Quantum Physics
Prof. V Balskrishnan
Department of Physics
Indian Institute of Technology, Madras
Lecture No # 24

Conversation between Student and Professor: Professor- The question rephrases the following

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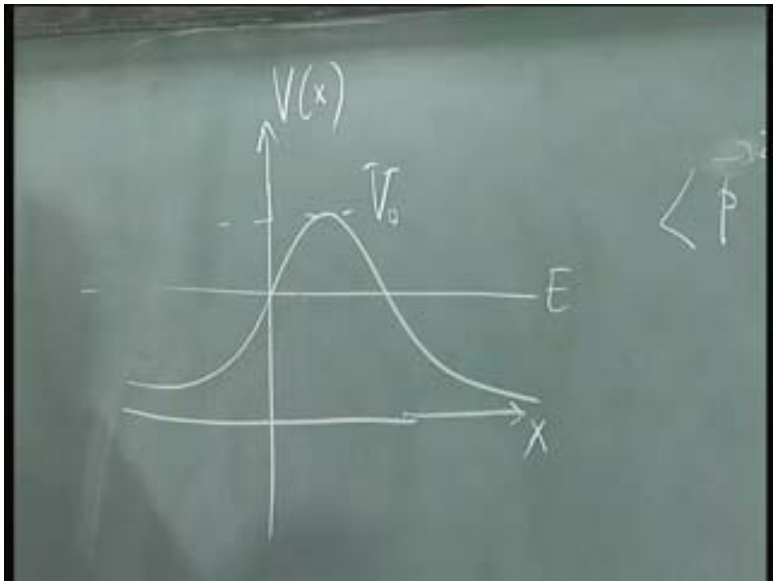

If you have potential of some a finite potential barrier of this kind for instance, then the question asked is if this is the height of the barrier V_0 (Refer Slide Time: 01:42), and this is V (x) verses x there is a finite probability that the particle is found in this (Refer Slide Time: 01:51) classically forbidden region even if its total energy is E. If this is the total energy of the particle, a classical particle shot on this potential is going to come down here. it is at best going to be able to reach this (Refer Slide Time: 02:05) point. it can't climb up this potential barrier beyond this point and then it goes back. And the reason is its kinetic energy cannot be negative. If this is the total energy and that's the potential energy, this would imply the kinetic energy is negative which is not possible. Now the question asked is now quantum mechanically also the kinetic energy's expectation value can never be negative.

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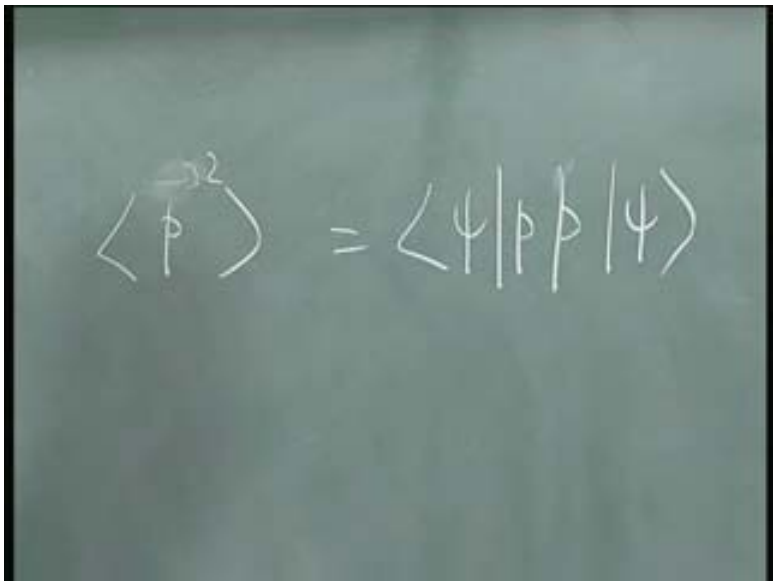

We saw that because the expectation value of p squared in any state of the system is essentially <psi| p p| psi> and of course this is the square of the norm of the vector p psi and therefore that can't be negative either. but the question is if we shoot a particle of energy E less than V_0 but greater than 0, greater than this (Refer Slide Time: 03:00) asymptotic value on this side, certainly there is a finite probability that the particle is in this region and there is a transmission probability to this side. The question is when the particle is in this reason does it imply that the kinetic energy is negative? So this idea doesn't exist in quantum mechanics. There is no concept as what's the momentum of the particle when it's here at this location. Therefore there is no question of saying when it's in this region, what's the kinetic energy because the total energy eigenstate is not an eigenstate of the kinetic energy or the potential energy separately. It's completely property of the state as a whole and that's this state has a way of function which is spread out from - infinity to infinity everywhere. so this is important to recognize that you cannot speak in quantum mechanics of what the value of the potential energy is when the particle is in a given region or at given point similarly for the kinetic energy.

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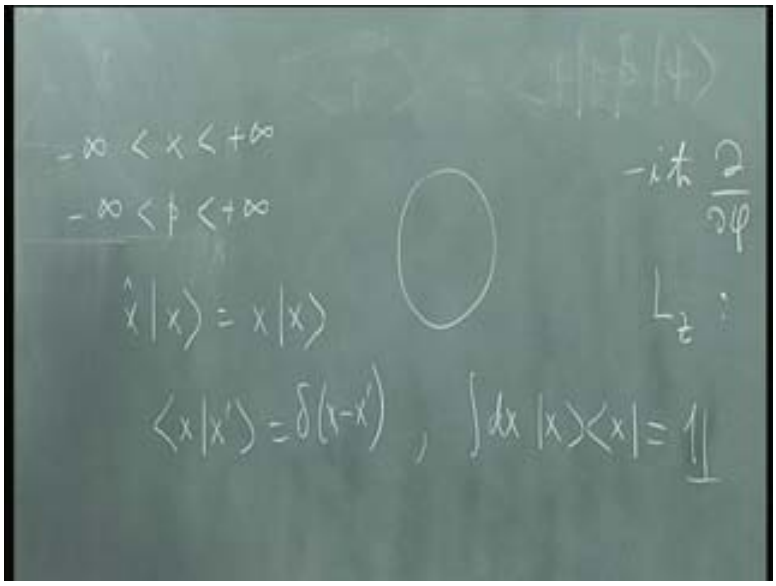

Let's go to the other question. The operators x and p have continuous spectra in classical physics. We know that - infinity less than less than x less than + infinity and similarly for p. when you convert these 2 operators, then in quantum mechanics automatically all the possible eigenvalues which were there classically and also all the possible values which existed classically or eigenvalues quantum mechanically and the spectrum is continuous. Of course if we put in boundary conditions and so on, then the operator might have a discrete spectrum. This is what happens for instance if you put this in a box and then say that the wave function must vanish at the ends of the box, then as you know just like waves on string, the eigenvalues could get quantized. For example the eigenvalues or the energy is get quantized if you are in a potential well or if you are in a confining potential.

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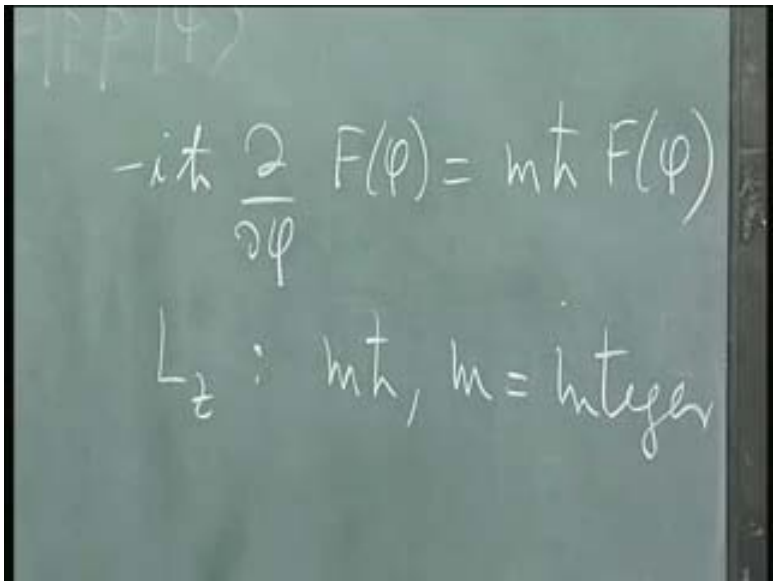

Even if you don't have, boundary if you have a circle, then the eigenvalues can get quantized. For instance if you look at the eigenvalues of the z component of the angular moment they must satisfy - ih cross delta over delta phi times the wave function F (phi) = m h cross F (phi). And then this wave function F (phi), if we require it to be single valued, the possible values of m are discrete. So the angular momentum component Lz has eigenvalues mh cross m = integer and that comes about by the single valuedness requirement or the periodic boundary condition requirement. Conversation between student and professor: Student – Still the variable x is continuous?

