Quantum Physics

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Lecture No. # 27

Today let me start off on a slightly new topic because I would like to spend sometime giving an application of quantum mechanics specifically in quantum optics and photon statistics. And this is one of the applications. There is a large number of such applications but i thought this would be useful and a good take off from were we have stopped. We studied a little bit about the harmonic oscillator and its properties as an exercise in quantum mechanics and we have also done a little bit about spin 1/2. I would to combine these and talk about 2 level atoms and models of atom radiation interaction. To set the stage for that, I would like to introduce you to where the harmonic oscillator plays a role in understanding the behavior of the electromagnetic field.

We saw that you could start with particle mechanics and then do quantum mechanics by converting Poisson brackets into commutators. The natural question that arises is what happens if you start with the classical field, like electromagnetic field and would it have a corresponding quantum version or not. This is really the purview of quantum field theory which is not part of this course but i would like to point out that there is a simple way of understanding how the harmonic oscillator plays a fundamental role in quantum field theory. So I am going to do the quantization of the electromagnetic field but I am going to introduce the results in a very simple physically appealing way. And then at some stage, we will make certain assumptions which will tell you how the quantization actually occurs and after that we will study the consequences of it. So what we intend to do is to understand how photons behave in the quantum mechanical way. So let me start by writing down Maxwell's equations and then imposing certain boundary conditions on it to get you familiar with the idea of quantized modes.

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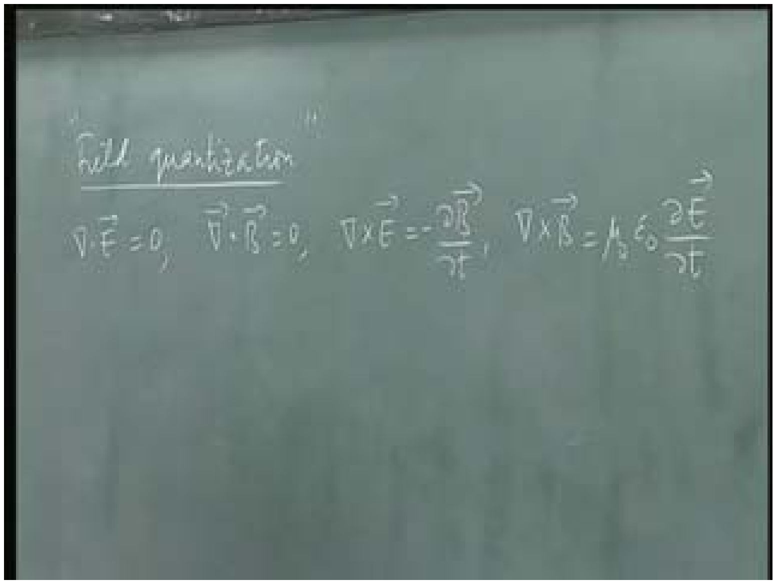

Classically, if I look at the electromagnetic field in the absence of sources, then the equations as you know are del dot E is 0, no sources and a region without any sources or currents. Del dot B is 0, del cross E is - delta B over delta t and del cross B is mu $_0$ epsilon $_0$ delta E over delta t. these are the source free Maxwell's equations in free space. And as you know, it's easy to show from here that both E and B obey the wave equation and the speed of these waves is given by 1 over square root of mu $_0$ epsilon $_0$. That's the speed of light in vacuum.

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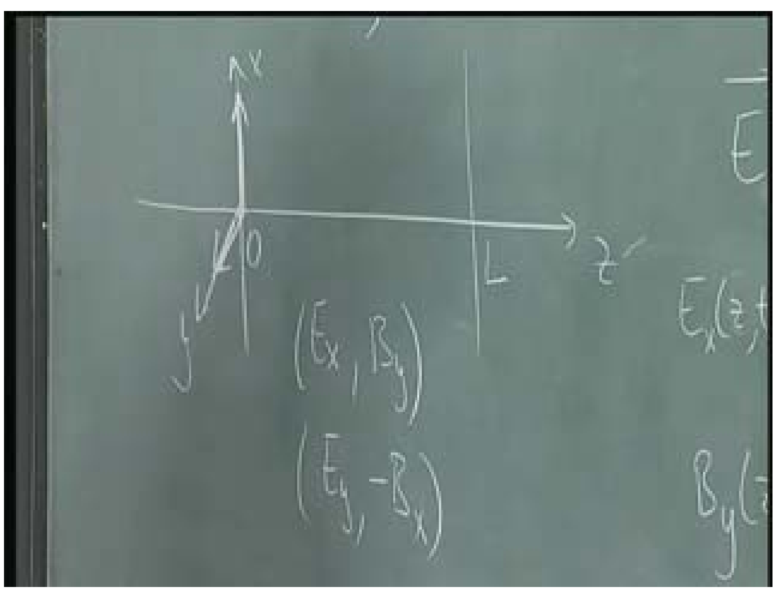

Now suppose I consider propagation of these waves along the z direction, between 2 conducting plates placed in the xy plane, and lets suppose that at z = 0 and z = l, i have 2 parallel plates which are conducting and look at what happens to standing electromagnetic waves in the region in between these 2 plates. The electric field must be 0 at the conducting plates they are earthed and now i am going to assume that the radiation is polarized along the x direction.

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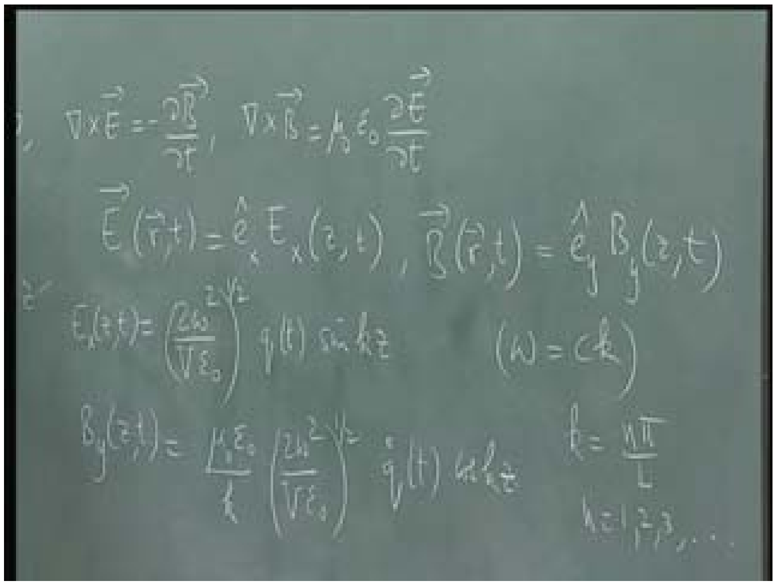

So I am going to assume that E, the electric field as a function of r and t is just e_x times the unit vector in the x direction. So the electric field is along this direction and since its radiation, the magnetic field could be along the y direction and the direction of propagation is z. as you know E B and K where K is a propagation direction for electromagnetic waves, they form a right handed triangle. The unit vector in the E and B directions, we take the cross product, you get the unit vector in the direction of propagation.

