Quantum Physics

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Lecture No. # 26

Conversation between student and professor: Student: I've read in the American Journal of Physics that there is a close classical analog of spin. He says that spin is equivalent to the angular momentum of the polarized electromagnetic wave. But in the class you mentioned that spin is totally associated with quantum mechanics and no classical analog. So isn't that a contradiction?

Professor- Good question! It's a deep question and the answer is in several parts. The statement I made was that spin was not like orbital angular momentum which has a classical counterpart but rather some kind of intrinsic angular momentum, intrinsic to elementary particles. On the other hand, it's been pointed out that spin is like the states of polarization of electromagnetic waves and since electromagnetic waves have classical counterparts, we see classical electromagnetic waves. Is there not a contradiction? Doesn't it mean that spin is also a classical object?

Now it's a valid question and the answer goes in several parts as follows. First of all, we believe that elementary particles really are the fundamental constituents of matter and radiation. We believe that quantum mechanics and relativity really are the guiding principles for understanding all of nature. Then, it turns out that once you accept quantum mechanics, everything is quantized. Elementary particles which you see around you are the real quanta of some field or the other. Electron is a quantum of the electron field. So to speak, the proton is a quantum of its own field. Actually it turns out today; we understand that nucleons like protons and neutrons are themselves made of quarks which have their own fields.

In the same sense, photons are the quanta of the electromagnetic field. So even though the electromagnetic field appears to be a classical object, the fact is, it too is quantum mechanical. It requires the right kind of experiments to probe its quantum mechanical nature. And it turns out that radiation too is in the form of quanta. But these quanta of radiation a slightly different from the quanta of matter like quarks or electrons and so on. The difference lies in the fact that the quanta of radiation has 0 rest mass. On the other hand, the quanta of the electron field or the quark field and so on have finite rest masses. The relation between the energy and momentum of a 0 rest mass particle is different from that for a massive particle.

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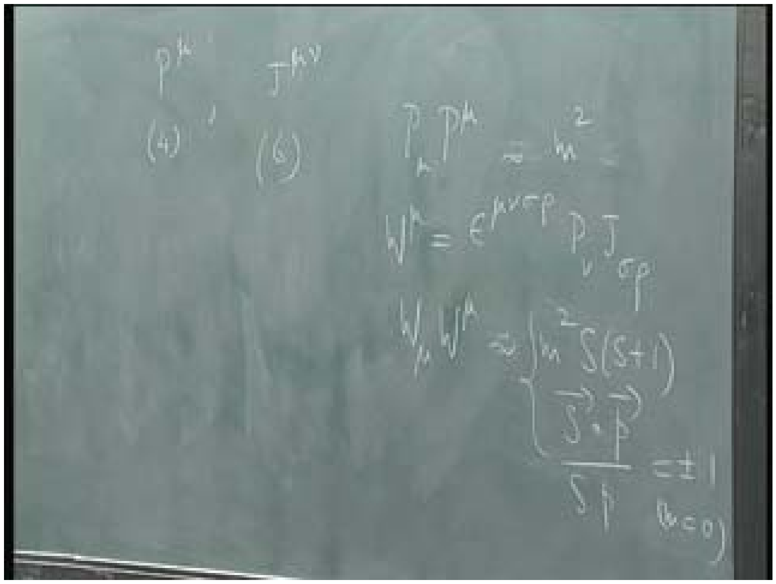

So on one hand you have E squared = C squared p squared + m squared C4. On the other hand you have E = cp when m = 0. This is a linear relation whereas the former is not a linear relation at all. This leads to a lot of differences in properties. That’s point 1. The second one is, we said that every elementary particle, when you impose the requirements of special relativity, the wave functions of these elementary particles would be characterized or labeled by 2 quantum numbers, 2 labels or eigenvalues of certain generators of the Lorentz group because you require these wave functions to transform in a specific manner under Lorentz transformations so that the laws of physics can remain form invariant under Lorentz transformations. And these 2 labels, if you like, they come from the values of certain operators which induce transformations belonging to the inhomogeneous Lorentz group which commute with all the generators of the Lorentz group. Those 2 operators are the following.

First of all, when I say the inhomogeneous Lorentz group, i mean that translations in space and time namely; shifts of the origin and shifts of time are permitted and there are 4 generators for it which I denote it by p mu. The 0 component here corresponds to the energy or the Hamiltonian under 3 special components correspond to the linear momentum. And then you have rotations in space rotations of the axis and velocity transformations. So there are 4 generators here. Then rotations of the coordinate system are specified in 3 dimensions by 3 possible Euler angles. Therefore there are 3 parameters and then you can also have velocity transformations in any direction what so ever. And therefore, there are 3 velocity components specifying an arbitrary Lorentz transformations or boost from a frame at rest to a moving frame. so those are induced by a set of 6 generators mu and nu run over the value 0,1, 2, 3 and this is anti symmetric in mu and nu and a 2 by 2 tensor in 4 dimensions has 16 components but if it’s anti symmetric, the diagonal ones are 0. That leaves 12 components and they are equal in magnitude, opposite sides of the principle diagonal and negatives of each other. So there are 6 independent generators here. 3 of these correspond to rotations and 3 more correspond to velocity transformations or boost. Together these 10 parameters form a ten parameter group called the in homogenous Lorentz group. And our belief is that the laws of physics are form-invariant under such transformations in flat space time.

Now these generators also lead to 2 combinations of these which commute with all of these generators and those combinations are $p_{mu} p^{mu}$ and the other one is a little more intricate. You start by defining a vector w^{mu} which is epsilon $^{mu\ nu\ sigma\ rho}$ $p_{nu}\ j_{sigma\ rho}$. They are written upstairs and downstairs for technical reasons which i won't go into now because these are Lorentz transformations in 3 +1 dimensions. W^{mu} is a 4 dimensional vector once again. It's called the Pauli Lubanski vector and this quantity $W_{mu} W^{mu}$ is also invariant. This operator commutes with all these generators and so does this. Therefore the irreducible representations of the Lorentz group are labeled by the numerical values of these quantities (Refer Slide Time: 08:50). And this here (Refer Slide Time: 08:54) in suitable units turns out to be m squared.

Let me put c =1 for the time being. So this is what leads to the rest mass of a quantum of a particle. This (Refer Slide Time: 09:08) here on the other hand does something interesting. This thing here is a scalar and you can look at ask what's it value is in the rest frame of a particle. And that turns out be a proportional to m square S(S +1). So this S here which comes out is the intrinsic angular momentum quantum number of a quantum and then of course the rest of whatever i said about angular momentum follows. So this is the origin of spin and mass for a particle. But now you could ask: what does this become when m is 0 because this looks like it goes to 0. That's where the difference comes between a particle with 0 rest mass and one without 0 rest mass. This turns out to be proportional to the component of the spin in the direction of the linear momentum of the particle because a particle with 0 rest mass always moves with the speed of light. If you stop it, it's annihilated. So free photons always move in the speed of light, the fundamental velocity and this (Refer slide Time: 10:20) thing here turns out to be S dot p over S p if you like, where these are the magnitudes of this vector.

