**Quantum Physics**
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**Lecture no. #04**

The space of square-summable sequences is denoted as $l_2$.

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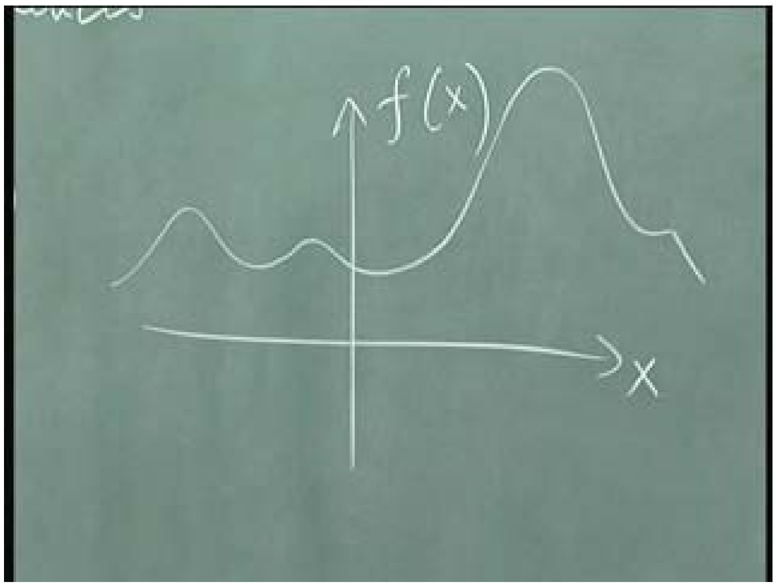


All functions f(x) is square integrable rather than square summable and they are square integrable in the sense that mod f(x) whole squared dx is finite and this space is called $L_2$.

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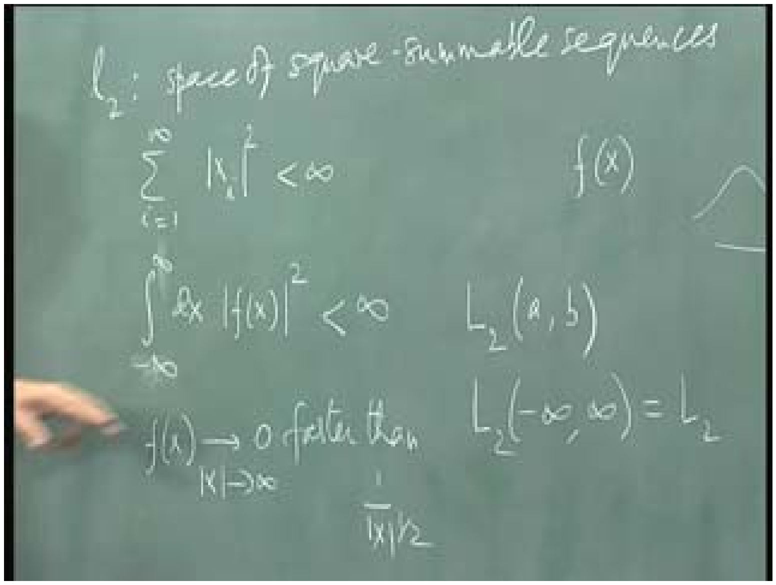


It's called $L_2$ (a, b) because the range is a to b but very often I am going to write $L_2$ of minus infinity, infinity as just equal to $L_2$ itself. So if I don't say what $L_2$ is, I don't tell you and its understood that it runs from minus infinity to infinity. So the space of square integrable functions is going to be of very great importance to us in quantum mechanics. Of course one can immediately generalize it to higher dimensions. So you talk about three independent variables, functions of x y z and then I would write d 3r here and mod f of r whole square less than infinity. That would be $L_2$ with three independent variables x y z.

So this space is going to be of crucial importance to us and its linear vector space. It has very interesting properties which we will examine but you have to understand that the only requirement we put on the function which belongs to $L_2$ is that it be square summable. It's obvious that if you have minus infinity to infinity that a necessary condition for a function f of x to belong to $L_2$ is that f of x should go to 0 as x tends to + or – infinity.

That's not sufficient but it's definitely necessary. So immediately it implies that this function would die down at infinity sufficiently rapidly. How fast should it go to 0 as mod x goes to infinity for this (Refer Time Slide: 03:16) to converge? f (x) should tend to zero as mod x tends to infinity faster than 1 over x only then will this converge. 1 over square root of x is enough because you are going to square it. So it should go faster than 1 over x to the half.

Of course in between it could have singularities but they should be integrable singularities. For instance, if f(x) at x=1 behaves like 1 over x – 1, this integral will not exist but if it behaves like 1over mod x - 1 to the power of quarter or something like that, that's an integrable singularity and things would be all right. So there is no requirement that it should be bounded. On the other hand, in physical applications when we apply to quantum mechanics, we will apply this to something called the wave function. We would require the wave function to be bounded. We are going to put an extra conditions on it based on physical requirements.

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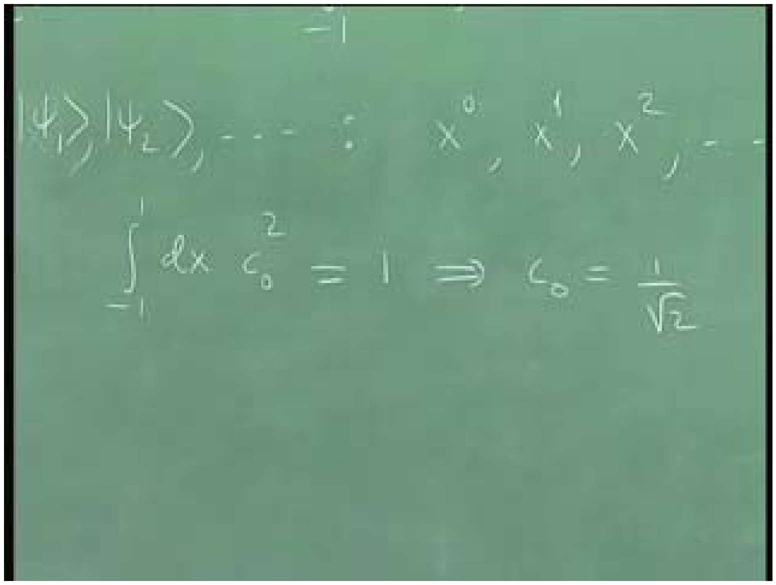


Let’s look at $L_2$. I look at this space of square integrable functions -1 to 1. Actually it doesn’t even have to be $L_2$.It could be $L_1$. It could be integrable and that’s sufficient. But the reason I use $L_2$ is because it’s important in quantum mechanics and it enjoys a property which is enjoyed already by function in $l_2$. This space is self dual. Its dual is also $l_2$. In exactly the same way the dual to $L_2$ is also $L_2$. So the dual vector space would be essentially the same space once again which is similar to the property enjoyed by Euclidian space itself. $L_2$ is a space of functions of a real variables x such that this condition is satisfied (Refer Slide Time: 05:56). What’s the dual to this space? It’s also a space of some functions. What are those functions? It’s a also self dual and therefore those functions must also satisfy exactly the same condition. How do you get those functions? It will turn out its fourier transforms. As you know Parseval's theorem actually in Fourier transforms tells you that this quantity is equal to integral dk - infinity to infinity mod f tilde of k whole squared where f tilde of k is a Fourier transformer of this function here.

So this Parseval's theorem tells that the norm of a function of an element of this vector space is independent of the representation. It’s exactly the same no matter how you write it. so we will get back to this and see how Fourier transforms emerge in a natural way. Let’s look at $L_2$ of -1 to 1. So we will look at those functions of some real variable x defined between – 1 to 1 which are square integrables. Now as soon as we say – 1 to 1, it suggests that you are looking at functions of the cosine of an angle because cos theta runs from – 1 to 1. That’s the range of cos theta. and therefore we are looking at those functions which are functions of the cosine of some angle theta but let’s write it as $L_2$ of – 1 to 1 of x and we look at all those functions which satisfy – 1 to 1 dx mod f of x whole squared less than infinity.