Professor – yes, the variable x i mean x become gets replaced by an operator. So let me, for the momentum, just to distinguish the operator call it x operator and that acts on eigenstate which are labeled by the eigenvalue itself. The eigenvalue we know is continuous. So we have an equation of this kind. This is the ket vector corresponding to the particle having a precise position x. now because it's a continuous spectrum, the normal equations you write down for discrete spectrum have to be modified slightly. For example, orthonormality would be something like delta(x - x prime) rather than a Kronecker delta and completeness would be an integral over dx |x>< x| = the unit operator. now as far as notation is concerned, this looks like a trivial change from kronecker delta you gone to this and from summation over all possible eigenstates, you have actually done an integration but it's more profound than that (Refer Slide Time: 07:24 to 07:31). It takes little more work to discuss what a continuous basis is. And i have slurred over those technical details and simply assume that you have continuous basis label by these here (Refer Slide Time: 07:45). But of course, you could raise many objections to it like what's the value of the delta function when the argument vanishes. Formally these become infinite and what's meant by this projector etc but we have not got in to those technical details. So we haven't really spent much time discussing the technical details of continuous spectra.

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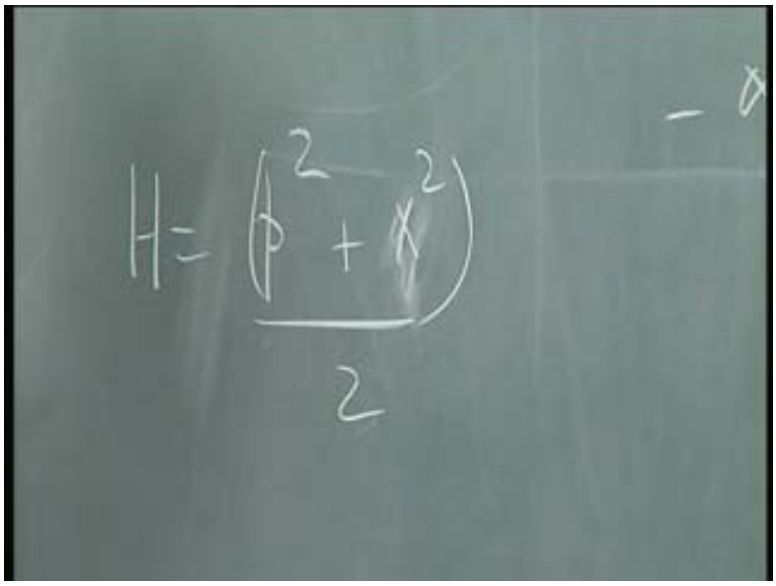

Now the Hamiltonian itself, if you look the harmonic oscillator which is p squared + x squared over 2 in suitable units, p and x have continuous spectrum but the square of x + square of p over 2 has a discrete spectrum and this is possible. Are these bounded operators, in the sense that do they have finite norm? no, because the eigenvalues can become as large as you please for each of these and therefore there are states in the system whose norms would become enormous and even if you normalize by dividing by norm of the states, it's clear that roughly speaking, once the eigenvalues of an operator become unbounded, the operators are unbounded too. The spectrum of this operator which as you know is 1/2 h cross mega 3/2 etc, is bounded from below but not from above. In the total energy, if it's not bounded from below, you are in trouble because it means the ground state is – infinity and things would fall to the ground state. It takes an infinite amount of energy to raise it from the ground state but this is exactly where quantum mechanics plays a role and helps you get a finite value.

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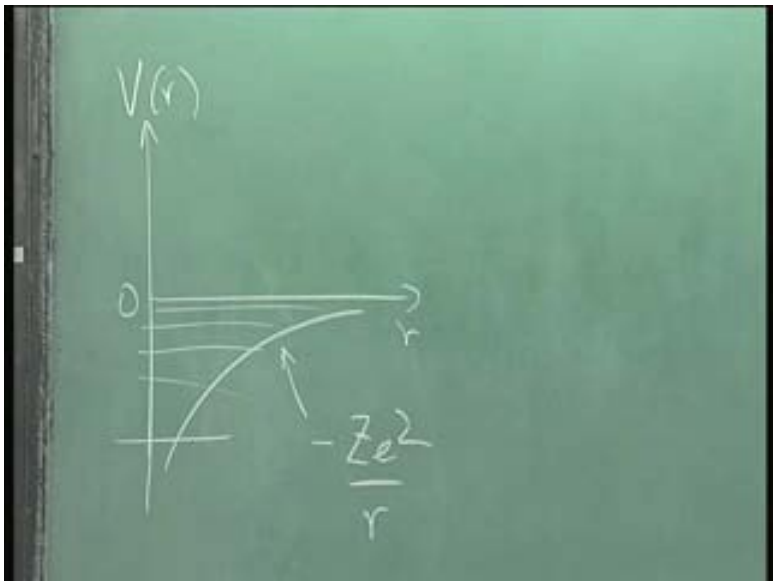

We are going to do that when i talk about the hydrogen atom because for the hydrogen atom the potential is central potential and its 1 over r potential these goes like - ze squared over r in suitable units. Now of course classically, the ground state of the electron, when it's orbiting round a positive charge, would be inside the nucleus itself. it would be at r = 0 because it will just fall down into potential but quantum mechanically we know that the answer is some -1 over n squared and then it keep increasing. so in Rydberg units its – 13.6 eV. What feature of quantum mechanics is responsible for raising the energy level from the classical - infinity to – 13.6 eV? What do you think it is?

Classically, the equilibrium state would be the particles at 0 momentum right at the origin and that violates the uncertainty principle. So you can't have both. Therefore quantum mechanically, the system compromises by saying you come too close to the origin, the potential energy gets too low, the kinetic energy gets too high and so on. It compromises by saying there is a ground state here (Refer Slide Time: 10:55) at this point. And of course, as the energy increases, you have more and more states and the fact at this potential goes to 0 very slowly like 1 over r is what is responsible for infinite number of bound state. Had it cutoff at some finite point, then you would only finite number of bound states.

For instance, you look at the attractive delta function potential in 1 dimension. You have just 1 bound state. You put 2 of them. You could have a second bound state. You take a finite well, you will always have a ground state but you might have1, 2, 3, etc. but you extend the range of the potential to infinity. Then the possibility that you also have an infinite number of bound states arises. And that's what happens in the case of the hydrogen atom. So if this potential goes to 0 sufficiently slowly at infinity, then you can have an infinite number of bound states. We will write down the criterion for this as well.

on the other hand if the potential goes to 0 too rapidly at the origin, then there is what's call collapse to the origin and the strength of the potential is so large that even quantum mechanics cannot rescue you do not have any bound states. Any potential that goes to 0 faster than 1 over r square at the origin become singular and apart from special cases, you don't have the conventional kind of bound state at all in such potentials. so going to 0 too fast at the origin is bad, going to 0 too slowly at infinity also can lead to an infinite number of bound states on the other side. So the coulomb potential is poised very nicely in between 1 over r and it has an infinite number of bound states. But the lowest bound, certainly the spectrum is bounded from below. There was another question on spin. How did the idea of spin arise? This i should have mentioned since i didn't give much about the history of the spin. This is perhaps little mysterious still. You see the spectra of atoms; the spectral lines correspond to transitions between the various energy states available for the electrons in these atoms.

Now long ago when quantum mechanics for first formulated and the hydrogen atom spectrum for instance was being explained in quantum mechanics, it turned out that the discrepancies between the predictions of the normal Schrödinger equation for these spectra lines and what was actually observed. And various resolutions were purposed, but one that turned out to be the right one had to do with the concept of an intrinsic angular momentum or spin. This was postulated by various people, in particular Uhlenbeck and Goudsmit. They specifically said that there is such a thing called spin and it is a 2 valued variable. And then the famous experiment of Stern Gerlach established what this meant. And the experiment goes as follows.