So you can get rid of this. It is just the normalization factor. It's the component of the spin along the linear momentum of the particle and it is called the helicity of the particle. now if you took an object with spin quantum number S and asked what are the possible values of S dot p divided by modulus p, then it's like asking what are the possible values of a single projection of the spin operator along any direction and there are 2 S +1 such values because S dot n or S dot any unit vector has 2 S +1 values when you apply quantum mechanics. So this (Refer Slide Time: 11: 06) thing here would have 2 S +1 values since i have normalized by S. these values would run between -1 and +1. If i don't have this (Refer Slide Time: 11:15), they would run from - S to + S in steps of unity. So now, coming to an electron, since S is ½, we only have the values - 1/2 and + ½. On the other hand, if you look at an object with spin1 for instance like a photon, and then S is 1 for a photon. Therefore you would expect that this quantity would have 3 possible values; -1, 0 and +1. However, because it has 0 rest mass, the definition of this (Refer Slide Time: 11:47) object is different in the case m = 0, not = 0. This object here itself can only have 2 values whenever m is 0.

That's the way it works out and this (Refer slide Time: 12:05) becomes = + or -1 when m = 0. +1 you would call one helicity and -1 you would call the other helicity. Now comes the connection between what happens classically and what happens quantum mechanically. A photon has S =1 but it has rest mass = 0. Therefore it does not have the S z = 0 projection. S z or S dot p can only be +1 or -1 and that's the connection with classical states of polarization. Less circularly polarized light corresponds to S =1 and S Z =1 or helicity = +1 and right circularly polarized light corresponds to helicity -1. So there is still an intrinsic angular momentum and it does arise from the spin of the photon. I would go the other way and say the states of polarization of radiation arise ultimately from the spin quantum number of the photon. Only these 2 are possible. You don't have S z = 0 projection at all.

That's true for every mass 0 particle. It is believed that gravitation is carried by the gravitons. We have not seen it. People are hunting for it. There are problems with it but if it exists, it would have spin S = 2. Therefore 2 S +1 would be 5. You would expect 5 possible projections or 5 possible eigenvalues of S, the helicity. You don't have 5. You have only 2. Once again + or -1. That's it, after you divide by the 2 there for the S. now what's the reason why you could see this polarization classically is buried in yet another subtlety. The fact is when a particle has integer values of its spin quantum number, and then collection of such identical particles obeys Bohr's statistics. When it has 1/2 odd integer, then a collection of identical particles obeys Fermi Dirac statistics.

In Bohr's statistics, it turns out that you put a collection of identical bosons together; there is no restriction on how many of these bosons you can put in a given state. In the case of fermions, there is a restriction which is called the Pauli principle which follows actually from the spin statistics connection from consistent quantization that you cannot put more than 1 such particle in given state at a given time. Since photons or bosons there, is no problem with putting a large number of them in exactly the same state. In particular there is no problem with putting large number of them in the same polarization state. Therefore you can observe a classical polarized beam of light because a whole lot of photons are contributing to the same state. This is also the reason by both gravitation as well as electromagnetism got observed long ago.

Gravitation got observed very long ago when the first apple fell down from the tree. The reason is again just this. These are classical fields. They are long range fields. They are massless. Now whenever forces caused are by 0 rest mass particles being exchanged, the corresponding force is a 1 over r force in 3 dimensions. That's long range. Otherwise if there is rest mass associated with the particle, then the corresponding force is proportional to e to the - mu r over r, where mu is the reciprocal of the Compton wavelength of the particle and it depends therefore on the mass. If mass is 0, you just get 1 over r for the potential. This is what happens in the case of electromagnetism and gravitation. but in more complicated cases where particles with mass are exchanged, then in the nonrelativistic limit, in the static limit, etc., you end up with a potential of this (Refer Slide Time: 16:01) kind. So this is why if you had massive particle exchange, those forces would be very short ranged. And that's the reason why week interactions are very short range. Radioactive decay is another word for week interactions. They too are mediated by bosons which have spin 1 but they are massive and because of that the range is 10 to the – 15 cm or 10 to - 18 m or less and this is the reason why you don't see classically.

The same is true in the case of the strong force which binds the quarks together into nucleons. These nuclear forces too are extremely short range. The reason being that, the forces are mediated by particles which are bosons which have spin 1 but which are massive and therefore range become short. So even though there is a potential for these things to become classical and so on, you don't see it in real life. So there are 2 effects playing a role here. One is the mass of these exchange particles and the other is the statistics they obey. Now you could have 0 rest mass fermions. In principle, it was believed for a long while that neutrinos are massless and fermions spin ½. So you could ask can i not have a force mediated by neutrinos which would be long range and so on and so forth but the fact is you cannot have a classical field of neutrinos. The reason is that you cannot have more than one of them put in the same state. You need large quantum numbers to observe it. Now the final point is all angular momentum is quantized.

The angular momentum of the ceiling fans is quantized but like I pointed out, the quantum number is so large that the discreetness doesn't play a roll at all. on the other hand, when you come to elementary particles and you have something like S times H cross and S is of the order 1 or 2 and H crosses in our units is 10 to the – 34, you see that as far as the comparison with daily life as angular momentum is concerned, this is completely negligible but the quantum effects become very significant. now having said all that, let me also say that if you took the classical theory of fields, didn't impose quantum mechanics, only relativity and you looked at that what these fields did, then the moment you have multi component to the fields like vectors, tensors, spinners and so on, then it is necessary to introduce the concept of an angular momentum carried by this in the field even though it's classical a field. In that sense, this is a classical analog of spin at a deeper level.

This is still a classical analog of spin. So it is not possible to have a consistent classical relativistic field theory of a tensor field for example without introducing the idea that this field actually carries angular momentum but that is also true for the electromagnetic field. The electromagnetic field carries an angular momentum. The classical electromagnetic field carries an angular momentum but the origin of that angular momentum, when you go deep is due to the spin of the photons. So in that sense, there is no classical versus quantum divide that classical mechanics is a limiting case of quantum mechanism and everything ultimately has a quantum origin. So i hope that explains this confusion. I look at classical physics as a limiting case of quantum mechanics. A very essential limiting case necessary, it's needed for interpretation and so on and so forth, but the fact is that we believe that the fundamental laws are quantum mechanical. Then let's get back to what we were discussing. We had stopped last time at the radial equation. i need to point out some features of this radial equation.