Let’s restrict ourselves to polynomials and let’s try to find out what these functions are. Now the simplest polynomial I can write down is of course just the constant x to the power 0 and then I would like to use monomials to build up all possible polynomials. So I would like to use as my initial set of vectors. So I would like to represent $psi_1$, $psi_2$, etc by x to the 0, x to the 1, x squared and so on. Let’s start with x 0 and impose this condition. in fact the condition that is imposed is slightly different for convenience. For instance let's impose this condition and ask what's – 1 to 1 dx $c_0$ and let me call it f $_0$ of x equal to some constant because x to the power 0 is 1 essentially. So what multiplied ah so some constant let's call it $c_0$ squared and let's choose the mod to be real and this is =1. Let me initially choose normalization 1. This would imply that $c_0$ = 1over root 2. I use Gram Schmidt orthonormalization to find $c_1$.

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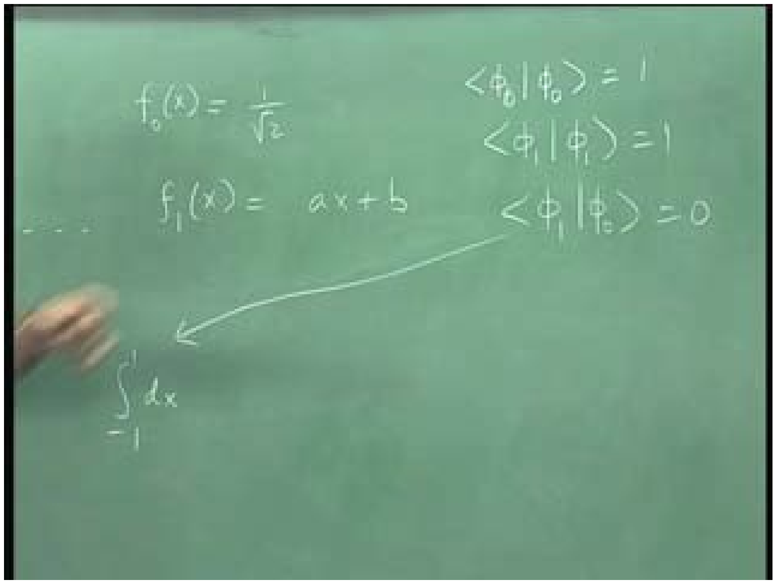


Then $f_1$ of x can be of the form some ax+b in general. It's going to be of this form it's a linear function we have choose ax + b. How do I find about a and b and the conditions I should impose? Since I am doing Gram Schmidt orthonormalization, I would like to have phi $_0$ phi $_0$ = 1, phi $_1$ phi $_1$ = 1and I would like to have phi $_1$ phi $_0$ to be 0. I would like to be orthonormal.

And I know so my initial vector in the function space is x to the power 0 the other next vector in the function space is x to the power 1 but the most general possibility I can have is a combination of x to the 0 and x to the 1. So I must project out the portion of x to the one along x to the zero and that means I must try to find a and b such that this condition is satisfied (Refer Slide Time: 11:16). So let's put that in and see what happens. I need to have – 1 to 1 dx a x + b the whole squared to be equal to 1. That's the condition here.

And I would do also like to have the inner product of this function with the original function f $_0$ of x to be 0. What's this translated to function space what does this look like? This will be integral – 1 to 1 dx $f_1(x)$ $f_2(x)$ = 0. Phi $_1$ is represented by f $_1$ of x and phi $_o$ of x is $f_0$ of x. i have restricted myself to real functions. Have I not done so, $f_1$ of x would have become f $_1$ star of x because I do complex conjugate transpose. This is the condition here but f $_0$ of x is just a constant its 1 over square root of 2. So we will remove that.

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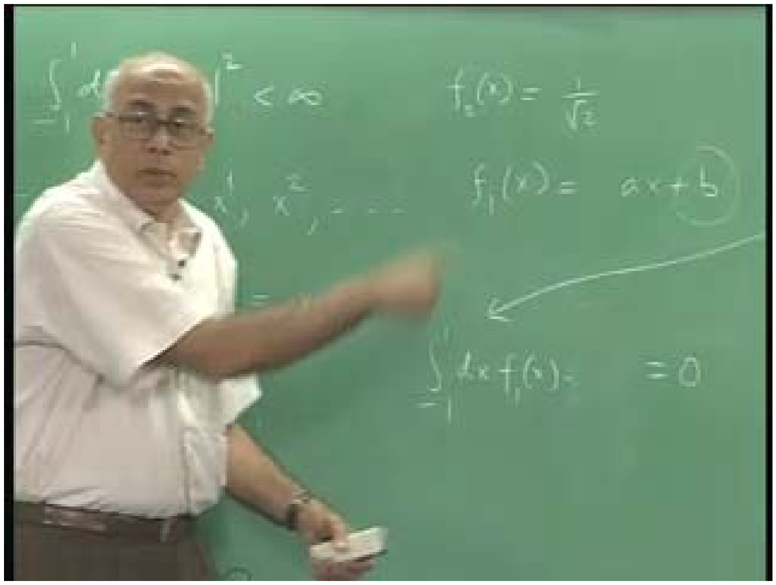


It’s got to be an odd function. so its clear that f $_1$ of x cannot have this portion the x portion would be integrated and give you a non-zero number.

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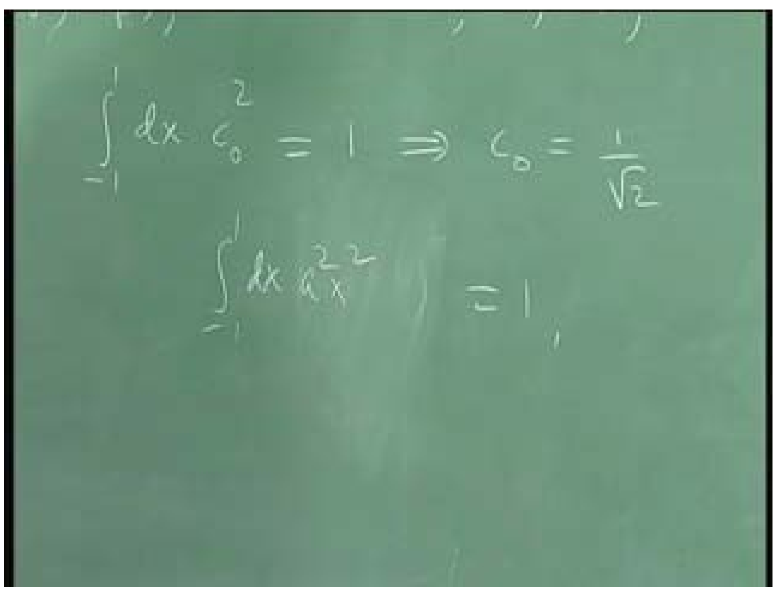


And therefore f $_1$ of x is strictly proportional to x and then this goes away immediately. Therefore you get a squared x squared = 1 and thus we get $f_1$ of x = square root of 3 over 2 x.

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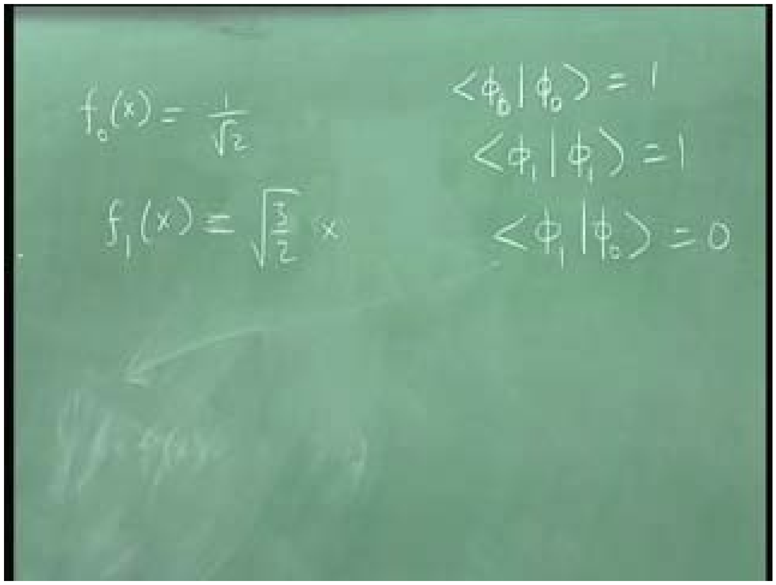


Now we look for $f_2$ of x and $f_2$ of x is going to be some constant plus something times x plus something times x squared and you need conditions to determine these three constants. You need three conditions.

