Quantum Physics

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Lecture No. #17

So let's continue with our study of charged particle in a magnetic field. There are a few comments i want to make about the system and then we will come back to this a little later after we talk about spin. So we will talk about realistic experiments which involve quantum mechanical particles in an external electromagnetic field.

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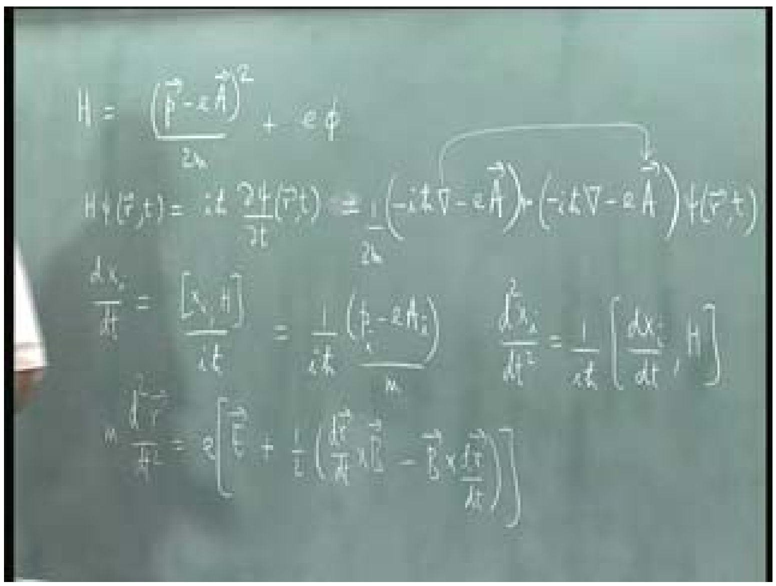

If you recall, there are just a couple of points I wanted to complete. if you recall, the Hamiltonian of a charged particle in an external magnetic field was written in the form p - eA whole squared over 2 m + there could be a scalar potential as well as a function of r and t. so that's the general Hamiltonian for a charged particle of charge e in an external electromagnetic field induced by the scalar potential phi of r and t and the vector potential A (r, t). if you wrote the Schrodinger equation down for this in the position representation, then the equation H psi (r, t) = ih cross delta psi (r, t)over delta t becomes in this case, when you have a scalar and vector potential present, remember that p is essentially represented by - ih cross the gradient. So this equation becomes = - ih cross del - eA dotted with - ih cross del - eA over 2 m on psi (r, t) + e phi(r, t) psi (r, t).

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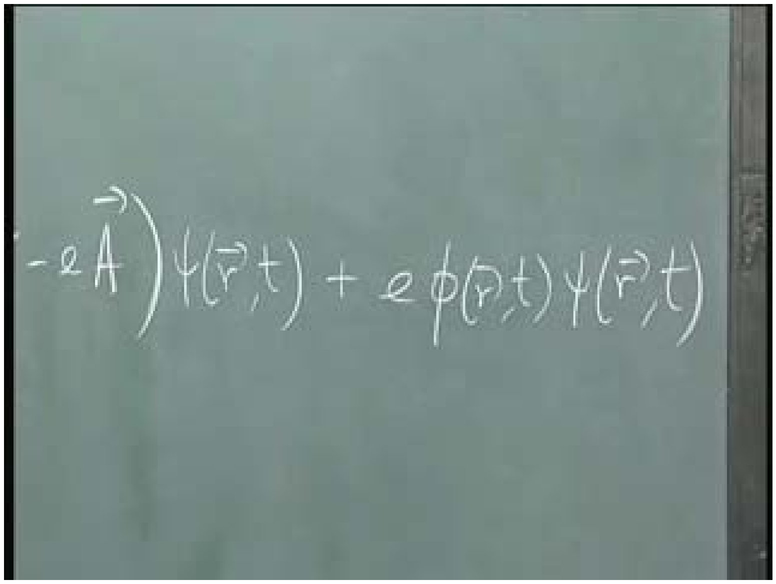

That's the general Schrodinger equation. And we looked at the case when there was no scalar potential, when you had a vector potential which was time independent and moreover, represented a constant uniform magnetic field. so that case was reduced to the harmonic oscillator case and we saw that you had Landau levels as the energy levels but this is the general time dependent Schrodinger equation and the point to note is that this del operator in general would also act on that (Refer Slide Time: 03:40) A because it sits to the left it and they would be terms which come about because this A changes form point to point. This is automatically implied always. So you have to be careful when you write a thing like this (Refer Slide Time: 03:50).