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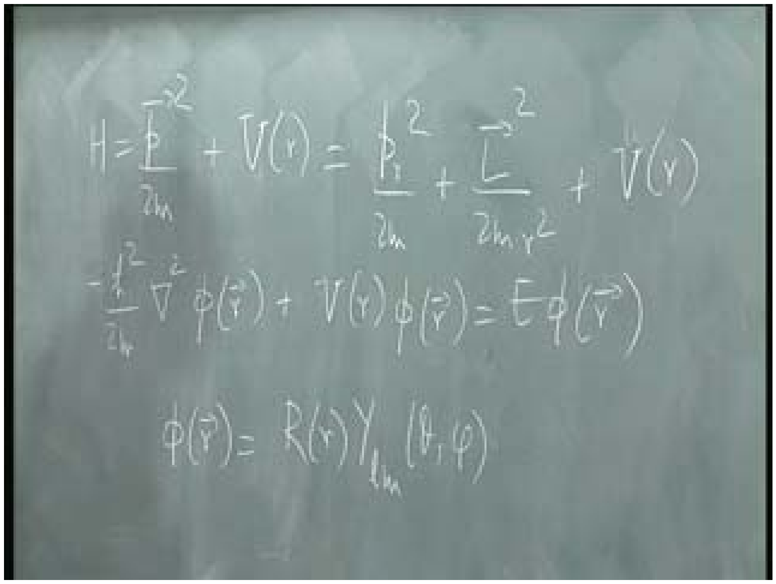

So to quickly recapitulate, our Hamiltonian is p squared over 2 m + V (r) which we wrote in the form pr squared over 2 m + L squared over 2 m r squared + V (r) which is an arbitrary central potential. we made some assumption and the Schrodinger a equation we wanted to solve was - H cross squared over 2 m del square phi (r) + V (r) phi (r) = E phi(r). And we discovered that this phi (r) could be written in the form R (r) and then the angular part was characterized by this spherical harmonic Y_{lm}.

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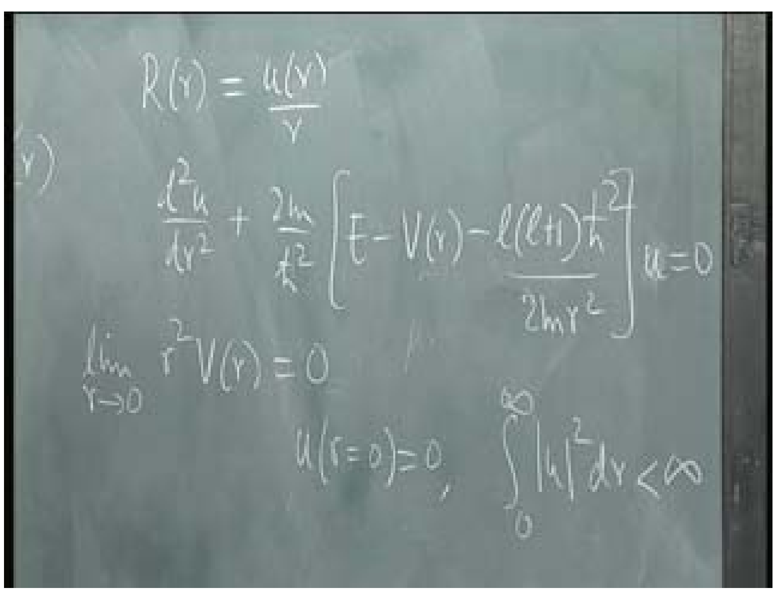

This radial quantity R(r), if i wrote in the form U(r) over r, then there was a convenient radial equation for this quantity here which was d 2 U over d r 2 + 2 m over h cross squared [E - V (r) –l(l+1) h cross squared over 2 m r squared] U =0. That was a centrifugal barrier on U = 0. In the boundary condition what we required was the following. In order to make this whole thing respectable, we made the assumption that r squared V (r) = 0 as r tends to 0. So it's not too singular at the origin then the boundary conditions where U at r = 0 was 0 and we required it to be normalized such that mod U squared d r less than infinity.

So this means you must go to 0 sufficiently rapidly. The r squared in the phase space factor was taken care of by this (Refer Slide Time: 23:01) division here and we want it to be 0 at the origin. Now this has become a one dimensional problem. Now what can the energy levels depend on? first of all, the angular part of the wave function is completely determined by this and it's specified by 2 quantum numbers, L and m. therefore, any state of the system that I'm talking about, already has 2 quantum numbers L and m in it and for given values of those quantum numbers, we are going to examine this and ask does it have solutions and so on. what we see immediately was that the effective potential is V (r) + a repulsive 1 over r squared potential which we call the centrifugal barrier and we also know that even if there are bound states supported, these bound state are going to get a little less tightly bound as soon as L starts becoming larger and larger due to the centrifugal barrier.

The question even arises whether for a given potential you may have a bound state or not. Because, if this becomes arbitrarily large, it could stop the potential from having a minimum value. It could become too shallow. So at once you begin to see the possibility, that in 3 dimensions, because of the presence of this centrifugal barrier so to speak, the number of bound states that are possible is actually going to be restricted. And you could even ask, for the given value of L and given potential, is there a bound on the number of bound states you could have and the answer is yes. There are such bounds. Now what would the wave function depend on? it would depend on the value of the eigenvalue of E. we have to solve this eigenvalue equation subject to these conditions and of course, the value of L and possibly on m too. But from this equation it's immediately clear there is no m dependence here (Refer Slide Time: 24:57) at all. The quantum number m has completely disappeared from this equation that at once shows that in a central force problem no single axis is distinguished from anything else. That is why there can be no dependence on m at all which is the quantum analog of the classical statement that in a central force problem, you have symmetry about all axis. You have spherical symmetry and therefore there is no particular axis singled out at all. The energy eigenvalues E therefore cannot be functions of m. now what can it be a function of? That depends on what V(r) is. it will turn out this is a one dimensional problem. So we could use our knowledge of what happens in one dimension. We know that in one dimension, (a).

There is no degeneracy and (b). The way functions are ordered in such a way that the ground state as no nodes, the first exited state has 1 node; second excited state has 2 nodes and so on. the same thing is applicable here except that it's as if you have a line in which you have an infinite barrier to the left of r = 0 because we have imposed the boundary condition that U 0 at r = 0. So this is like a 1/2 line problem. Some potential on the right hand side which may support bound states and on the left, you have an infinite barrier. then the ground state has no nodes excluding r

= 0 which is the boundary point and then the first state has 1 node and so on, exactly like a particle in a box from 0 to L. the ground state was sin pi x over L which was 0 at the end points by boundary conditions but no node in between. The first excited state had exactly1 node and so on. so i expect the same thing to happen here and therefore, i expect that this E would be a function of 2 quantum numbers. One of which would be a radial quantum number which would run from 0,1, 2, 3, etc and label the energy eigenvalues of this 1 dimensional problem for a given value of L and of course, if i change L, this would also once again change the energies. So therefore i expect that these energy eigenvalues be a function of some radial quantum number and the L quantum number but not m. This is what we expect to start with.

Now we need to examine and ask for a given value of L, are there bound states at all possible or not. i have to solve and find out the normalizable solutions, etc. Let’s first settle what happens at the origin. Now in order to make sure that we don’t run into technical difficulties, i have assumed that V (r) doesn’t go like 1 over r squared but slower than 1 over r squared. When it does go like 1 over r squared, then we have to reexamine in the problem and i will do that very briefly. But lets put this assumption in and ask what happens here. What kind of physics do i expect?

