Quantum Physics
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Lecture No. # 18

In the previous class, you recall we had discovered that the eigenvalues of the operator J squared are of the form J (J +1) h cross squared, where J = 0, 1/2, 1, 3/2, etc.

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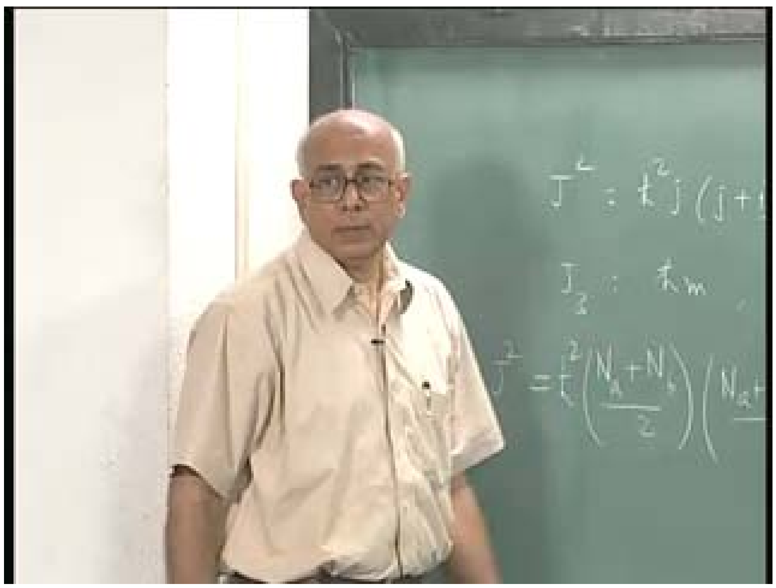

Those were the allowed values for J squared. Similarly, for any component J_3, the eigenvalues were h cross m, where m = - J, - (J -1),… (j -1),j. There are (2j +1) values. This is where we had got to, yesterday. the way we constructed these states was to point out that J squared could also be written in the form h cross squared N_a + N_b over 2 (N_a + N_b / 2 +1), where Na and Nb are the number operators for 2 independent simple harmonic oscillators which can therefore take on the values 0,1, 2, 3, etc. and the relations we had were j = n_a + n_b over 2 where these are the eigenvalues of these number operators and m = the eigenvalue of j_3 apart from h cross, this was n_a - n_b over 2. This immediately implies that n_a is j + m and nb is j – m. It is obvious that you have quantization of the eigenvalues of j squared and of j_3 and that the values of m are bounded by those of j and run in integer steps from - j to + j. this is the origin of the (2j +1) kind of degeneracy which you are familiar with in other contexts. Now the question is what are the states corresponding to these eigenvalues.

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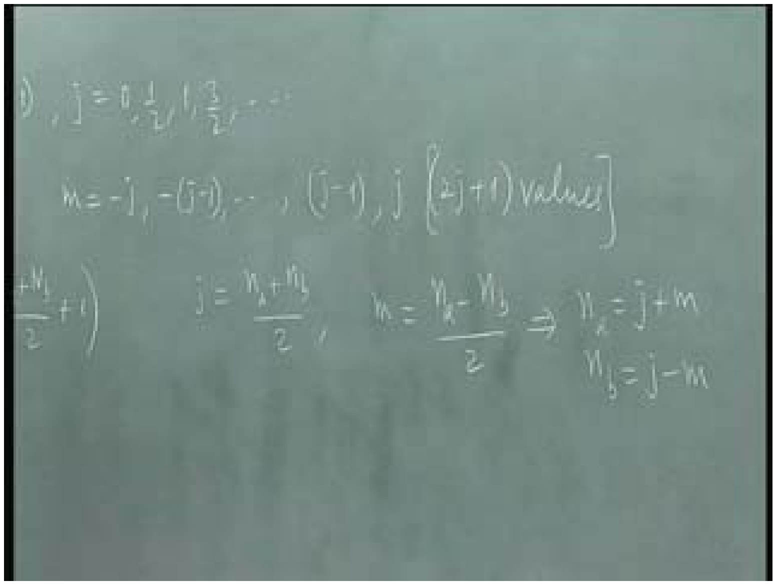

Of course you could construct these states very easily by starting with the harmonic oscillator states. The harmonic oscillator, in the space of the harmonic oscillator number operator states, remember for a single harmonic oscillator, we had abstract ket vectors of this kind (Refer Slide Time: 03:47) such that a dagger a acting on this n gave you n times n. these were the normalizable states in that Hilbert space.

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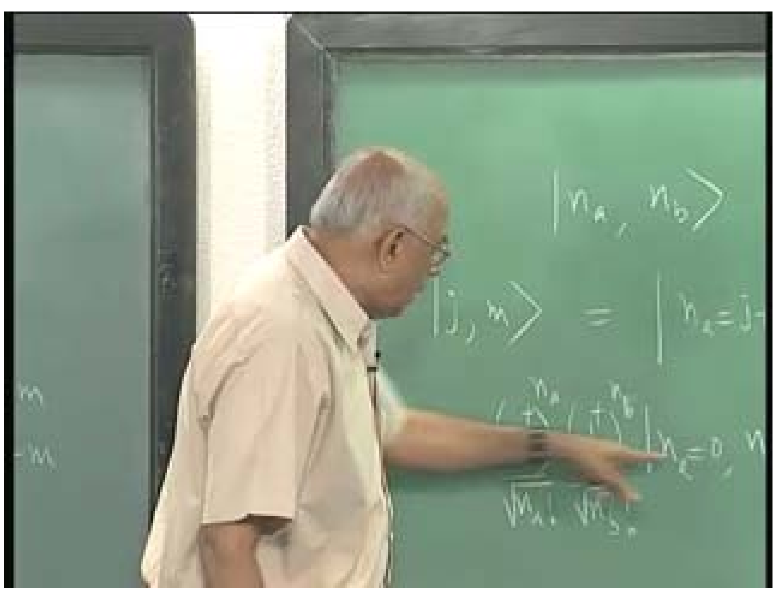

Now we have 2 oscillators and therefore one should really write these state vectors as |na , nb> to show that they are really 2 independent oscillators. And you must have eigenvalues specified for each one of these terms. So i will just simply use this notation |na, nb> to write this abstract state vector corresponding to these number operators states. The state we are interested in are angular momentum states labeled by a particular j and the particular m and the notation used for it is |j, m>. this a is j squared acting on it and this gives you h cross squared j(j +1) and j_3 acting on it gives you m h cross times the same state. This of course same as $n_a = j + m$ and $n_b = j – m$. they are really the same state. It's just that either i could work in terms j and m or i could work in terms of n_a and n_b and in the angular momentum problem, obviously j and m are the relevant variables. I will use the same notations for both of them. Now of course you could start with a state $n_a = 0$, $n_b = 0$; that's the ground state. And if you want to construct a state which has $|n_a, n_b>$, then the way to do this is to take this state and operate on it with b dagger to the power n_b, a dagger to the power n_a and the normalizations are root n_a factorial root n_b factorial. And if you recall, this was in fact the state $|n_a, n_b>$.