This stands for the dot product of this with itself and so on. So that's the time dependent Schrodinger equation and i am not about to solve it because it's a mess and you need to know exactly what these potentials are. in general, this is a hard problem to solve but a much easier one would be to ask if we can work in the Heisenberg picture. Could we simply write down the equations of motion for a physical quantity such as the position of the particle? So the simplest thing one would write down is what happens to i th component, dx_i over dt, where x_i is the i th component of the position of the particle. This quantity, if you recall this operator in the Heisenberg picture is 1 over ih cross [xi, H] in the Heisenberg picture and this commutator is easy to find because it's the commutator of x_i with p_j - eA_j whole squared.

And then of course, since this is position and that's position dependent, this term commutes with that and it doesn't contribute to anything at all. so it's easy to see that this is 1over ih cross times twice this (Refer Slide Time: 05:14) term. so its pi – eAi. That's the derivative of that quantity divided by m because the 2 cancels out when you differentiate. So it says the velocity of the i th component of the particle if you like is related this way.

But now you could ask what the acceleration is? Can we derive the Lorentz force equation and i leave that to you because now you have to write d 2 x_i over dt 2 = 1over ih cross [dxi over dt, H]. and you got to be a little careful because it's this quantity you have to now commute with, find the commutator with the Hamiltonian and in general this (Refer Slide Time: 06:03) is position dependent, there is a momentum sitting here (Refer Slide Time: 06:05) and therefore there would be contributions. And surprisingly it would involve derivatives of i and so on. I am going to leave this as an exercise. Finally the outcome should be something like the Lorentz force because that's what the mass times acceleration is. you should get an equation which says d 2 r over dt 2 = m times E; i am not going to write this out but it's the electric field which is related to the vector potential and phi; + the normal term would have been V cross B but remember this vector V or mV is p - eA and it doesn't commute with a function of the position like B. so one has to worry about these commutators and the outcome is something which looks like (dr by dt cross B - B cross dr over dt).

These 2 quantities (Refer Slide Time: 07:20) don't commute with each other. This is a function of r in general. What you need is this antisymmetrized product because this is Hermitian. Remember the cross product and the interchange of these 2 quantities. They are all operators. When you take the adjoint, this (Refer Slide Time: 07:36 to 07:46) term would come here and this term would go there although they are Hermitian. So to take care of that, you need this extra term here. So this is the Lorentz force. It's exactly what you have classically. Classically, of course you are commuting variables so V cross B is - B cross V and therefore these 2 add up and simply give you V cross B. but this is the correct generalization.

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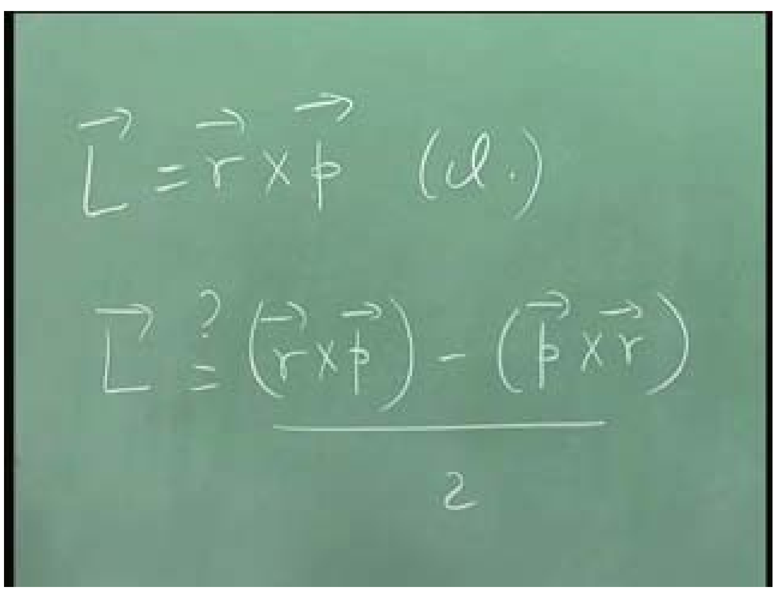

In exactly the same way as when you define the angular momentum, when I write the angular momentum in quantum mechanics, I write L = r cross p. I am going to talk about properties of the angular momentum. Now you could ask r and p don't commute with each other, so how come I wrote it as r cross p? Why shouldn’t i write it as - p cross r and r and p are different from each other? So the normal guess would be to say perhaps, this is the wrong definition and this is classical. Quantum mechanical would perhaps be (r cross p) – (p cross r) and then a factor 2 to take into account, the fact that this is equal to - of that classically. This is guaranteed to be Hermitian. So one could ask why don’t i do this, but we don’t need to. Why is that?

