Quantum Physics
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Lecture No. # 28

Today, what I would like to do is to go to the states of the radiation field of the electromagnetic field which would correspond respectively, to black body radiation on the one hand which is like what's called the degenerate photon gas or thermal radiation and the opposite extreme to it namely coherent radiation like in a single mode ideal laser. We will see what the photon distribution properties are in the 2 cases. So let me start with black body radiation and I recall to you something from elementary thermodynamic which all of you know, namely that the energy density of the radiation inside a black body cavity is proportional to the 4th power of the temperature. This is the Stefan Boltzmann law. What do we mean by a black body cavity?

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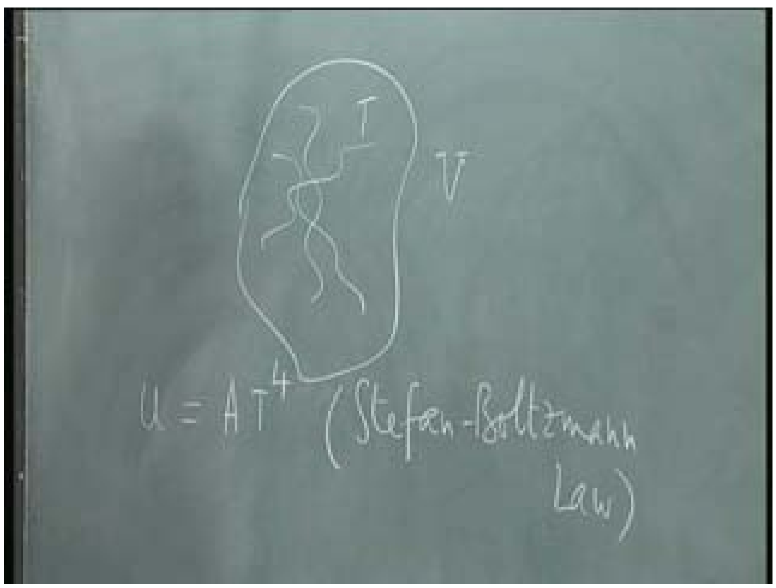

Well, it is some container of volume V in which there is radiation of all possible wavelengths going back and forth in thermal equilibrium. Therefore, it is essentially a photon gas of all possible frequencies, polarizations and directions of the wave vector. These photons are imagined to go back and forth between the walls. They get absorbed by the atoms in the walls and reemitted by these atoms. If the system comes to thermal equilibrium at some temperature T, then we know the Stefan Boltzmann law says that, "The average energy per unit volume inside this cavity is proportional to some constant A times T to the power 4." Now what i should like to do is to ask a slightly different question namely, what's the probability that the number of photons inside this cavity at any instant of time, the number of photons is a certain number n? But once I say a photon, I have to also specify the frequency of the photon. I have to specify its wave number and so on. So we will focus on photons of some given frequency, omega and some fixed polarization and ask what the probability is that the number of such photons is n. now one can do this problem vigorously by the following method. The gas inside is a gas of photons and we know that photons of 0 rest mass. So, these are essentially particles which move such that the energy is proportional to the momentum, E = cp where is c is the speed of light in vacuum. But there is more to it than that. These photons have a spin quantum number equal to1. Therefore there are bosons and it turns out that when you have a collection of identical bosons, the total wave function of this boson system must be symmetric under the exchange of any 2 bosons. That's part of rules of quantum mechanics namely, the fact that integer spin particles obey Bohr's statistics and they are indistinguishable and the total wave function or the state of the system is symmetric under the exchange of any 2 of these particles.

Conversation between student and professor: what do you mean by total wave function?

Well, in a simplistic picture, if I would like to describe the state of the system by some ket vector psi, then under physical exchange of the labels of any 2 particles, this state vector should be invariant. In the simplest instance, when you have 2 material particles and they have coordinates r1 and r 2 and the wave function in the position basis for these 2 particles is some psi (r1, r 2), then psi (r1, r 2) must be equal to psi (r 2, r1). But then, you must also exchange all the other labels including the spin label. In the case of photons, the states are supposed to be symmetric. In the case of half odd integers spin particles like electrons, the states are supposed to be anti-symmetric.

Now, the consequence of this is something I haven't gone into because we didn't have time to do quantum statistics, is that the number of possible particles in a given state (1 particle state of a collection of such particles), the number of possible particles in a given state is 1 or 0 in the case of fermions and it can be anything in the case of Bosons. That's all that we need to remember at the moment. Now what I will do instead of going into quantum statistics, I will derive this result in particular something about black body radiation. So to speak by the back door and I will do this by heuristic method. All I am going to do is the following.

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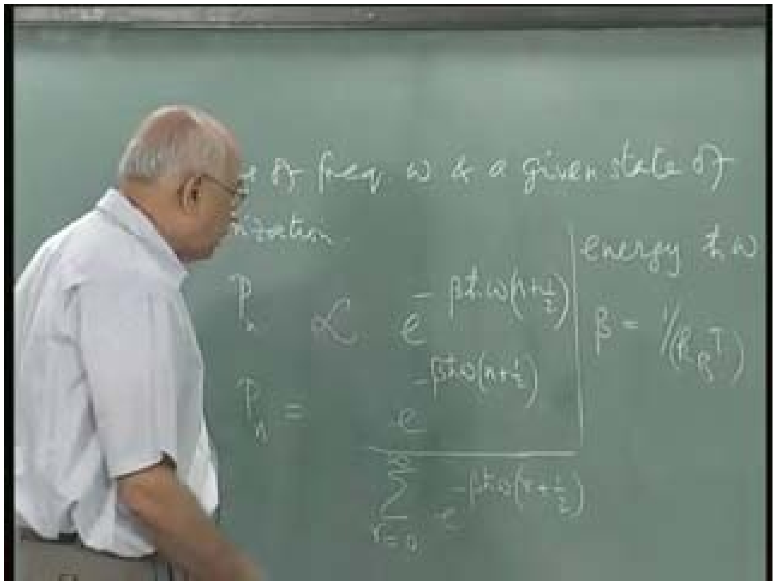

I am going to look at a photon of frequency omega, and a particular state of polarization, say left circularly polarized or right circularly polarized. It doesn't matter. Now I am going to ask the following question. What's the probability that in thermal equilibrium at this temperature T, this cavity contains n such photons at any instant of time? And it is denoted as P_n. Of course it's going to contain photons of all frequencies and all wave numbers in both states of polarization. But I am focusing as I said, on just this single frequency. Now what is this going to be proportional to? Well, each photon of frequency omega has got an energy h cross omega. Because for a photon, the energy momentum relationship is linear.

The moment you give me its angular frequency omega, the energy is h cross omega. So n of these photons would have an energy nh cross omega. But now comes the crucial point that the photons don't interact with each other very strongly. a gas of photons may scatter of each other in some sense but the scattering cross section is really very weak. They don't interact with each other. They do interact with the walls of the cavity. That's how the system gets into an equilibrium. I put in some radiation inside black body cavity and I wait for sufficiently long time. The walls then absorb these photons; re-emit them at various frequencies until an equilibrium distribution is reached.