So this is Ex and that is a function of z and t alone and similarly the magnetic field B (r, t) is Ey By and that's a function of z and t alone. So I am looking at plane polarized light. The plane of polarization is the xy plane and the direction of polarization is the x direction for the electric field. now these fields have to obey these equations here (Refer Slide Time: 06:54) and its easy to see that suppose i put this whole thing in a cavity of volume V, I haven't told you what the boundary conditions on these sides are but we will assume its very large container of some kind with some volume V then, Ex (z, t) is of the form sin kz, where k is the wave number of this propagation and its going to be such that sin k sin kz is 0 at z = 0 and 0 at z = l, those are the boundary conditions times the normalization, so let me write down the solution. If the angular frequency is omega and the wave number is k then, the solution is 2 omega squared by V epsilon $_0$ to the power

1/2, that’s the normalization factor and then some time dependent factor which i called q (t). So these are some constant factors, omega is the angular frequency of this. Mod k is the wave number and the 2 are related as you know by omega = ck. V is the volume of this container, epsilon $_0$ is the constant that appears in these Maxwell equations. q (t) has the physical dimensions of a length. That’s how I chose my normalization constant and you can check that this quantity has the right dimensions for the electric field and this is dimensions of length here. Then if you put that in into this (Refer Slide Time: 09:00) equation and solve, then delta e over delta t will have a q dot of t and that’s del cross B and B we already know is in the y direction. So if you work it out, the del cross B is going to bring out a factor k.

So it’s By of z and t turns out to be mu $_0$ epsilon $_0$ divided by k. that’s because the del operator is going to produce a factor k that comes down in the denominator times 2 omega squared over V epsilon $_0$ to the power 1/2 q dot of t and then this is a cos kz because it involves a spatial derivative and it becomes a cosine. Now it’s a simple matter to check that these solutions actually satisfy the Maxwell equations here. Now what are the possible values of k? It’s clear that k must be such that sin kl vanishes at this (Refer Slide Time: 10:11) end. So k is = n pi over L, n =1 2 3 etc. It’s like waves on a string clamped at 2 ends.

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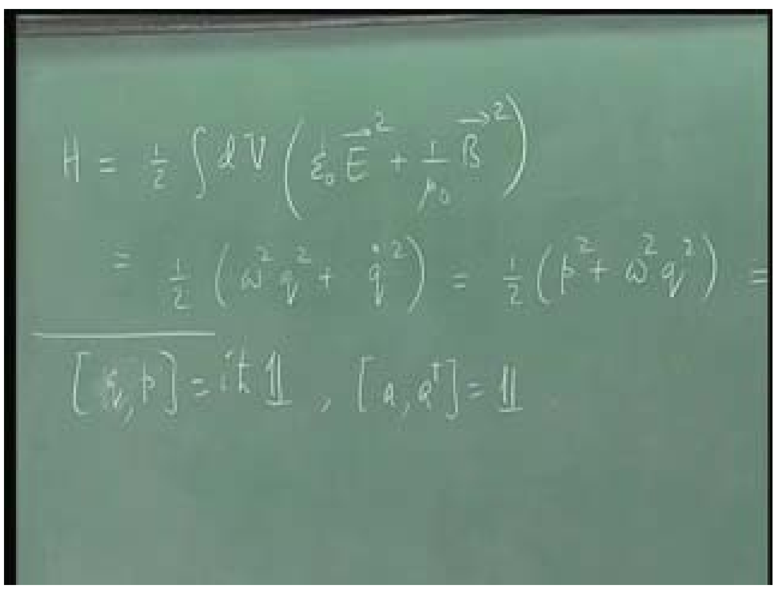

We just have a classical solution of Maxwell’s equations but if you put this into the energy of the system; the Hamiltonian, now what’s the energy density of the electromagnetic field? What’s the energy per unit volume? It’s 1/2 epsilon $_0$ E squared + over mu $_0$ B squared. so the total energy is =1/2 integral over the volume the energy density and the energy density is epsilon $_0$ E squared +1 over mu $_0$ B squared. And you put this in and perform the volume integration. We have conveniently chosen a thing so that you get1 over B to the 1/2 here (Refer Slide Time: 11:21) and when you square it you get a 1 over V which cancels the volume integral there. And then when you do this, you end up with 1/2 omega squared q squared + q dot squared. That reminds you of the harmonic oscillator.

In fact, this problem is identical to that of the harmonic oscillator. Omega is any1 of these numbers. It is ck where k is n pi over l. so you have many modes possible and let's look at any one mode. Some allowed value of k, and then the energy in that mode in that mode is given by just a harmonic oscillator. In fact it's a harmonic oscillator with mass =1 m =1 effectively. So instead of q dot, let me just write this as p because m q dot is p. so 1/2 p square + omega q squared. So you can see where the harmonic oscillator is appearing by the backdoor in the problem of the electromagnetic field. E squared + B squared is essentially like q squared + p squared. This is the way the field is quantized.

I haven't quantized the field i haven't started with canonical commutation relations and quantized it. But I am just motivating the fact that the oscillator would appear naturally in this fashion. It really appears from the fact that E and B, for a given mode of the electromagnetic field, act like the position and momentum respectively. So let's accept that that's the way field quantization appears and now what I am going to do is to take a big jump and say that I am going to take p and q to be operators as we do for the harmonic oscillator and impose the canonical commutation relation. So from now p q and p are operators which satisfy ih cross that's the unit operator. What happens then? It implies that the electric field and the magnetic field are operators. Ex and By are themselves operators. We could ask what's the commutation relation between Ex and By for example.

That would be related to the commutation relation between q and p you can write down explicitly what it is. Now we solve the harmonic oscillator by going over to operators a and a dagger which was essentially q + ip and q – ip. Then we obtained a number operator, a dagger a, we found the eigenstates of a dagger a, they were just natural numbers and we associated a number operator. Exactly the same thing will happen here except that this number operator now has a physical meaning. It is the number operator corresponding to the number of field quanta or the number of photons. so straight away from the oscillator, we go over to a picture where the electromagnetic field is pictured as a collection of oscillators. One oscillator for each mode of the electromagnetic field and then the number operator corresponding to that oscillator is just the number of quanta of the field. Number of photons with that particular frequency and wave number and state of polarization. We haven't talked about the polarization. I will come to that in a second.