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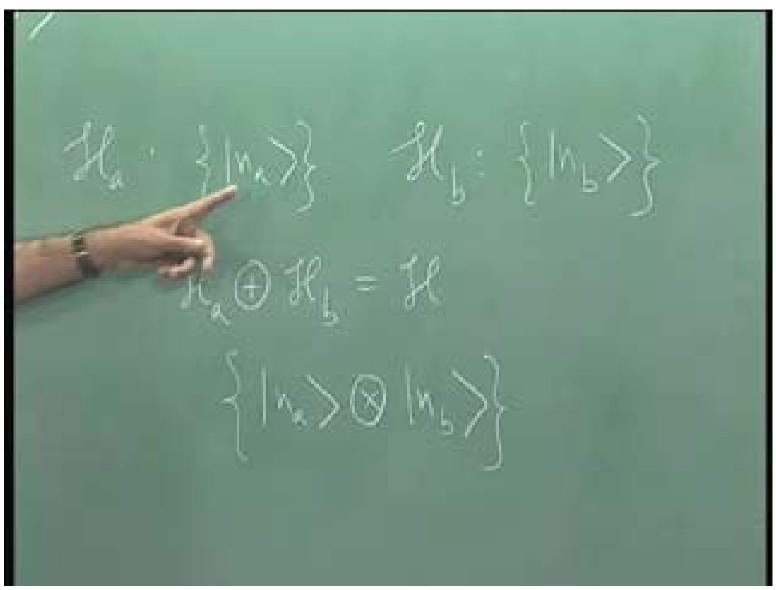

For a single oscillator, i recall to you that if you took this (Refer Slide Time: 06:07) and you did a dagger to the power n and you normalized it, you got |n>. this (Refer Slide Time: 06:22) is actually 2 ket vectors, so the question is what do i mean by this? What i mean is direct product of these 2 Hilbert spaces. So the idea is the following. It's a good question. The idea is there exist for the oscillator a, a Hilbert space H_a which has states in it. And this has states which are labeled by n_a= 0, 1,2, 3, etc. that forms a Hilbert space by itself. For the other oscillator, there's another Hilbert space H_b and this is spanned by n_b. These form a basis in this Hilbert space. So they span the space, they are linearly independent and so on.

Similarly for the other oscillator in its Hilbert space of states, this forms the basis here (Refer Slide Time: 07:22). Operators a, a dagger and so on act on states in n_a and operators b, b dagger etcact on states in n_b. What i have in mind here is the direct product of these 2 states. So i have Ha, this is called the direct sum. It gives H with H_b. this is the Hilbert space H which form which says take one member from n_a and one member from n_b and form a state here and that will span n_a direct product n_b. so just for argument, if these are all infinite dimensional spaces but if this had 10 states possible and that has 15 states possible, the direct product space has got a 150 states possible; one from here and one from there. And then all the a operators act on the terms from H_a and b operators act on the terms from H_b. Now for this direct product state, i have used this (Refer Slide Time: 08:38) short hand notation. So all the a's would act on whatever is sitting here and the b's would act on whatever sitting here (Refer Slide Time: 08:40 to 08:45).

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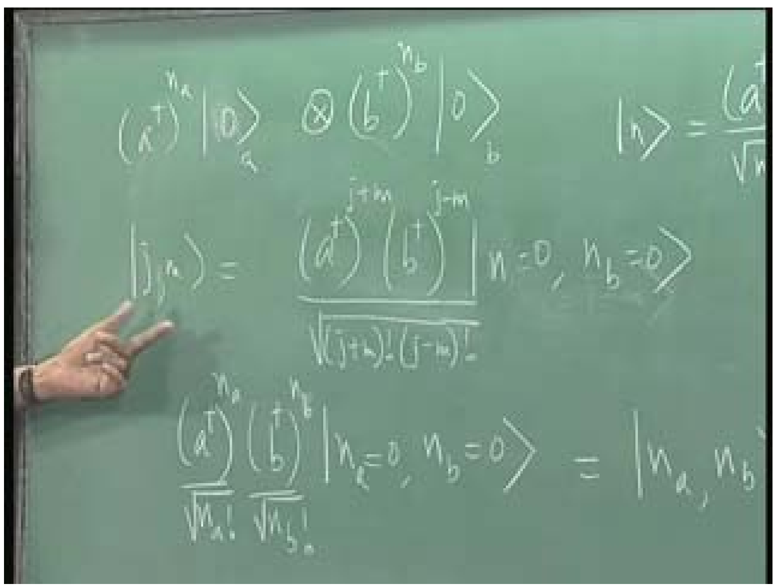

This is the same as saying take a dagger to the power n_a acting on the state 0 in in the space a, let me denote it as in this fashion and take the direct product of this with b dagger to the power n_b acting on the vacuum or the ground state in that space and that state is what i have call $|n_a, n_b>$. And that's this (Refer Slide Time: 09:26) apart from the normalization factors. The word direct product is also sometimes called the tensor product or the Cartesian product. It's a very straight forward concept. These are completely independent of each other. So the total number of states is just the product of these 2 states. Just like the plane is the direct product of r and r. every point on the plane is labeled actually by 2 coordinates (x,y).

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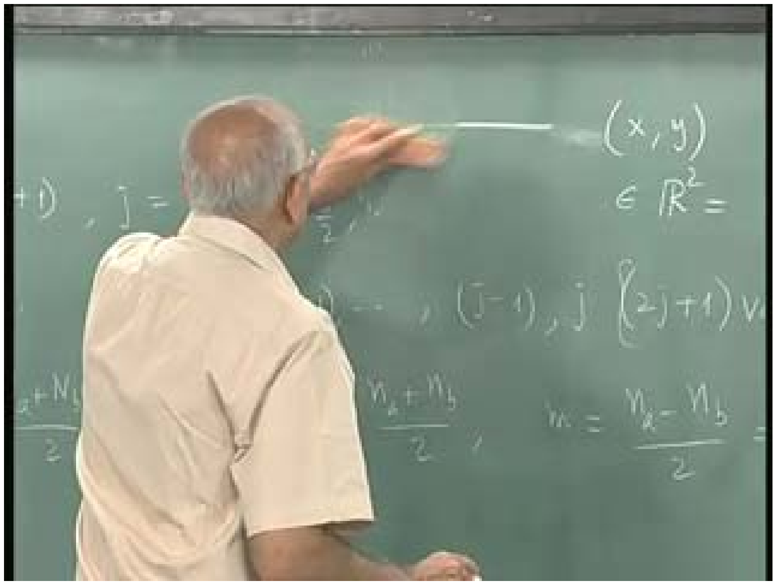

So x is an element of the real, y is the element of the real and these ordered pairs are elements of R 2 but that is just R cross R. it's like saying take any point on it and at every point, you associate another line with it so that you can have coordinate in that direction. Then you go to the next point and then you associate a line with it and so on. You have the plane by doing this. So the plane is just Cartesian product of R cross R. it's in the same sense that i have used this notation here. So i know that if i operate in this fashion on this state, i end up with $|n_a, n_b>$. So the question is how do i find j , m? j, m is found by taking the vacuum state the $|n_a = 0, n_b = 0>$ and you act on it in n_a times but this n_a is j + m. so you really acting a dagger to the power j + m and then b dagger to the j – m over square root of (j + m)! (j – m)!. So if this is a state normalized to unity, you are guaranteed this state is also normalized to unity because you have taken those square roots in the denominator.

