever it was to the eigen vector psi 1. So definitely it's made a change. On the other hand, if the system was already in the eigen state psi 1, I make a measurement it remains where it is.

So measurement in general changes the state of the system quite drastically and it's called the collapse of the wave function in quantum mechanics. And one interpretation of quantum mechanics assumes that, by some mysterious as yet unknown process, any measurement of physical observable on a system collapses the state of a system instantaneously to one of the eigen states after which the Schrodinger equation will take over and system will evolve once again. So the idea is that if you make an instantaneous measurement immediately after you made a measurement and got the eigen value lambda 1, you will still see lambda 1. How long it will take to decay here and become something else is dependent of the Hamiltonian of the system.

On the other hand, there are interpretations of quantum mechanics which don't talk about instantaneous collapse at all but use totally different premises to explain the measurement process. So the details of the measurement process itself are something we will push under the carpet. We will not talk about it here because that's the part of quantum mechanics which is tricky and is opened to interpretation. Student- Do all states of the system have to be eigen states? Professor- Absolutely not!

Let me give you a specific example of what will happen and this is in fact going to tell us how to calculate expectation values. It will also lead us to the probabilistic interpretation of quantum mechanics. We could have started with the way it's done in the final lectures and then work back to expectation values. But I would like to start from here and go back because I am going to assume that it's easier to understand classical probability theory and then I point out where quantum probability is. It differs from classical probability. The whole thing can be looked at in yet another way which is a small change in the rules of probability theory. Now that takes you from classical to quantum mechanics.

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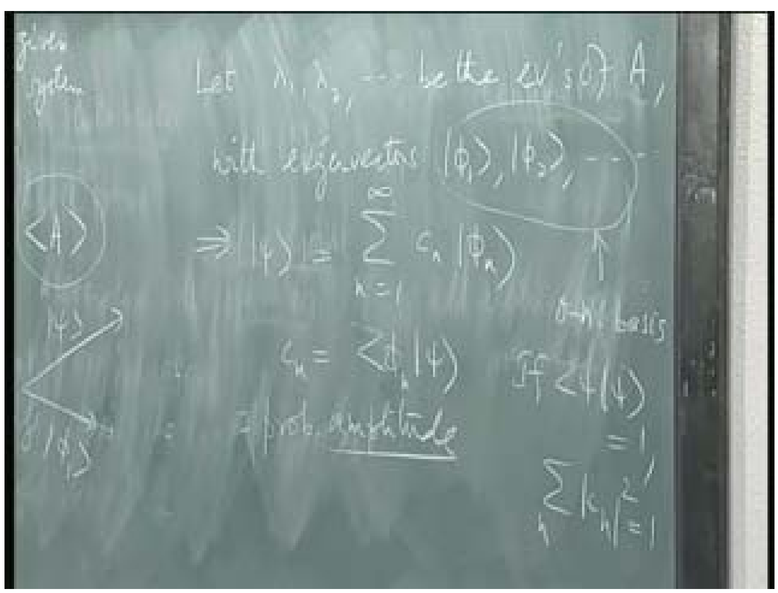

Suppose you have a physical observable, associated with which is the operator A and the state of your system for the moment. Let me assume everything is being done at one particular instant of time. So I won't put in the time variable. So let's suppose this is the operator and the state is psi. Then the question is, what's the average value that I am talking about which you get when you measure A repeatedly over an ensemble or collection. I call that the expectation value. I am going to denote it by 2 angular brackets. And as I mentioned, quantum mechanics gives you a formula for this quantity.

Now how are we are going to calculate this quantity. Buried in here in this formula is in fact the probabilistic interpretation of quantum mechanics. Let's assume that this physical quantity, corresponding to the operator A has a set of eigenvalues lambda 1, lambda 2, etc for the moment. Let me assume the eigenvalues are countable. So let lambda 1, lambda 2, be eigenvalues of A with eigenvectors phi 1, phi 2 and so on. For ease of notation, let me assume these are all non-degenerate. They are real numbers because A is a physical observable, so the eigen operator is self adjoint. And let me go a step further and say that these phi's can be made into an orthonormal basis set. Now the immediate question is, given a physical observable, can you always make an orthonormal basis set out of its observables? For ease of presentation, let me assume I can for a moment. We will see what happens when you can't later on.