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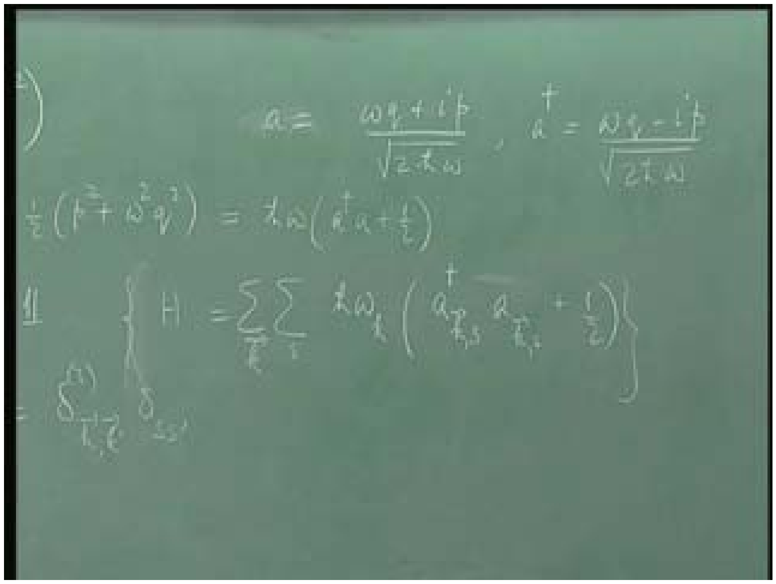

So if you go ahead and say, at this stage let a = m has been set =1, so its omega q + ip over root 2 h cross omega and a dagger = omega q - ip over 2 h cross omega, then this goes over into h cross omega into a dagger a + ½ exactly as it happened in the harmonic oscillator. Together with, of course we have adjusted those constant such that a with a dagger = the unit operator. Now the rest is a just a copy from the harmonic oscillator. All we have to do is to play around with these operators a and a dagger and we have a complete set of eigenstates of this and so on. Now please notice that this whole thing is for one mode of the electromagnetic field.

a mode that's propagating in the z direction with the particular wave number k which is 1 of these (Refer Slide Time: 16:56) numbers and a fixed state of polarization. In reality, if you wanted to describe the electromagnetic field, you have to be a little more careful than this. And that's not the total Hamiltonian. We could write down a better Hamiltonian than this. And what's that going to be? Well first of all, that even in this (Refer Slide Time: 17:17) direction of propagation, there could be 2 states of polarization. You can have 2 independent states of circular polarization for the electromagnetic wave. One of them, if you resolve it into plane polarization states, you have an Ex and By.

So the electric field is along that direction and the magnetic field is coming out of the plane of the board. There is another linearly independent state of polarization which would correspond to Ey. so the electric field points along this direction and the magnetic field along – Bx such that E cross B still gives you the direction k. these are independent states of polarization. You can combine these 2 and form left circularly polarized and right circularly polarized corresponding to the electric field vector making a circle either in the positive sense or in the negative sense. So if I denote these 2 states of polarization and i pointed out earlier that the 2 states of polarization correspond to 2 helicity states for individual photons, then what would the Hamiltonian actually look like? Well, the total Hamiltonian really would be a summation over the states of polarization which could be either left circularly polarized or right circularly polarized, let me use a symbol little s for the state of the polarization, and then the summation or an integration over all allowed values of the wave number k and that's a summation over all allowed values of n. but then we have assumed the radiation is propagating in the z direction. It need not. It could propagate in any direction. And therefore you must really sum over all possible k vectors as well and for each of these, you would have an h cross omega subscript k because for a given k omega is ck, multiplied by a dagger k vector s a k vector s comma s + ½. For each mode, each wave number k, you would have an a a and for each of the s values you would have an a dagger and a. so those operators would have 2 labels and this would be the total Hamiltonian of the electromagnetic field. What would this commutation relations look like? These are independent modes completely.

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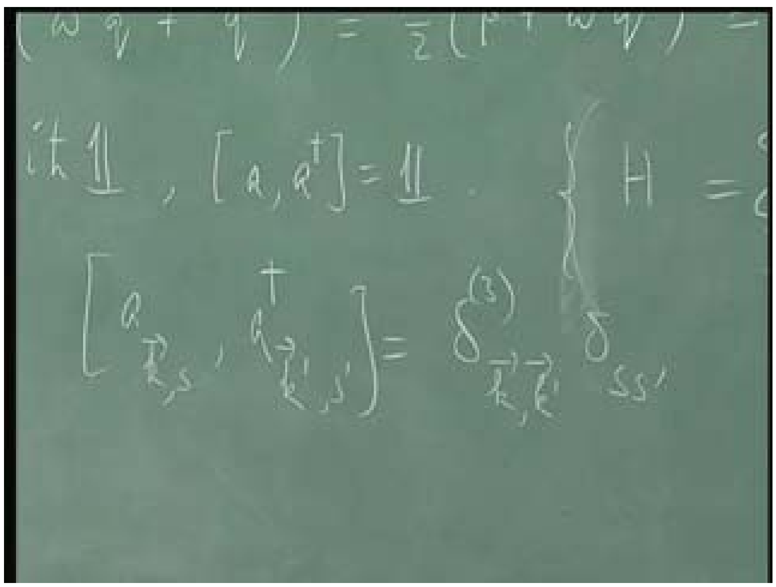

So it's clear that these commutation relations would go over into ak s with a dagger k prime s prime is, independent modes these operators would completely commute. But if they both correspond to the same mode and the same s, then the answer should be1. So it's clear that this would be = a Kronecker delta, a 3 dimensional delta by the way of k k prime delta of s s prime times the unit operator. So this is the start of field quantization. So you start by postulating these commutation relations and then look at what the consequences would be. I am not going to do because that's really the task of quantum field theory. We are not going to talk about that at all. We are going to come back here and i ask what consequences can be deduce from just this (refer Slide Time: 21:40) thing here for a single mode, what does it look like what happens if you put this in equilibrium with the thermal heat bath at some temperature which will take us to black body radiation. But the crucial physical ingredient is that the electromagnetic field is also quantized. We are not talking about it in interaction with anything at the moment. The free electromagnetic field acts like a collection of a harmonic oscillators in the quantum mechanical sense.