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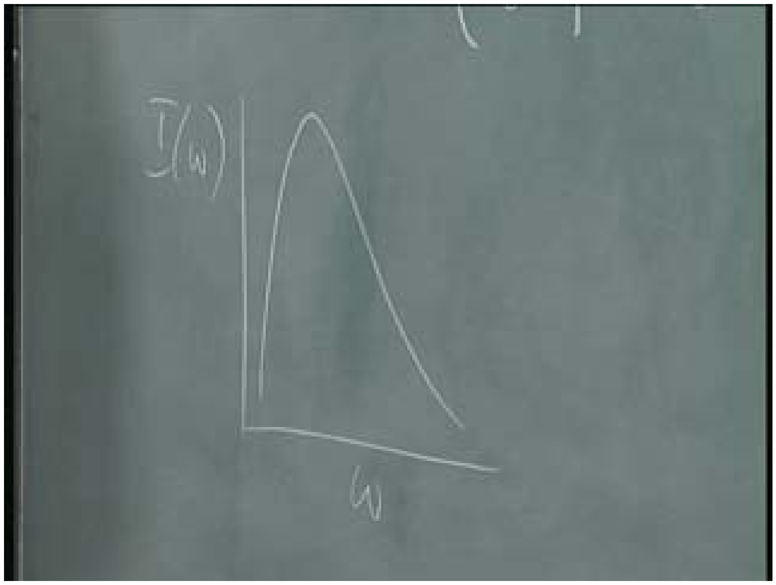

All of you are familiar with the Planck distribution which essentially says that the intensity curve at some frequency looks like this (Refer Slide Time: 09:11). This peak is dependent on the temperature. As the temperature of the cavity increases, the peak frequency and the average energy increase. This is the famous Planck distribution law. We are not going to quite derive that. I will get it by the back door as you will see. But now, we are asking a different question. We're saying we fix the frequency and ask what's the probability of a certain number of a photons. Now, I know that if you have n of these photons, the energy is essentially nh cross omega. These photons form a subsystem inside the black body cavity. The probability that a subsystem has any energy E when it is put in equilibrium with a thermal reservoir at temperature T, is proportional to the Boltzmann factor, e to the - the energy over kT. So this probability is definitely proportional to e to the power - beta h cross omega where beta is 1 over k_B T.

We know that from Maxwell Boltzmann factor. No matter what statistics these photons obey, the actual probability is proportional to e to the - beta h cross omega. There are n of them. So the this must be e to the – beta h cross omega n. Now of course this is just a proportionality. The actually probability must be a number between 0 and1. Therefore you must normalize this probability. The way you normalize the probability is to sum over all the possible values of n. But let me be a little more accurate in this. We know that the radiation field inside this cavity can be regarded as a collection of oscillators corresponding to different frequencies. What is the energy of an oscillator in which there are n quanta? it is h cross omega and n + 1/2. So really we must keep track of that 1/2 too. it will turn out that it won't matter in this calculation but that's the correct answer. What's P_n actually equal to? It's equal to e to - beta h cross omega into n + 1/2 divided by a summation over all possible states of the system. In this case, it's all possible numbers. So this runs r = 0 to infinity, e to the - beta h cross omega into r + ½. So that's my probability. It's a function of n. So let's work it out. It is a simple geometric series. As you can see, this 1/2 factor here and this ½ factor cancels out because this comes out of the summation and cancels on top and middle. So it doesn't really play any role at all.

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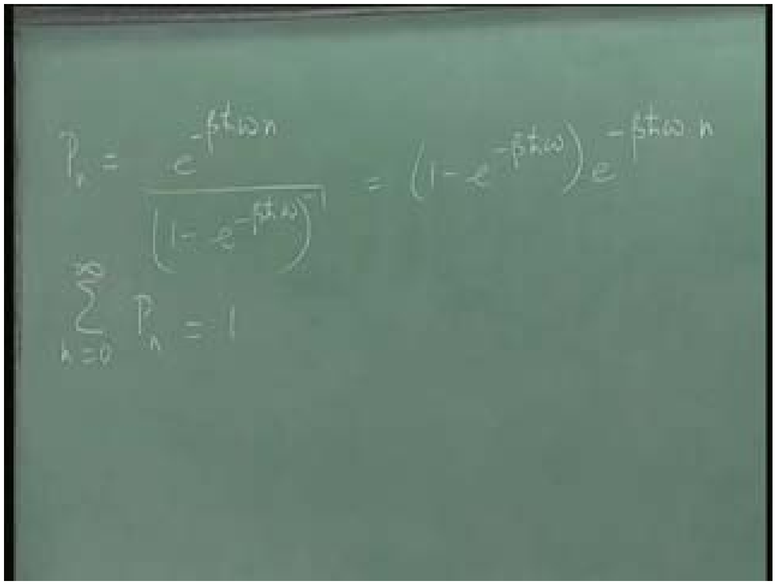

So P_n = e to the - beta h cross omega n and then geometric series. e to the - beta h cross omega is a number less than 1 because beta h cross omega is positive. So this number is less than 1 and the geometric series is a number raised to the power r and therefore this is divided by 1 - e to the - beta h cross omega inverse. So we actually have a close expression which says this = 1 - e to the - beta h cross omega e to the - beta omega h cross n. so it tells us in thermal radiation for photons of a particular frequency and state of polarization. This is the absolute probability. It depends on the temperature. You can see what the behavior of this is going to be as a function of temperature. Now what is the average number of photons?

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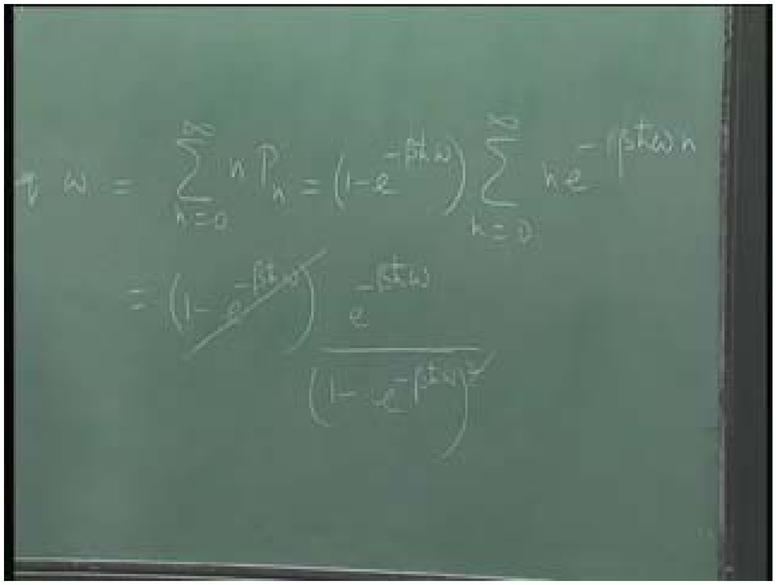

It’s easy to see that this is normalized to 1. We must check that right away. n = 0 to infinity P_n, this is also sum from 0 to infinity and this will cancel this exactly (Refer Slide Time: 14:25). So it is clear that P_n =1. So it is a normalized probability. What’s the average number of photons? Average number of photons of frequency omega and a given state of polarization otherwise you have to multiplied by 2, this = summation n = 0 to infinity n P_n, by definition. The first moment of this distribution. What is this going to turn out to be? Well, this =1 - e to the - beta h cross omega summation n = 0 to infinity n e to the - n beta h cross omega n. i have to sum this now but with an n inside. What kind of sum is that?

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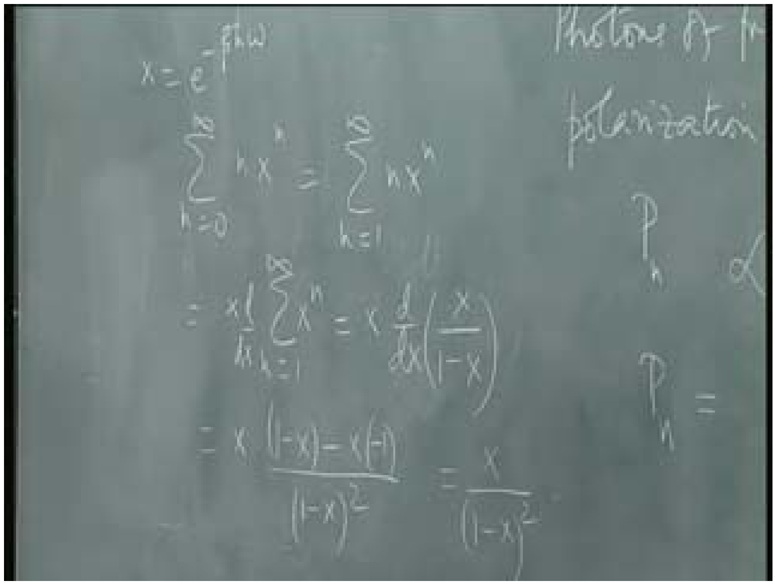

If you put x = e to the - beta h cross omega, you have to do summation n = 0 to infinity n x to the power n, where x is less than 1. This is an arithmetic or geometric series. So this = summation n =1 to infinity n x to the n, because n = 0 term doesn't contribute. let us write this as equal to x summation n =1 to infinity n x to the n -.1 But this is d over dx of x to the power n. so i pull the d over dx outside x to the power n which is x d over dx x over1 – x. This is = x times 1 - x the whole squared, if i differentiate that i get a 1 - x out here, - x times, there is a -1 if i differentiate this. So this = x over1 - x the whole squared. So this =1 - e to the - beta h cross omega. That is 1 - x and then x which is e to the - beta h cross omega over 1 - e to the – beta h cross omega the whole squared.