We don’t need to do that because different components of r and p commute with each other and in r cross p, you never have a product of the same component unlike r dot p where you would have xp x + y py + z pz. So this is not needed. This suffices and its Hermitian by itself.

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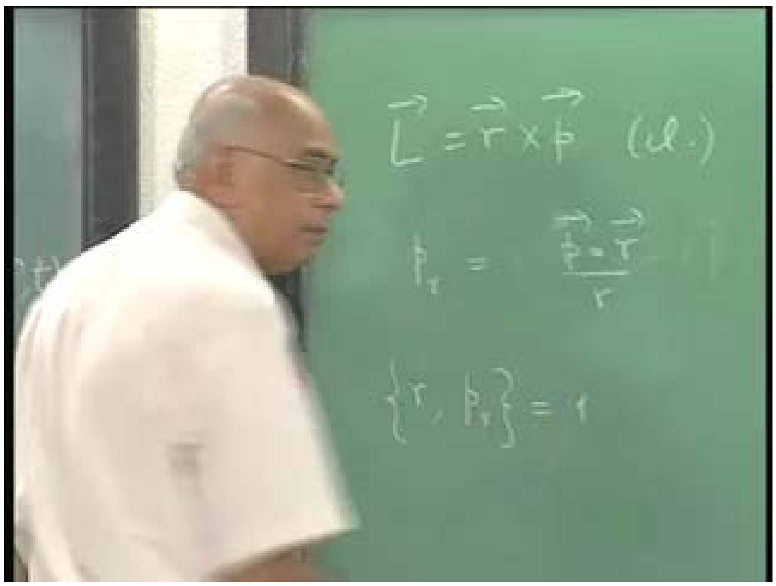

On the other hand, if you look at the radial momentum p_r, this is equal to the normal way of defining the radial momentum classically. It is to say it’s the component of the momentum along unit vector in the radial direction. This is the normal definition of the radial momentum and the angular momentum is the other component of the total momentum. Now the question, is this valid quantum mechanically or not? No, it’s not because we have no guarantee why should it be p dot r. why not r dot p or why not any combination of p dot r and r dot p?

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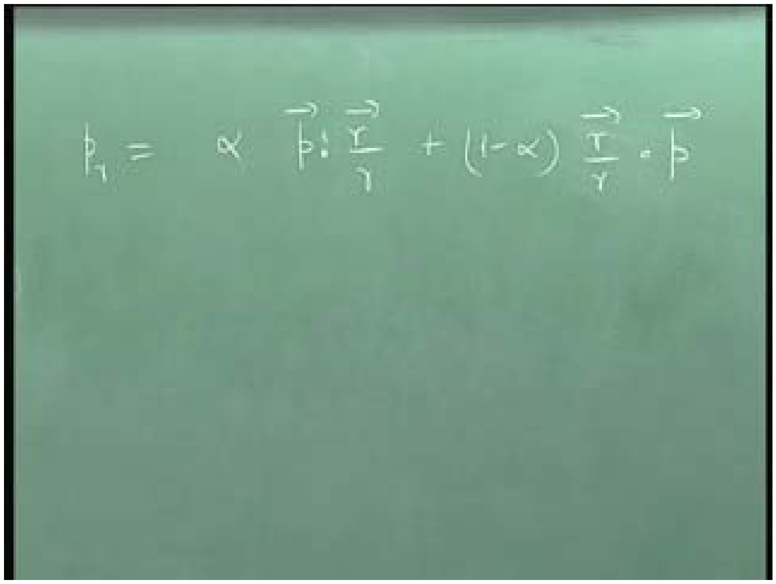

Why not p_r equal to some alpha times p dot r over r + 1- alpha times r over r dot p, where alpha is a number between 0 and 1? What should alpha be? What would be the criterion that you use for this?