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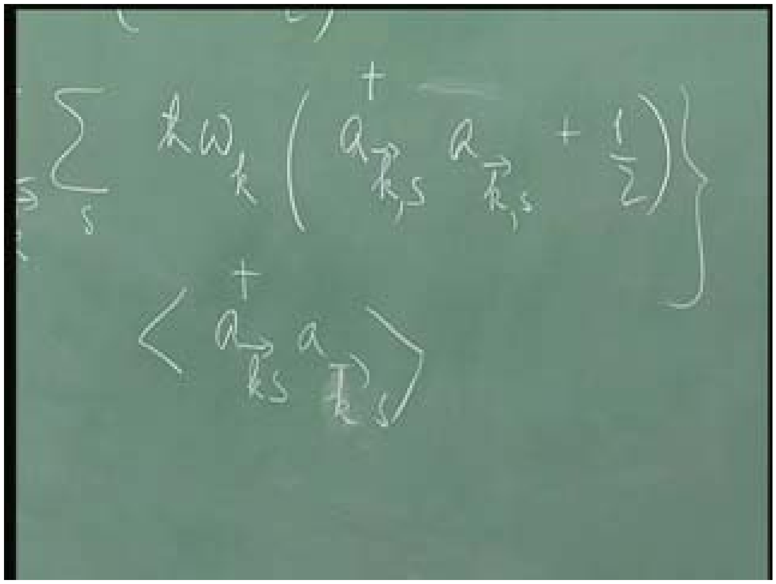

In this picture, in any state of this field, what would this correspond to? <ak dagger ak s>. What would this expectation value correspond to? What would be the physical meaning in any state of this field, what would be the physical meaning of this? It would give you the number of photons. the average number of photons, it’s a number operator of wave vector k and state of polarization s. and it would therefore, if you would now integrate this over some range in k, if its continuous then you get the intensity of this electromagnetic field in that range. But that’s proportional to the number as you can see. Now that we have this, let’s focus on one mode and ask what happens.

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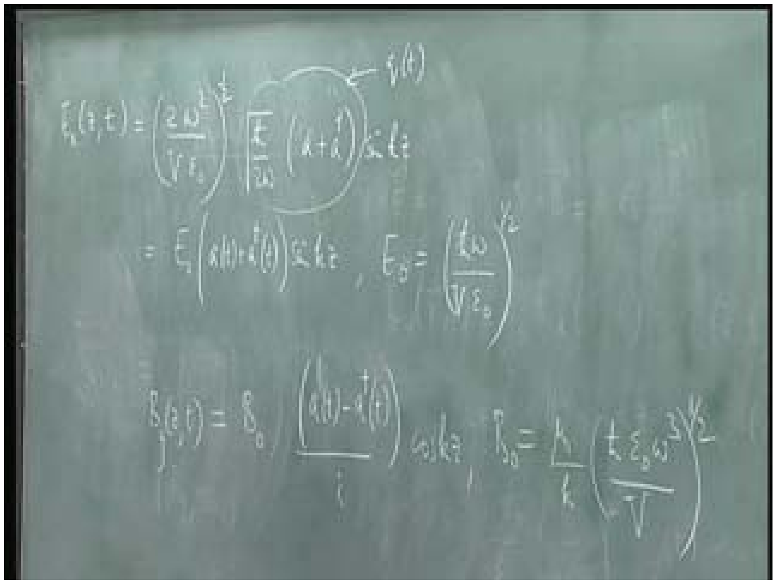

So let's write down for example what E does. Notice that we already have Ex (z, t) = 2 omega squared over V epsilon $_0$ to the power 1/2 q (t). Now what is q (t)?

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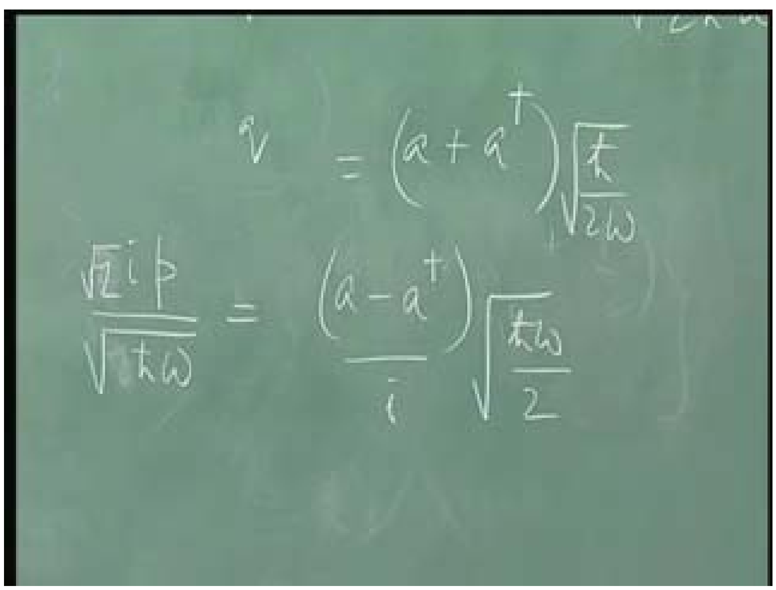

We already have these equations. So this is 2 omega q over squared, root of 2 h cross omega = a + a dagger and this is a root 2 omega there over h cross. So this multiplied by h cross by 2 omega is q. so q (t) is a + a dagger root h cross over 2 omega. so lets put that in. um yes root h cross over 2 omega a + a dagger sin az which is = some amplitude, let me call it just E_0, the operator part i want to retain sin kz. This (Refer Slide Time: 25:33) factor came from q of t. the t dependence of classical field; I put in this factor q of t. so that's true even here they must be t dependence. So this means we are working in the Heisenberg picture. These operators are time dependent. So we must be careful. Let me write that down properly. So this is a (p) + a dagger of t and this is some constant what's the value of this constant. E_0 is, it has dimensions of electric field so h cross omega over V epsilon $_0$ to the power ½, just for reference. and similarly the magnetic field By of z this is the y direction field so By of zt is = there was a mu $_0$ epsilon $_0$ over k and then there was this 2 omega squared over V epsilon $_0$ to the power 1/2 and then q dot or p. so what was p? 2 ip over root of 2 h cross omega was = a - a dagger. So the 2 becomes root 2 here and over i root h cross omega over 2.

So if I put that in for p here, this was a - a dagger over i and then root of h cross omega over 2 cosine kz. And if I put this (Refer Slide Time: 28:09) in this becomes omega cubed and the 2 cancels, I can put this h cross inside here so that goes away and that's what it is. so let me call this whole thing B_0 and write this as a of t - a dagger of t over i cosine kz and call B_0 this whole amplitude mu $_0$ epsilon $_0$ over k h cross epsilon $_0$ omega cubed over V to the power 1/2 and this goes away. That's B (t). So now we have our quantized fields. Ex is a function of z and t. for this mod of propagation is a + a dagger and B is a - a dagger. As you can see they play the role of a position and momentum so to speak, apart from some dimensional constants.

We can compute the commutator of E with B. that's like the commutator of x with p and you can see the commutator is a with a dagger is going to give you1 and a dagger with a is going to give -1. The question is what's the time dependence like? What's the solution like? That's not hard to find because there are many ways of finding this time dependence and in this problem, it's completely trivial.