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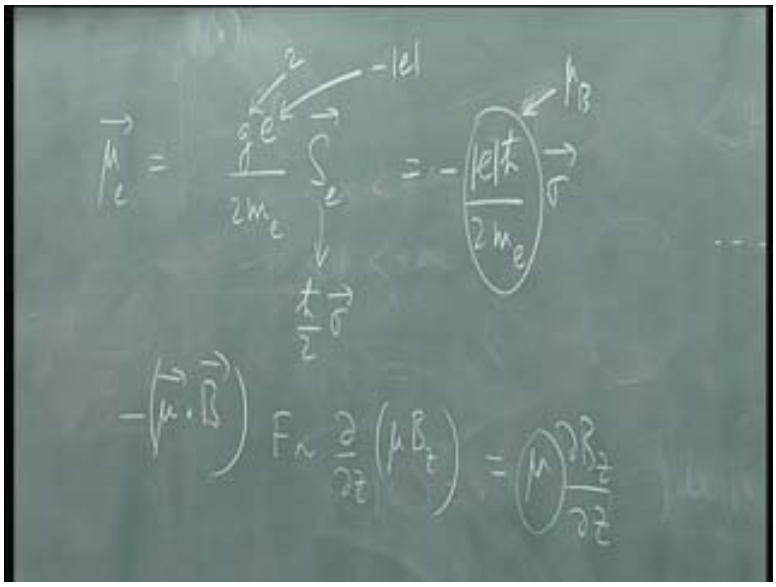

You recall that i mentioned that the intrinsic magnetic moment of the electron was = some gyromagnetic ratio which is g times a charge of the electron over twice the mass of the electron multiplied by the spin operator of the electron and of course, we also said the g was = 2, e = - modulus of e and the spin operator was h cross over 2 times of Pauli matrix and this was an operator which had eigenvalues + or -1 along any direction. So this whole thing became = e h cross mod e h - over 2 m e sigma and this (Refer Slide Time: 15:44) was called the Bohr magneton.

Now you could ask: what's the consequence of this and how do you measure this directly? What was the done by Stern Gerlach was to show that this has a real measurable effect once you place these electrons in a magnetic field. But of course, placing free electrons in magnetic field is quit a trick. So what they did was to take silver atoms and silver has 47 electrons, 46 of them lie in close shell essentially contributing nothing to the magnetic moment. And this whole shell is essentially spherically symmetrical. so might assume that in a ground state, these things are actually in a total angular momentum = 0 state. The 47th electron, the 5s electron is also in an orbital angular momentum = 0 state but it has a spin.

The full silver atom essentially acts like a single magnetic moment due to the intrinsic magnetic moment of the electron or the spin of the electron. Now once you have a magnetic moment, you place it in a magnetic field then there is mu dot p potential. And if the magnetic field is inhomogeneous, there is a force and the force is - the gradient of this potential. So the idea they had was to prepare a beam of silver ions it's monochromatic and so on. The sense that they made it's collimated; it's made monoenergetic and so on. and then by methods which we won't go into, so here is the path of that and then you put it in the path of an magnetic field who's pole pieces are like this (Refer Slide Time: 14:42). This is an inhomogeneous magnetic field.

You can see that there is a drastic change in the magnetic field. It's not parallel lines of force at all but it's inhomogeneous and changes as you go along any of the direction say the z direction. and then what happen is if you look at only the z direction of this field the force is proportional to delta over delta z mu times Bz where mu is the magnitude of this mu dot B is = mu delta B z by delta z. therefore if along this direction, the z component of the magnetic field changes substantially. You would have, due to this (Refer Slide Time: 18:33) term here you do have different forces depending on whether you had the + eigenvalue or the – eigenvalue. So in one case the force would be upwards in the other case of force would be downwards. therefore the path of this would either go like that or like this (Refer Slide Time: 18:52) and if you put a screen here and measure the intensities here, you would discover how many such sates there are in fact to you find out if this spin is 1/2 or 3 ½ or whatever.

You can find out what this spin is because there may be that many spots depending on what the allowed values of this mu r. So this is how it was established that indeed the spin of an electron is ½. but we know from the general theory of angular momentum that whatever be the eigen origin of this angular momentum, the allowed eigenvalues of the

angular momentum operator itself are in fact 0, ½,1, 3/2, and so on. And then it was recognized that the spin was in fact one of the 1/2 integer valued representations. Now one could ask what the deeper implications of this spin ½ are, where did it really come from, why did the angular momentum quantum number itself turn out to have either integer values or 1/2 integer values. We saw that it came out of the algebra of the angular momentum operator themselves.

But orbital angular momentum takes only integer values and i give a sort of hand waving arguments saying that's because it has got to have single value wave functions. You could ask what about the spin wave function. Does it not have single value wave functions and so on, what does it imply. And where does this 1/2 really come from. i will spend a few minutes and tell you where the 1/2 integer comes from and why you have double value representations for the rotation group. But this was the historical origin of spin.

So it will be affected by the orbital angular momentum. so if for example you have n = 2 then the total angular momentum would have values in L - 1/2 to L + ½ and then it could go from 3/2 to 5/2 so something like that and each of those would get split into 2 z +1 values and it you would have a large number of lines here. This is why they chose very cleverly an atom where it was guaranteed that the total angular momentum would be that due to the intrinsic angular momentum of the outer most electron of a single electron and that unambiguously established that it is spin ½. But of course in more complicated items you have much more complicated spectra. Now in the hydrogen atom, notice that I mention there was thing called as spin orbit coupling where there was a coupling between the magnetic momentum of the electron and the orbital motion of the electron.

The magnetic moment which the electron sees as a result of the proton going around it if you like and forming a current loop. that leads to and affective Hamiltonian which is proportional to L dot s and that breaks the central force nature of the potential which electron sees and then the degeneracy that you have of the L levels being degenerate is completely removed. You can also remove the degeneracy of the coulomb potential by applying a magnetic field. A magnetic field applied to any atom is going to lead to a splitting of the different time levels and it's called the Zeeman Effect and you could also break it by applying an electric field and that's called the Stark Effect. So these are spectroscopy effects which the early days of quantum mechanics established the reality of spin and so on in various ways. So the essential picture is correct. Now for few minutes on why we have spin ½.

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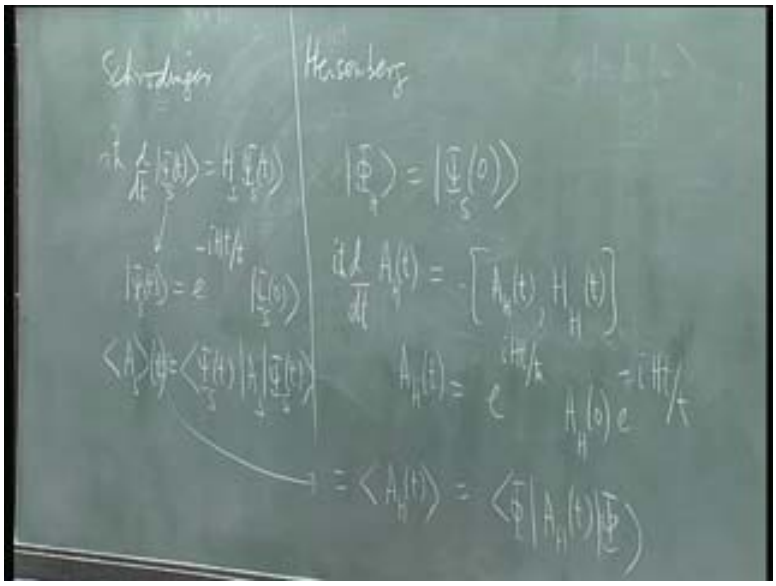

Just to recall to you what the 2 pictures are in the Schrödinger picture, as opposed to the Heisenberg picture, you had the Schrödinger equation ih cross d over dt psi (t) = H psi(t) and let me now distinguish between the pictures by writing H_S here. Then the state vector was time dependent and physical operators were assumed be time independent including the Hamiltonian. There is no explicit time dependence. So we look at all those operators which don't have explicit time dependence. Then the state vector involves in time. psi (t) is e to the - i ht over h cross psi(0). And the expectation value of any operator <A>at time t is given by psi (t) A psi (t) and this is in the Schrödinger picture. The time dependence for physical measurable quantities comes about because a state vector changes with the time. Now one can make a unitary transformation on this picture and arrive at Heisenberg picture where the state vectors are supposed to be time independent.