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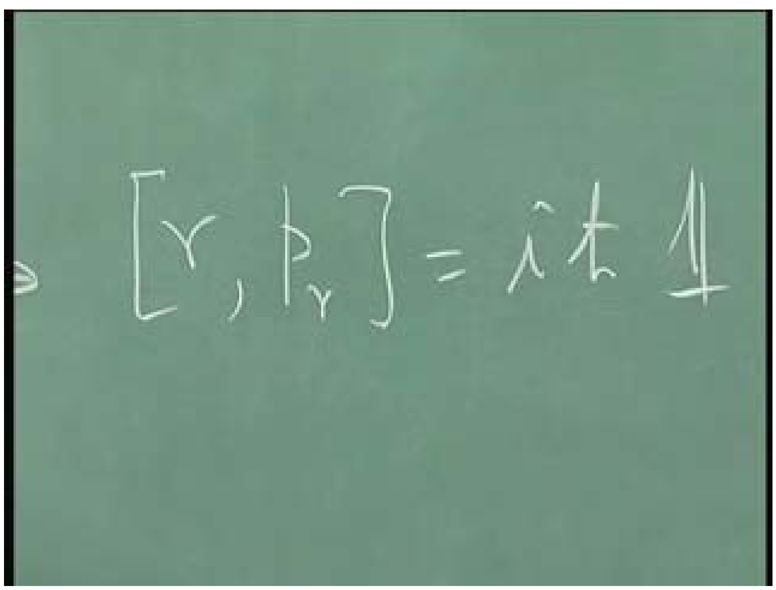

Well, the first thing you need is this. We know that classically, the Poisson bracket of r with pr must be equal to 1 because a generalized coordinate with its conjugate momentum, the Poisson bracket by definition is 1. So this must go over quantum mechanically to $[r, p_r]$ = ih cross times the unit operator. That's the rule to go from classical to quantum physics.

So you need that you need this commutation relation to be valid. So whatever that is we need to check that this is really true. What else do you have to impose on p_r? It's a physical observable. Therefore it must be a Hermitian operator. p and r are Hermitian. All these (Refer Slide Time: 11:50) quantities are Hermitian operators and alpha is some real number. So when is this Hermitian? What happens if I take its Hermitian conjugate? This term comes here and this quantity goes there (Refer Slide Time: 12:00) and therefore when is it Hermitian? It is when alpha = half.

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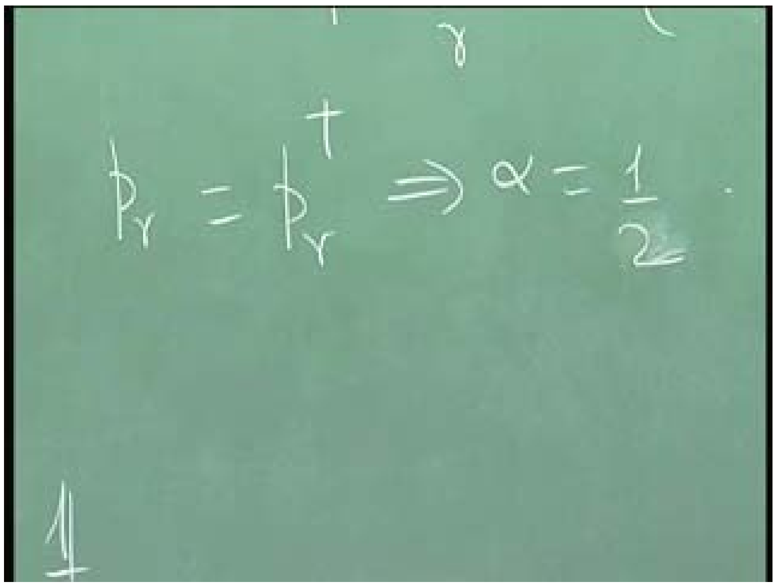

So p_r = p_r dagger implies alpha = ½. So that's the correct definition of the radial momentum and then you are guaranteed indeed that it's the Hermitian operator that satisfies the required commutation relation. so whenever you go from classical to quantum physics, you have to make sure that the commutation relations are respected, the physical quantity is respected so that you get real eigen values for that. Similarly in the definition of the Laplace–Runge–Lenz vector remember this was for the Coloumb problem, this vector was of the form p cross L and then there was a portion which was mkr over r or something. now again you have to make sure the p cross L is Hermitian and since p cross L doesn't enjoy the status as r cross p and the commutaion relation of L with p is fairly complicated, you must write it as 1/2 p cross L - L cross p when you do the quantum version.

We will talk about that later. Now that we have the Schrodinger equation and I also have the Lorentz force, let me make a couple of more comments on gauge transformations. You see this whole business works because we have said that no physical quantity should change when you make a gauge transformation of the electromagnetic potentials. In the quantum case, it employs that physical quantities like probabilities, mod psi squared and so on shouldn't change when you go to a different gauge in the vector and scalar potentials. One has to ensure that and ask what' the actual change when you go from one gauge to another? Let's do this in the following way. Let's ask what happens if i switch on an electromagnetic field to the probability current and if you recall the probability current was the following.