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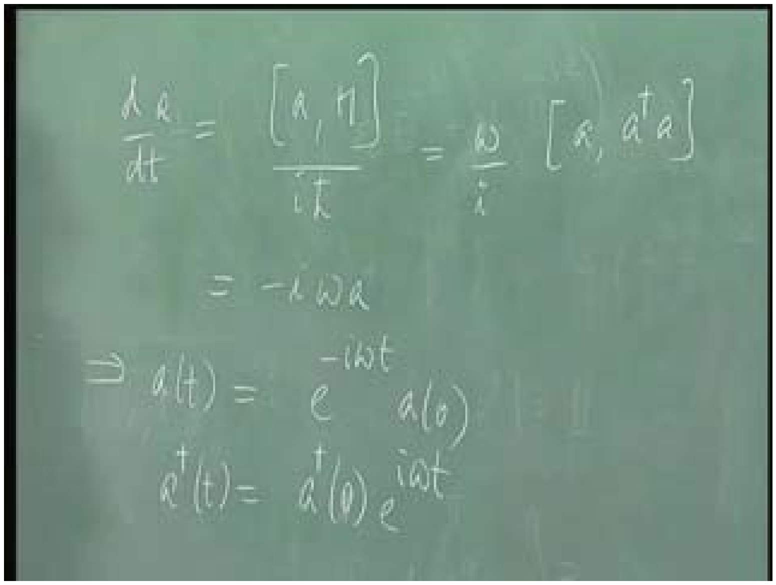

The Heisenberg picture operator, da over dt is = [A, H] over ih cross at any instant of time. That’s trivial to find because this is =1 over i or rather omega over i because H is h cross omega a dagger a. so this is the commutator of a with a dagger a. the 1/2 part doesn’t contribute to anything. And that = - i omega a because a with a dagger is unity and there is an a which comes out. What’s the solution to this? This implies a(t) = e to the - i omega t a of 0. Similarly, a dagger of t can also be computed but it’s just the Hermitian conjugate of this. So it’s a dagger of 0 e to the i omega t. this is just a number. So it doesn’t matter where I put it but just by shear force of habit, the moment I take a Hermitian conjugate, I invert the order here. So all you have to do is to put that in here and you have the exact the solutions where a and a dagger are Heisenberg picture operators. By the way, you can solve in another way. You know the formal solution to this equation. Now the question is: what is the expectation value?

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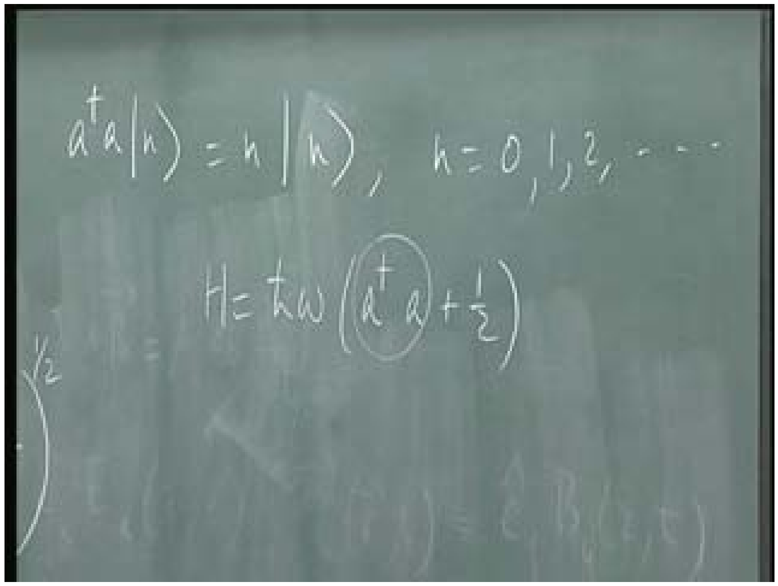

I am going to say a dagger a on n is n | n> and n is 0, 1, 2, etc. the eigenvalue n I identify with the number of photons in the number state n. The state of this radiation field in which the number of photons is given, is that an eigenstate of that electric field? So, the question being asked is the Hamiltonian is H cross omega a dagger a + 1/2 .a dagger a is the number operator. These are the eigen states of this number operator. So the question asked is, in a state in which the number of photon is sharp, is that also a state in which is electric field is known precisely. No, because a and a dagger don’t commute with a dagger a. similarly the magnetic field is not known precisely. In fact, you can ask: what’s the uncertainty in a state in which the number of photons is fixed?

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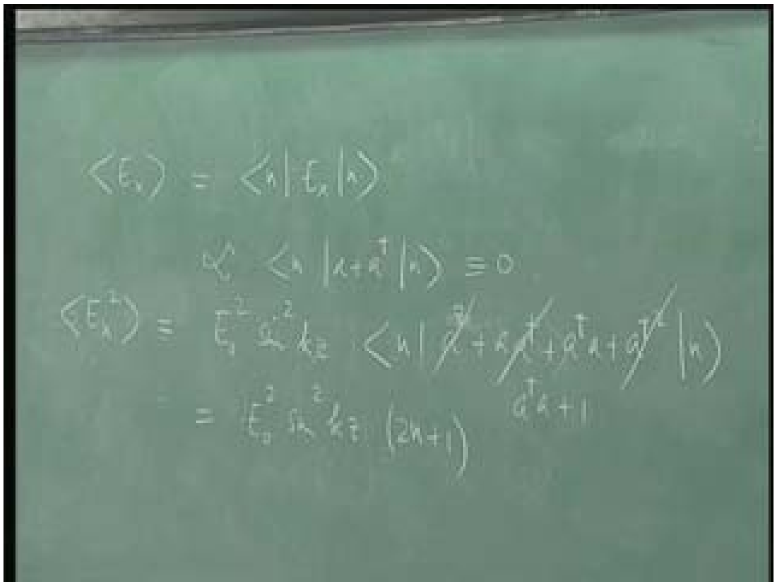

Well, all you have to do is to compute the expectation value of Ex in whatever state you want. This = <n|Ex|n>. What's the expectation value of a or a dagger in a given number state n? So this is some constant. this is proportional to n a + a dagger on n. now this is 0 because a acts on n and lowers it to n -1. a dagger acts on n and raises it to n +1. So a and a dagger now can be called photon creation and photo destruction operators. Because a dagger acts on the number state n and makes it n + 1.

In other words, it creates a photon and a destroys a photon. It reduces the number of photons by 1. Since these have no diagonal elements at all, this is identically 0. So the average value of the electric field is certainly 0. But the mean square value is not 0. What's the mean square value? Ex squared = E 0 squared sin squared kz times a + a dagger squared. So n a + a dagger squared on n now what is this equal to (Refer Slide Time: 36:20)? So you should be able to compute these expectation values fast. It's got to become proportional n + 1/2 because this (Refer Slide Time: 36:37) term here is a squared + aa dagger + a dagger a + a dagger squared on n. a dagger squared has no diagonal elements. That's 0. a squared has no diagonal elements and that is 0 as well. aa dagger is a dagger a +1. So this can be written as a dagger a +1. But a dagger a on n is just n times n. therefore this works out be = E_0 squared sin squared kz times 2 n +1.