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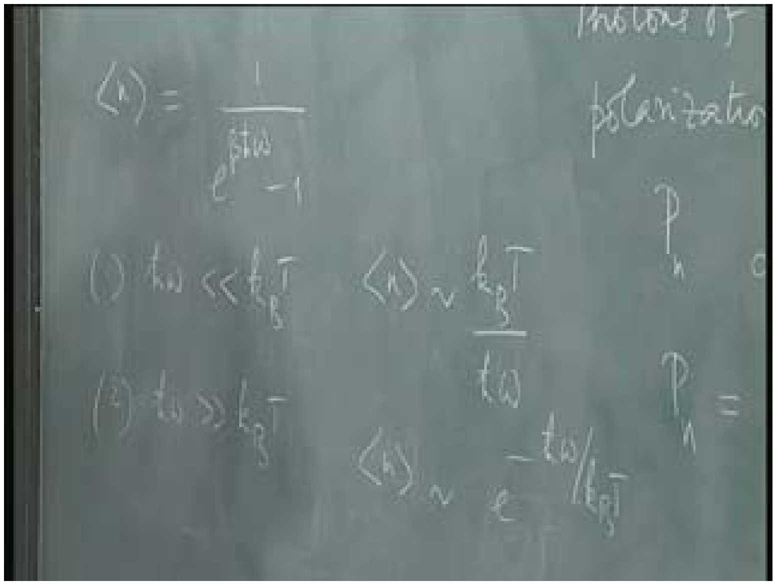

So this (Refer Slide Time: 17:39) factor goes with that and if I multiply through by e to the beta h cross omega, we get our result which says the average value of n =1 over e to the beta h cross omega -1. i just multiply the numerator and denominator by e to the - e to the beta h cross omega. this is a non-negative number as it should be here but please notice that this number could become very large, it could be very small, etc. so we have 2 regimes. One of them is beta is1 over kT i.e., if you have h cross omega much less than kBT. So the frequency you are looking at in energy units is much smaller than the thermal energy average thermal energy. What would this correspond to? This is like e to the x where x is much less than1. Therefore, you can write it as 1 + x + x squared and so on. The ones cancel out and the leading term is just 1 over x. so n average goes like kT over h cross omega.

On the other hand, in the other limit h cross omega much greater than k Boltzmann T, this exponential is very large compare to unity and you can drop it. So n goes like e to the - h cross omega over kT. This is a small number. So it is clear that very high frequency photons are very improbable. You don't have too many of them on the average. If the temperature is not high enough, it is going to cut off fast exponentially fast. This number can get exceedingly small.

For example, if you ask: what's the probability of having optical photons at very low temperature black body cavity, this number can become extremely, very much smaller than unity. On the other hand, in the other extreme near the peak for example you would have a very large average number. So remember this distribution has this funny shape. It goes up and comes down exponentially fast. Now there is a better to write this distribution and that's the following. So let's use n average as a parameter. Let's denote is as n bar.

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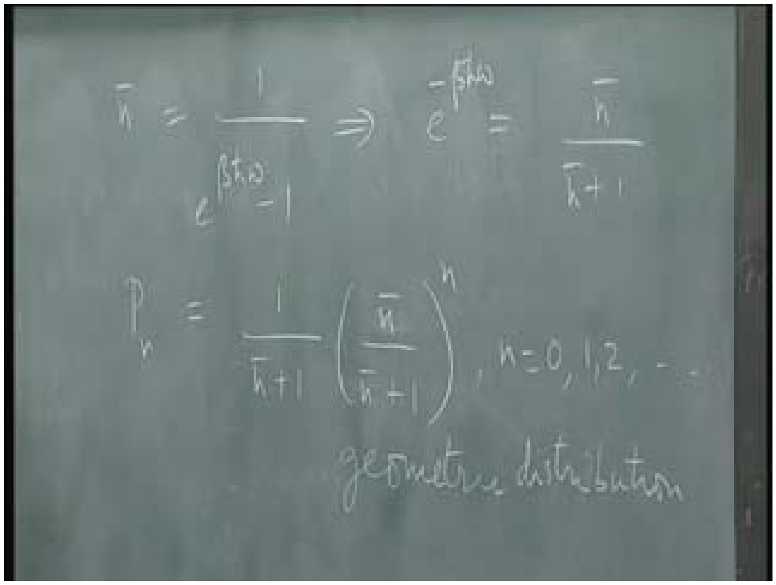

n bar =1 over e to the beta h cross omega . This implies that e to the beta h cross omega = 1 +1 over n bar which is n bar + 1 over n bar. So e to the - beta h cross omega is n bar over n bar +1. So that helps us to write down this Pn in a very convenient fashion. Pn = 1 over n bar +1 n bar over n bar +1 the whole n where n = 0, 1, 2,…so that's my distribution. This is the photon number distribution for photons of a given frequency and polarization in a black body cavity. What kind of distribution is this? It's some number which is less than1. It's a fraction raised to the power n. It's a discrete distribution. It is a geometric distribution because the different terms in the sequence Pn are like the different terms in a geometric sequence.

Remember, that n bar is a function of the temperature and the frequency. So the relation is very clear. n bar is 1 over e to the beta h cross omega -1. So at any instant of time in thermal equilibrium, if you ask what's the average number of photons of a given frequency, this number could be smaller than1. It means 0 is dominating. Remember 0 is also an allowed value. So this distribution has 1 over n bar +1 and then P_1 is smaller than P_0 which is smaller than P_2 etc. so as the Pn increases, the P decreases in this geometric fashion and P0 is the largest out. So the average could well be between 0 and1. It could be a fraction. We already saw earlier that the probability can become vanishingly small depending on the temperature regime. So at any given temperature, this is the way the distribution looks.

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If you draw the histogram here, what would it look like? P0 would be here, P1, P2, P3, P4, etc. it would go down in this fashion like some power (Refer Slide Time: 23:51). So it's conceivable that the average is sitting between 0 and 1. What's the variance of this distribution? That is the interesting question to ask. So what is that scatter about this average value. The shape of the distribution already suggests that it's a very broad distribution. They could be a large dispersion about the mean value. The mean value for many distributions is not enough to specify much about the distribution. We need to know the variance, the higher moments and so on. So that requires the mean square. Now let us compute mean square value.