So let me put psi_S everywhere. This in the Heisenberg picture could be taken to be the Schrödinger state vector at some fiducial instant of time which i will choose to be 0. It doesn't have to be but at some origin of time. When the physical operators are supposed to obey the equation of motion A_H, now let me put in explicit time dependence here. The operator supposed to be a time dependent operator. This is $H_H(t)$ in principle. in principle, this doesn't change at all. What is true in both cases is that if you solve this (Refer Slide Time: 25:47) equation, this says A_H (t) is e to the power iHt over h cross, $A_H(0)$ e to the I H t over h cross. The way you match these 2 pictures is by saying, for physical quantities i want exactly the same answer on both sides. So I require that this (Refer Slide Time: 26:29) also be = the expectation value of $A_H(t)$ which is the same as phi $A_H(t)$ phi. I require these 2 to be exactly the same. That implies all the other things working backwards. So it turns out that according to the solution here, if you look at H itself on both sides, since H commutes with itself, these 2 factors come right across when i substitute for A_H.

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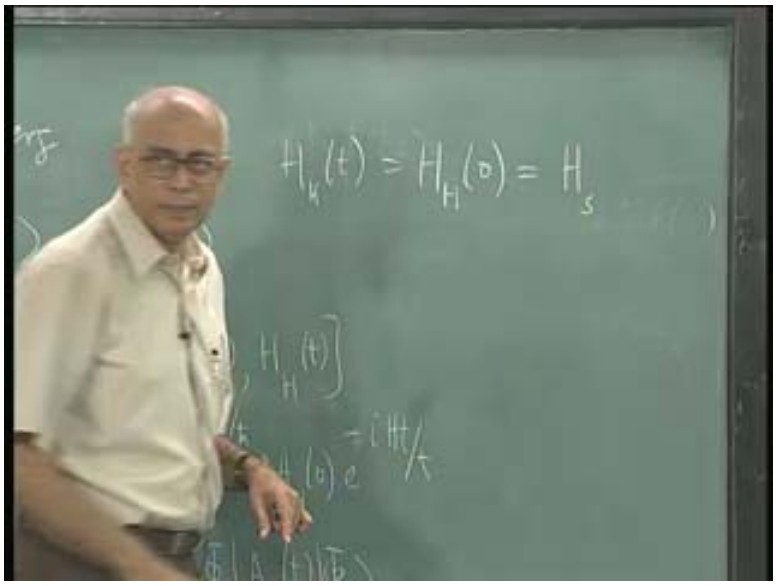

And I get H of $H_H(t)$ is the same as H_H (0) which by definition is S_S. this is what I get provided S_S did not involve time. But of course there are problems where the Hamiltonian itself involves time. For instance if i am pumping energy in to a system, then the Hamiltonian explicitly time dependent. its not in autonomous system. What happens then? Well, the Schrödinger equation continues to be true.

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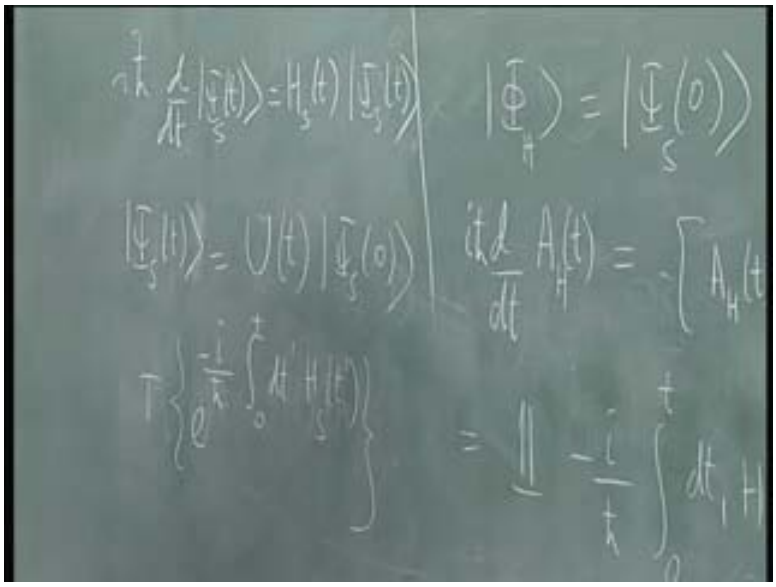

And now the solution to this is not so trivial. $psi_S(t)$ is not an exponential but a more complicated operator, some U (t) with acts on $psi_S(0)$ and this has got the form of what's called time ordered exponential. It's more complicated operator and if time permits in this course i will derive this expression here. It's not too difficult to drive an expression for this. This is a unitary operator which is not e to the i ht nor is it the very simple e to the - ih over cross, integral 0 to the t dt prime Hs (t prime). This is what one would expect naively because if this was not an operator, this is just a function here. this certainly would be true and in the case when it becomes time independent, you just get Hs and then t out here which was the original solution but this is not true because there is no guarantee that Hs (t prime) commutes with Hs of any other time.

And since e to the A, e to the B is not e to the A + B you can't write it in this fashion here. Instead you write it in what's call the time ordered exponential denoted in this (Refer Slide Time: 29:30) fashion which is formally like an exponential but it involves certain time ordering inside here. In any case, the solution is some unitary transformation acting on this. Then you can go to the Heisenberg picture from the Schrödinger picture for any other operator using the fact that the 2 would coincide with each other at some specific instant of time like 0. So that's always permitted. i can take an (Refer Slide Time: 30:15) exponential here and an exponential here, use for it the operator at some instant of time, Hs(0) for instance.

That will give you the unitary transformation from one picture to the other. so you can still go to the so called Heisenberg picture. Nothing is going to be different except in technical detail but it would not be the original very simple exponential that you wrote down but this is just a matter of convention here that the 2 pictures coincide at t = 0. that's all I need but now if you ask what does the evolution itself look like when you have a slightly different view point here, in such cases notice that the Hamiltonian continues to generate time translations always because that's what a content of this Schrödinger equation is. But if this for example, had explicit time dependence, then there is an extra term here (Refer Slide Time: 31:13) which has got a partial derivative term and so on.

So this is a simple exercise to work out what's the shift transformation to the Heisenberg picture when you have explicitly a time dependent Hamiltonian. Let me write this down.

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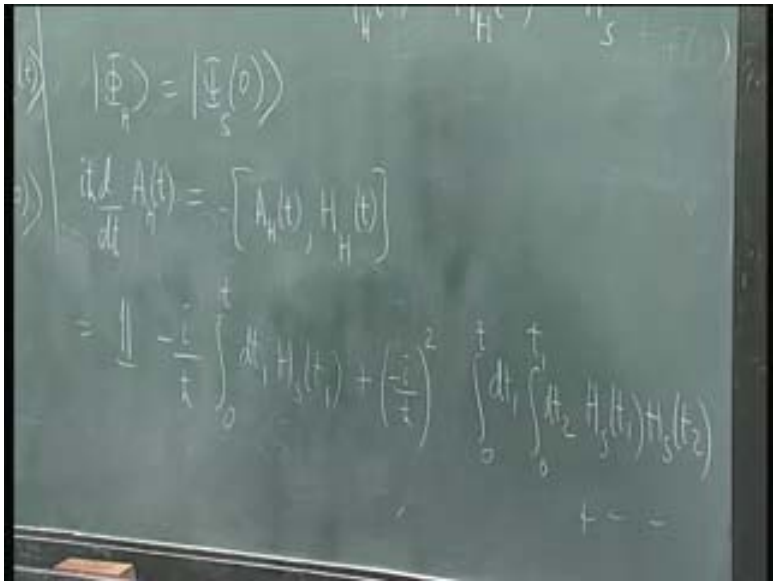

This is the unit operator - i over h cross, 0 to t, dt_1 H_S (t_1). the second term would be + (- i over h cross whole squared) 1 over 2 factorial, if you didn't have any problems with time ordering, this would be integral 0 to t dt_1, integral 0 to t dt_2 H_S (t_1) Hs (t 2) + higher orders. But, because you have this problem with time ordering, the 2 factorial goes away and this becomes 0 to t1. So the later time appears on the right and the earlier times appear on the left in this fashion and that neatly cancels the 1 over 2 factorial here. Because this (Refer Slide Time: 32:49) quantity here, if they are classical commuting variable is a symmetric function of t1 and t 2. And what you are doing is in the t1 t 2 space, you are integrating over this square (Refer Slide Time: 33:03) but the integrant is symmetric and therefore the integral over this triangle is = the integral over this triangle (Refer Slide Time: 33:17). So you can get the rid of the factor 2 and write in this fashion here. And you have taken care time ordering; earlier times to the left and later times to the right.