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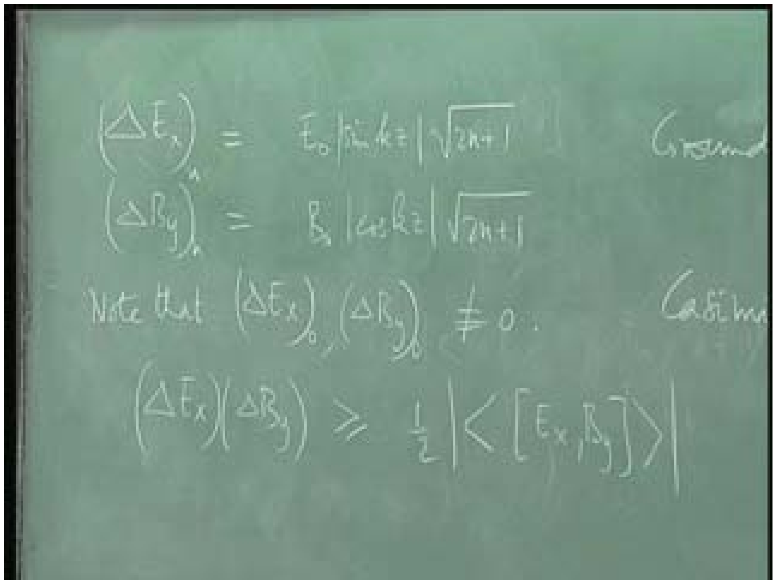

Therefore, in this number state, delta Ex in the number state n is the standard deviation. So this is = E $_0$ modulus sine kz = the square root times root of 2 n +1. What's By going to be? Well it's clear that once again the square of these is going to give a 0 and let's quickly compute what the expectation value of delta By is.

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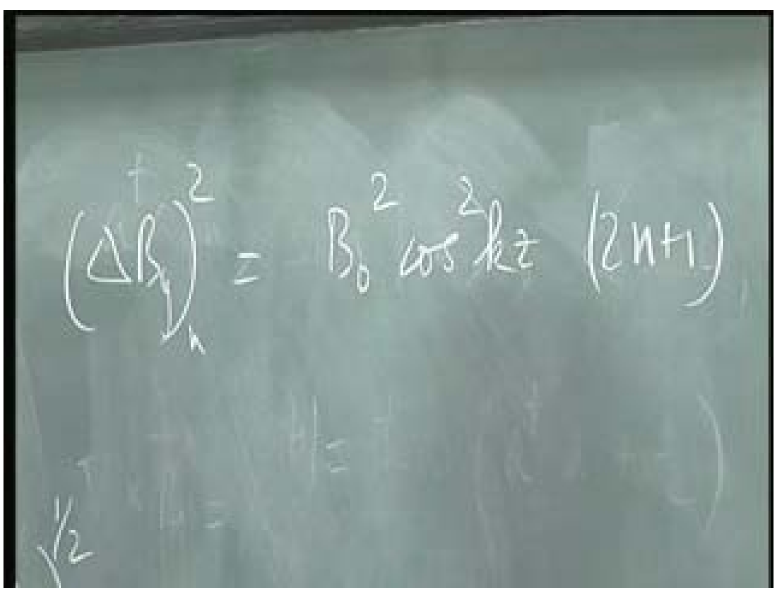

Delta By whole squared in this number state n = B $_0$ squared - cos square kz then expectation value of aa dagger and a dagger a both with – sign. So this - sign goes away and that's going to give me 2 n +1. So delta By in this number state n = B $_0$ cos kz root 2 n +1. and the important thing to see is that even in the ground state, the uncertainty in the eclectic field and magnetic field is not 0. So there are vacuum fluctuations these are called vacuum fluctuations. Because in a quantum field it is very subtle. Even in the state where there are no quanta present and you have a vacuum. There is still a fluctuation of the electric and magnetic fields and they have physical consequences. This has a physical effect which is measurable. So note delta Ex at 0 and delta By at 0 is not identically = 0. They are still vacuum fluctuations. One of the physical effects is the following. if you took 2 parallel plates as we done here at z = 0 and z = l conducting plates, then even in the absence of an electromagnetic field in the region between them, there would still be a force between the 2. This is called the Casimir effect and it happens because of the 0 point energy. Notice that there is 0 point energy in this problem and it's formally infinite which is completely crazy but then this is an artifact of quantum field theory.

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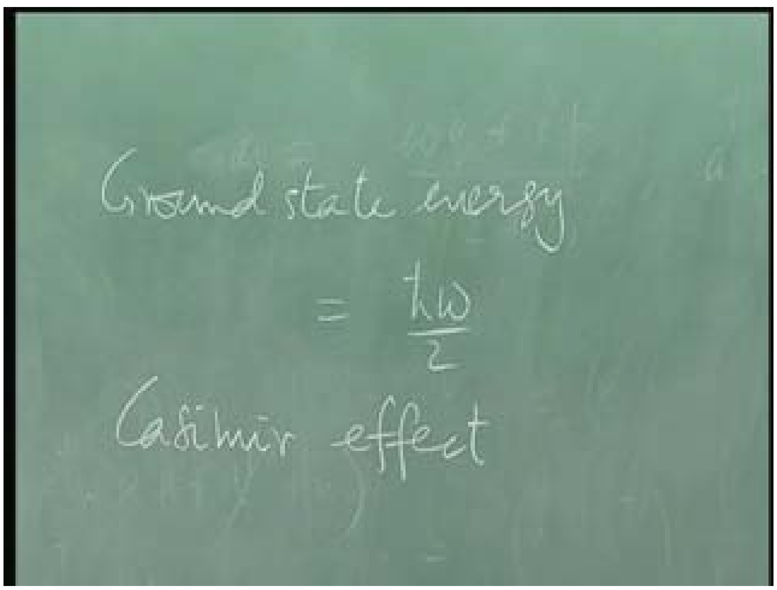

The ground state energy = when n is 0, we know the energy is h cross omega into n + ½. So it's actually h cross omega over 2 per mode. Every mode contributes an energy h cross omega over 2. So every wave number contributes an energy H cross omega over 2. 2 polarizations, the factor 2 goes away because there are 2 such contributions. Then, you have to sum over all possible omegas and that's infinite because the number of k's is actually infinite. So the 0 point energy of the electromagnetic field in a vacuum is actually infinite. But you could ask for this force between 2 plates, how do I compute it?