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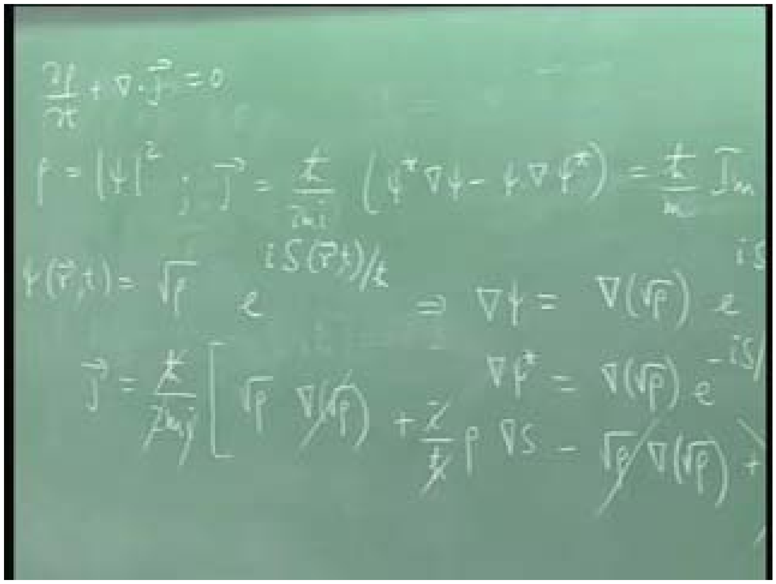

Delta rho over delta t + del dot j = 0, where rho was equal to mod psi squared. That's the probability density. And the corresponding proability current j was equal to h cross over 2 mi (psi star grad psi - psi grad psi star). This was the definition of the probability current. This came straight from the Schrodinger equation.

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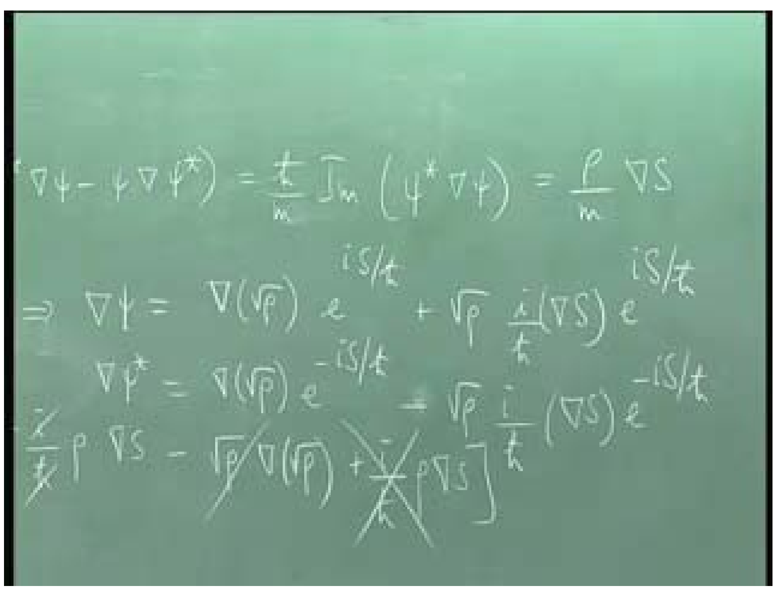

You can also write this as equal to h cross over m, the imaginary part of psi star grad psi. Because its that number - its complex conjugate divided by 2 i. so that's a compact way of writing the probability current. There is yet another way of doing this which is important in other applications of quantum mechanics and that's the following. Observe that you can always write the wave function psi(r, t) not just as a real part + i times and imaginary part but as modulus times e to the power i times the argument. So one could write this in standard notation as square root of rho because that's (Refer Slide Time: 15:43) mod psi, this is the function of r and t, this is the probability density here. This is square root of rho e to the i S (r, t) over h cross which is the standard symbol for phase. This S has the connotation of the action.

That's the reason i use this word S and it comes from classical physics. This S has got a specific connotation and then in quantum physics, we can always write the phase in this form. Now what happens to psi star grad psi - psi grad psi star? so this implies that grad psi = the gradient of root of rho e root of power i S over h cross + square root of times i over h cross gradient of S times e to the power i S over h cross. I just differentiate this; take its gradient on both sides. So that immediately says j = h cross over 2 mi psi star times the gradient of psi. So let's write gradient of psi star as well. So what does psi star grad psi become? so it says root rho grad root rho, if i multiply this (Refer Slide Time: 18:04) by psi star, this phase factor cancels out + i over h cross rho grad S and that cancels - root rho grad root rho + i over h cross rho grad S.