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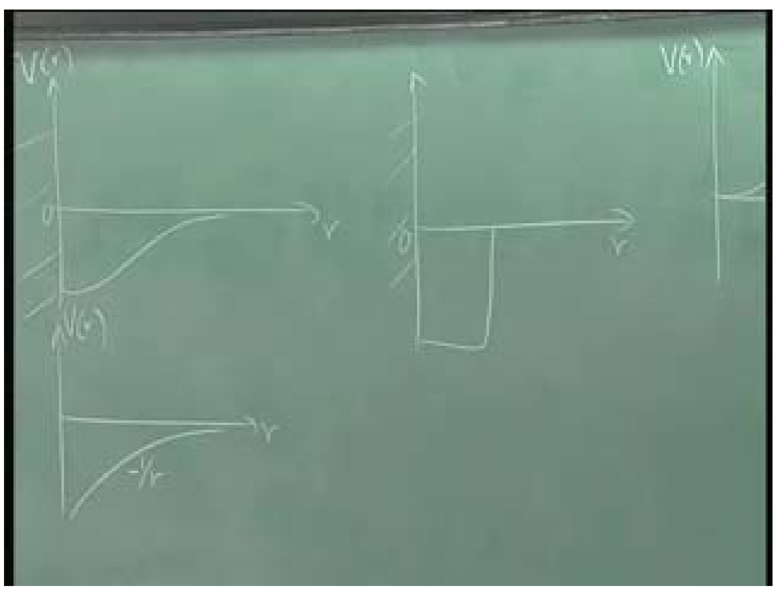

I expect that heres r, heres is V (r) (Refer Slide Time: 28:21), a typical potential which would support bound states would be there is barrier here (Refer Slide Time: 28:26). So to the left of this, the potential is infinite and then on the right hand side, i expect maybe a well like this or that (refer Slide Time: 28: 50).

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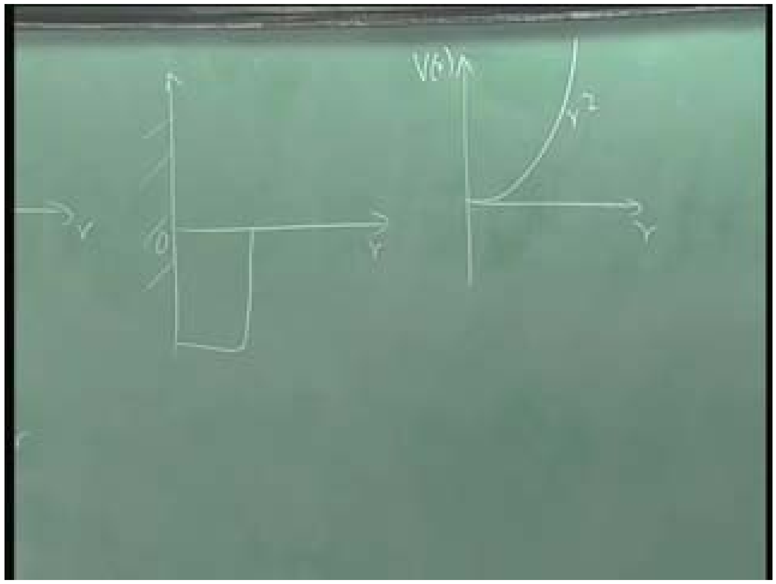

Even the 3 dimensional oscillators which is just r squared, this is an infinite well in which you have only bound states. So these are the possible kinds of shapes that I am thinking of. Notice that you also have the possibility of Coulomb potential which is like a -1 over r. This also could have bound states. So it's this kind a problem that i am trying to address at the moment. Now let's first settle what the possible behavior of U (r) can be. We've said we will impose boundary condition U = 0 at the origin but what kind of solutions come out from this equation? Well, let's look at that equation here. Near R = 0, this is some finite number hopefully. If V (r) doesn't go to 0 doesn't explode as fast as 1 over r squared, then the dominant term near r = 0 is just this (Refer Slide Time: 30:14) term.

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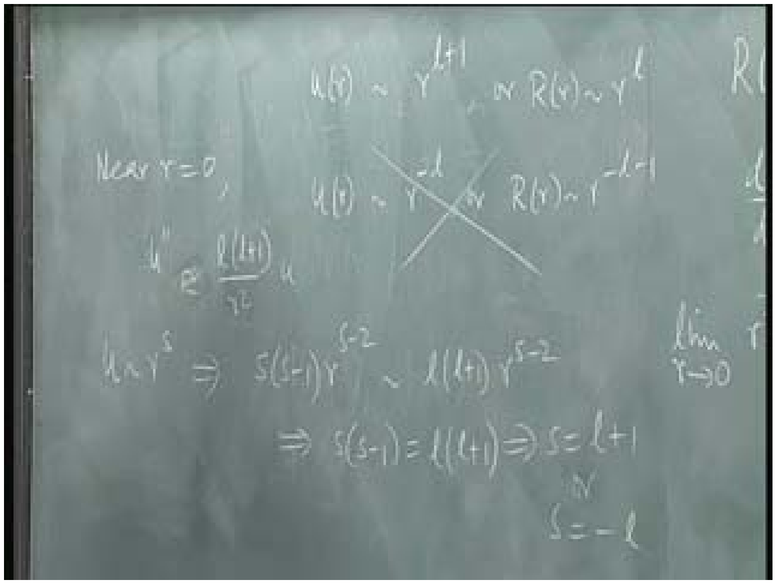

And then you have to solve an equation which says near r = 0, U double prime is like 1 over r squared times U. that's what eigenvalue equation tells. Now that's a simple equation to solve. A simple way of solving it is to assume that the trial solution is of the form U as some power of r. so U ~ r to the S implies S times S -1 r to the S - 2 and that goes like on the other side, it is in fact l times l +1 over r squared. So this is an exact equation sufficiently close to be origin. So this is l times l +1 r to the S - 2 which implies S times S -1 is l times l +1 and the solutions are obvious. So it immediately implies that U (r) goes like r to the power l +1 because S is l +1.Therefore S -1 is l. R (r) goes like like r to the power l because R(r) is U (r). But there is another possibility. It's a second order differential equation. So you must have 2 linearly independent solutions and the other solution implies S = l +1 or S = - l.

So U (r) ~ r to the –l or R(r) ~ r to the –l-1. But that's a not a regular solution. That violates our boundary condition. Even at l = 0, it violates the boundary condition. You would like U to vanish. It violates the solution so it's not physically acceptable. You see I have a differential equation but i also have a boundary condition. Those are the physically acceptable solutions. Just as if i give you a differential equation, you may have a non normalizable solution but the physically acceptable wave function is normalizable one. So this (Refer Slide Time: 33:30) solution is not an acceptable one. It's not that it doesn't play any role whatsoever. The fact is this is only near r = 0. They are always 2 linearly independent solutions to the general equation and the general solution is a superposition of the two. But you have to also satisfy the boundary conditions. So this is the physically acceptable solution (Refer Slide Time: 33:55).