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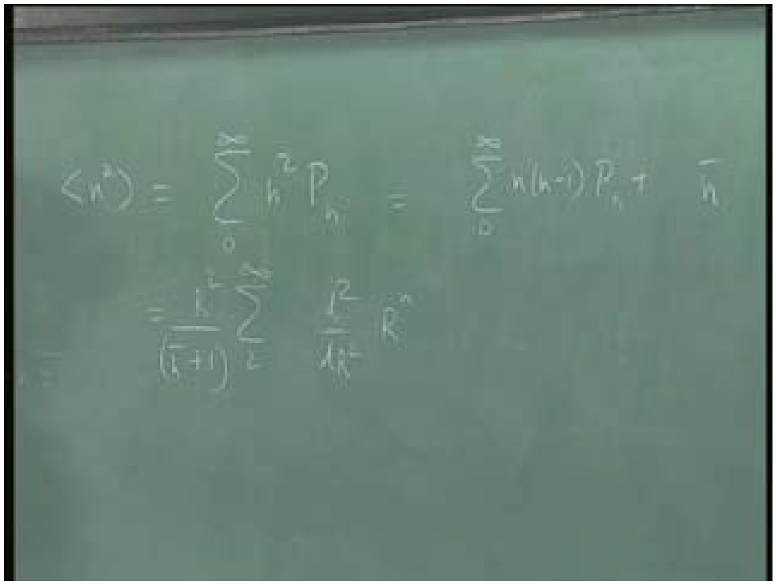

It is n squared = summation over 0 to infinity n squared Pn. That's equal to summation 0 to infinity n times n -1 P_n + 0 to infinity nP_n + n bar. 0 and 1 don't contribute here. So that is a summation from 2 to infinity n n -1 Pn but Pn is 1 over n bar +1. And then it is this quantity (Refer Slide Time: 25:36). Let me call it R for now. So its R to the power n. we can write this as the second derivative of R. so put an R squared and make it R power n-2. Then it is d 2 over dR 2 R to the n. this factor here (Refer Slide Time: 26:15) becomes twice n bar squared + n bar.

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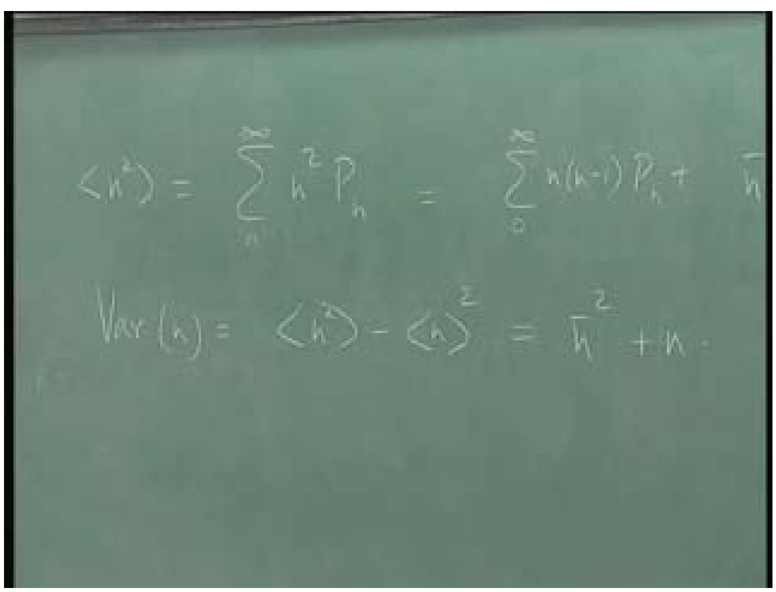

Therefore the variance of n = n squared - n average squared = n bar squared + n. so it is actually quite large. It is larger than the mean value. and the standard deviation is the square root of variance. Remember that all these of functions of temperature and frequency.

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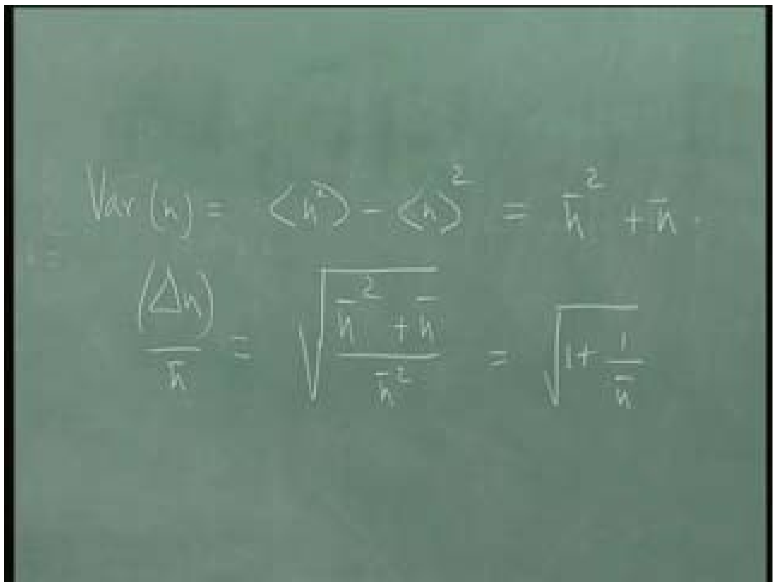

So given this you could ask what's delta n over n that = square root of n bar squared + n bar divided by n bar squared = square root of 1 +1 over n bar. So that is the relative fluctuation and if n bar is small, you can see that this can become extremely large. Even otherwise it tends to1. This means the standard deviation is becoming comparable to the mean if the mean is large. But if the mean photon number is small, this variance can be actually quite large. So although the mean value can be very small, the scatter about the mean is enormous. So number fluctuations are very significant in this case.

This is not surprising because photons are continuously absorbed and emitted. Therefore, the number of photons is going to be a random variable unlike material particles where the number of photons is fixed. So this system has very large number fluctuations. So this is a an alternative way of looking at black body radiation namely in terms of the photon statistics. By the way, once you have this result, then it is easy to compute what the internal energy is. <n> is the number of photons of a given frequency and a state of polarization. Now I ask what is the average energy. Well, here is quick way to do this.

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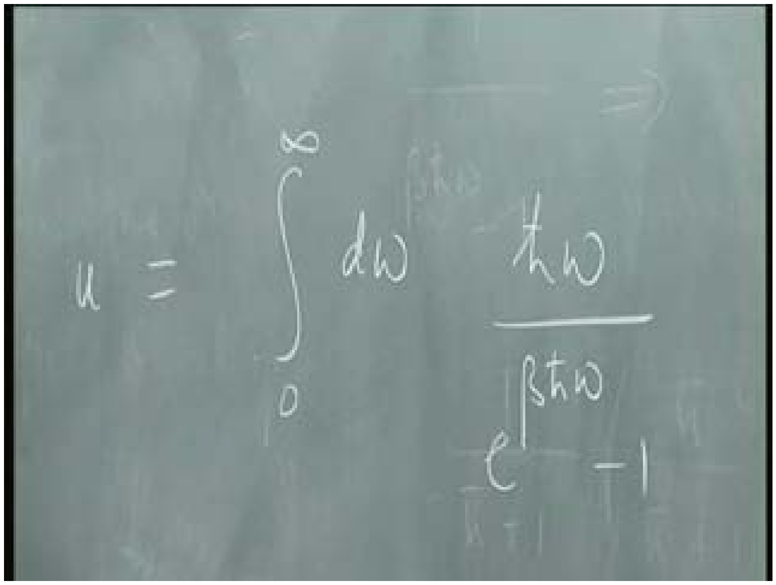

the internal energy of this gas is going to be an integral over the volume of the gas etc. so lets look at the energy density, the energy per unit volume. this is going to come by integrating over all possible frequencies from 0 to infinity d omega. this is the average number of photons of frequency omega. therefore the average energy contribution to the average energy from photons of this frequency is just h cross omega multiplying this (Refer Slide Time: 29:55). so it is h cross omega multiplying e to the beta h cross omega -1. and you must multiply it by the number of photons that you can have in a frequency window d omega. because you are integrating over this d omega. what's that equal to? Well, lets go back to our old phase space argument. in phase space, if you regarded these as ordinary material particles, the energy is going to be related to the momentum by linear relation, E =cp. remember that the phase space had a 4 pi p squared dp.