So in the space of angular momentum states |j , m>, what's the orthonormality relation? It's clear these states are all mutually orthogonal. Different na's are orthogonal to each other and different nb's are orthogonal to each other. So it is obvious that< j, m| j prime, m prime> must be delta j j prime delta m m prime. That's the orthonomality relation in this space. Now you fix the little j and i say i have a system for which the total angular momentum quantum number is specified. Then what's the dimensionality of the space? It's only m that runs from - j to +j. what's the dimensionality of that space? It's 2 j +1 and therefore you can represent everything by matrices. so in a space in which you have a fixed little j, the angular momentum states are labeled by just the m values and it's a 2 j +1 dimensional space and therefore finite dimensional. Once its finite dimensional, i know how to write the basis and i know how to write the natural basis in this space. I use finite 2 j +1 dimensional matrices to represent spin angular momentum operators. So we will do that very shortly but first let’s ask, what’s the effect of various operators acting on this |j, m> states?

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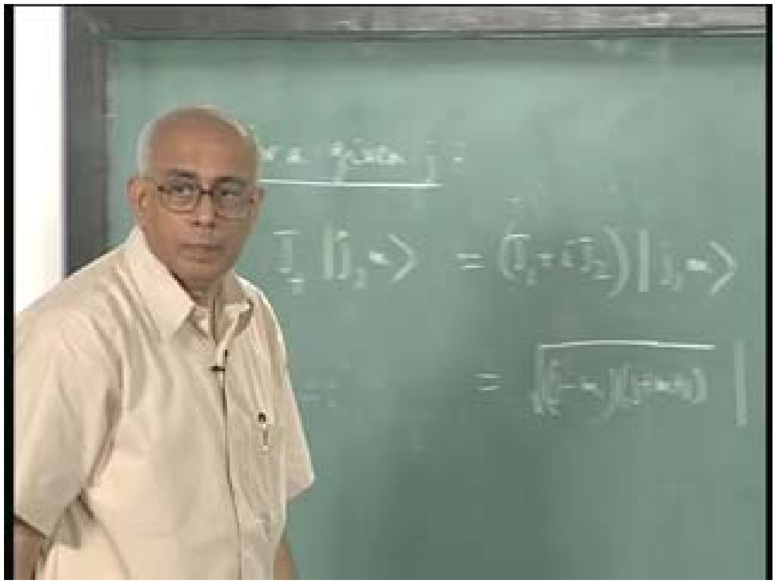

For a given j, it could be 0 then which case everything is trivial but it could be half1 3 1/2and so on. And for a given j what is the effect of taking this state |j , m> acting on it with the following operator j +? What would it do, is the question. Recall that j + = j_1 + i j_2. And now we ask what does it do when it acts on |j, m>? little j is fixed once and for all but little m runs for - j to + j. so you have 2 j +1 of these states and i ask what happened if j_1 + i j_2 act on that state.

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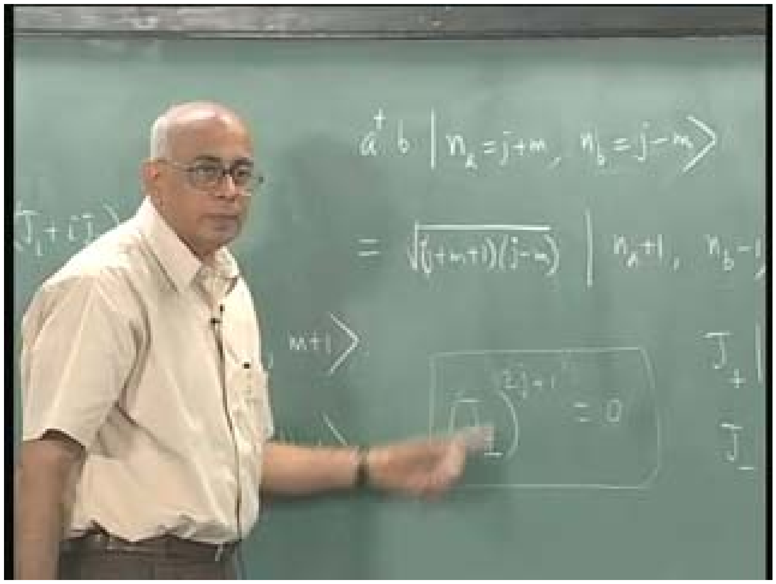

The way to answer is to say that this term here (Refer Slide Time: 14:50) is exactly the same as saying take a dagger b; that's what j_+ is in the harmonic oscillator basis, and act on the state j that's <na + nb over 2, na - nb over 2> and I take this and act on that state and what would be the result? What happens when you take a dagger b? so you are given an n_a and you are given $n_{b,}$ so in the harmonic oscillator basis, this term n_a was j + m and n_b is j – m. in the harmonic oscillator basis these are the values of n_a n_b for given j and m and you are asking what happens if i apply a dagger b on that. now go back and recall that in the oscillator language, a dagger acting on a state n gave you square root of n +1| n +1>. It raised the state and a acting on n lowered it. It was root n |n -1>. So all we have to do is to use that. This gives you square root of (j + m +1) (j – m). That's what n_b is

And the resulting state is n_a +1. And what happens here (Refer Slide Time: 16:57) ? This n_b -1 in the oscillator basis. This is the value of the first oscillator's number operator and this is the value of the second oscillator's number operator. So in the j m basis what does that give you? That becomes square root of (j – m) (j + m +1) times j, m. this is why we have called it j + because it raises for a given little j. it raises the value of j_z or j_3 eigenvalue by unity. So now we are ready to write down what these states look like. For a given j, you have 2 j +1 states.

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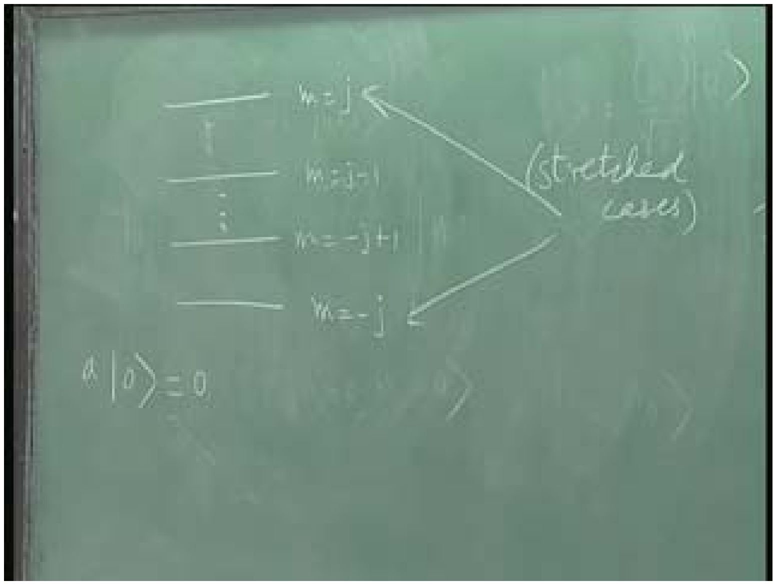

I am just representing them. These are not energy levels. These are just different states that i am representing. So this is m = - j and it goes on from –j to +j. this is the highest level and this is the lower state. And what this j + does when it acts on the state is to move it up here with this normalization factor. And similarly j - on |j, m> gives you what?