Well, I put the whole thing in a box. So I take the side walls of the box to be very large and the distance between the 2 plates to be small compared to the linear dimensions of the side walls and I ask what is the energy it takes to bring the second plate from infinity up to a distance l. The difference in these two 0 point energies will be the work done in bringing this plate out here and from that if you take a gradient, you end with the force. And you can actually compute this force and it was experimentally measured in the 1950s and there is such a force. It's a delicate experiment because you have to make sure that you got rid of all the other effects and the conditions of the experiment have to satisfy the assumptions made deriving this expressions. But this has been done. The 0 point energy leads to a real force called Casimir effect. It has been experimentally seen. There are other quantum electrodynamic effects.

One of them is the famous Lamb shift which occurs in this spectral lines of atoms including that of hydrogen and that too has been experimentally verified to high degree of accuracy. So we are confident that the quantization prescription is actually a correct1 and this is the way the electromagnetic field behaves. Now I leave you to figure out what the uncertainty products are for n, Ex Ey and so on and so forth. Notice that you must definitely have, in any state delta Ex delta By is greater = 1/2 the magnitude of the commutator of Ex with By of the expectation value the commutator in that's state. That's easily verified in this case.

Now one thing is very clear that, when you make a measurement on any physical observable, the postulate of quantum mechanic says the answer you get is one of the possible eigenvalues of this operator. The expectation value we have is in the number operator state, we haven't said anything at all about the actual eigenvalues of a and a dagger or eigenstates of a and a dagger. But we know the classical harmonic oscillator that the spectrum of p is actually continuous. So these operators, Ex and By actually have continuous spectra. You make a measurement; you are going to get one of those answers. What we are now saying is if you prepare the system in a state of a definite number of photons, then you measure what the uncertainty in the electric field is, even in the ground state. Even if there are no photons, the answer is nonzero.

Now of course, you could ask, in a real black body cavity, what kind of radiation do you have? It is certainly not standing waves operating in 1 direction. There is a whole admixture. So let's do that. Now there are several other questions which I will come back to here, which we need to answer. One of them is, in the case of a classical electromagnetic wave, I have an amplitude as well as a phase and then interference of it happens because of this phase. What's the quantum analog of the phase? This is a very important question and we will come and talk about it a minute. The second question is, Ex and By in some sense are like the position and the momentum and therefore, they are conjugate variables. The commutator of Ex and By is like the commutator of x with p and gives you ih cross times 1 and so on. What about a dagger a? What's the conjugate variable to a dagger a? In other words, what operator should it be such that if you take a dagger a, commutator with that will give you unity? Now the square root of a dagger a is taken to be the amplitude. The classical intuition tells us that there is the amplitude and the phase. Therefore, there must be some phase operator somewhere and maybe this is the conjugate of a dagger a. Let me show that's not as simple as it seems. We will then do black body radiation, comeback and talk about the phase of this operator because I would like to introduce the idea of coherent states and go on with quantum optics. So let's try and see what happens if you try to define a phase. You run into trouble as follows.

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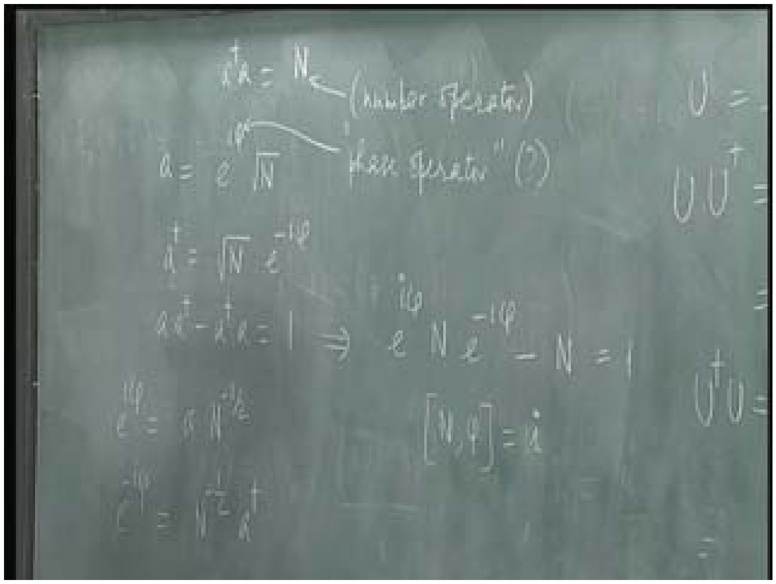

a dagger a is the number operator N. so just as I take a complex number z and write this as r e to the i theta, where r is really the square root of zz star. in exactly the same way, instead of z i have a, instead of z star i have a dagger. And i have this combination mod z squared which is a dagger a and that is a Hermitian combination. Let me see what the standard notation is for this. a = e to the i phi square root of N. Let me say I do a polar decomposition of a just like I do a polar decomposition of complex number. I do a polar decomposition of this operator a by writing it in this form. And then what's a dagger? Phi is supposed to an operator and N is an operator. Now this becomes square root of N e to the - i phi. So let's see if this works or not. And the commutation relation is aa dagger - a dagger a =1.

This implies that e to the i phi N e to the - i phi - a dagger a which is N. because a dagger is root N e to the - i phi e to the i phi root N and e to the - i phi and e to the i phi commute with each other and give you a 1. And this must be =1(Refer Slide Time: 51:35). Let me call phi as a phase operator. I put a question mark to see if this is really sustainable or not. This was Dirac's idea originally. If you do this polar decomposition and see if you can consistently define a phase operator of this kind. This commutation relation between a and a dagger which came from the original xp = ih cross goes over into this.

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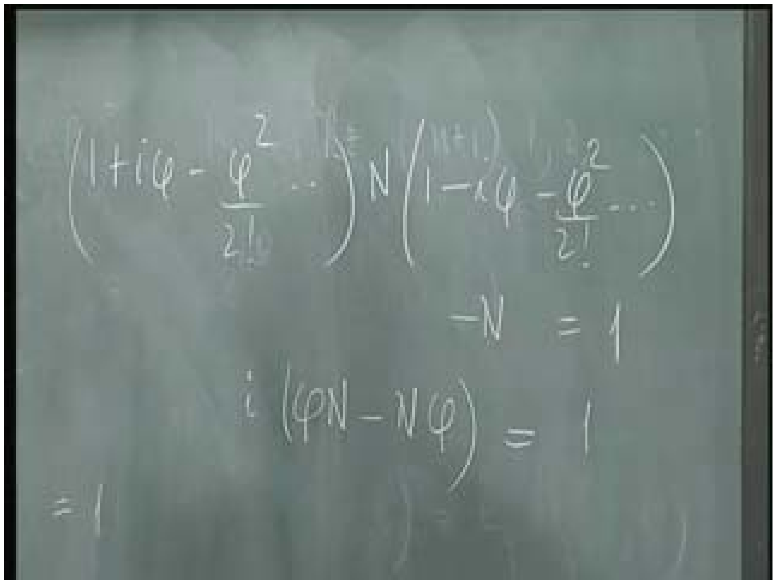

e to the i phi is1 + i phi - phi squared over 2 factorial … with N and 1 - i phi - phi squared over 2 factorial … - N = the unit operator. i(phi N - N phi) =1. It looks like this is sustainable if i define [N, phi] = i. so it looks like we are almost there. Because just as I had a with a dagger =1, i have N with phi = i times the unit operator. You can check that if i have this commutation relation, then this (Refer Slide Time: 53:42) relation itself is satisfied. So it looks like we are in business. you can write a in this form, a dagger in the other polar form and then N and phi are complimentary to each other, in the sense that the commutation relation between the number operator and the phase operator is in fact just i.