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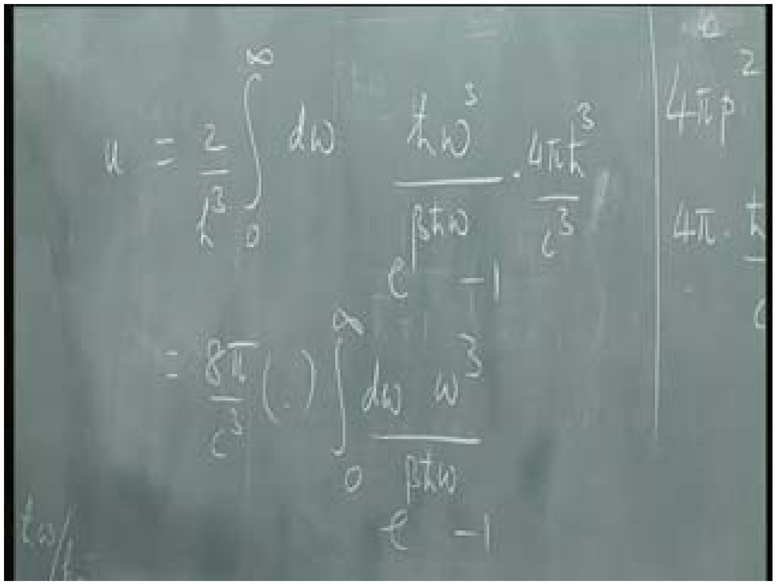

This is because the direction the momentum doesn't matter. The whole thing is spherically symmetric. so in phase space the d 3 p, the volume integral in momentum, that volume element had a 4 pi p squared dp. And for photons E is h cross omega = cp. so this says h cross omega by c is p and you need a p squared dp. so p squared dp is 4 pi into h cross squared omega squared by c squared dp is h cross by c d omega. That's multiplied by4 pi h cross cubed over c cubed omega squared that becomes an omega cubed here, d omega. And then you have to multiply this by factor 2 because that's the 2 states of polarization that any could have.

The actual internal energy is this multiplied by the volume but divided by the cell size in phase space, which in 3 dimensions is h cubed. So you put all these factors in. So its constant times d omega omega cubed over e to the beta h cross omega -1 0 to infinity. That's the crucial part. They will contribute to this Stefan Boltzmann constant. Now the integral is well defined because as omega goes to 0, the denominator vanishes like omega because e to the x goes like x but there is an x cubed here (Refer Slide time: 33:19). So the integrand is completely convergent at the origin. At infinity, this integral cuts off very fast. This integral can be done by contour integration for example. Its value is some finite number. Our interest is in finding out what is the temperature dependence of this without doing the integral.

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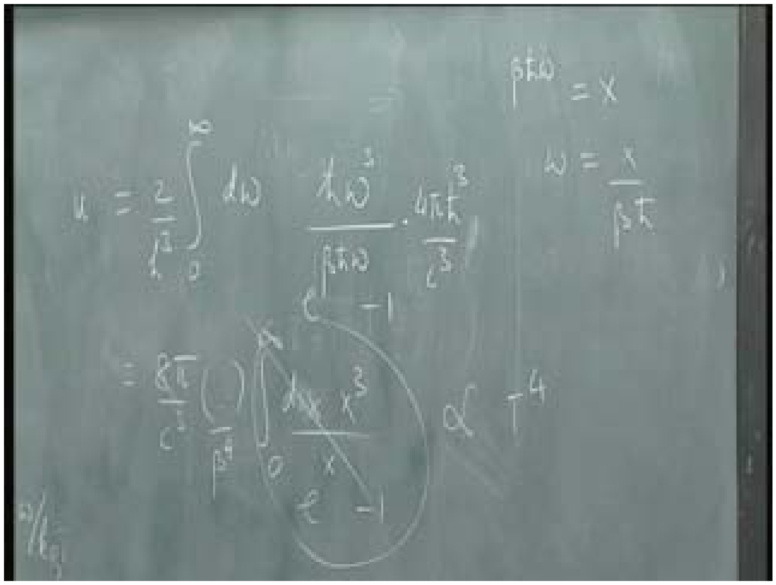

Well, that's not hard to do because we can put beta h cross omega = x for example, then it says omega is x divided by beta h cross. so this d omega is going to give me a 1 over beta and this becomes dx and then omega cubed is going to give me 1 over beta cubed. So its beta 4. This becomes x cubed e to the x -1. Finally, it is proportional to T to the 4 because there is a beta to the 4. So that's how the Stefan Boltzmann law emerges. Notice how different it is from the classical ideal gas of material particles where the energy is proportional to T. U is 3 half's kT per particle. U over n is proportional to T. It's the first power of T but here, it is the 4th power of the temperature. It's very different from what happens in the case of material particles.

Two things contributed to it. One of them was the fact that the energy was a linear in the momentum or the frequency as opposed to the formula p squared over 2 m which is what you get for the kinetic energy of the material particle of finite rest mass. in this case, the energy is proportional to the momentum directly. The second thing was Boltz' factor, the 1 over beta h cross omega -1 which arose completely from quantum statistics. It's a non-classical factor. Together these 2 conspired to make energy proportional to T to the 4. Therefore, the specific heat would be proportional to T cubed. This is responsible for the T cubed behavior of the specific heat of photon gas. You can also go on to calculate the pressure, the equation of state and so on. It's not hard to show in this case that P is 1 third U.

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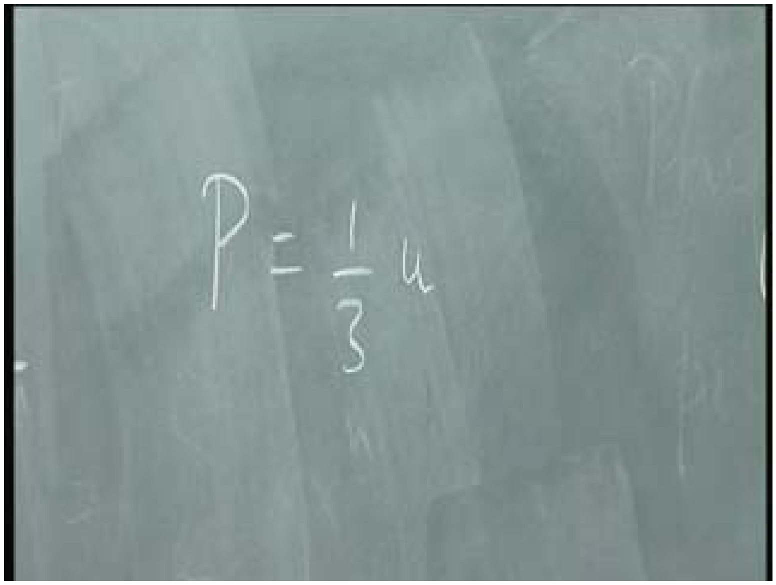

The pressure of this gas is proportional 1 third its energy density. Therefore, it is also proportional to T to the power 4 contrast this with PV = RT for the material particles in this case. It's also an ideal non-interacting photo gas. But, because its quantum statistics and the energy momentum relationship is different, you get completely different behavior. So now that we know how to do this problem, let's turn back and ask what happens in the case of coherent states of radiation? So let's go back and ask what if I took these oscillators and put them in a coherent state. Well, it turns out that ideal single mode laser light is coherent light in this sense. It's well described by what we looked at earlier in the context of the harmonic oscillator as oscillator coherent states.