And this term cancels out (Refer Slide Time: 18:42). You get a 2 i that cancels the 2 i against this and this goes away here and the h cross cancels out obligingly. So that gives us our result. it says that this probability current that I wrote down could also be written in the form rho over m gradient of S. so there is a rho there is an m here and then grad S (Refer Slide Time: 19:14). So the physical significance of this phase of the wave function S is that its gradient gives you the probability current apart from this factor rho. So you see this is almost like the classical interpretation that you have. for a particle that's moving, here is the charge density if you like and the current density would be the charge density multiplied by the velocity of the particle (Refer Slide Time: 19:35 to 19:40). And this grad S plays the role of this velocity.

It's the probability current. So in this sense, there is a close analogy between classical physics and quantum mechanics but you must remember this is probability current and that's probability current density here. Now the question is what happens if I put this in an electromagnetic field with a vector potential and a scalar potential. What happens to this probability current? Well, that's not hard to answer but we can write the answer down. We could painfully sit down and compute this quantity. But you can actually write this answer down because all you need to know is what does the current do.

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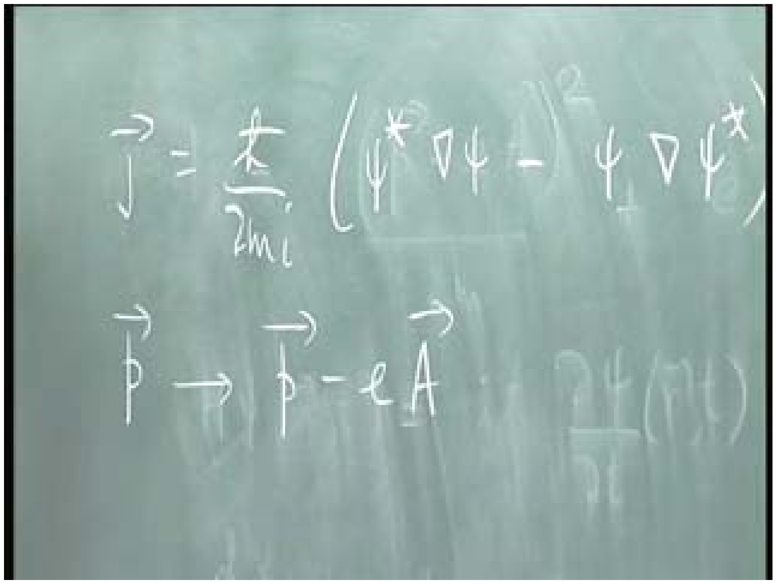

j = h cross over 2 mi (psi star grad psi - psi grad psi star). And now i only have to recognize the fact the p goes to p - eA in the presence of a an electromagnetic field. What i originally call p, i know call p – eA.

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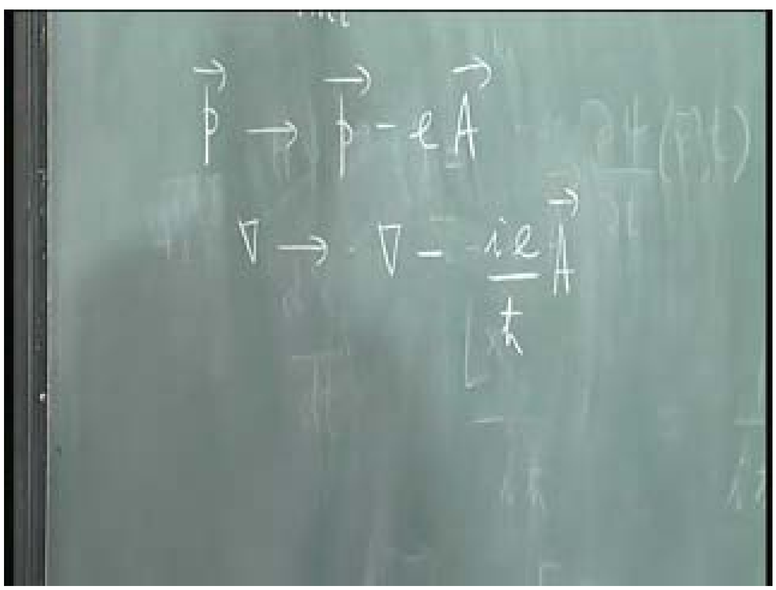

So this means - ih cross del goes to, in the position representation, is replaced in this fashion (Refer Slide Imte: 21:13). So let's make this + everywhere and divide by ih cross. This is the replacement that you have (Refer Slide Time: 21:33).