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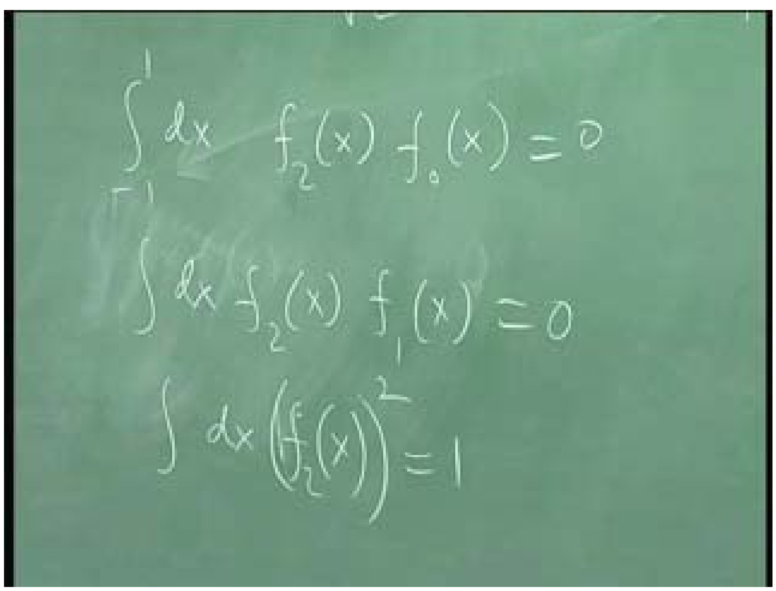


So they are integral – 1 to 1 dx $f_2$of x $f_0$ of x to be 0, you want – 1 to 1 dx $f_2$ of x $f_1$ of x to be 0 because I want phi $_2$ with phi $_0$ to be 0 and phi $_2$ with phi $_1$ to be 0 and in addition you want integral dx $f_2$ of x the whole squared = 1. And general form of $f_2$ of x is ax squared + bx + c and you have to determine a, b & c and these three conditions are sufficient to determine a, b & c.

It's obvious from here that $f_2$ of x is an even function of x it's immediately clear that $f_2$ of x therefore will have only the constant term and the x square term. Finally it is going to be something times x square and some constant here. You let me change the normalization.

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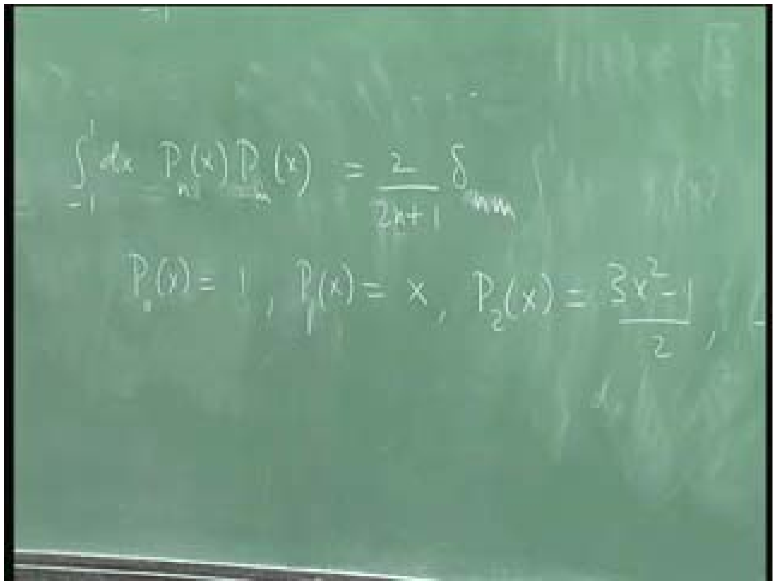


Let's call these functions $P_n$ of x. They are the Legendre polynomials and let's write these functions such that the normalization is – 1 to 1 dx $P_n$ of x $P_m$ of x, where n and m multiple integers should be zero when n and m are different and when n and m are equal, it is 1 and you get a normalization constant. The standard normalization for historical reasons is 2 over 2n + 1 delta nm. This is the normalization factor for the Legendre polynomials and if you recall $P_0$ of x = 1, then $P_1$ of x = x itself and $P_2$ of x = 3 x squared - 1 over 2 and so on.

So these objects act as unit vectors in the space of functions which are square integrable between – 1 to 1 and therefore any respectable function which satisfies these conditions can be expanded in a Legendre function series of Legendre functions. So this in orthogonal polynomials is just a special case of the expansion of this arbitrary function in a linear vector space in terms of a basis set. And we ensured the basis set is actually orthonormal. The normalization is arbitrary and you could choose other things and so on. There are several things that have to be specified. The first is the range whether is it a to b, - 1 to 1, 0 to infinity, - infinity to infinity and so on. The second is the normalization factor. This factor here (Refer Slide Time: 18:03) that is chosen for convenience in specific manner.

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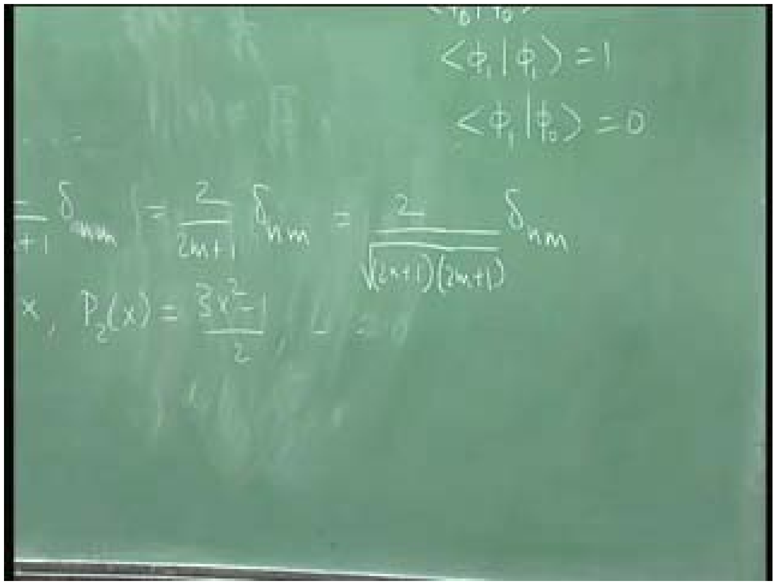


It is symmetric. i write it like that right (Refer Slide Time: 18:50). It's symmetric in n and m. [Conversation Between Students and Professor (00:19:05)] I i start by saying I am going to choose a basis out of polynomials so I am deliberately choosing the polynomial basis [Conversation Between Students and Professor (00:19:18)] any function which is not a polynomial it if it shouldn't have singularities if its unbounded and so on this won't work right if it's a nice function a complicated function it doesn't have to be a polynomial itself like e to the power x in idea one plus x plus x square and so on and so forth so it doesn't have to be a polynomial itself but the interesting thing is we we will come across more examples of this

So the factors that you have to look at in this range are the weight factor here and finally you might want to have functions we find in - infinity to infinity. Then of course the question of writing polynomials in there doesn't arise because integrals will block.

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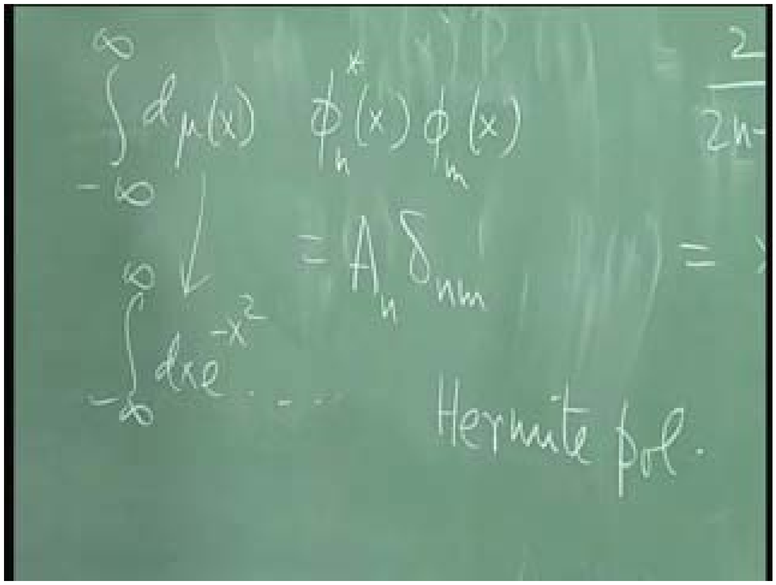


So you put a weight factor so its possible that you have a set of functions. it looks like - infinity to infinity d mu of x phi $_m$ of x phi $_n$ of x star = some $A_n$ delta nm. So there should be some weight factor here and in then some normalization factor here and there is a measure here which ensures that the integrals are finite(Refer Slide Time: 21:43). A popular measure is a Gaussian measure. So you could put integral - infinity to infinity dx e to the - x squared that ensures that this thing goes to 0 multiplied by these polynomials here. You again start with x to the 0, x to the 1, etc and impose these conditions and you get a unique set of polynomials called Hermite polynomials.