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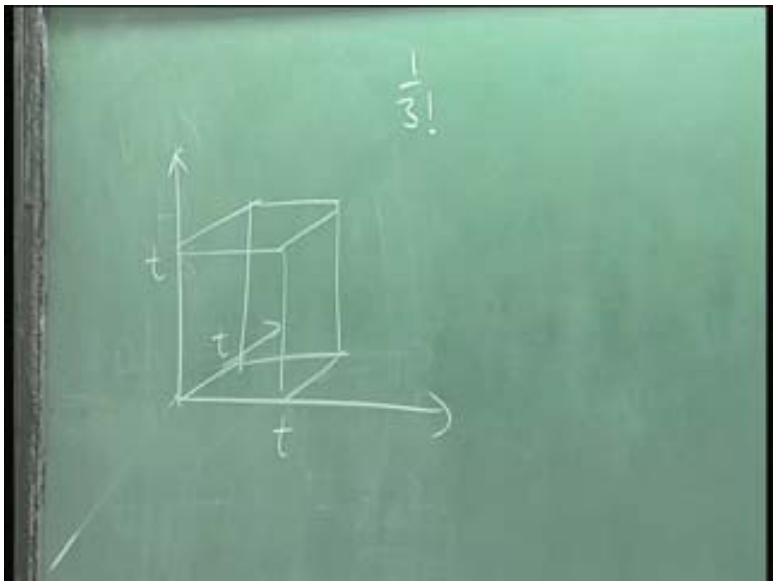

In the next term, the cubic term, you have a 1 over 3 factorial. if you take 1 axis in this fashion, it would be over this cube t1 t 2 t 3, each of side t. there are now six ways in which you can order t1, t 2 and t 3. And there is exactly one way in which you have t1 greater than t 2 greater than t 3. So the 3 factorial cancels in the denominator and gives you an integral 0 to t dt1, 0 to t1 d 2, 0 to t2 dt 3, H(t1) H (t 2) H(t 3) and so on. And it exactly cancels. so each time you have this hypercube and the 1 over n factorial goes and you have an ordered prescription. That's what is meant by this time ordered exponential and that turns out to be right solution for this operator for this unitary evolution. So this is not difficult problem at all. Although you must remember that explicitly time dependent Hamiltonian means you don't have stationary states any longer. Now, a couple of statements, since I would like to do the radial force problem but let me start on that tomorrow. Let me spend the rest of today telling you why this spin 1/2 arose, where did this come from really. So let me do this quickly and let me also bring on the connection between SO(3) & SU(2).

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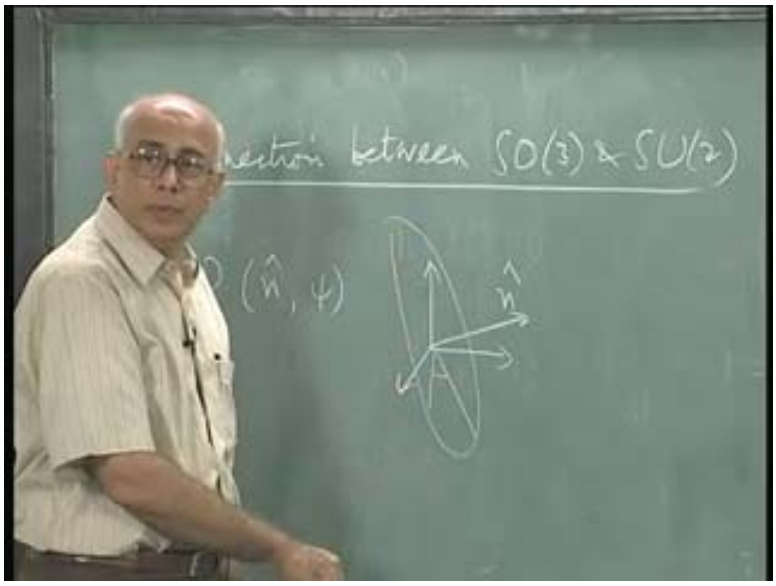

It goes as follows. You know that in 3 dimensional space I would like to generate rotations and what's my definition of a rotation? It's a linear transformation of the coordinates which leaves a point unchanged; the origin in this case and is homogeneous. That means the origin is map to be origin and everything else changes and it has determinant +1. This implies that the handedness of the coordinate system is not changed; the right handed coordinate system remains right handed coordinate system. those 3 suffice to fix rotations and they form a group this group is called as SO3, S stands for determinant +1, O stands for orthogonal because a transformation have to be orthogonal in order to ensure that the distance between any 2 points is not changed under the rotation and it's in 3 dimensions, so it's called as SO 3.

The set of 3 by 3 matrices which are orthogonal which have determinant +1 form a group. The unit matrix is the identity element and these matrices form a representation of the abstract group of rotations. so the rotations are operations in they are abstract but they are explicitly represented by the set of 3 by 3 orthogonal matrices with determinant +1. Now the next question is, what's the parameter space of this set of rotation? In other words, what are the values of the angle that specify the possible rotations? Now these can be specified in many ways. As you know you can go to one coordinate system to a rotated one by specifying 3 Euler angles. but you can specify those Euler angles in many different ways. They all turn out to be equivalent to each to other but it's nothing unique about it.

The most convenient way of specifying these rotations is to say that rotation occurs about some axis in space with respect to some fixed coordinate system through a certain amount of rotation. so to specify the direction in space, i need a unit vector and to specify the amount of rotation i need one more angle which can take on values from 0 to pi. So

the rotation is specified by a unit vector and then angle take values from 0 to 2 pi and let me call my angle psi, I don't want to confuse with theta and phi which are used for polar coordinates for n here. i fix a coordinates system to start with first in the lab and then I say I am going to make a rotation about this unit vector and of course, there is a perpendicular plane to this unit vector. Once I draw reference a line on that, i am going to rotate about a certain angle from 0 to 2 pi. So the parameters i need to specify the rotation are n and psi.

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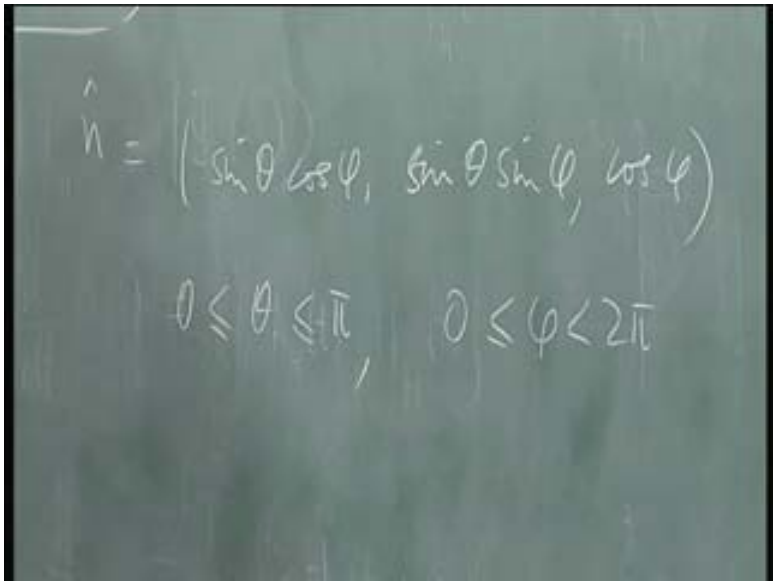

n is a unit vector, so it is specified by 2 polar angles. In Cartesian coordinates this would be (sin theta cos phi, sin theta sin phi, cos phi). There are 2 independent variables here because the squares of the 3 components of n add up to unity. the physical range of theta is 0 less than or equal to theta less than or equal to pi that goes from north pole to the south pole and the physical range of phi is 0 less than or equal to pi less than 2 pi. That's the azimuthal angle in the xy plane and theta is a polar angle. In spherical polar coordinates, what are the surfaces; x = constant y = constant z = constant? They are planes. What about r = constant? They are spheres. What about theta = constant? They are 1/2 cones. If a cone has an acute angle, then theta is less than pi over 2. if it's got an obtuse angle, then theta is between pi over to and pi. What about phi = constant? They are 1/2 planes because the pass through the origin and you distinguish between phi and phi + pi.