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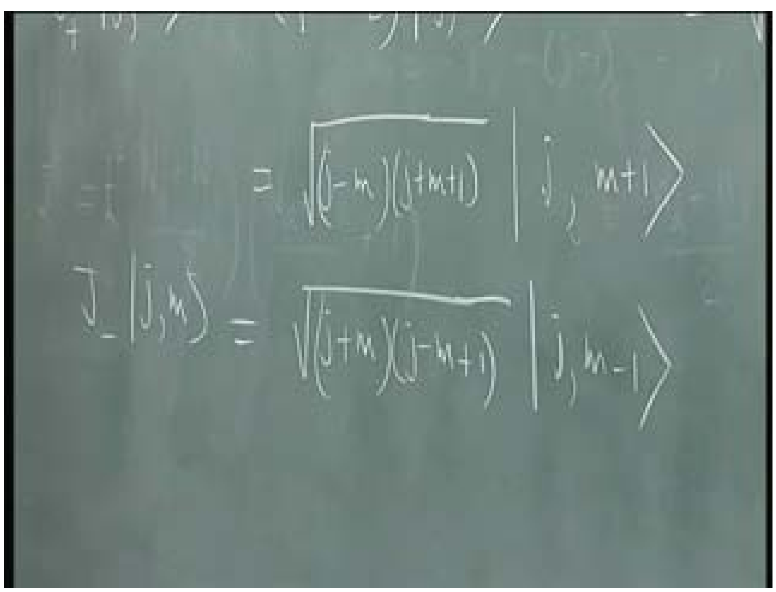

Not surprisingly it would give you (j +m) (j - m +1) multiplied by j, m -1. It would lower by 1. Now just as in the harmonic oscillator case, we had a ground state such that a acting on |0> was 0. You couldn't go below that but you could raise it up all the way to infinity. Here you don't have that freedom. The number of states is bounded from below and from above. What would happen if you took the raising operator and acted on m = j? You should get 0 because it can't go up anymore but this corresponds to m = j. so you must have j + on |j, j> must be = 0 and that is true if i put m = j. it can't raise it up anymore. Similarly if i took j_- and act along the lowest state it can't go any further and if j = - and this (Refer Slide Time: 20:39) is gone to 0 anyway.

So j - on |j, - j> = 0 and these 2 states (Refer Slide Time: 20:54) are called the stretched cases because you can't go beyond that. The lowering operator acting on the lowest state is going to be 0 and the raising operator acting on the highest state is also going to be 0. That's just something interesting about these operators. Therefore in this space, the operator j_+ must be represented by such a matrix that if you raise it to the power 2 j +1, you should get 0. So as an operator statement you must have j_+ to the (2 j +1) must be 0. You can raise it up to power 2 j but you can't go beyond that. And that must be true for - 2 because you start on top in such a case. You know there exists no state in the system on which j + to the 2 j +1 can produce anything other than 0. Therefore it must be represented by the null matrix. So we will keep these in mind that these operators have useful properties. What do you call an operator or a matrix which when raised to a certain power gives you null matrix? It's called nilpotent where beyond a certain power; it's just gone to 0. So these (Refer Slide Time: 22:47) would be nilpotent matrices. Are j + and – Hermitian? No, of course not. They are adjoints of each other. They are Hermitian conjugates of each other; j + dagger = j - and so on. j_1 and j_2 are Hermitian. These terms themselves are but not those operators. So now we have enough machinery.

We can very well we could write down for example, what is this state in terms of this state? So i could regard this lowest state as some kind of ground state and i could go on applying and find out what it is. It is j_+ raised to the power 2 j apart from this (Refer Slide Time: 23:33) normalization factor. So this square root here plays the role of the square root of n and square root of n +1 which we had for the harmonic oscillator.

Now we got to start applying it to some system or the other. The first thing we should do is to see what happens in the case of orbital angular momentum. what's the extra input which says that orbital angular momentum quantum number, the corresponding j which is denoted by l in that case, can only take values 0,1, 2, 3, and not ½, 3/2, etc. that has to do that requirement that the wave function be single valued. On the other hand that's not an absolute requirement. There are many instances where that may not be the case at all and then you have the so called 1/2integer representation of the rotation group. Before i do that, i would like to point out one very interesting fact. And that is, these 3 quantities j_1, j_2, j_3

are really matrices for a fixed little j. they are 2 j +1 dimensional matrices but yet i call them j_1, j_2, j_3 and act as if they are vectors. So this is a bit of a strange thing that i have these 3 matrices but i am calling the collection of 3 matrices the components of a vector. How is that justified? Why is that so? i started off by looking at l_x l_y l_z and they are generalized to the commutation relations. And now we are ending up with these terms who act as matrices but i want to assert that j_1 j_2 j_3 transform like the components of a vector under physical rotations of the coordinate system. Let's see this is true or not.

So remember that in a vector space, you have the concept of multiplication by a scalar and there exists a null vector which when added to anything gives you nothing else. it gives you the same vector as before. So because of that property, 0 times any vector gives you the null vector. I use the same 0 on both sides for the null vector but this is not the null vector in the space. And this is some vector which has got some finite norm. So in exactly the same sense, j_+ acting on this (Refer Slide Time: 27:23) gives you actually. There is nothing else in the state. It annihilates this state. And similarly j_- annihilates the ground state. So now why did i call these vectors? The answer lies in the following. If i make a physical rotation of a coordinate system, i should check if j_1, j_2, j_3 really transform like the components of a vector or not. In other words, we should have the following. Pretend for a minute that you are looking at an ordinary vector of some kind. It has got components v_1, v_2, v_3 or something like that.

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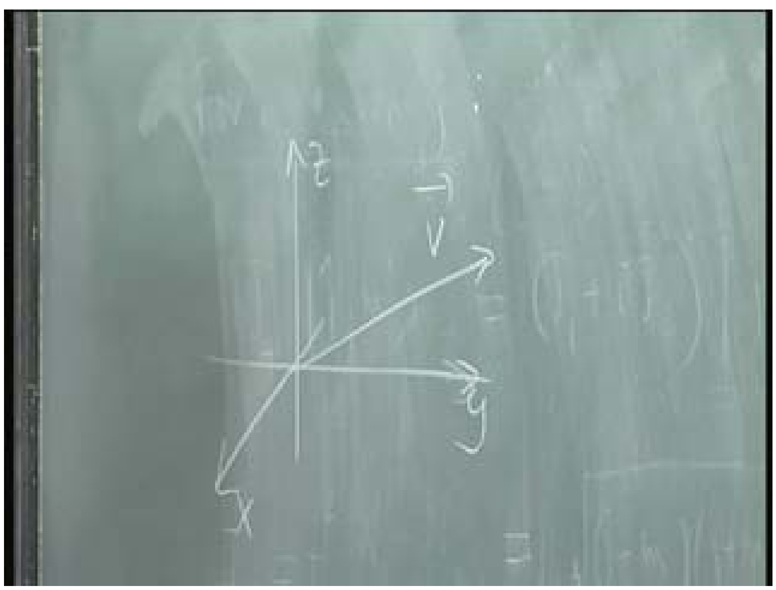

Then in a coordinate system in which i choose the x axis here, the y axis there and the z axis there (Refer Slide Time: 28:10). This is a vector v with components v1 v 2 v 3.