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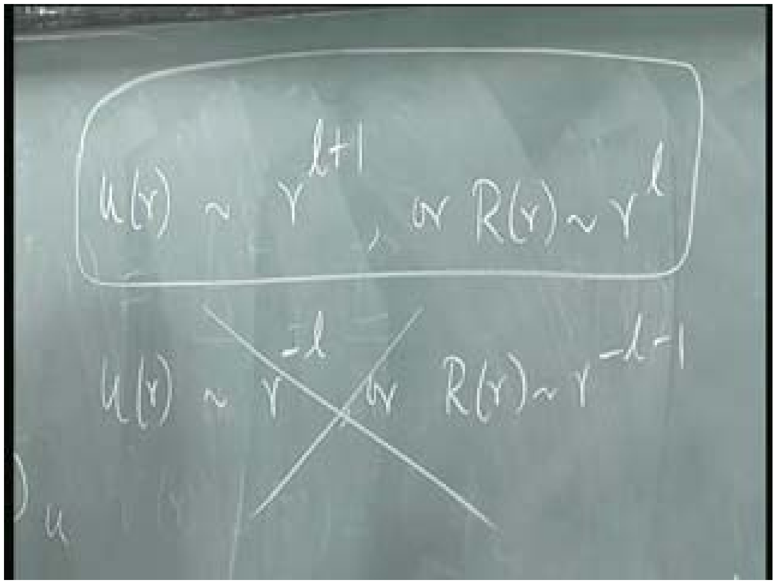

And this incidentally answers the question of what's the actual behavior of U near the origin. The behavior is U goes like r to the l +1. So in the ground state, if l is 0, U really goes like r and this (Refer Slide Time: 34:11) goes like a constant. So that's our first lesson. In fact, you could do something better. You could say suppose V (r) was 0, I am looking at free particle motion in spherical polar coordinates, what would it look like? Normally, they would depend on the initial state. If I have any initial plane wave state, then of course I must expand the plane wave in spherical waves and then look at what happens to each one of these. That happens in scattering theory write now I am worried about bound states here. So if you have no potential at all, you can't support any bound state anyway.

But it tells us that if you had this (Refer Slide Time: 34:54) equation, then if the potential behaves sufficiently well near the origin, then this (Refer Slide Time: 35:02) is the way the physically acceptable wave function behaves near the origin. so you manage to extract that piece of the information. Now you could look at difference classes of potential. For example, suppose this is - V_0 (Refer Slide Time: 35:20), it's a well till the point a and 0 after that. One could put that in here (Refer Slide Time: 35:30) and say V(r) is - a constant V_0 till r = a and after that, it's 0 and ask what do the normalizable solutions look like. Now what would the wave function look like as a function of r? What should the wave look like in this (Refer Slide Time: 35:50) region? It should clearly die down exponentially otherwise you can't normalize it. And in this (Refer Slide Time: 35:56), it could be oscillatory out here and this is the problem of a particle in a spherical well. Because now you have said the potential V = 0 till a certain radial distance and after that it's completely 0 and it's in an attractive potential. so it turns out you can solve this problem and I'm going to give it as a problem and tell you to show that you have a bound state if the well depth is sufficiently large, unlike the one dimensional case where no matter what the well depth was, as long as there are finite depth and width, you always had at least one bound state which is the ground state. And then you may or may not have had excited states. In this problem in 3 dimensions, it turns out that you need to have a sufficiently deep well in order to have a bound state. Otherwise it's not going to work because it's a 3 dimensional problem. And then if the well is deep enough, you may have excited states. Our interest now is really to look at the Kepler problem, the 1 over r Coulomb problem. Let me explain where that comes from.

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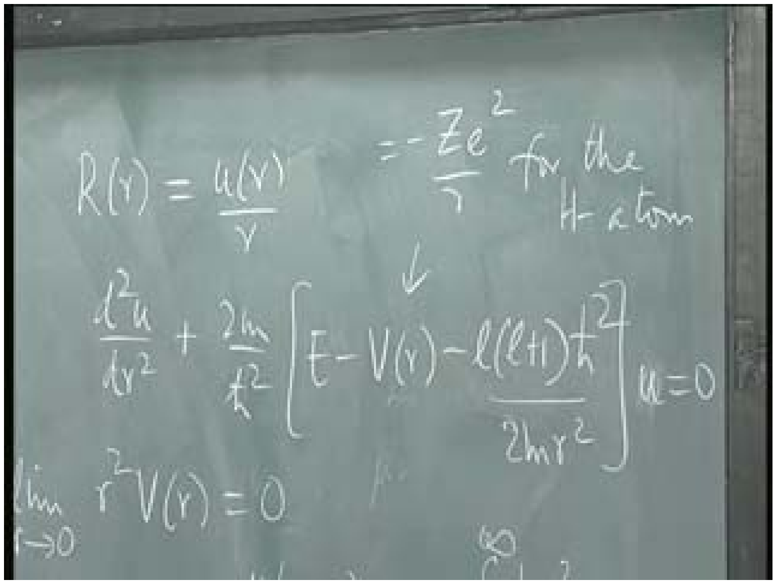

If this (Refer Slide Time: 37:18) quantity is = - z e squared by r for the hydrogen atom, then one is faced with the task of solving this equation explicitly and there are changes of variables which lead you to a special function called the Laguerre function and one can solve this equation completely, put it back and write down the exact solutions. I am not going to go through that here.

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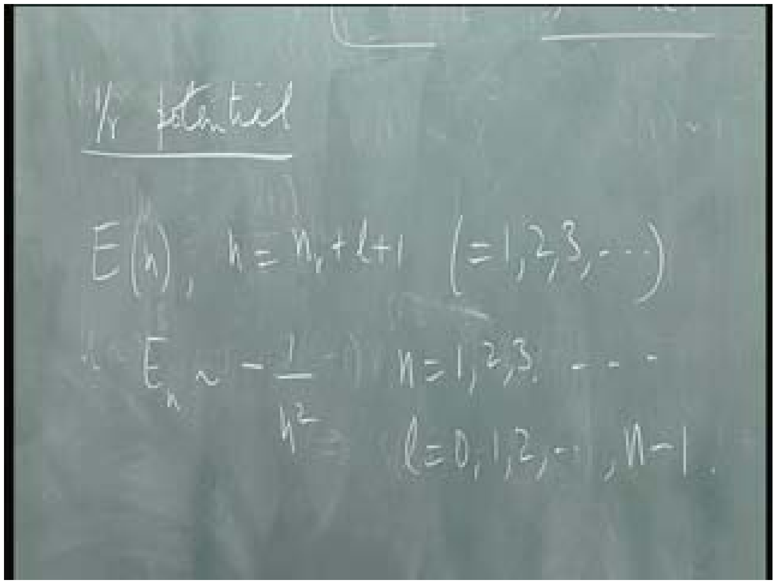

It's a standard piece of algebra in text books but it turns out in that case that the energy levels E for the 1 over r potential, are functions of the following combination. They are functions of only n = n_r + l +1. since this radial quantum number goes 0,1, 2, 3 as i said earlier, l goes 0,1, 2, 3 and n itself goes 1, 2, 3 etc and is called as you know the principle quantum number. E_n goes like -1 over n squared in suitable units. There is no explicit l dependence. This is a consequence of the fact that the potential is 1 over r. it has extra symmetries. It is not true for a general central potential for which the energy eigenvalues are functions of 2 quantum numbers but in this special case, it can be reduced to a single combination. This is called accidental degeneracy and as i said, on several earlier occasions, it is a peculiarity of the coulomb problem.