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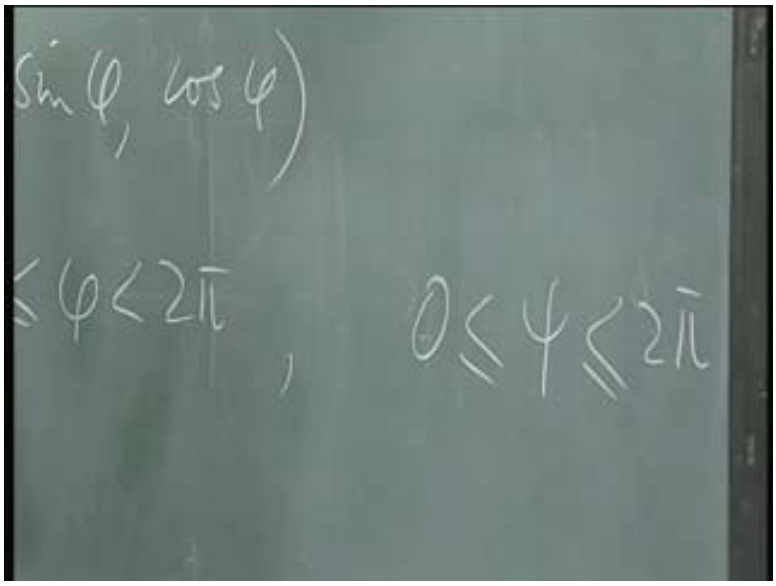

What's the range of variation of psi? It's obviously 0 less than or equal to psi less than or equal to 2 pi less than 2 pi. That's the amount of rotation; you can do in psi direction. So now we have 3 variables which specify ranges and we can pretend that we can put them as points in a certain space and we can model that 3 dimensional space, we have 3 variables.

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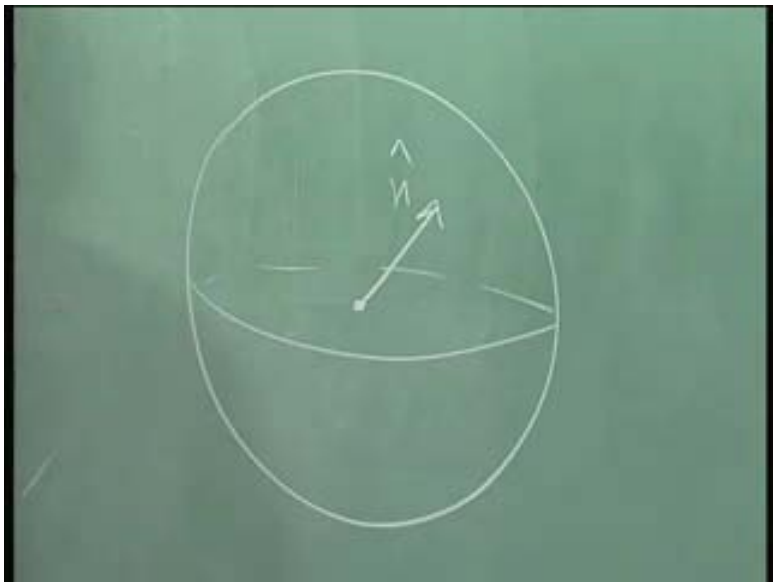

The way it's done is to say I will use a sphere as my model and the direction from the origin on this sphere to any point inside or on this sphere is going to be this specification of the unit vector n. so if i say this is n (Refer Slide Time), then the polar angle of this n and the azimuthal angle of this n correspond to the theta and phi here of the rotation that have in mind. and since i must have a rotation going from 0 to 2 pi, i could take the solids sphere to have a radius 2 pi and then say that by convention, the distance from the origin to the point i am interested in specifies psi. So you see I have 3 angles modeled in solids spheres of radius 2 pi but I like to make very sure that every point in this space corresponds to only one rotation. I don't want any double counting and I don't want to leave out any rotations either. But in 3 dimensions, it's fact of life that if i rotate an object through pi about an axis it's a same as rotating it about pi through the opposite axis. That's a fact of 3 dimensional life. So rotating about a certain axis through pi is the same as rotating about - that axis through pi once again. If I rotate through an angle that is a little lesser than pi, this coincidence doesn't happen. So happens only with pi.

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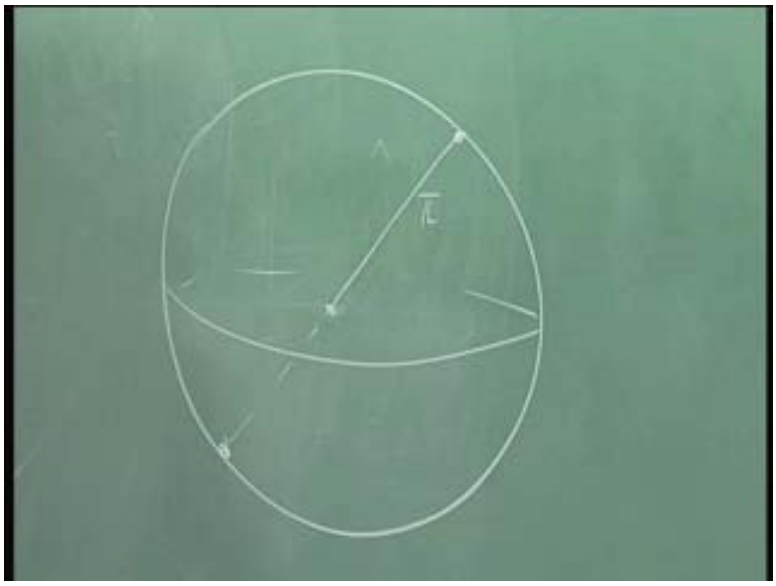

So this means that this (Refer Slide Time: 45:05) space need not have a radius = 2 pi. Pi is enough because if this is pi that sufficient because i also have the possibility of rotating about that diametrically opposite direction. But there is a further complication and the complication is that this (Refer slide Time: 45:23) point is mathematically the same as that point in this parameter space because they both correspond to rotation by pi about an axis or its opposite. They physically correspond to the same rotation to the parameter space of SO 3 and it's complicated. It is a solid sphere. Its radius is pi but it has also got the property that you must mathematically identify every point on its surface with its antipodal point. It's as if there is an invisible connection between opposite points. And of course such a space cannot be represented in 3 Euclidean dimensions.

So its respectable space but you cannot represented in 3 Euclidean dimensions but you can look at all its mathematical properties given this property. Now this space is connected. The connected space is one where you can go from any point in this space to any other point in the space continuously without leaving the space. So this space is certainly connected. There is no doubt about it but it is not simply connected. a simply connected space is one where any continuous closed path in the space can be continuously deformed or shrunk to a point without leaving the space.

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For instance, if I took the plane of this black board, any point to any other point I can go by an arc wise path. Every close path of this kind can be shrunk continuously to a point without leaving this black board. So that space is certainly connected and simply connected.

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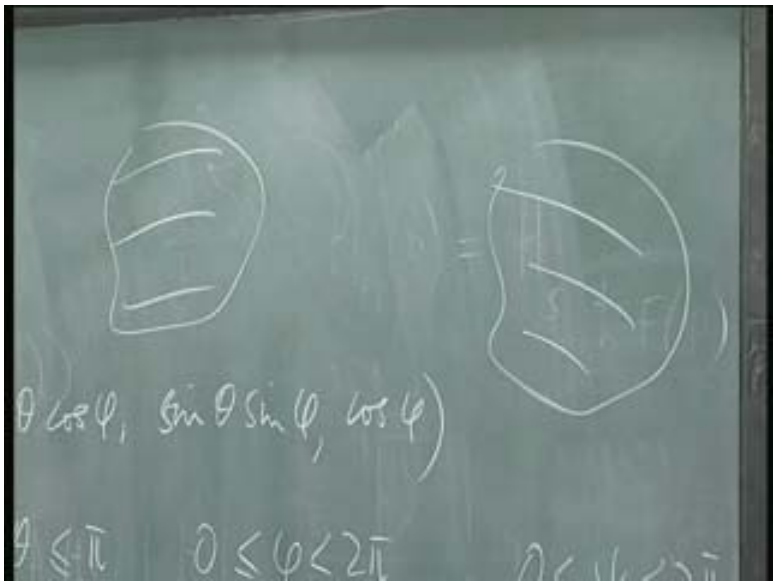

A space which connects which has for example, one piece here (Refer Slide Time: 47:26) an another piece there and this part does not belong to the space is not connected because there is a point here and point here (Refer Slide Time: 47:34) and you cannot join them continuously by arc wise path which doesn't leave the space.