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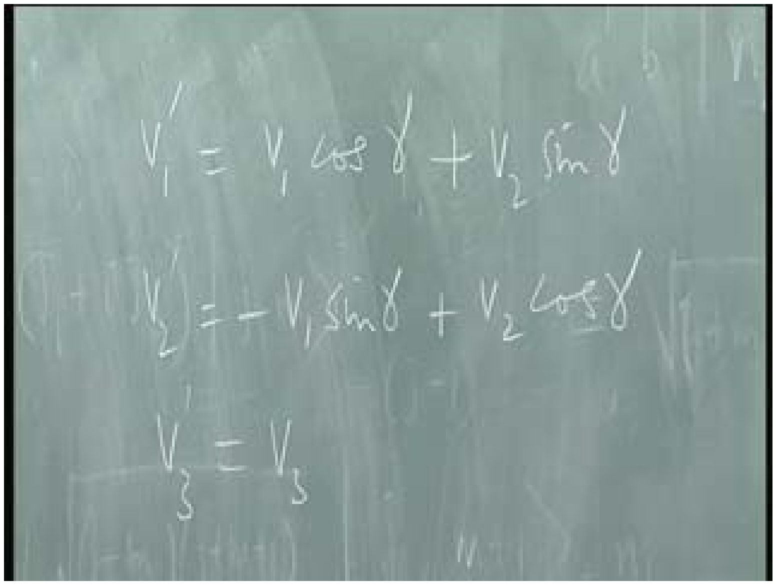

Then if i make a rotation of the coordinate axis, say about gamma in the x y plane for example, then is it not true that v_1 prime will be v_1 cos gamma + v_2 sin gamma, v 2 prime = - V1 sin gamma + v 2 cos gamma and v 3 prime = v 3 because anything along the z axis is not effected by rotation in the x y plane. So this is my definition of a vector.

A quantity v made up of an order tripped v_1, v_2, v_3 is a vector under rotations if and only if, under a rotation, each of these new components is a homogeneous linear combination of the old component with determinant +1. That's my definition of a vector. So if this is true for j1 j 2 and j 3, then it is clear that under the rotation of the coordinate axis, j1 j 2 j 3 must go to j1 prime j 2 prime j 3 prime which act exactly in the same way and what manner of this is irrelevant to me. If these are matrix valued, so be it. But it would still transform like a vector under rotations. They could be operators. I still don't care after all you are familiar with a such a thing.

The gradient operator; that's a differential operator but the fact is it is got 3 differential operators. They form a combination which transforms like a vector under rotations. So the fact that these are not numbers or functions is irrelevant completely. So now we have to test if this is really true for the j's. The j is themselves generate rotations but they themselves transform like a vector under rotations. Now let see how the magic comes about. So consider for example consider what happens to an operator under a transformation. A state vector would just undergo a transformation U for example, but an operator would undergo that transmission U operator U inverse so that expectation values are unchanged.

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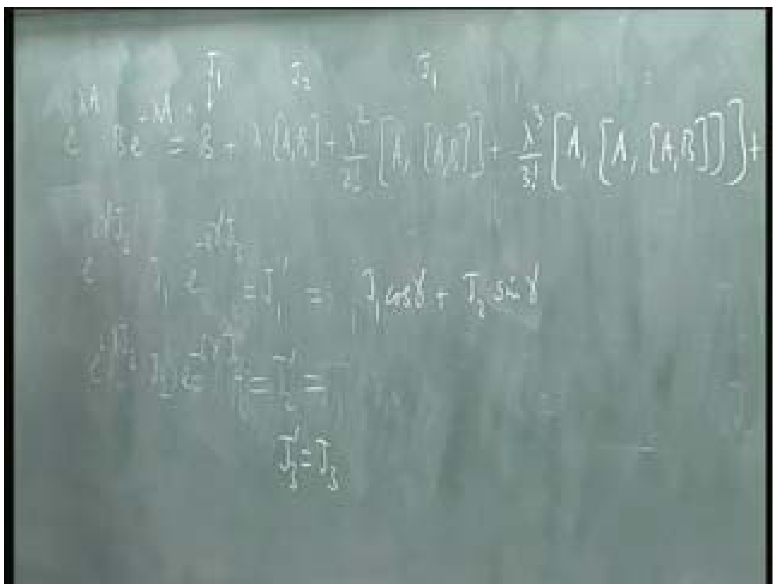

So let's consider this e to the power lambda A B to the power e to the - lambda A. so now in the earlier problem sets, i had given this identity and asked you to establish what this identity is and what this expansion is. if A and B don't commute, then this in general is B + lambda times [A, B] + lambda squared over 2 factorial, lambda cubed over 3 factorial [A,[A,[A,B]]]+… So let's ask what happens if i rotate about the z axis through an angle gamma. Then that's equivalent to saying what's e to the i gamma j_3 acting on say, j_1 e to the - i gamma j3? This should be my j1 prime.

And the question is what does this reduce to. Similarly e to the i gamma j 3 j 2 e to the - i gamma j 3 = j 2 prime and we ask what it is on the right hand side. And finally with j 3. but j 3 prime is obviously = j 3 because it commutes with itself and therefore this goes right across and gives you j 3 prime as j 3. One has to therefore find out what these 2 quantities are. Now j 3 and j1 don't commute with each other. And that's what happens here. A and B don't commute with other. Similarly j 3 and j 2. One has to therefore apply this formula in which the operator A is j 3 and the operator B is j1. The commutator of j 3 j1 is j 2. So what's going happened when you apply this formula, not surprisingly it's going to be this term which is j1 A with B and that is going to be j 3 with j1 or that proportional to j 2 apart from some - i h cross and so on. This is j 2 and that's commuted with A which is j 3 and that gives you j1 once again. so this is again j1 and this is going to give j2 once again and so on. So it's immediately clear that this quantity j1 prime is a liner combination of j1 and j 2 (Refer Slide Time: 33:53 to 34:23). And that's what we expect for a rotation. And in fact with the suitable i h crosses and these factorial and so on it would be precisely sines and cosines on both sides. So verify that this (Refer Slide Time: 34:45) is indeed j1 cos gamma + j 2 sin gamma.

Student - Why does this (Refer Slide Time: 35:00) combination represent a rotation? Professor - If i have a transformation under which a system is invariant, then physical quantities don't change under this transformation.

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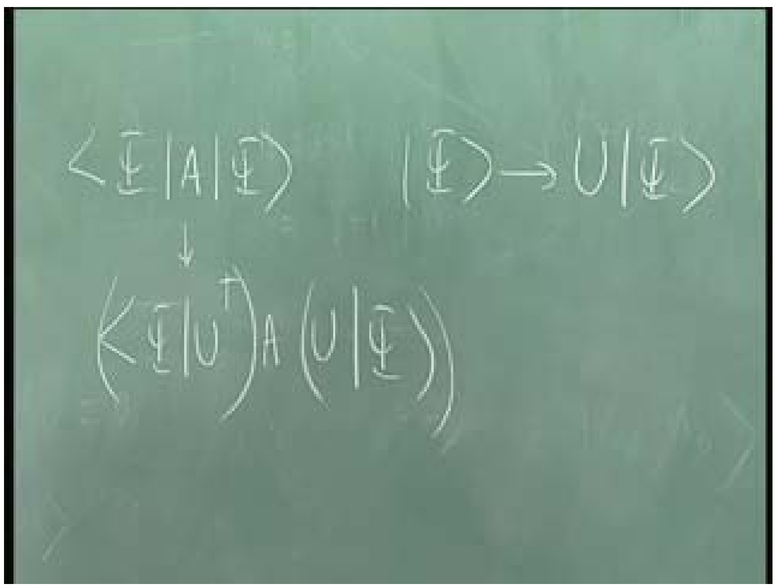