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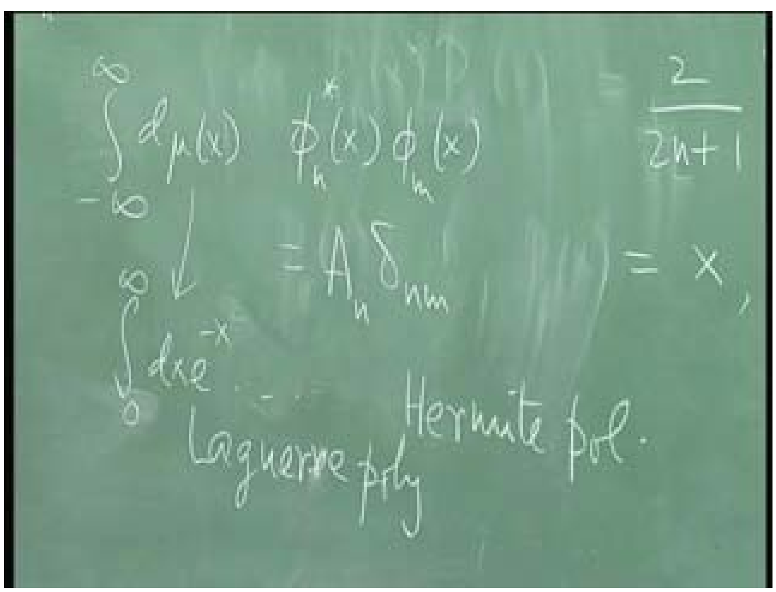

There might be instances where you would like to work from 0 to infinity. For example, it could be a radial coordinate or something from 0 to infinity. So you could have 0 to infinity n e to the – x. There is no need for - x squared now because it's only going from zero to + infinity. Whereas in the other case you need a - infinity so you could have x squared and these polynomials are called as Laguerre polynomials.

I start by saying I am going to form this basis set out of polynomials. So the way to construct polynomials is to use the monomials x to the 0, x to the 1, x to the and so on and I will form linear combinations of these to give me my orthogonal vectors. The statement is that once you are given a function, a linear vector space and a basis in that space, then the statement is a function can be uniquely expanded in a unique manner.

Every element of this vector space can be expanded in a unique manner in this basis set. That's the theorem in the vector space. Let's now look at general expansions. i just want to give you instances of these basis sets. By the way even the Fourier transform of a function is really writing a function in a basis set.

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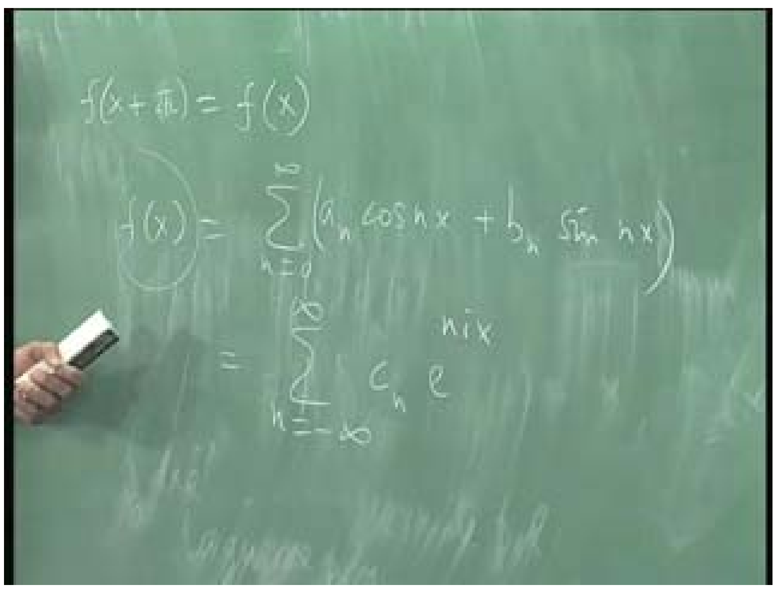


If you have a periodic function of a real variable such that f of x + period lambda equal to f of x. the lambda is generally taking to be 2 pi or something like that. Then you write this (Refer Slide Time: 24:26) whole thing in a Fourier series. You can always expand it in the form f (x) = summation n = 0 to infinity ($a_n$ cos nx + $b_n$ sin nx). So let's take two pi to be the period. now what you are doing here is to say that a function which is periodic can be expanded in a Fourier series such that the basis set consists of the functions cos nx and sin nx, where n runs over 0,1,2,3,etc ,the coefficients are the components of the vector which is represented by the function here. So that's the way to understand it.

This is the vector in a certain function space which is a linear vector space. these are the unit vectors in that space they are the basis set and this and that are the components of this vector(

Refer Slide Time: 25:12). Now just as the components uniquely determine the function, in exactly the same way, if the idea here, subject to certain conditions, once you can make this expansion instead of specifying the function f (x) I could specify the coefficients $a_n$ and $b_n$ and that specifies the function. Why do I choose n = 0 to infinity and not - infinity to infinity? It's the same this because sin and cosine are definite parity properties and therefore it's exactly the same thing. I could have written this also as summation n = - infinity to infinity, some coefficient $c_n$ and e to the power n i x. Then I combine $c_n$ and $c_{-n}$ in an even and odd combination and I call them $a_n$ and $b_{n.}$

So this is so for periodic functions. If the function is not periodic, if f of x is defined from - infinity to infinity and is not periodic then I cannot expand it in a Fourier series but I can still expand it in the Fourier integral. So you could still write even without this condition.

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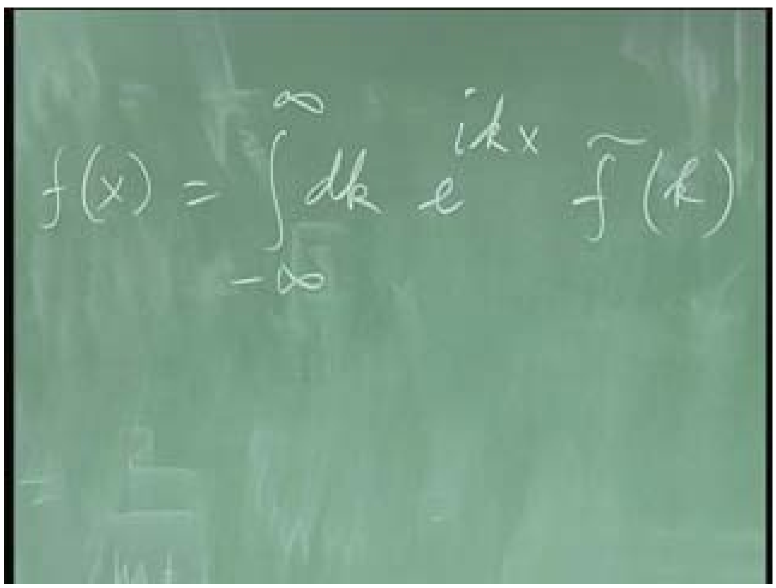


This can be written as f (x) =integral – infinity to infinity dk e to the ikx f tilde (k). Now what's the basis set of vectors? It's a continuous basis labeled by k. it's labeled by n here (Refer Slide Time: 27:08) but it's a continuous variable so you need an integration over k and these (Refer Slide Time: 27:19) are like the unit vectors. Those are like the unit vectors and then f tilde of k are like the components which itself is a function and turns out that the components of f (x) is in $L_2$ .then f tilde of k is also an $L_2$ but for writing a Fourier transform you don't need the function to be in $L_{2.}$ The condition for a function to have a Fourier transform of this kind is f of x should be integrable. It should be in $L_1$. But then there is no guarantee that f tilde of k is also in $L_1$ because it's not self dual in general.