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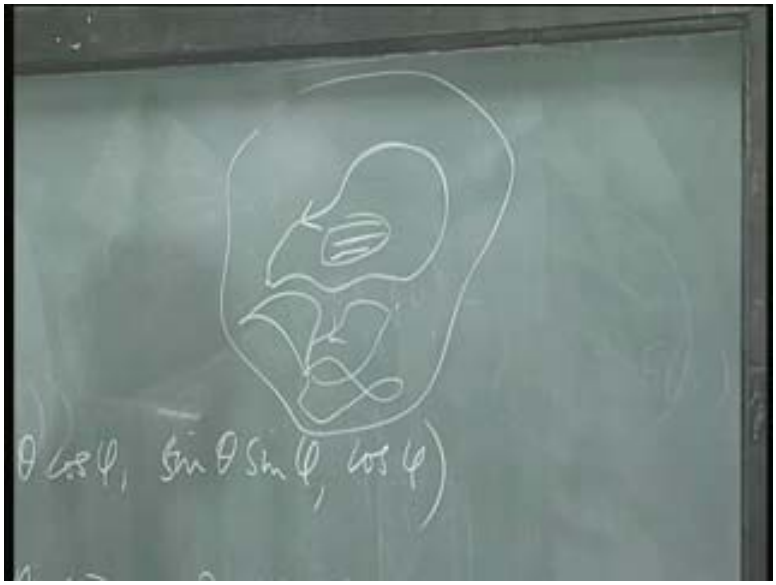

On the other hand, if that's your space (Refer Slide Time: 47:52) with a hole punched out in it is connected because you can go from any point to any point continuously without living the space but it's not simply connected because although you can close a closed path like this and shrink it down to a point. a close path like this (Refer Slide Time: 48:14) cannot be shrunk to a point because there is no way you are going to able to cross that hole and this path will get stuck at the periphery of this hole. So here is a space which is connected but not simply connected. In fact, even if you exclude one point that is good enough for the purpose of avoiding a simple connectivity. Even one point is punched is remove from the space, then the space is other simply connected.

So a sheet with a hole punched in it, even a single point is not a simply connected space. Then the next question is, if it's not connected what kind of connectivity does it have and this is precisely answerable. One says paths are equivalent to each other if you can deform one to the other continuously. So in that sense, this path is completely equivalent to this path which is completely equivalent to this path, etc (Refer Slide Time: 49:15). They can all like rubber bands be deformed to each other.

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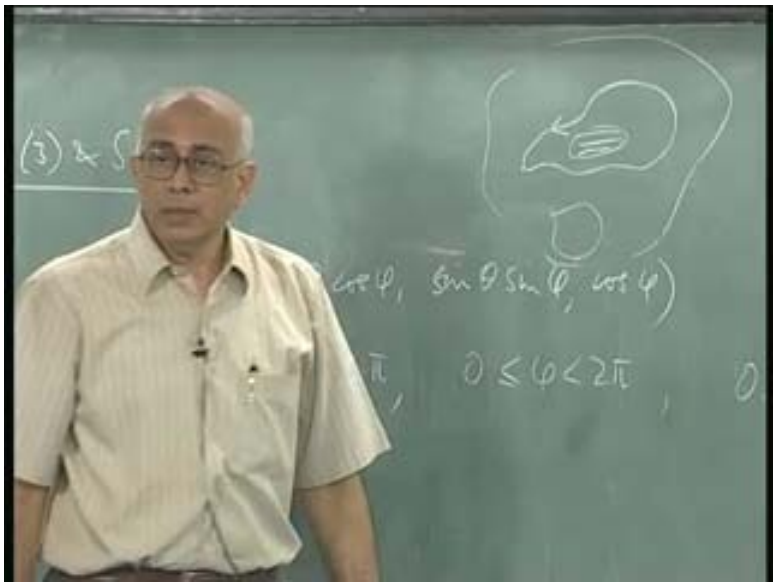

But this path here cannot be deformed to this path because that path because that path cannot be shrunk to a point where as this can be shrunk to a point (Refer Slide Time: 49:24 to 49: 29). So what one asks for is what are the classes of paths which can all be deformed to each other. All paths which can be deformed to each other form what's called an equivalence class. Then the question of what are the different equivalence classes of paths that you have in a space arises. It turns out that by an obvious rule of composition of paths of joining paths, these equivalence classes form the elements of a group. We started with closed equivalent paths and we said these close paths can all be put in different equivalence classes and these equivalences classes form elements of a

group and the group composition law is just the composition of paths. So for instance, if you had the path like this (Refer Slide Time: 50:35), this path is composed of 2 paths, one of which is just this and this here (Refer Slide Time: 50:45) and together they form the element of a group. They form a third path. So equivalence classes of paths from the elements of a group and this group is called the fundamental group of a space. Its called the fundamental homotopy group of this space.

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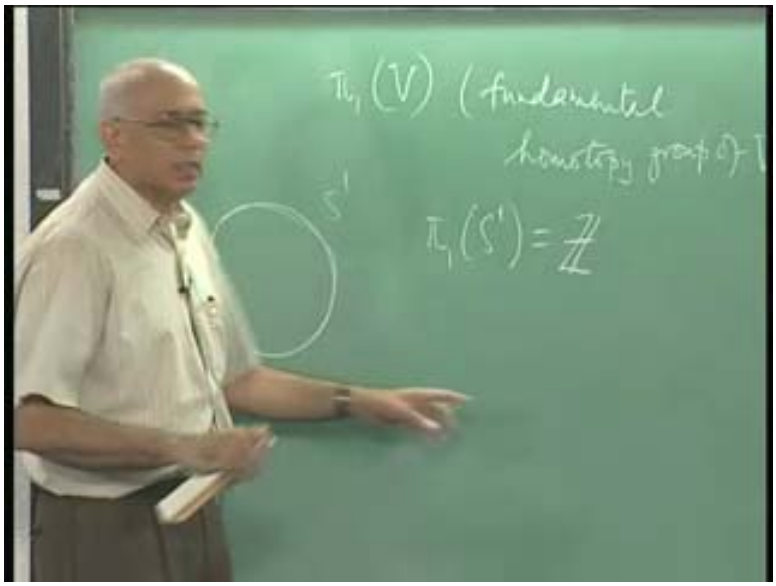

It's called $p_1(V)$ and as a group, it may have interesting properties. For instance, taking a simpler example. Suppose this space is just S1; the rim of a cycle wheel. That's my space. i have to live on that space. All paths are on that space. Then what are the equivalence classes of paths? It's clear wherever i start, if i move about like this, then the only thing i can do is to go back to from a closed path. Now all such paths can be shrunk to point continuously. So I could even come here and go round here and then go back. that would still be shrunk to a point but the moment i complete the path by coming back to this point (Refer Slide Time: 52:21 to 52:33), then i have actually taken a rubber band or rubber tube and covered the rim and there is no possibility without cutting this band, of shrinking that band to a 0 any more beyond this point.

And I can then do it twice or thrice and you can see none of these can be shrunk to each other. Or I could have done it in the opposite sense because closed paths must always specify sense or direction. And you can easily see intuitively that the number of inequivalent ways in which you can take a rubber band and cover this cycle wheel is just the set of integers. There is a one to one correspondence with the set of integers. If the path winds on once in the positive direction, call it winding number 1. If it winds twice, call it winding number 2. So in this case, it's easy to see that pi_1 (S1) is in fact the set of integers under addition because when you compose paths, all you are doing is adding

winding numbers. Clearly, if you go around 3 times in the positive sense and twice in the negative sense, you have gone around once in the positive sense. So it's the group of integers under addition and it has implications. What is $pi_(S\ 2)$?