It implies that something like psi A psi must remain invariant under a transformation. Suppose the transformation acts on the states themselves, so psi goes to U psi but U is an unitary operator. then this (Refer Slide Time: 35:39) term, if i take the passive view point and say the states change or the distribution changes, then this would be = psi U dagger A U psi. But i could also take the active view point and say that it's the operator that changes and the state don't change. In which case, A goes to A prime where A prime is precisely this (Refer Slide Time: 36:11), but since U is unitary, its U inverse A U and that's exactly what i used here. So i have tacitly taken the active view point and said that the observable changes. And they would change in this root. So this is the way rotations are represented and this thing here is in fact proved now that these j's transform like the components of a vector. They themselves generate transformations. They generate rotations but what it acts on depends on the object you are looking at always. How do i represent a rotation about some axis in 3 dimensional space? If i want act on points in 3 dimensional space, what would i do? Now it depends on how you represent these points? If you choose to represent them by a column vector with the x_1 x_2 x_3 components, then the rotation is represented by a 3 by 3 orthogonal matrix with determinant +1. on the another hand, i choose to represent it by some other means like a vector with an arrow or something like that, then a rotation is represented by something else altogether.

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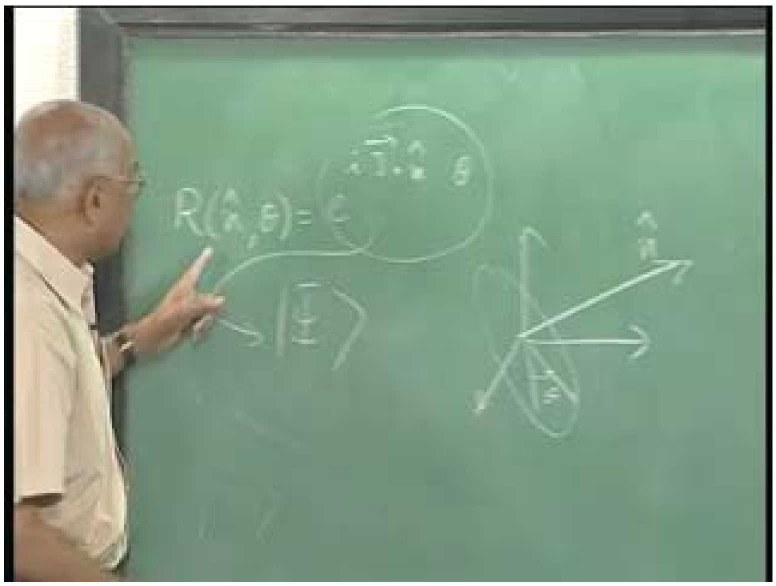

The actual rotation is e to power I, the rotation generator j and then n j dot n theta. This is my general representation of a rotation of the coordinates system. J generates rotations and if i am 3 dimensional space and it's the coordinates system in 3 dimensional space that's rotated, this set of 3 operators, the angular momentum generators they transform themselves like the component of a vector
And that is dotted with the direction about which i make the rotation and theta is the amount by which i rotate. So the representation of the abstract operator of rotation about an axis n, through an angle theta is represented by this unitary operator (Refer Slide Time: 38:55). J is Hermitian and theta is a real number. Therefore this is a unitary operator (Refer Slide Time: 39:03). How you represent this is dependent on what you act on. if i act on physical points in space and take R to R prime, then the representation i need for that j is 3 by 3 matrices such that you end up with the orthogonal matrix. After you exponentiate this i, j dot whatever it is, you end up with an orthogonal matrix and that will act on your physical points. On the other hand, a rotation of the coordinates system induces a change in the states system. Suppose i am describing a quantum mechanical particle and its state is given by ket vector psi, and now i suddenly change the coordinate system, that would show up in the action from this psi gets taken to some other vector in general. This depends on the dimensionality of psi and how i write the representation of whatever acts on it. What i can say is that this operator

would acts on it (Refer Slide Time: 40:20). And i need a unitary representation of this rotation of the same dimensionality as the dimensionality of the Hilbert space so that it acts on it and gives you another vector possibly. So, depending on what abstract transformation acts on, the representation of the transformation differs but it's the same physical transformation always. We will see that in a 3 dimensional space, there's is another interesting way of writing ordinary vectors in 3 dimensional space. It need not be written as column vector x y z. you could write it in some other way. And we are going to do that very shortly. And then you have a very deep connection between rotations and the unitary group as U 2. And that's important to understand because we want to understand what are fermions, bosons and so on. So any physical transformation of the coordinate system, say you go from a right handed to left handed or you rotate it or you translate or something like that would be represented by an abstract operator of some kind. the actual representation of this unitary operator would depend on the object you are going to act on.

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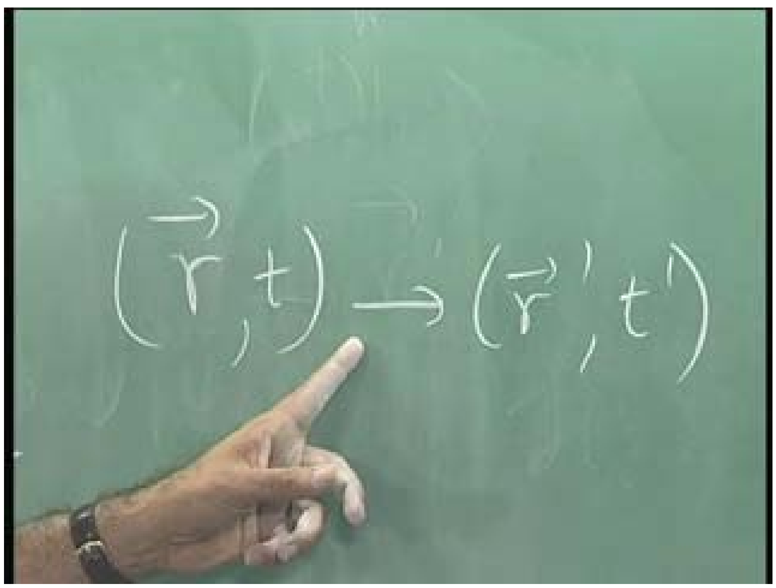

But i want you to appreciate this point right away that if you start with in some frame with r and t and it goes to some r prime t prime, this transformation induces transformations in the states of a quantum mechanical system in general. And the way these states transform depends on the way this transformation is represented in the Hilbert space of states. So this is why representation become completely crucial.

So now let me do the following. We are going to do several representations of the rotation group. One of them for the orbital angular momentum acting on physical wave functions as a function of (r, theta, phi) for example. We will have to see what happens there. The other thing we are going to do is to look at a given value of little j and ask what do rotations look like in this space, how do they transform, how do they act and so on. Let me do the second problem first because it's a little easier. it doesn't involve differential operators. So let's look at the simplest case and ask what do the angular momentum operators look like. Because, once we have an explicit representation, then things will become much easier to visualize. So let's look at the case j = ½.