In fact we know from the theory of Fourier transforms that if f of x is very compact in a finite range then f tilde of k will have all possible k's. It will be a function which is not compact. And the tighter f of x is, the broader f tilde of k is. The ultimate of course is reached then f of x has a support only at one point, a delta function. What happens to f tilde of k? From - minus infinity to infinity it's just constant.

So Fourier transforms enable you to actually expand functions which are more than even generalized functions like distributions like theta functions, delta functions and so on. But if this is in $L_2$ then so is that also in $L_2$. What's the orthogonality relation in the space of functions which are integrable? Let me give some examples.

Suppose I have a set of functions $f_n$ of x where n =0,1,2,3, etc which forms a basis in some function space say $L_2$ is 0 to – 1,-1 to 1,etc or whatever. Let us suppose this is an orthogonal basis orthonormal basis what's the orthonormality relation? Let me write down the general cases here phi $_n$ phi $_m$ = delta $_{nm}$ and summation of n phi $_n$ phi $_n$ =1.

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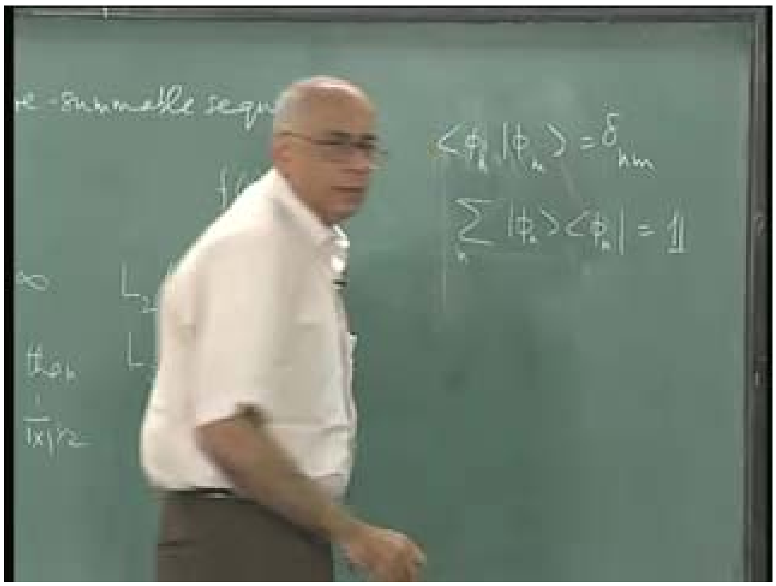


The former is orthonormality and the latter is completeness. I need to impose those conditions in the function space and the function space has a basis f $_n$(x) and say some range a to b. what's the orthonormality relation clearly it will be an integral dx $f_n$ *(x) $f_m$ (x) from a to b = kronecker delta.

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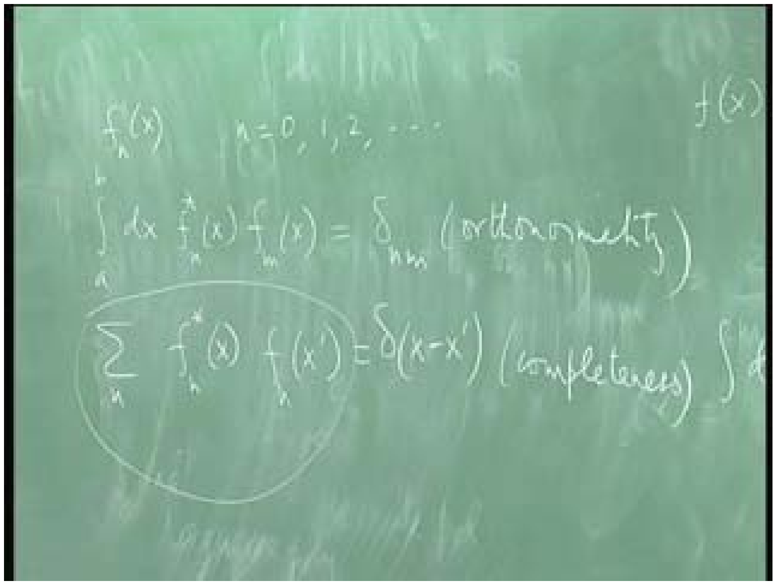


This is orthonormality. What's completeness? Well remember in completeness you have to now not integrate it I mean this quantity here (Refer Slide Time: 31:10) was defined in this fashion here as an integral. I need to have the unit operator appearing but the unit operator in function space is such that when it acts on any function it reproduces the same function at every point. So what should be the analog of this (Refer Slide Time: 31:34) relation here? It will be summation over n $f_n$*(x) $f_n$(x prime) = delta (x-xprime). If I take the right hand side and integrate over an arbitrary function psi of prime, I am going to produce psi of x. so it is the analog of the unit operator isn't these two

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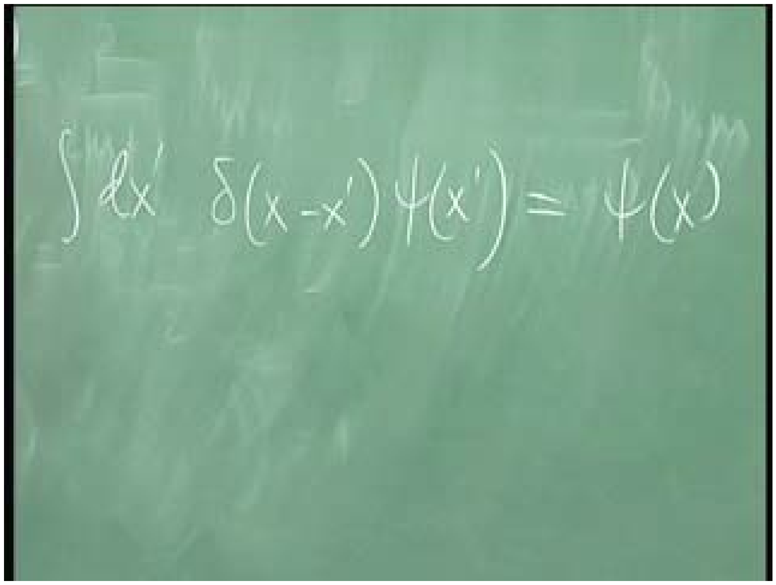


If I take an arbitrary function do dx prime delta x - x prime, this is meant by saying a unit operator acting on the function space produces the same function once again. So this quantity here plays the role of the unit operator and function space. This is completeness. The summation is over the argument here (Refer Slide Time: 33:34) and the summation is over the index here (Refer Slide Time: 33:36).

The arguments are different and they produce the analog of the unit operator in function space. So you have to check both these relations when you have an orthonormal basis, then I mention that these two are different properties but when translated to function space this is what they look like. You might have noticed that when you look at the orthogonal polynomials not only do you have this condition but you also have something where you have sum over n for example so summation $P_n$ (x) $P_n$ $P_m$ of x prime gives you something times the delta function on the right hand side. From the theory of differential equations of second order, that itself is part of a more general relation and this is absolutely beautiful. If you look at the Legendre differential equation for example, you know it has two independent solutions. One of them is a Legendre polynomial and the other one is not a polynomial at all. The other one is got a singularity -1 and 1 and has got logarithmic singularities.

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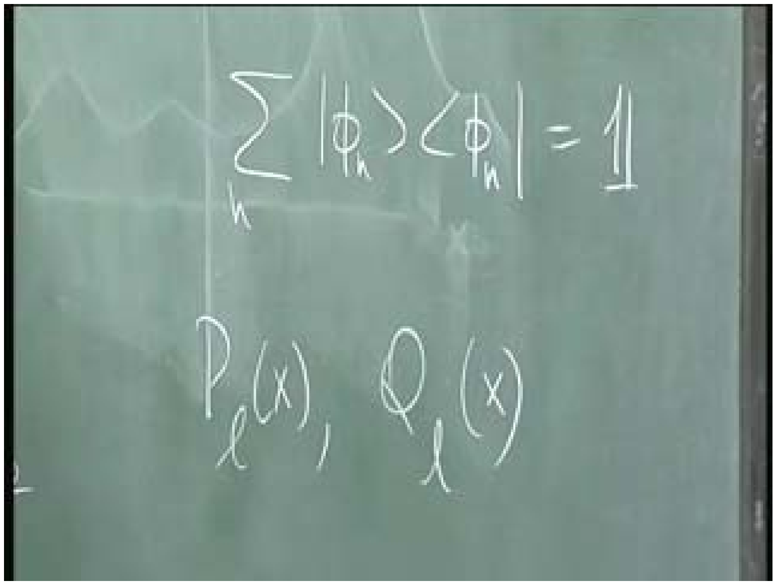


So this $P_l$ of x and the second solution of the Legendre function are called $Q_l$ of x. It's a polynomial plus a portion which looks like log 1-x over 1+x. So it has logarithmic singularities either two end points.