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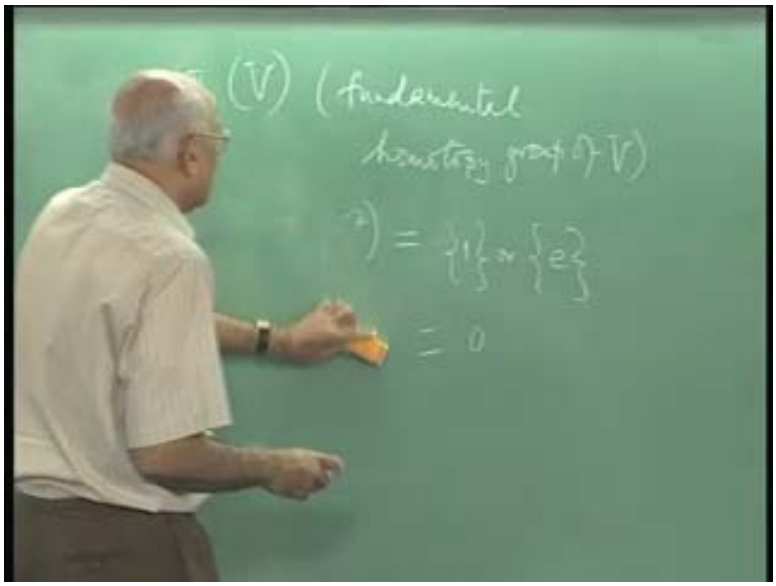

Now we have the surface of a sphere in 3 dimensions and i put a closed path on it; a rubber band. What's $pi_1(S\ 2)$? all paths can be shrunk to a point on it without leaving the space and this is graphically stated as by saying you cannot lasso a basketball because things will slip off and therefore pi1 of S 2 is in fact not just 1 element. So there are many ways of writing it. People write it as {1} or just the identity element or just 0. That means a trivial group. You have to be little careful with notation. A group = 0 means it has only 1 element. There is no other element in there. Similarly $pi_1(S\ 3)$ is 0 and $pi_1(Sn)$ is alos 0.

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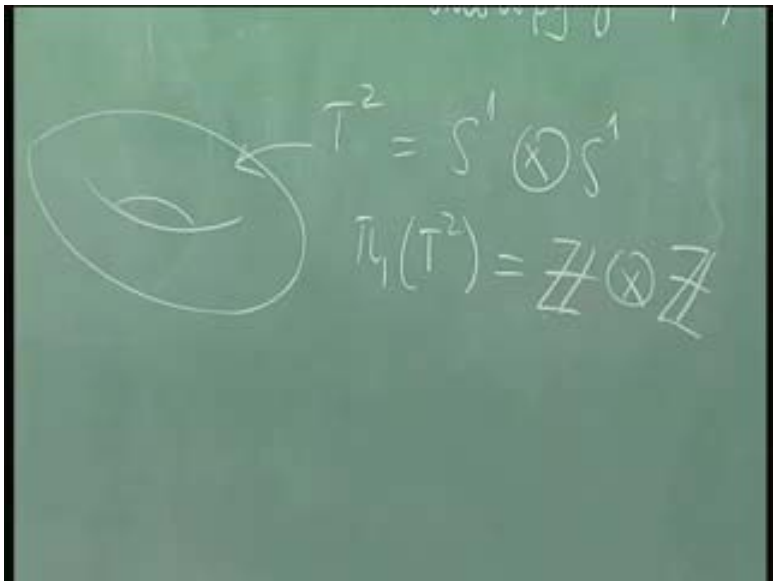

Now coming to our space, we ask what pi1 is but before that, this is the T 2. It's the 2 torus. And that is form by taking a circle direct product circle. For every point on it in one direction, you associate a circle in the other direction too and you get S1 cross S 2 which is T 2 which is 2 torus. Now what are the possible closed paths on this? This is just going to be z cross z because any closed path on it can be converted to going around this larger radius certain number of times and winding around this a certain number of times. So closed paths on the 2 torus can be specified by 2 winding numbers. The group we are the concerned with for the rotation group is somewhat different.

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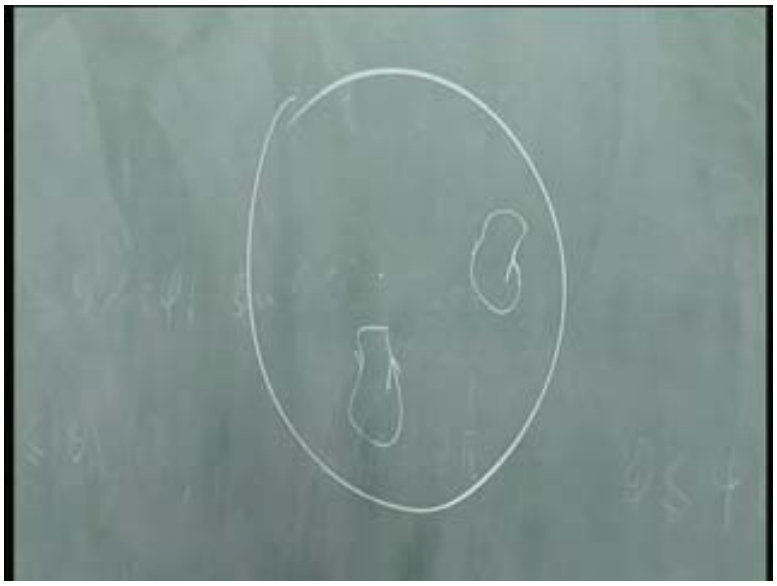

Now we have to discover what all the possible closed paths are when you have the origin here such that these points are connected to that. Well, it's quite clear that if you took any path inside or anything lying on the surface, going back, etc, those would all come back to same point. They would all be shrunk to a point. So there is one class of close paths which is just the conventional class of close paths.

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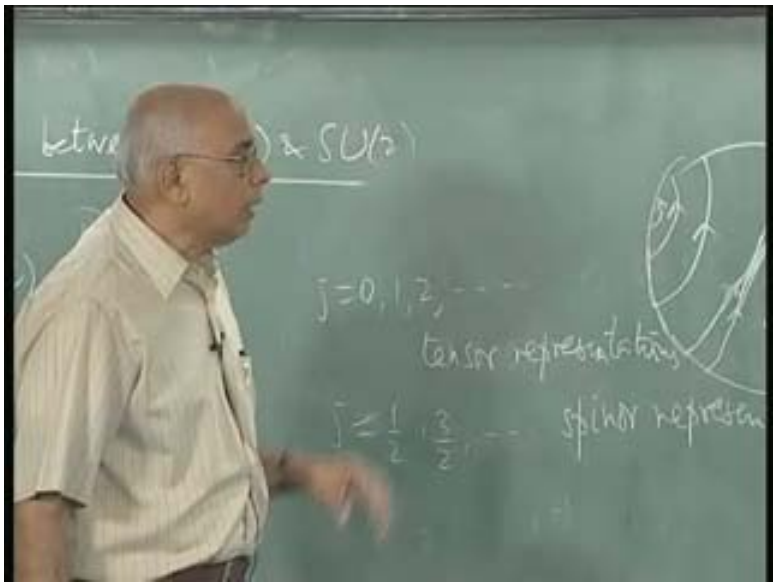

But there is a second class of close paths which corresponds starting here (Refer Slide Time: 57:05), going to the surface but then that point is same as this (Refer Slide Time: 57:11). Therefore this is a close path in that space, although it's hard for you to imagine that this is so. And it is completely distinct from the other class of close paths which can be shrunk to a point because this cannot be shrunk to a point. if you try doing it and you try moving, then this moves perversely. If you want bring this closer and start moving in the other direction, this is not the way to close this path. on the other hand, the trick is you start from the center and you go out, you ended up here and you come back do it again right on top of old path; i am just showing this separately for convenience, and you come here (Refer Slide Time: 58:00 to 58:14). This can be shrunk to a point because this path is entirely equivalent to doing this little trick (Refer Slide Time: 58:25). And of course, we move this thing, and then this one is going to move there. So after some time, i have done this and that is moved here and this thing is moved there. Whether i move this point here, this point moves.

That's very good and so both are gone (Refer Slide Time: 58:30 to 59:00). So it's clear that by doing this close path a second time, i have actually been able to come back to the original. It's equivalent no rotation at all but that implies a rotation of 4pi. Instead of a rotation of 2 which is a complete rotation, i do a rotation of 4 pi and this object comes back to itself. So this means there exist in this parameter space, 2 classes of objects. Those that come back to themselves after a rotation of 2 pi and those that come back to themselves after rotation of 4 pi. This is the origin of the 1/2 integer valued quantum representations of the rotation group. So J = 0,1,2 etc would be called the tensor representations and J = ½, 3 /2, etc is called the spinor representation.

So you know that in the normal tensor representations, a tensor of rank 0 is a scalar and then you have a vector, then you have a tensor of rank 2 and so on. The spinner so to speak, interpolate between these. So you don't have the normal properties that you have for vectors, tensors, etc namely; when you rotate everything by 2 pi, they comeback to themselves. Here, there is a change of sign and the second rotation brings you back to the original value. It turns out these are the only 2 things possible. in fact the way you write this is to specify this that SO 3 is doubly connected. pi_1 (SO 3) is z 2, a set of integers modulo 2. Just 2 elements in the group. Then you could ask is there a way of changing from pi1 to some other group, which is single valued such that there is mapping from that group to this and the answer is yes. And that's very important for quantum mechanics.