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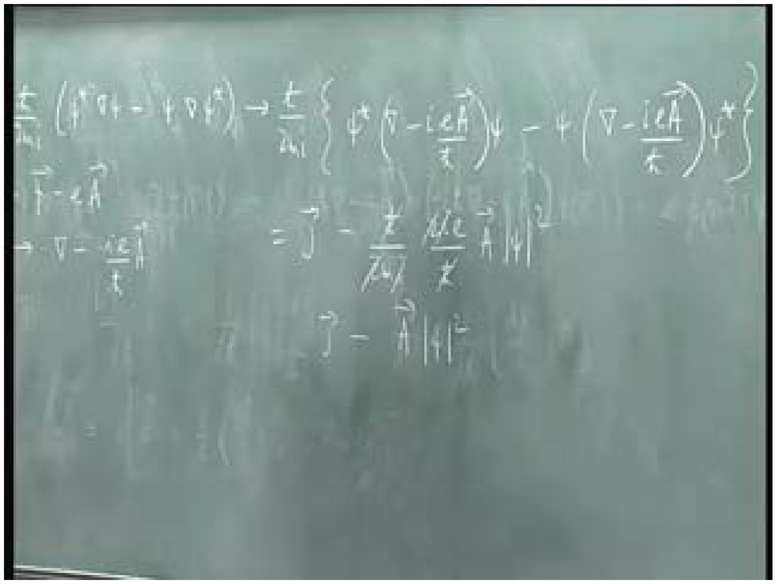

So this goes to h cross over 2 mi psi star del - ieA over h cross on psi - psi and is del - ieA over h cross on psi star. so this = j, the original current which was in the absence of the electromagnetic field and then you have - h cross over 2 mi. then i am supposed to get 2 ie over h cross A mod psi squared. So this 2 i cancels and the h cross cancels. So really this goes to j - A mod psi squared. This is what i am supposed to get. I need to take the gradient of psi star. So check this out from beginning. I made a sign mistake somewhere.So there is something which changes here (Refer Slide Time: 23:17) and that can be interpreted in terms what happens to gradient of S.

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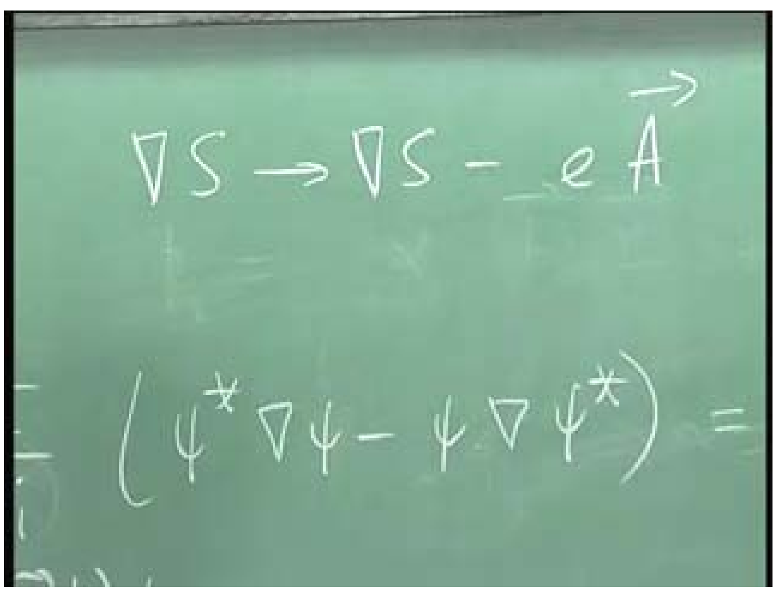

So to cut a long story short, mod psi square is just rho so gradient of S goes to gradient of S – eA. So this is the all that happens when you have an electromagnetic field. The gradient of this phase function S is shifted by the vector potential A. therefore you see the possibility that all sorts of quantum interference effects can happen. Interference happens when phases add or subtract in some strange fashion when they superpose. Now S is the phase of the wave function. So gradient of S changes by A, so S changes by a line integral of A dot dl. that leads to the possibility that, perhaps the phase of this wave function in an external electromagnetic field is path dependent because you have an integral of A dot dl which is going to appear there when i integrate that equation to find S. we will see how this comes about.

This precisely what happens and therefore sometimes one says that one has a magnetic field, you have a non-integrable phase, in the sense that it is path dependent. We will write this out explicitly and see how it works in the Bohm-Aharonov effect. What I want to do now is to back track and go back to angular momentum, work out the angular momentum algebra, the eigenfunctions and so on. So we study the quantum theory of angular momentum in some level and then come back and apply it to a charged particle like an electron sitting in an electromagnetic field. For that, we need to know the properties of angular momentum. So that's what i am going to focus on. So let's now switch to angular momentum.