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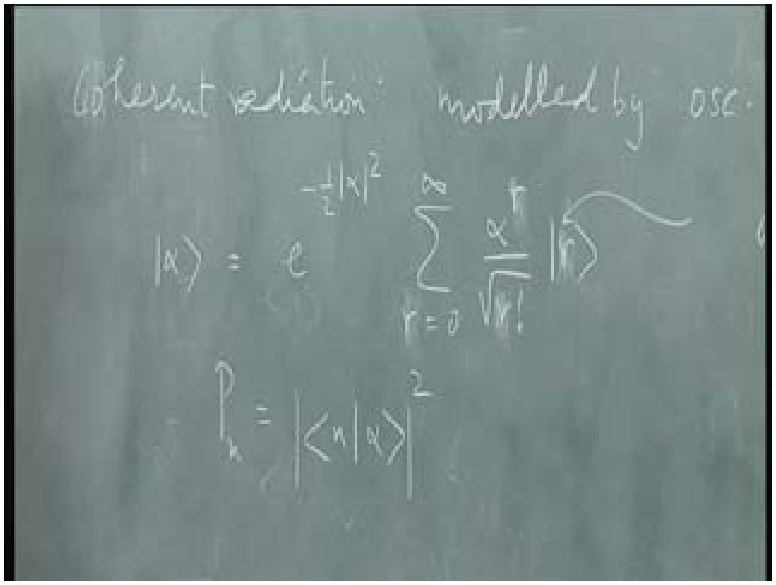

So I am going make this assumption without proving it too carefully that if you have and a single mode ideal laser light, then you have coherent radiation and this is modeled by oscillator coherent states. i will explain in a minute what is coherent about it. Let's recall the oscillator coherent states. We denoted alpha as an oscillator coherent state which was e to the – 1/2 mod alpha squared summation n = 0 to infinity alpha to the power n over square root of n factorial n. this state here was a number operator state. So it is a dagger a on n = n |n>. So now I am claiming that if in an ideal laser cavity, the single frequency omega and a single state of polarization, the radiation in that cavity is not in a state of thermal equilibrium. It's a very far from thermal equilibrium. It's in a coherent state. These are number states.

In other words, this state |n> now you must think of as a state corresponding to a fixed number of photons. When you superpose all these, you will end up with coherent radiation. What's the meaning of this quantity alpha? Well, we immediately see that this alpha, which is a complex number, has both real and an imaginary part or a modulus and a phase. The modulus of it is in fact the n bar that we talked about earlier. Because what's the probability Pn in this case? What's the probability that this state of the radiation field has n photons at any instant of time?

So let's put R and the coefficients in the superposition are the probability amplitudes that the number is some R. so Pn of course this n alpha. This is probability amplitude. So what's that equal to? It's just the coefficient of that. The coefficient is this e to the - mod alpha squared mod alpha to the power 2 n over in factorial. This is it. So all you have to do is to put the ket and the bra n here, use the fact that nr gives you a kronecker delta, that will pick out just 1 coefficient and then you take its modulus squared. That's it. What does this distribution remind you of? It's a Poisson distribution.

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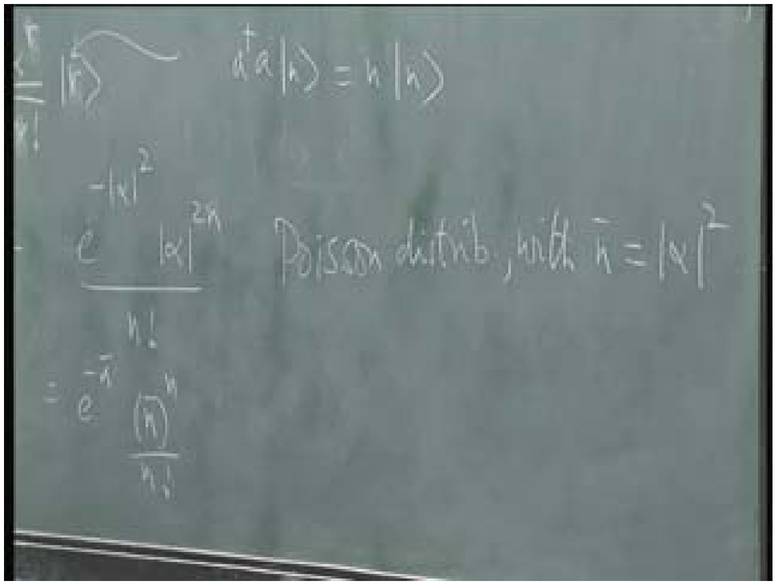

It's very different distribution all together. You have already seen that the average value of this quantum number n in this coherent state is in fact mod alpha squared. So that's the average value of photons whereas the other one is Geometric distribution. It was n bar over n bar +1 to the power n. We write this as e to the - n bar n bar to the power n over n factorial. That's the Poisson distribution. What's the variance of this distribution? It's n bar. For a Poisson, this has an interesting property that the variance = mean. On the other hand if you have coherent radiation, variance of n is n bar.

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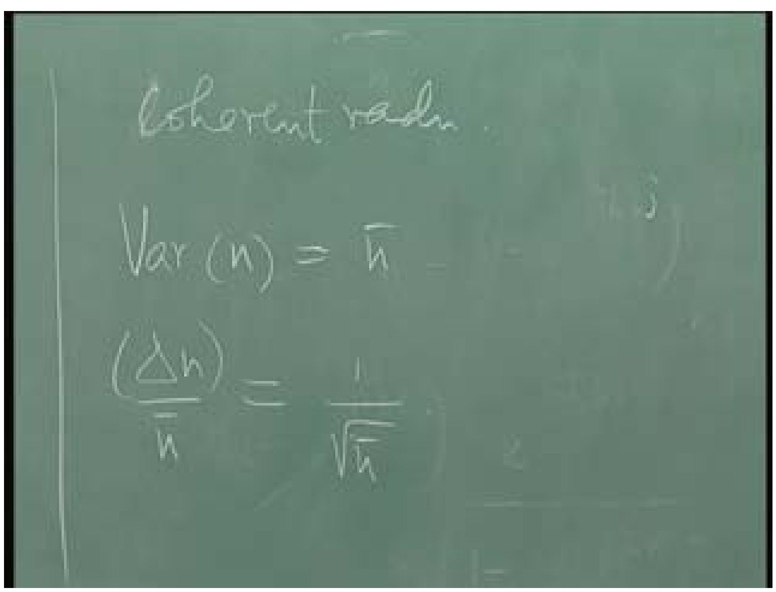

Therefore delta n over n bar =1 over square root of n. and it's a typical behavior of a Poisson distribution that the relative fluctuation goes like 1 over the square root of the average. Now this alpha is at your control. You can actually change the alpha depending on the kind of laser and the kind of source you have. You can make this anything here but the point is that, it is very different from the other state of radiation. This has got a definite state of polarization. It's single mode, incoherent, random superposition of all possible frequencies and all possible wave numbers and polarization is just the opposite here. So that's the reason why it's called coherent. But this is a classical state of radiation as classical as you can get, as oppose to thermal radiation. Because when you have blackbody radiation, it's a quantum statistical gas. You use the rules of quantum statistics whereas in this case, although it looks like you started with the quantum oscillator and so on. This is as classical as you can get because, recall what we have said earlier about the connecting the electric field to the oscillator.

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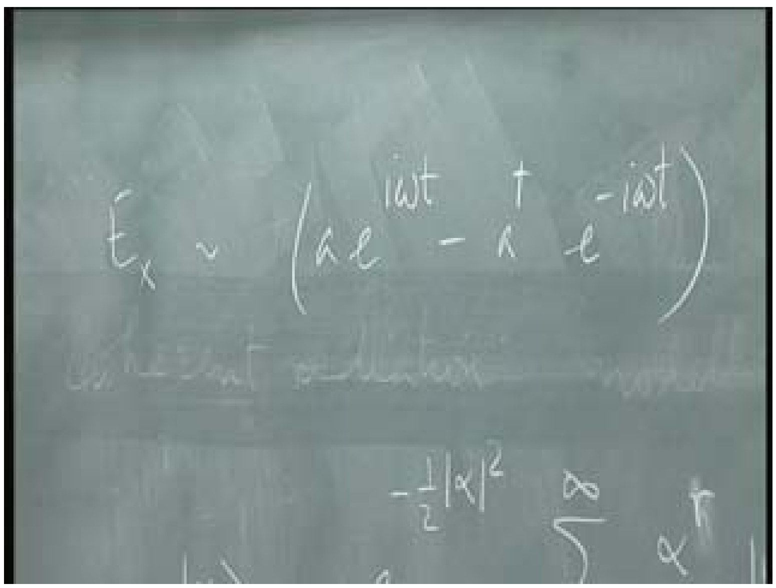

We call that the electric field E in that case had just 1 direction, Ex. this went like a e i omega t - a dagger e to the - i omega t. then of course to construct the field, you have to sum over all possible wave numbers. Now you could ask: what's the average value of this field in a coherent state? And what's the value of the mean square etc. Then it turns out this is a simple exercise to do that if you compute the mean square value of this, which is like finding the mean intensity that looks like classical wave its as close to a classical wave as you can get. The phase is uniformly distributed it's a completely classical and the phase is distributed in a Gaussian pattern i will show that and it's as classical as you can get. We already know that this is a minimum uncertainty state. I would like to recall to you some things we did in the context of the oscillator. Remember, that i pointed out that the electric field was like the x of this oscillator and the magnetic field was like the p of this oscillator.