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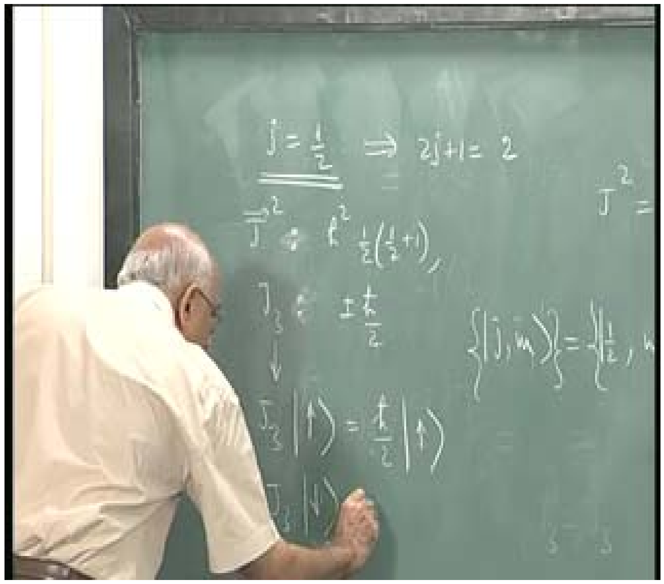

It's a very important crucial case. The most trivial value of j is 0. of course there is nothing to represent. Everything is one dimensional representation because 2 j +1 is just 1 in that case and you get nothing. Under rotations, scalars becomes scalars nothing happens but let's look at j = ½. so we are looking at a system whose total angular momentum squared has the only possible value h cross squared 1/2times 1/2+1. This is the famous spin ½ problem. It turns out that particles like the electron, proton and the neutron for instance have a property called intrinsic angular momentum.

Even in the rest frame of the particle, there is an angular momentum. It's clear this is not classical because all the anglular momentum you are used to thinking about is about some axis or a point and there is a physical rotation. But that's not true because for point particle like the electron, you still have associated an angular momentum and intrinsic angular momentum even in a frame in which the linear momentum of the electrons is 0. Therefore this is beyond classical concepts but it's like the charge or the mass of the electron. This property is an intrinsic property. There is an angular momentum and experiments have shown us that the value of this angular momentum quantum number corresponds to little j = ½.

So we will take for that granted and see what its consequences are. So everything should be represented by 2 by 2 matrices. this implies that J squared, in this case is h cross squared 1/2into 1/2+1 and j3 has eigenvalues +/- h cross over 2 because it must run from - j to + j in steps of 1.

No, unfortunately it doesn't because first of all, to best of our knowledge this electron doesn't have a structure. Things like the proton and the neutron do have a structure. They are made up of quarks inside. However the origin of the spin is in relativistic quantum field theory. It's not a non-relativistic concept and it's not a classical concept at all. Every elementary particle turns out to have an intrinsic property called spin just like it has a rest mass. These are properties dictated by the requirement of Lorentz invariance; namely the invariance of the laws of physic under Lorentz transformations.

No mechanical model is possible of these concepts at all. So it has nothing to do with the spin of a physical rotation about some axis of an extended object. Now a photon also has a spin but since the photon has 0 rest mass, the idea of spin, the origin and the meaning of spin for 0 rest mass is slightly different from what it is for particles with non-zero rest mass. I will talk about that also. But it yes it does have an intrinsic angular momentum. The spin quantum number for a photon is denoted by s without a vector instead of little j. that happens to be1 for a photon. And for an electron, proton and a neutron, it happens to be ½.

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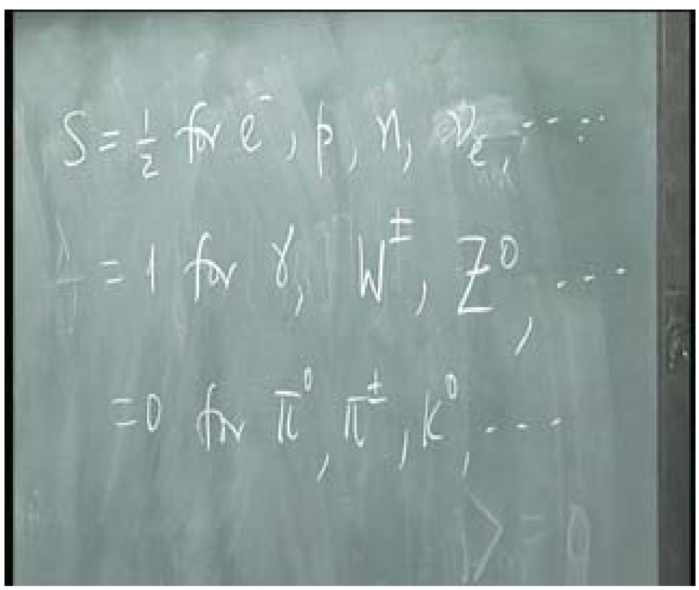

So let me even write that down. S =1 for the electron, photon, gamma etc. there are other massive particles which have got spin 1.There are many of them in the elementary particle zoo. There are particles called W $^{+}$ or $^{-}$, there is a particle called Z^0 and so on. all these particles have got spin 1. Particles with spin 0 are pi ^{0}, pi $^{+}$,pi $^{-}$, k ^{0}, k $^{+}$, k^- and so on. These elementary particles have all got spin 0.

Actually we know now that these are not elementary particles. In the sense, they made up of quarks themselves. But the electron is an elementary particle. So is the neutrino, the photon the W +/ -, the z ^{0}. All the particles with non-zero rest mass are broken up into quarks and leptons. And then there are other particles which generally have spin 1 like the photon, the W, Z gluan's and so on. So this is the so called standard model of particle physics. I will come back and talk about this a afterwards. But right now i want to emphasize that this spin of an elementary particle is no way associated with any mechanical motion.

In fact for particles of non-zero rest mass and when they have non-zero spin quantum number, they have an intrinsic angular momentum even in the rest frame when p is 0. So it cannot be of the form r cross p but this operator transforms like an angular momentum and it's called the intrinsic angular momentum. So now the question is how do we represent things in this space? now it is clear that the j squared operator itself must commute with j_1, j_2, j_3 in this space because j squared commutes with j_1, j_2, j_3 for any fixed value of j. so what can it be?

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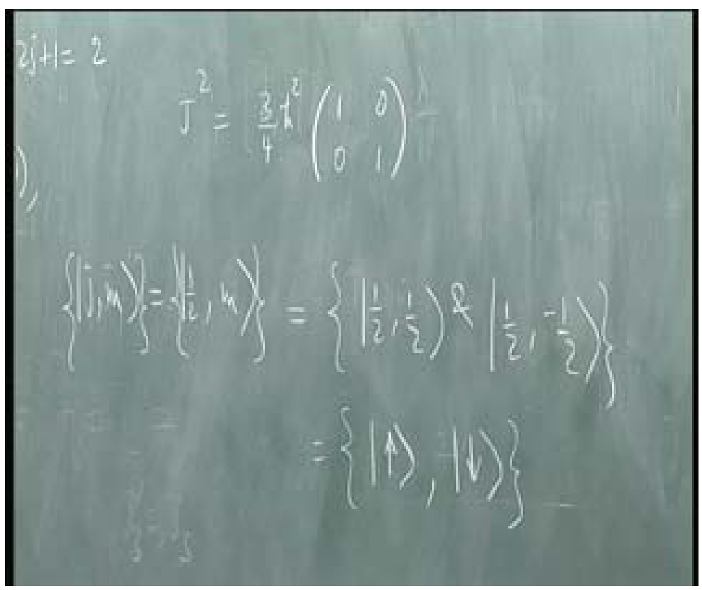

It turns out that j squared is = this is the 3 quarters h cross squared and it must be the unit matrix because that's the only thing that commutes with everything else. The space is 2 by 2 matrices. All operators are represented by 2 by 2 matrices in this space. And the all states in this space are linear combinations of the j, m states. So the set of these (Refer Slide Time: 52:25) j, m states is a set of these possibilities and this is just = {|½, 1/2 > and |1/2, - ½>}. That's all that's possible. j 3 is either + 1/2h cross or - 1/2h cross. Since little j is also fixed1 uses a little short hand for this. I also write this as spin up and spin down. Very picturesque but it doesn't have anything to do with things spinning. so in case you have heard that the electron has spin + ½ and it's spinning counter clock wise and - ½ and it's spinning clock wise, please put that out of your mind because you only have to look at it from below in order to revert whatever you said earlier. So that's total nonsense.