So this set here (Refer Slide Time: 57:20) forms a basis set. Now what does that imply? That immediately implies that psi can be written as summation n = 1 to infinity, some coefficient say c_n phi $_n$. So I am going to assume that they form orthonormal basis. Then I can write this psi in the form c_n phi $_n$. psi is the state of the system. Then this c_n we know is equal to phi n psi. Now key question is what the interpretation of cn is. It's just a coefficient in the expansion of a vector. So it tells you how much of the vector psi is pointing along the vector phi n. that's all it is I mean if symbolically if this is psi and this is for example phi 1, then this quantity here is nothing but the projection of the psi into this direction here(Refer Slide Time: 59:04 min). so it is this vector if I multiply by phi 1 on the right hand side that will in fact give me the component of this vector along this(Refer Slide Time: 59:09). That's mathematics it just tells you that you projected this and only the coefficient tells you how much operate this along that.

Now the interpretation in quantum mechanics comes by saying this quantity is the probability amplitude that when the system is in the state psi, it is the state phi n. So it's the probability amplitude and not probability. That's the term that's introduced and it's a non-classical term. It's a specifically quantum mechanical term that's introduced to say that it's the probability amplitude that the state is actually phi n. The next postulate in quantum mechanic says the mod squared of this quantity is the probability and that's the postulate. But this itself is a complex number but you are guaranteed for any complex number the mod squared is always a nonnegative real number and therefore that is associated with the probability. So this is one more way of introducing quantum mechanics. You could start by saying that these quantities and inner products (Refer Slide Time: 01:00:44) are probability amplitudes such that it's the probability amplitude that this event is in fact and it will work even with time. so if I have a state at some instant of time and I have another state at another instant of time, this inner product the initial state here and the final state here is the probability amplitude that this guy has an overlap with that and that will tell us compute these numbers. So let me do this next time. We will start at this point and I will show you how this definition of this quantity, as a probability amplitude immediately tells you what the probability distribution itself is.

Yeah [Conversation between Student and Professor - Not audible ((01:01:30 min))] Yes mod Cn squared [Conversation between Student and Professor – Not audible (01:01:38 min)] Yes [Conversation between Student and Professor – Not audible (01:01:39 min)] [Noise] so there is a sum of nonnegative numbers so each mod Cn squared must be less than 1 and therefore is a probability and that's the whole point. [Conversation between Student and Professor - Not audible ((01:02:07 min))] Yeah so the whole idea of quantum mechanics says the state of the system is some abstract object and everything about the system is buried in there. [Conversation between Student and Professor – Not audible (01:02:22 min)] ah the state of a system there is nothing that's the way nature is

The state of the system is not an observable by itself. It is something from which you derive expectation values of observables but it's not an observable itself. It's a false analogy but it gives you a good picture to say the state of the system plays the role of something like the probability distribution of a random variable. The probability distribution is not an observable of a random variable. It's an not observable only the mean square etc are observables but the distribution helps you compute these quantities.

[Conversation between Student and Professor – Not audible (01:03:07 min)] anything you can see [Laughter] anything you can measure [Noise] [Conversation between Student and Professor – Not audible (01:03:14 min)] yeah I will write this down clearly and then you will see what Cn is it's the probability amplitude that indeed or better still I should call it an overlap between the state psi and state phi n which is what the geometrical interpretation is but the postulate of quantum mechanic says the mod Cn squared is the probability that the system when it is in the state psi is in the eigen state phi n. that's the exact statement yeah [Conversation between Student and Professor – Not audible (01:03:48 min)] pardon me [Conversation between Student and Professor – Not audible (01:03:50 min)] that's the only quantity which has the desired properties they are real number they are between 0 and 1 they normalized to unity [Conversation between Student and Professor – Not audible (01:04:00 min)] [Noise]why not mod Cn itself that's the way quantum mechanic says [Conversation between Student and Professor – Not audible (01:04:07 min)] that's the way it is. it's not mod Cn its not mod Cn to the power 1/3.no its not mod Cn squared. that's the way it is. that's the only consistent way we can write down quantum mechanics. [Conversation between Student and Professor – Not audible (01:04:24 min)] yeah because of the inner product equation but yeah surely but you know the point is you could have started with the different formulism all together and that's not the way it operates this is the way it is.

Thank you!