And in this coherent state of the oscillator, delta x and delta p turned out to be as small as they are in the ground state of the oscillator itself which is a minimum uncertainty state. So just to recall to you delta x delta p was proportional to h cross into n + ½.

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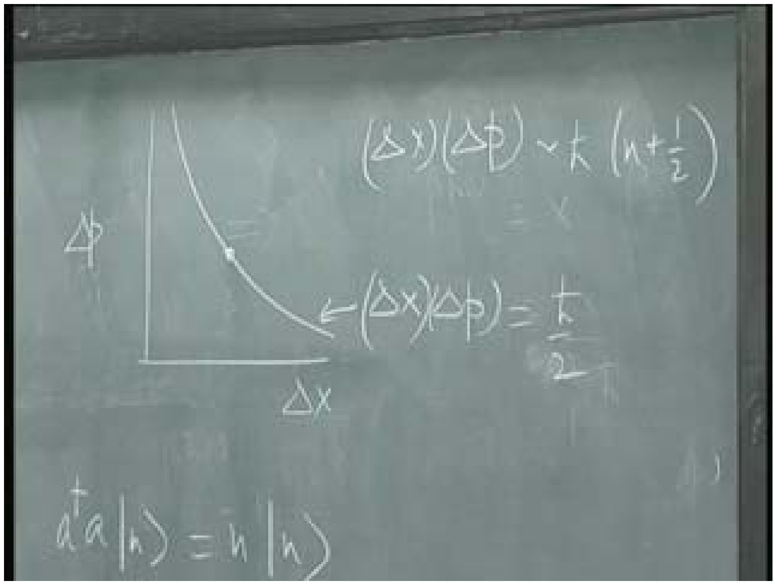

When n was 0, this is as small as you can get. Those are the ground states of the oscillator. For alpha, plot delta x here and delta p here. This is the curve delta x delta p (Refer Slide Time: 47:14) = h cross over 2, the minimum uncertainty curve. in suitable units, delta p and delta x were both here for the ground states of the oscillator and also for any arbitrary alpha. So in that sense, the coherent state is classical state quote unquote "minimum uncertainty". Besides, the mean square value of the electric field, the magnetic field, etc behaves like classical waves. The phase of the state is something we haven't been able to define. Because recall when I talked about the phase operator in the last time I had difficulties because the number operator didn't have a proper inverse. We need to overcome that and ask what the phase is and here is how one does it. So let me define a state of definite phase as follows.

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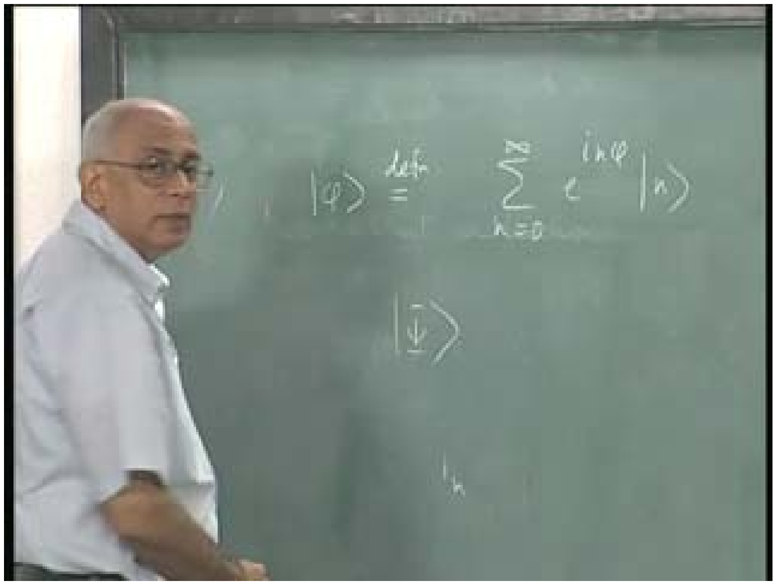

Let me call it phi. This is a definition as equal to summation n = 0 to infinity e to the i n phi. So this is a number. It’s a phase angle between 0 and 2 pi. I define it by using these coefficients because if phase were phi for this state e to the i n phi would be the factor that appears here summed over n. this is not a normalizable state as you can see. Because the coefficients in front a modulus. So it’s not a normalizable state but we have not interested directly in these states itself. We are interested in asking given a state of the radiation field, what’s the distribution of the phase. So if you give me an arbitrary state of the radiation field, then I will compute this quantity here (Refer Slide Time: 49:57) and take its modulus and this = the probability distribution of the phase of the state. This can be made well defined.

If you do this now for the thermal state as well as for the state of blackbody radiation, then in the case of blackbody radiation you discover that this p (phi) is uniform which is physically acceptable. It is all possible phases. It’s completely random between 0 and 2 pi. on the other hand, if you did for this the coherent state then the phase distribution is Guassian and if alpha is mod alpha e to the i theta, it’s an arbitrary complex number, then this becomes proportional to e to the - phi - theta whole squared. So it’s a Guassian peaked about the phase of this complex number alpha. So that’s the significance of what this alpha is in a coherent state. Its modulus square gives the mean number of photons in that state. And its phase gives you the peak of the phase distribution of the state of the radiation. So this is how one defines a distribution in the phase angle. If you go back to what I talked about with regard to the phase last time, you note that we had difficulties because we couldn’t define the inverse of the number operator. So this was a problem. We ran into difficulty defining a unitary e to the i phi. There are many ways of overcoming in and one of them is as follows. I define the following. I know that a dagger a has eigenvalues 0,1, 2, 3, etc.

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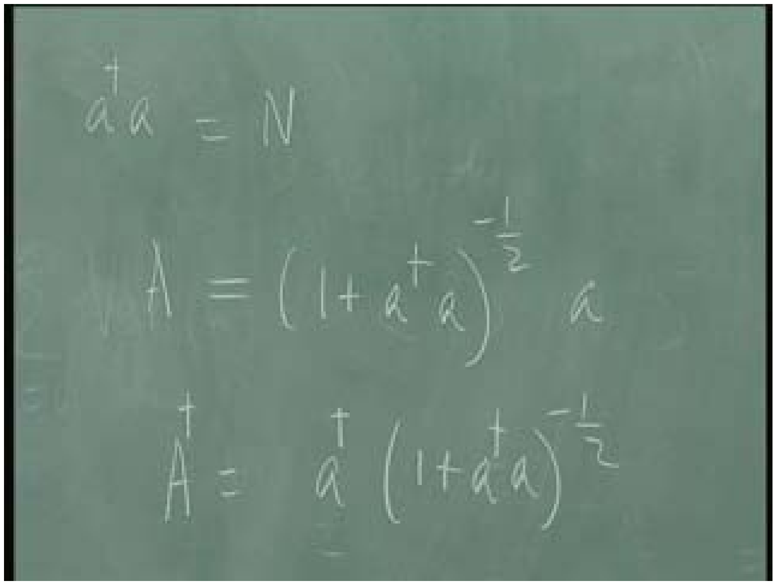

This is the number operator. It doesn't have a respectable inverse and so on. but if i consider this operator A which =1 + a dagger a to the – 1/2 times a and its Hermitian conjugate which is a dagger 1 + a dagger a to the – ½, i consider these 2 operators. This operator has an inverse. You can define **–** 1/2 etc because this spectrum is a a dagger. Its spectrum is 1 + the spectrum of a dagger a. so it's 1, 2, 3,4 etc instead of 0,1, 2, 3 and then the inverse exists. So what is A A dagger?