In a very short while, I will tell you where this extra symmetry is. n_r is 0,1, 2, 3, etc and now if you work it out in the required square intagrability etc, then not only is n 1, 2, 3 all the way to infinity but l runs 0,1, 2 up to n -1. So as all of you know, this leads to the counting of the degeneracy of hydrogen atom state. when the principle quantum is n, then the possible number of linearly independent states corresponding to this quantum number n have to be computed by summing over from m = - l to + l summation l = 0 n -1 and when you include spin, then the electron can have 2 possible spin projections along any direction. Therefore, there is an extra factor 2. And it's a trivial exercise to show that this is = 2 n squared. this is 2 l +1 and summation l = 0 to n -1 up to l +1 is n squared and when you multiply it by 2, you get 2 n squared.

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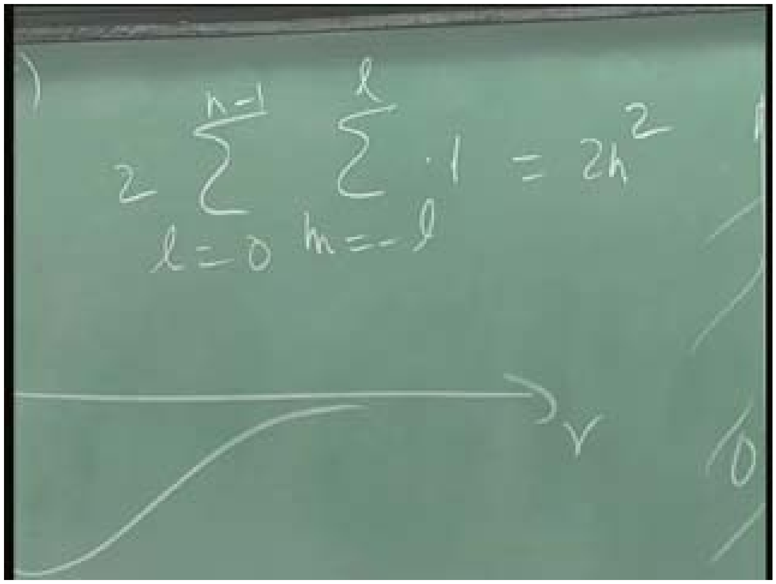

So this is what happens in the case of the hydrogen atom. We will come in a minute to what the reason for this extra degeneracy is. But I emphasize once again except for these problems a special symmetry and there are just 2 of them. One of them is a 1 over r potential and the other is 3 dimensional harmonic oscillator. Except for these two, in general for a central potential, the energy will depend on the orbital angular momentum quantum number as well. That degeneracy is not lifted. No dependence on m, the magnetic quantum number for any central potential is due to spherical symmetry but there is dependence on l. what happens in the case of the harmonic oscillator? Let's do that also quickly.

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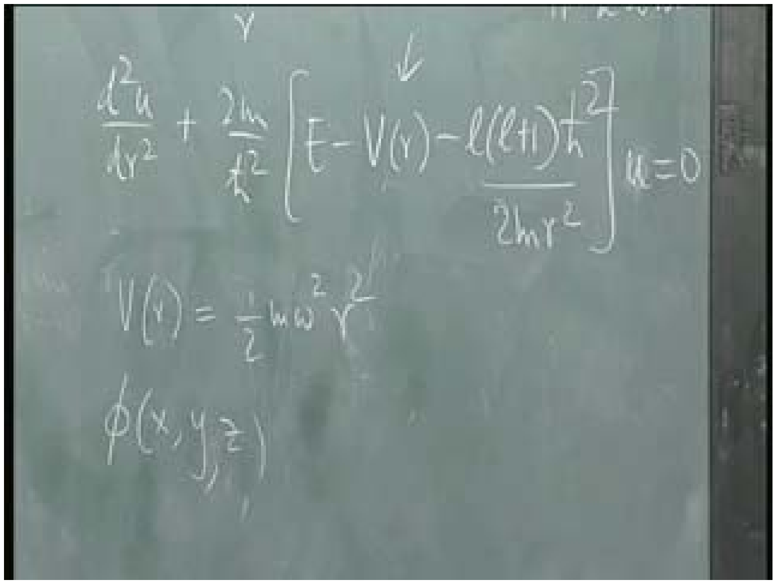

This Hamiltonian V (r) = 1/2 m omega r squared. And that problem is very to solve. In fact, we don't need spherical polar coordinates at all because r squared is just x squared + y squared + z squared and del squared in cartesian coordinates is just the second derivatives with respect to each of the Cartesian coordinates. The problem separates in Cartesian coordinates and we know what the solutions are for the 1 dimensional oscillator. and now you simply have to multiply it by the corresponding wave functions in y and z. so for the 3 D oscillator, it's very easy to find phi as a function of x y z. let's put all these quantities =1 or something and let's write down what the solution is.

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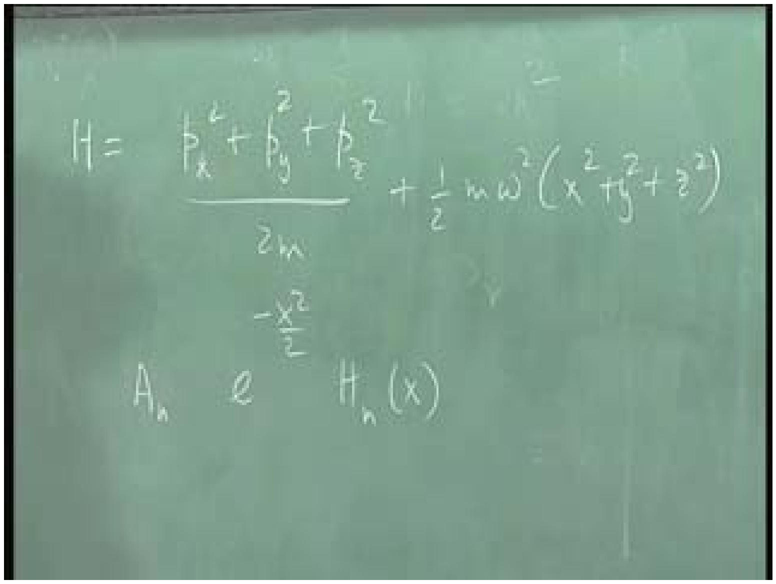

So here's my Hamiltonian. It is p_x squared + p_y squared + p_z squared over to m + 1/2 m omega squared (x squared + y squared + z squared). then its completely trivial to write this solution down because we know that in 1 dimension, the solution would be e to the - x squared over 2 in units of square root of h cross over m omega. So let's put all those factors. H_n (x) and then a normalization constant. This is the wave function and n runs 0, 1, 2, 3. Now what do you think is the wave function in 3 dimensions? Since it is solved by separation of variables in Cartesian coordinates, it's just the product.