Now if you want these to be physical operators, you want N and phi to be Hermitian. in other words, you want e to the i phi to be a unitary operator. Otherwise it doesn't make any sense. so you certainly like this to be a unitary operator. If it's a unitary operator, it means U U dagger is =1 and U dagger U =1. look at what happens. e to the i phi = a N to the - ½. if i operate with N to the - 1/2 on the right hand side, i get e to the i phi is a into the - ½. And if I do the same thing here, e to the - i phi = N to the - 1/2 a dagger. If this is an operator U, a unitary operator, i expect this to be U dagger or U inverse. This is U inverse and we want this to be U dagger for it to be unitary.

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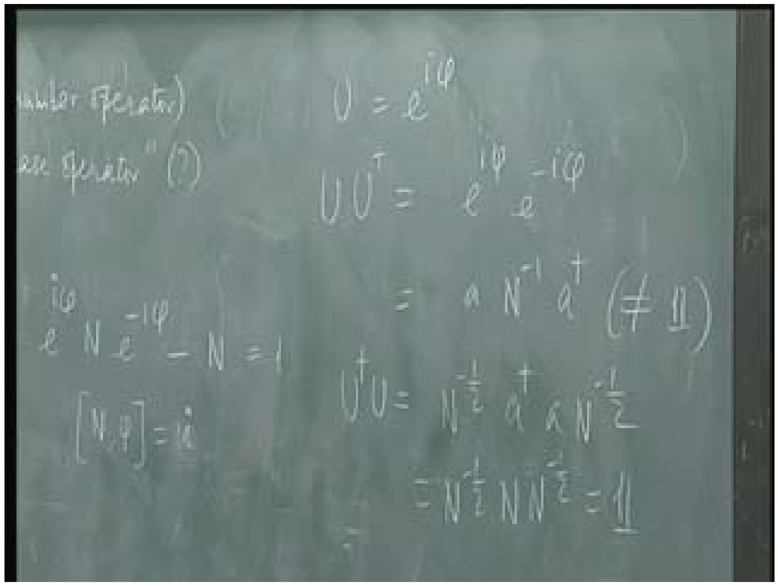

So U U dagger = e to the i phi into e to the –i phi. This gives me a N inverse a dagger and that’s not identically equal to the unit operator by any means. On the other hand, if i do you U dagger U, you get N to the - 1/2 a dagger a N to the - ½. But this is N to the - 1/2 N N to the - 1/2 = the unit operator. So here’s our first indication that everything is not all right because these are infinite dimensional Hilbert spaces. There is no guarantee that if there is a left inverse, there is also right inverse. If something is unitary, you require not only that U U dagger be = i, but also U dagger U be equal i. but with this set of operators, you end up with U U dagger is not 1 but U dagger U is the unit operator. So you begin to see there is already a problem this is an infinite dimensional Hilbert space and you have to be very careful here. There is another way of seeing this. Its 1 over N doesn’t have any meaning. 1 over an operator doesn’t have any meaning. By that you mean N inverse, the inverse of the operator.

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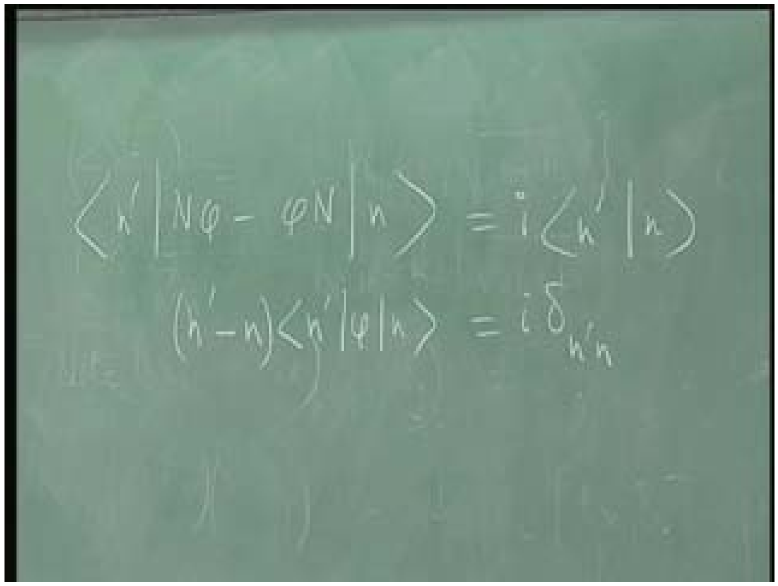

So we have said N phi - phi N and i take this between the state n and a state n prime and that's = n prime the unit operator n and there is a factor i. But this quantity i know is i delta n prime n because these are orthonormal states. On the other hand, when this N acts on this n prime, you get an n n prime. So there is an n prime from here. When phi N acts on that, you get an n. then the matrix element is this (Refer Slide Time: 01:00:34). If this commutation relation is true, then that is true. but what happens if n is = n prime, (n-n prime) is 0 and <n prime |phi| n> = i. so you immediately see this commutation relation is not sustainable in this Hilbert space spanned by the states n. so although it looked as if you had no difficulty in writing a phase operator in this fashion, really this is not sustainable. Phi cannot be a physical Hermitian operator because if it were, this is unitary operator but we see explicitly it's not a unitary operator. Besides, the commutation relation between N and phi is not maintainable. This leads to an inconsistency. It says 0 = i, to start with. So there is a serious difficulty in defining the phase operator in quantum mechanics.

Classically, there is no difficulty at all. And this has arisen for a variety of reasons. Firstly, it is an infinite dimensional space. Secondly, it's not very clear how to define a physical phase operator here. There are several resolutions that have been proposed and we will talk about just 1 of those things to get over this mathematical difficulty. It brings out the fact that these operators in this infinite dimensional space sometimes could have a left inverse but not a right inverse. So we won't spend too much time on this. The reason it's important is because you do need to know what's the phase of the electromagnetic field. The first thing I do next time is to talk about what happens if you have a radiation field inside, like black body radiation and we will talk about the photon number distributional in a black body radiation and then go over to the interaction of atoms with radiation with matter. we will look at some simple models for it which will help us make a connection with the spin 1/2 problem. So let me stop here today.