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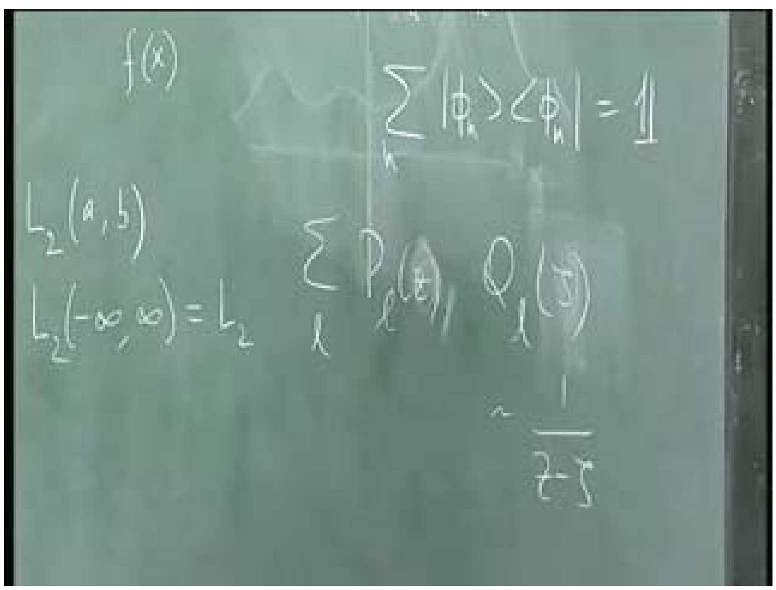


Then it turns out that if you do a summation over l $P_l$ of z and $Q_l$ of zeta where l and zeta are complex variables, this quantity here (Refer Slide Time: 36:30) is 1 over z -zeta which is called the Cauchy- Kernel. Now if you take this function, it has got branch points between -1 and 1 and you take the discontinuity across the branch point, then it becomes the $P_l$. the discontinuity of a $Q_l$ is a $P_l$ and so on. I write zeta = (x + i epsilon) (x – i epsilon).

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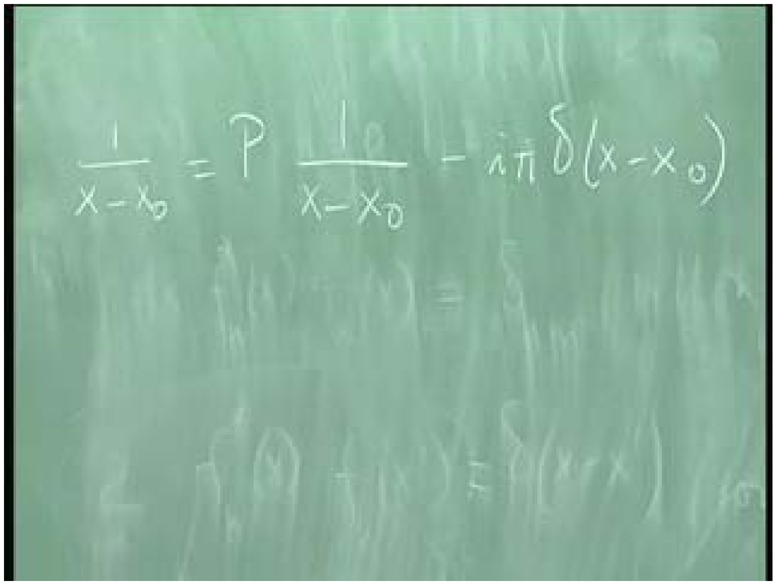

We can always write 1 over $x-x_0$ = the principle value 1 over $x- x_0$ - i pi delta $(x-x_0)$. A function which of a complex variable f of z where z = x + iy and I write f of z as u + iv and u and v must satisfy certain conditions for it to be an analytic function. then those conditions are called

All these functions you studied must all be regarded as functions of complex variables. That's the right way to look at it. Then things become easy for example I start with the exponential function, I take its symmetric and antisymmetric parts. Those are cos hyperbolic and sin hyperbolic. Its analytic continuation to pure imaginary values gives me cosine and sin. So the trigonometric and hyperbolic functions are just analytic continuation of each other. Tan inverse in the log functions are all related to each other etc.

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This stands for principle value (Refer Slide Time: 39:34). So you have integration over some function and that happens to be a singularity at $x_0$ you could come along here and if you go either this way and avoid it or you could go this way and avoid it and they would correspond to these two points here. (Refer Slide Time: 39:49)

So the imaginary part is in fact this because you see if I do this or this, I either get +i pi or – i pi by Cauchy's rule. So the difference between the two is like going around fully and that's equal to 2 pi i times delta. So it's a trivial way of understanding why you get a minus i pi there but if I did that and took the imaginary part here, the discontinuity I get a delta function which gives me that completeness relation between $P_l$. so this gets replaced by a $P_l$ and this is called the Miller formula. This is called the Cauchy- kernel (Refer Slide Time: 40:26). This kind of mathematics is classical mathematics but it is a very interesting and very important in various applications.

so these are all there is a very unified theory of these things now and so well established over a hundred and fifty years ago more than that but they help you a lot this so all these electrostatic problems you solve fluid mechanics and so on and so elasticity and so on there all very easily very comfortably understood if you have someone ah analysis complex analysis. Now let me come to expansions. We will come back to the general case and state a few results.

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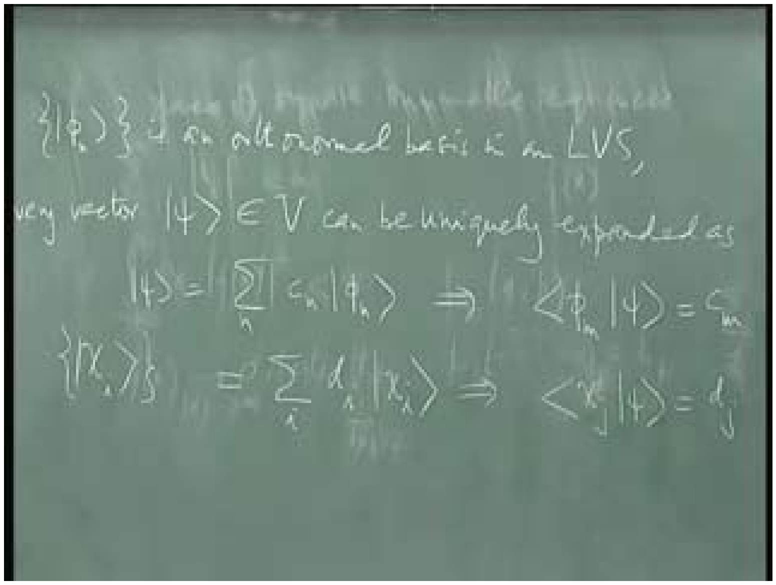


If phi $_n$ is an orthonormal basis in the linear vector space given a basis every vector psi which is an element of V can be uniquely expanded as in the following form. Psi = summation over n $c_n$ phi $_n$. so that's the statement and this could be an infinite dimensional space so in general in linear combinations. The inversion formula would be to find the coefficient $c_n$ in terms of this psi.

His question is is it guaranteed that each of these for arbitrary cn's is inside. So we will look at an infinite dimensional space and then put a condition on this. So all I have to do to find $c_n$ is to put a scalar product on this side with some phi $_m$ and when n=m, it will result in $c_n$. So this will immediately imply that phi $_m$ psi = $c_m$. Please notice we are going to look at complex vector spaces therefore the vector you expand is the ket vector. So the coefficients will have this as the ket and this as the bra here (Refer Slide Time: 44:00). It's a complex conjugate if you write it the other way around. So specifying these coefficients is equivalent to specifying this psi. But of course there is nothing that tells you that you must have the unique basis. There are many basis. So let us suppose that there exist another basis on the same space which are chi $_i$ is also a basis in this space. Then the same vector psi could be written as summation over I, a different coefficient $d_i$ chi $_i$. All you must make sure of is that the dimensionality of the space is fixed. so if its finite dimensional say 24, then this will run to 24 values this will also will run to 24values (Refer Slide Time: 44:55). This will imply immediately that that chi $_j$ with psi equal to $d_j$.