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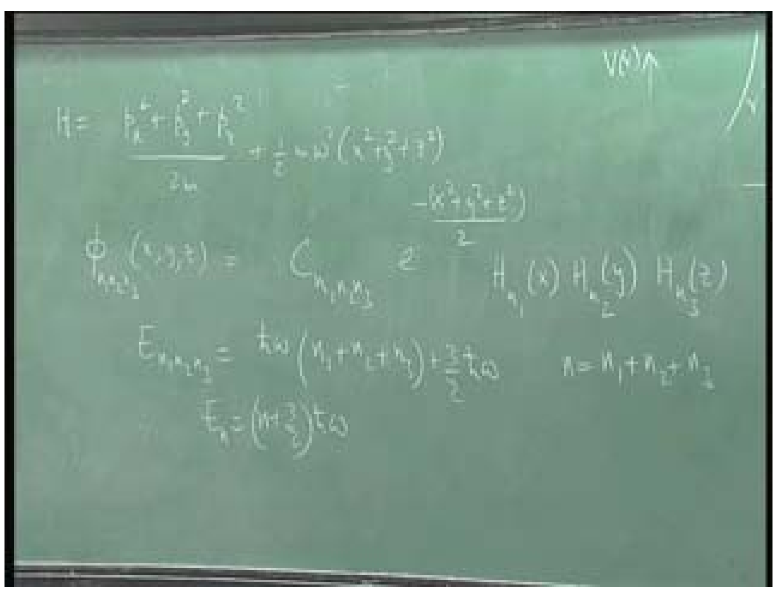

Therefore the actual wave function phi as function of x y z would be some normalization constant multiplied by e to the - x squared + y squared + z squared over 2 Hn (x) Hn (y) Hn (z) except that for each degree freedom, you need a quantum number. So it's n_1 n_2 n_3. This C would be a function of n_1 n_2 n_3 of course, that's what the solution is. What are the energy eigenvalues and therefore this is labeled by 3 quantum numbers n_1, n_2 and n_3. And what are the energy eigenvalues? E as function n_1 n_2 n_3 = h cross omega times n_1 + n_2 + n_3 + 3/2 h cross omega. Each degree of freedom gives a 0 point contribution 1/2 h cross omega and for 3 dimensions we have 3/2s. What are the allowed values of n1 n 2 n 3? It's 01 2 3 etc. so could write this you could write this as n equal n1 + n 2 + n 3 and then you have En = (n + 3 /2) h cross omega. What's the degeneracy of the state n? it's the number ways in which you can write a non negative integer n as a sum of 3 non negative integers n1 n 2 n 3 and what's that? So the ground state is 0 0 0.

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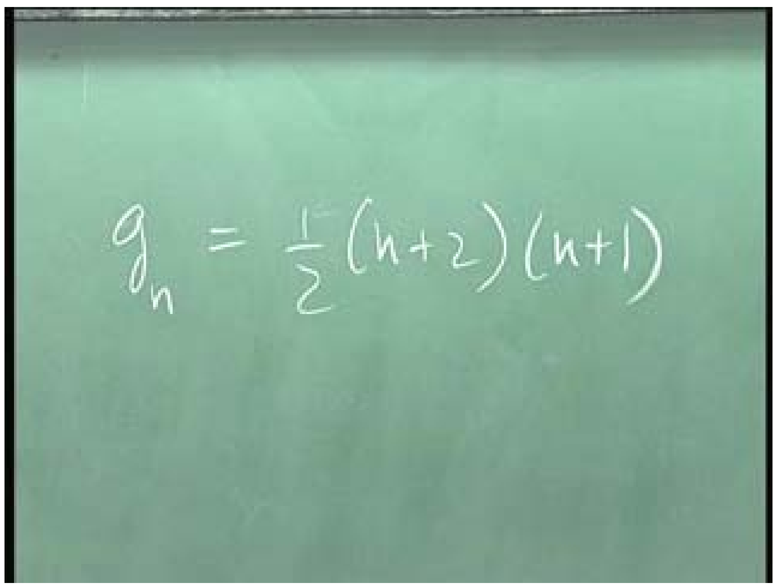

So g_n = ½(n + 2) (n + 1). the ground state is just 0 0 0. All 3 quantum numbers must be 0. n =1 can be done in 3 ways; 1 0 0, 01 0,0 01 and so on. So you have n and you got to put them in 3 boxes, you need 2 partitions. So actually you have n + 2 objects. You can permute them as you like. The number of permutations is (n + 2) factorial but the 2 partitions can be permuted among themselves which is 2 factorial and the objects can be permuted among themselves, n factorial. So when you divide n + 2 factorial over n factorial 2 factorial you get that. So that's the degeneracy of the state and the wave functions of course would depend on n1 n 2 n 3 and the ground state has no node at all. It is just the Gaussian e to the - r square as you expect. So this is it in Cartesian coordinates. Now what would happen if you solve the same problem in spherical polar coordinates? I can still do that. I should get exactly the same answer.

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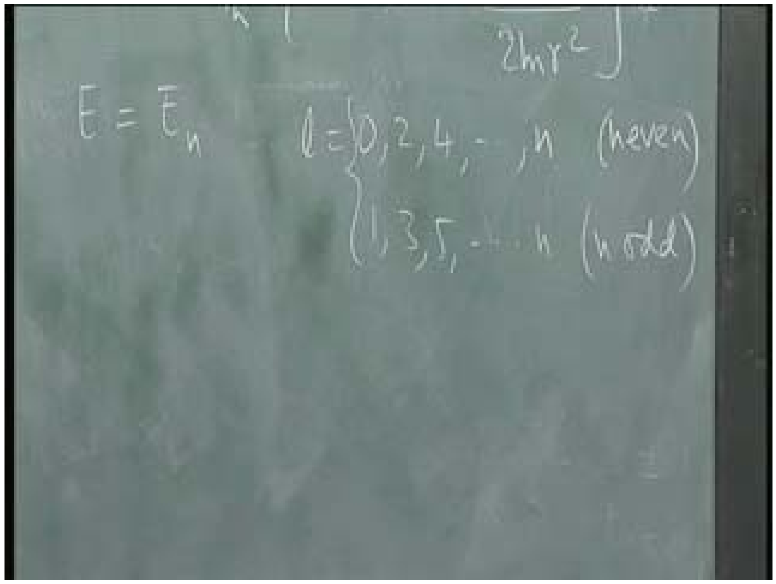

Then I go through this l, m etc and it will turn out the energy values are exactly the same but once again in this case is turns out that E = En doesn't depend on l, m once again in this problem too but here l runs 0, 2, 4 for n even and 1, 3, 5 for n odd. It can run all the way up to n. so if you work this out this is what happens. I leave it you as a trivial exercise to show that if you sum 2 l +1 over these allowed values of l then indeed you end of with that degeneracy, not squared but this quantity (Refer Slide Time: 49:14). But again this problem has an extra symmetry, exactly as the Kepler problem has. Now what's the reason why the Kepler problem has this extra symmetry? We saw in classical mechanics that a particle making an orbit or moving under the influence of 1 over r potential in addition to the angular momentum, it has another vector constant of the motion, the Laplace Runge Lenz vector. And that carries through in quantum mechanics too.