For convenience, these states i call up and down in this fashion. this comes from the fact that1 is used to electrons which have this property and when you apply a magnetic field, then the magnetic movement points either along the directions of the magnetic field are opposite to it. It has 2 possible eigenvalues which you would like to call + and - or up and down. So I will frequently use this notation. This is for m = + 1/2h cross, this is a m = + 1/2and this is for m = - ½. How do you represent these 2 states? So we would like to have j_3 acting on up must be = h cross by 2 on the same state. It's an eigenstate, after all that's the meaning of this statement and J 3 acting on down is - h cross over 2.

And these 2 states must be orthogonal to each other. This scalar product must be 0 because j m bra ket j prime m prime is delta j prime delta m prime. Now we are working in the subspace in which little j is ½. So it's only m that you are worried about therefore up with down must be 0. So my question is how are you going to guess what ji, j2 and j3 are? Well the answer is very obvious. All we have to do is to write a representation for these. This operator is already diagonal (Refer Slide Time: 55:40). It has 2 eigenstate. i want 2 linearly independent eigenstates in a 2 by 2 space. What do you call them in the x y plane? i call them unit vector long x direction unit vector long the y direction. Or if i want it as column vectors i would write 1 0 or 0 1.

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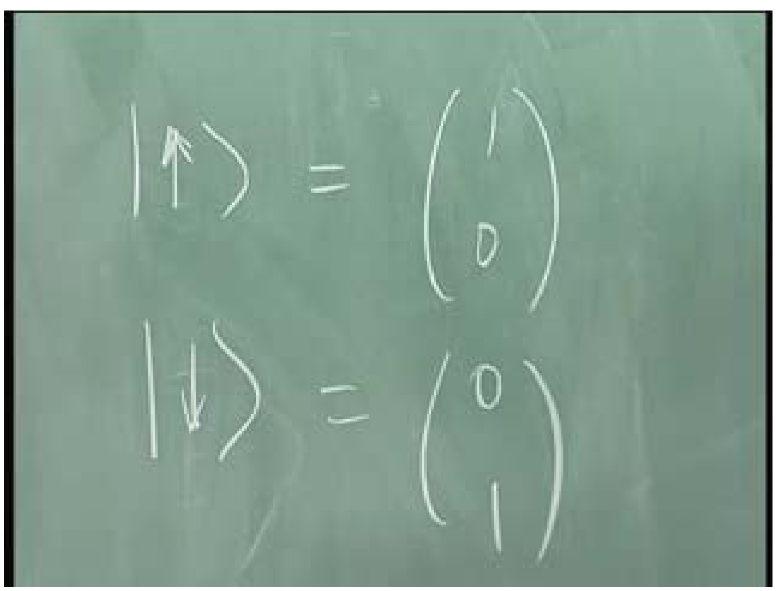

So the up state is represented by 1 0. The left hand side is in some abstract Hilbert space the right hand side is in the actual representation as 2 by 2 matrices. The states are represented by column vector which has 2 elements. And similarly this (refer Slide Time: 56:37) is the natural basis in this space. So what's j 3 in this space? It’s h cross over 2. That’s for sure. those are the eigenvalues and when it acts on the up; 1 0 it must give you + 1/2h cross and when acts on the other1 it must give you - 1/2h cross. So what does j 3 become? It’s it going to be1 0 0 -1. These are the 2 eigenvalues. And it’s clear that j 3 commutes with j squared because of the unit operator. by making it a unit operator, we have ensure that everything else all the other operator which are j1 j 2 j 3 and the linear combinations and so on function of a they all commute with j squared.

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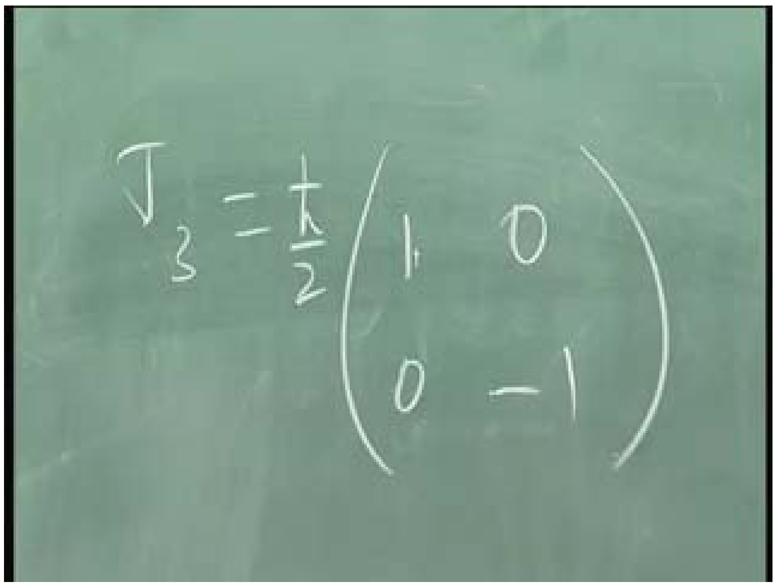

There is a theorem in mathematics called Schur's Lemma says that if you fix little j and work in that 2 j +1 dimensional subspace, then any operator which commutes with all the operators in that space must be some multiple of the unit matrix. It can’t be anything else. By the way the j1 j 2 j 3 angular momentum operators form what’s call a Lie algebra. In this case the Lie algebra of the rotation group. And the operator j squared which was a function of j1 j 2 j 3 commuted with all the generators. Any operator in a Lie algebra which commutes with all the generators of that Lie algebra is called a Casimir operator. So this operator J squared is a Casimir operator for the rotation group for this algebra. The number of Casimir operators is called the rank of the algebra. The number of Casimir operators for the angular momentum lie algebra is just 1. So you are guaranteed that there is nothing else.

In this case there isn't a complicated function of j1 j 2 j 3 which commutes with the rest. Can J1 and J 2 also be diagonal matrices if j 3 is given? 2 diagonal matrices will commute with each other. So it's clear immediately that J1 and J 2 are not diagonal matrices in this representation only J 3 is there. Of course, we could have changed the representation so that any particular component of j is diagonal and everything else is not. We have chosen j 3 already and that's it. This matrix called it's a third Pauli matrix. Since Pauli matrices have fantastic properties, we will be able to do this whole thing completely.