You may very frequently want to change from one basis to another. What's the relation between the cm's and the dj's? Well what I would do is to take this basis and say each of these (Refer Slide Time: 45:40) fellows can be expanded in that basis obviously. So let's do that.

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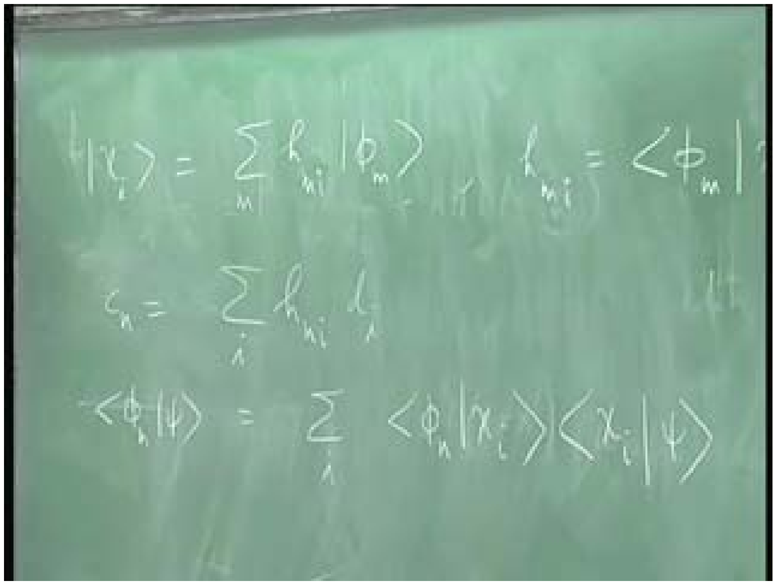


Let's write chi $_i$ itself to be = summation over m ket phi $_m$ and a coefficient h for which should depend on i and m. then it's clear that h $_{mi}$ is nothing but the overlap by phi$_m$ with chi $_i$.

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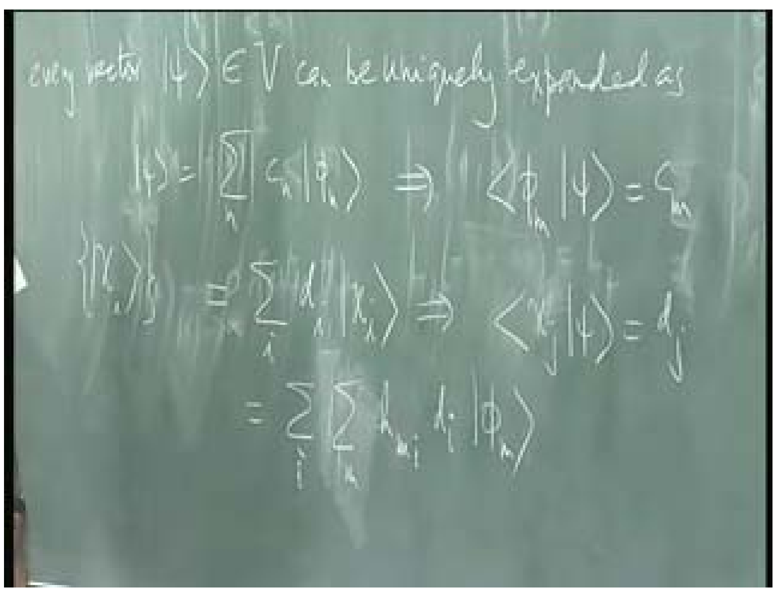


Now if I plug it into this place here, chi $_i$ will be summation over i summation over m h$_{mi}$ d$_i$ phi $_m$. all I did was to substitute for chi i from that thing. Well I would like to compare that with this here Refer Slide Time: 47:45) and they are just the sums with respect to phi $_m$'s and therefore I can equate coefficients by coefficient.

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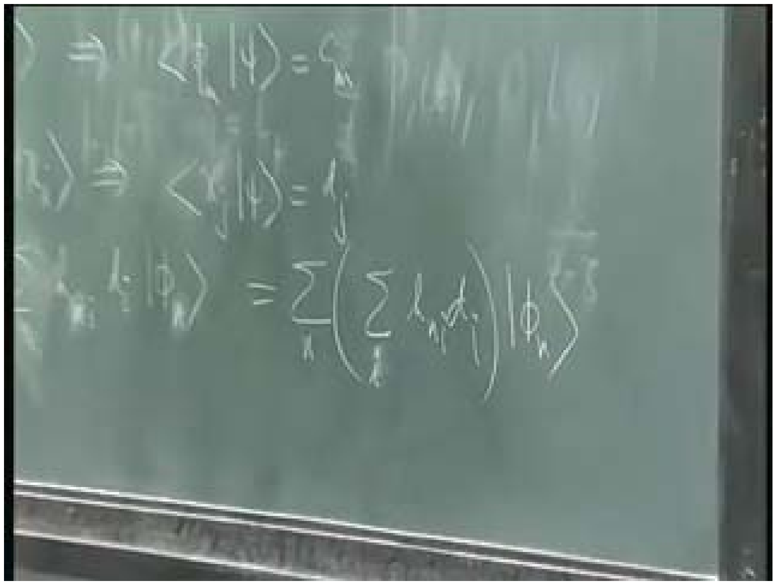

I have phi $_n$ here. So let me change this dummy index to n and I could write this a summation over n summation over i h $_{ni}$ $d_i$ .this whole thing is sum function of phi $_n$. So I take an arbitrary vector in this space, expand it in the phi basis and the chi basis but each of the chi basis vectors can be expanded in the phi basis.

I put that expansion in and now since this (Refer Slide Time: 48:43) is an orthonormal basis set, this is equal to that implies that coefficient by coefficient they are equal. So it immediately says that $c_n$ = summation over i h $_{ni}$ $d_i$. So it relates one set of coefficients to another set of coefficients but that relation could have been written down by inspection. It says that $c_n$ is phi $_n$ with psi and that's = summation over i h $_{ni}$ but h $_{ni}$ is phi $_n$ chi $_i$ with $d_i$ but $d_i$ is chi $_i$. But there is no i here there is no I, there this i here. Therefore I can take this summation inside.

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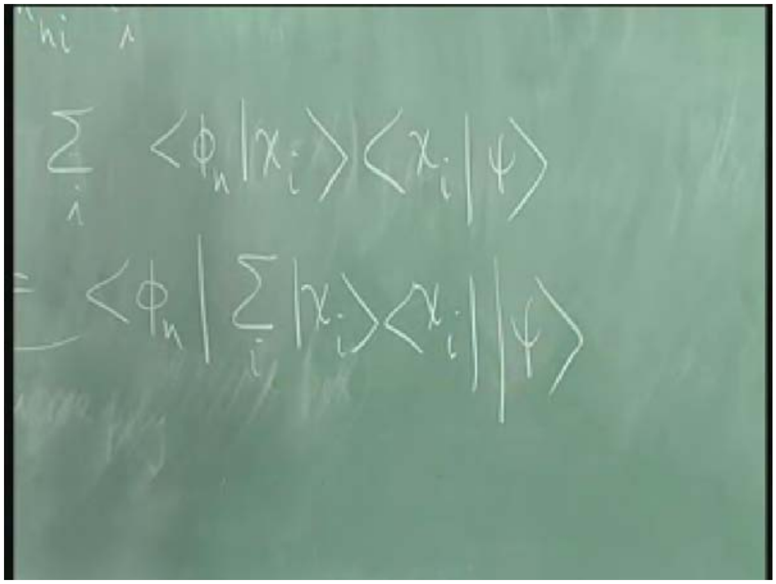

This chi is the unit operator by completeness which therefore this is equal to phi $_n$ and psi. It's an identity. So if we want to go from one basis to another, all you have to do is to insert the identity in between. We work this backwards and then you arrive at this relation. So it tells you how powerful this insertion of the identity is any time I want to change a basis or insert the identity in that basis.

I do it repeatedly in a long calculation in 8 different places I inserted different basis and so on but I can't go wrong because everything is in terms of the unit operator and this is completely a self-correcting kind of notation. You can't make a mistake in doing this but only thing you have to remember is that when you expand a vector, please remember that the coefficients have the vector as a ket and not as a bra is otherwise it complex conjugates and that's the only crucial step you have to remember. Now suppose this is in L $_2$ space and it's clear that these things here must have finite norm.