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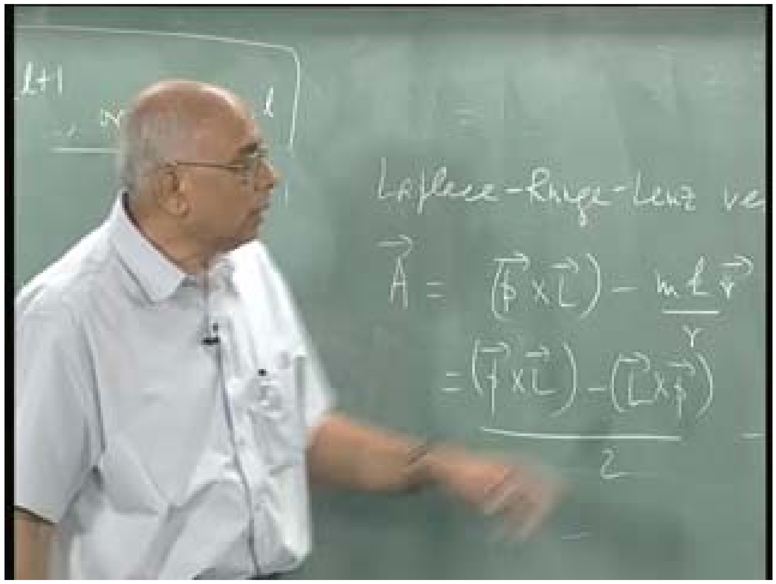

V (r) = - k over some constant over r. this vector classically was = p cross L - m k r over r. and this quantity is constant where L is the orbital angular momentum, r cross p. and just to refresh your memory, if the classical orbit was like this (Refer Slide Time: 50:54) in an ellipse with1 of the foci being the center of attraction, then the direction of this vector must be constant in time. so you can evaluate at any instant of time in particular if you evaluated at this (Refer Slide Time: 51:09) instant of time, its easy to see that its along this (Refer Slide Time: 51:15) direction because at that instant of time the momentum is perpendicular to the radius vector and then its easy to see that its actually along the r direction itself.

So the direction of the semi major axis and this direction doesn’t change which means the ellipse doesn’t precess and that’s a characteristic of bound motion in the1 over r potential. Quantum mechanically, this would be an operator but it has to be Hermitian operator. It’s made up of observables. It is the summation as its stands. r of course is a Hermitian operator. p is1 and l is. p and l don’t commute with each other. l is r cross V. so there is an r part inside l and that won’t commute with p. so this is not a Hermitian operator. How should i fix it? What should i do to this? P cross L - L cross p over 2 – m k r over r. And then indeed you can show that this comminutes with the Hamiltonian and the presence of this extra symmetry leads to this accidental degeneracy. Actually, that’s not a very satisfactory way of saying it. You could ask i have another constant of the motion, so why should i have an extra problem symmetry here. For this I have to take you back to classical mechanics and recall to you that the Hamiltonian remains invariant and the equations of motion remain invariant under a group of transformations which have to be canonical so that they are symplectic transformation and they don’t change the structure of Hamiltonian’s equations and Hamiltonian has to remain invariant. So you look at that sub group of the symplectic group Sp 6 in this case which keeps the Hamiltonian invariant. And therefore the set of solutions goes to the set of solutions. And that subgroup for this Kepler problem is So4 it has 6 generator and there are from by linear combinations of A and L.

So this is the role played by A and L. Their linear combinations act as generators of infinitesimal transformations under which the Hamiltonian as well as the equations of motion don't change in phase space. So that's what constants of the motion like this do. Similarly in the case of the harmonic oscillator, there are large number of constants of motion here not all functional independent of each other but there is a symmetry group. The 3 dimensional dynamical symmetry group happens to be Su 3. It has 8 generators and they can be formed from these combinations. Actually it turns out that this combination here, pi pj + qi qj where ij runs from 1, 2, 3 are actually constants of the motion. Of course you put i = j and sum over I, that gives you just the Hamiltonian itself. That's one of them but that is a tensor of rank 2. That is the symmetric tensor. It is a constant of the motion in suitable units. The angular momentum is a constant of motion. So qi pj - pj qi are 3 of these combinations.

They are also constants of the motion and they together form a complicated algebra. The algebra happens to be that of Su 3. So that's the reason why that problem has a degeneracy. Quantum mechanically you can write down the corresponding expressions for these generators. They commute with the Hamiltonian and you expect this extra symmetry but there is one more way in which you can predict when a problem would have this kind of symmetry and that is when the Schrodinger equation is separable in more than1 coordinate system. This tells you that there is some symmetry. This Hamiltonian is separable in 2 coordinate systems. What are they? Clearly, they are Cartesian and spherical polar coordinates. Therefore, you would expect certain degeneracy here. the hydrogen atom problem is not separable in cartesian coordinates because this 1 over r is 1 over square root of x squared + y squared + z squared in Cartesian coordinated but this problem is separable in spherical polar coordinates and what are called parabolic cylindrical coordinates. So there are 2 coordinates system under which there are separable and then you have this symmetry. Incidentally, if you had just free particle motion, just del squared in Hamiltonian which is p squared if you like, then there are 11 orthogonal curvilinear coordinates systems in which del squared is separable. So if you had not potential at all you will expect that in these 11 coordinates systems.

You would have some special features but the moment you put in a potential V (r), that's gone automatically. now of course we know from classical physics to that the only that r square then the1 over r of the only 2 central potentials for which all bounded motion is closed orbits. And that carries through to quantum mechanics to. So both the oscillator and Kepler potential are very special. They are the 2 solvable cases. As I pointed out, even later in relativistic quantum physics and field theory, the 1 over r potential place a very special role. It's intrinsic to nature itself for a variety of reasons. So I haven't solved these problems in a tedious way but solutions are available in text books so that you have exact expressions for the wave functions.

But I thought I would give some idea why these solutions look the way they do and that's more or less where i would like to stop. What we need to do now is to ask what happens if the problem is not exactly integral? What happens if I add QA and arbitrary Hamiltonian and its no guarantee at all that i can solve the Schrodinger equation? Then I would try to do it by what called perturbation or approximation methods of various kinds of which perturbation theory is very crucial and i would like to introduce that.

The other point is, very often you have quantum system and you do something to it, from outside you apply time dependent perturbation and then it induces transition between the levels of the system. That comes under the purview of time dependent perturbation theory. So I would like to introduce in this course, at least the rudiments of both time independent and time dependent perturbation theory. There are simple rules including famous a called Fermi's golden rule which I would like to definitely talk about because that's absolutely crucial to applying quantum mechanics anywhere. And after that, the very last topic we will talk about is what I have been mentioning throughout namely; the spin statistics collections of identical particles and this will help us to terminate the course.