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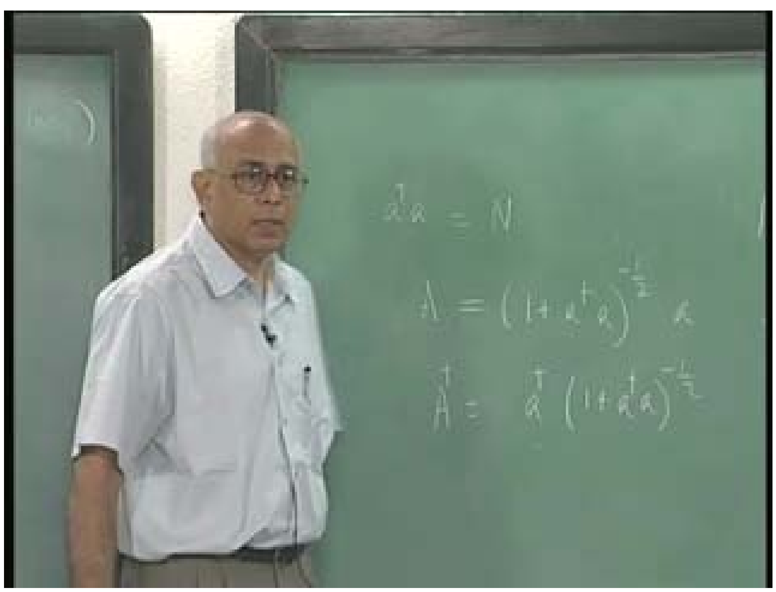

This =1 + a dagger a and – 1/2 a a dagger which =1 + a dagger a. it is the unit operator. on the other hand, A dagger A has a problem which is a dagger 1 + a dagger a inverse and then an a. this of course, you can’t write as a unit operator. It’s not true. But you can go back and ask how close it is to the unit operator. If it were unity, then i could take this to be respectable exponential operator and it would satisfy the commutation relations needed for the operator e to the i phi. But it misses it by a little bit because although a a dagger is unity, the unit operator a dagger is not. But you could go back and ask what does this actually looks like. What is the representation of this operator in this number basis? It’s clear that this = summation n n, by completeness. That is certainly true.

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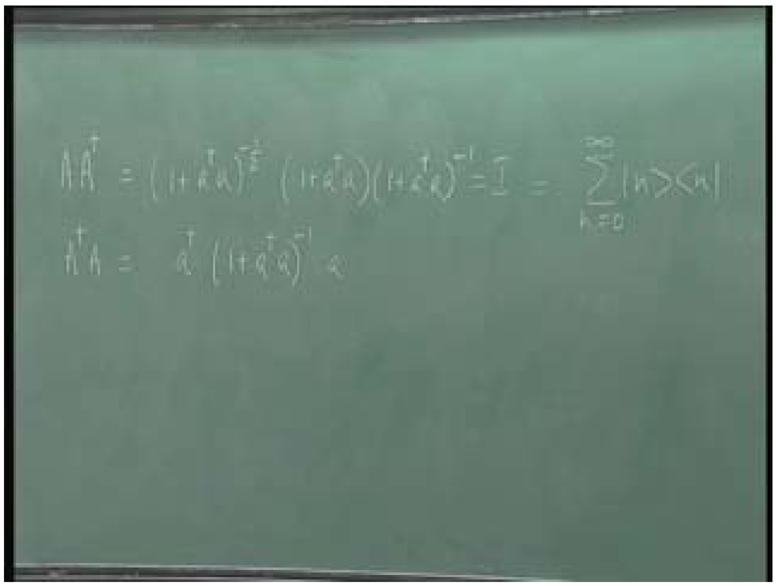

So work it out and find the matrix elements. So all you have to do is to find the matrix elements of this operator on both sides. When it acts on 0, the state ground state, this actually ends up with a 0 here because this annihilates the ground state. When it acts on 1, what would it do? When it acts on a general n, what would it do? So let’s find a dagger1 + a dagger a inverse a on a general n, not equal to 0,1, 2, 3, etc.

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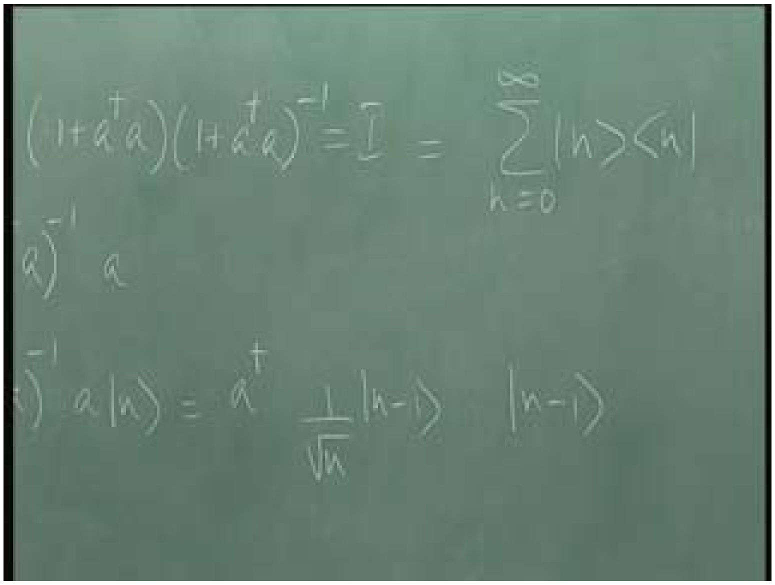

This gives you a n = a dagger1 + dagger a inverse and then there is a root n and then n -1. You have a dagger a acting on n -1. a dagger a acting on n -1 is just n -1. 1 + that is1 + n - 1. So that's n and there is an inverse there. So this whole thing becomes 1 over root n -1. Now a dagger acting on n -1 gives you a root n with the same with n. it's well defined. there is no 0 here. In fact, it is a power series and we take term by term. you have to be a little careful to make sure this series convergence so there are 2 expansions you need to resolve to but when do that it ends up with just1 over n. a dagger acting on n -1 is going to give you root n on n and therefore this is also diagonal operator expect when n equal to 0 otherwise it gives you 1.

So this whole thing this operator can actually be written as i - just the projection corresponding to the ground state. So it's not quite unitary but it comes very close. In particular, if the contribution from this (Refer Slide Time: 58:11) is small compared to the photon number, then you can neglect it and it's essentially unitary. So these operators a and a dagger play the role of e to the i phi. We can use those to define reasonable exponential of the phase operator. I don't want to get further into quantum optics. I just wanted to point out 2 salient features.

One was that in the case of thermal radiation blackbody radiation which is an ideal photon gas at some fixed temperature T, the photon number distribution of photons of a given frequency is a geometric distribution. In the case of coherent radiation which is very close to a classical state of the radiation field very far away in character than the thermal radiation, you have the Poisson distribution and the statistical properties are very different from each other. In fact a way of testing to what extent radiation is coherent is actually, take a cavity and make a small hole in it and then count the statistics of the photons that come out. Depending on what distribution you get can deduce the amount of admixture of thermal and ideal radiation in this. This is a part of the theory of photon statistics which I don't want to get into but I thought I would show you how fairly straightforward and simple ideas are starting with the harmonic oscillator. It actually helps us to get into quantum optics. So I don't want to say anymore about this except to now look at a model of atoms interacting with radiation. We talked about in the blackbody cavity atoms are absorbed and emitted. We would like to know is there is a simply model of interaction between atoms on the one hand and radiation on the other. That will take us to perturbation theory and that's the last topic I want to do.