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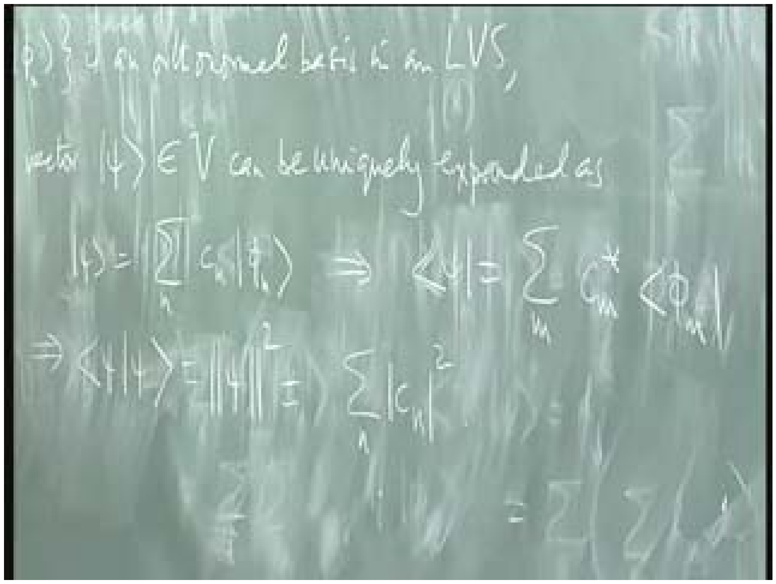


Once I make this expansion, this implies that psi with psi = norm of psi whole square = summation over n $c_n$ squared. Bra psi implies immediately that this is = summation over m $c_m$ * phi $_m$. now I want to find what is psi with psi so I take this and I apply to the left of ket psi and I get a phi $_n$ phi $_m$ which is a Kronecker delta. So only n = m is going to be picked out and this is equal to summation over n $c_n$ whole squared.

So this is equivalent to saying that if I have a linear vector space in which I look at all vectors with finite norm, that's the same as saying that is I expand it in a discrete basis of this kind then the coefficients in discrete basis belong to $l_{2.}$ They must be in square summable. Now this tells you the importance of the space $l_2$ because every time I have an arbitrary linear vector space and I say I am going to look at all vectors with finite norm, then it automatically implies that when I expand these things in an orthonormal basis the coefficients would belong to $l_2$.

They have to be finite and it has to be square summable. That's the condition you need and this statement is independent of basis because this (Refer Slide Time: 54:04)will also be = summations over i mod $d_i$ whole squared.

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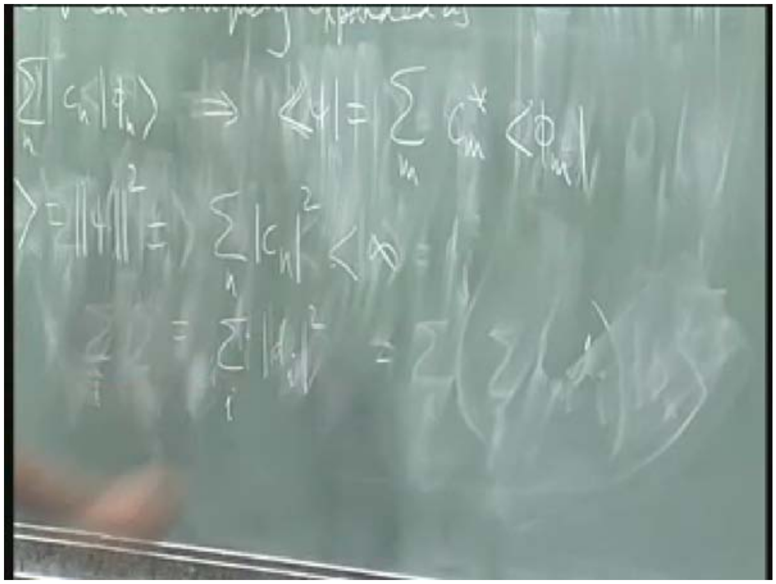

In fact this is Parseval's theorem which says if you expand a function or you expand it in terms of Fourier coefficients, both of them would have exactly the same norm or the function would have to same norm.

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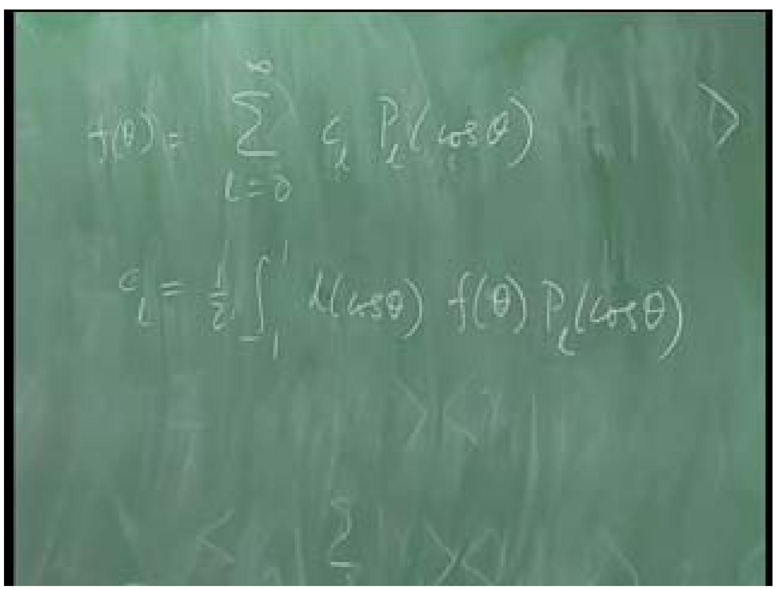


If you have a function space which is $l_2$ of -1,1and I write some function of theta = summation over l=0 to infinity, some $c_l$ $P_l$ cos theta where theta is running from 0 to pi and then I expand it in terms of the Legendre polynomials, then this is the expansion and the inversion formula here which gives you the coefficient $c_l$ = one half integral -1 to 1 d (cos theta) f of (theta) $P_l$ cos theta. I should really write $P_l$ star of cos theta because its bra vector but this is a real function and therefore we write this as $P_l$ cos theta. That's the inversion formula. The Fourier inversion formula is also trivial. Here is the function expanded. I would like to invert it. On the other side this is an integral - infinity to infinity dx e to the power – ikx. And you put a 1 over 2 pi. The 1 over 2 pi comes from the orthogonality relation.

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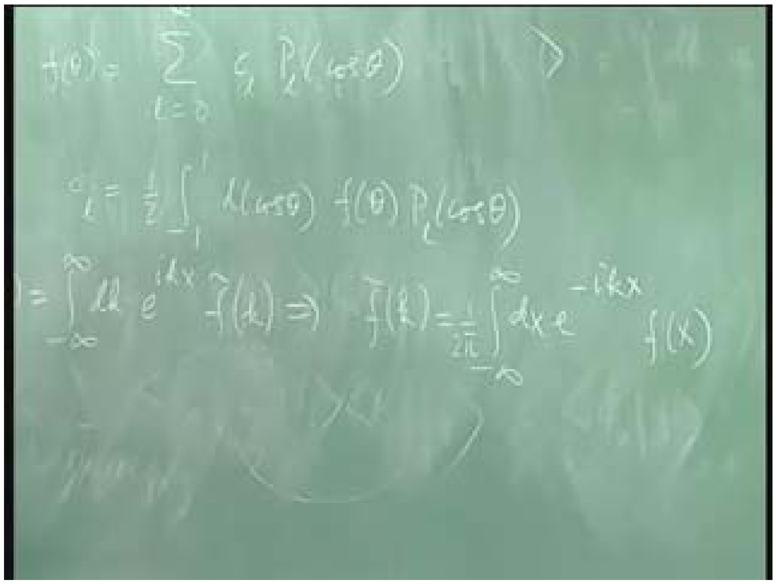

Different books follow different conventions. You would sometimes have a minus here (Refer Slide Time: 57:47) and plus there but it doesn't matter. You sometimes have a 1 over 2 pi here and nothing there and sometimes have a 1over square root of 2 pi here and one over square root of 2pi there to make it look symmetrical and so on but it doesn't matter. You have stick to one convention to do this.

The next thing to do is to look at operators but that's really where the real stuff is. So we will define tomorrow what's meant by linear operators in a linear vector space and then we have to talk about the size of an operator, norms of an operator, the domain, and the range and so on. Thank you!