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Quantum mechanical angular momentum turns out to be somewhat different from classical angular momentum, but it takes off from there in exactly the same way that we are used to in classical mechanics of a single particle. If you recall, for a single particle the orbital angular momentum was simply defined as L = r cross p and i pointed out that quantum mechanically also this definition is still true with these being replaced by the corresponding operators. Now, for a moment let's looks at it classically. We know that $\{x_i, p_j\}$= delta $_{ij}$. This is the classical Poisson bracket between a coordinate and the conjugate momentum. When you apply it to a particle of this kind you can ask what are the commutation relations of the L's among themselves. So this would imply that $\{L_i, L_j\}$, i mean the x, y, z components = L 3 on the right hand side. So this turns out to be equal to epsilon $_{ijk}$ L_k, where epsilon is equal to + 1 if these 3 are even permutation of 1,2, 3; - 1if it's an odd permutation and 0 in other cases. So that's a compact way of writing all the 3 computation relations. Quantum mechanically, whatever be the angular momentum, it is defined by these commutation relations.

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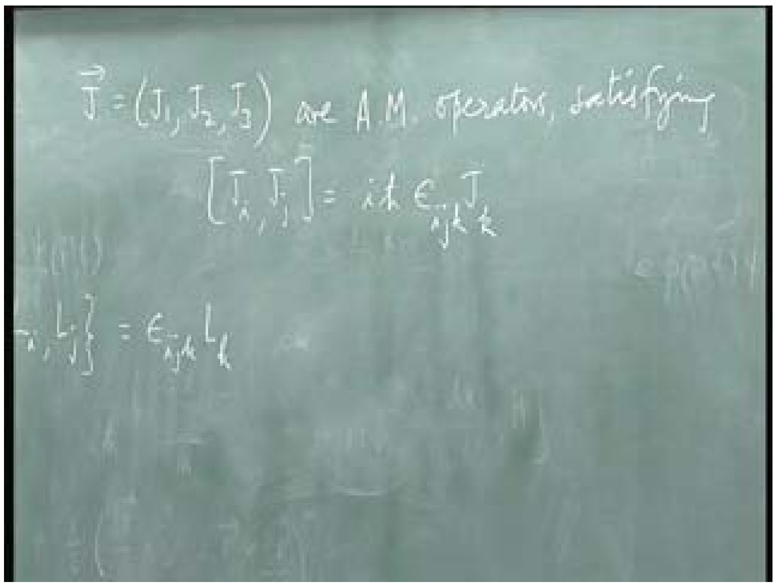

It's defined by the relations J = (J_1,J_2, J_3) are angular momentum operators satisfying the commutation relation [J_i, J_j] = ih cross epsilon $_{ijk}$ J_k. so any quantum mechanical angular momentum of any system whatsoever, is represented by 3 Hermitian operators; J_1,J_2, J_3 which satisfy this commutation relation.

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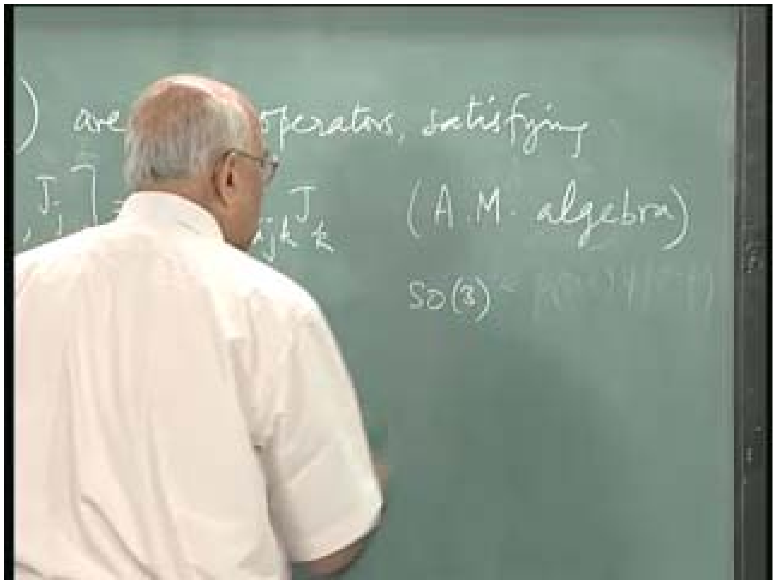

And of course you can see this has got physical dimensions of angular momentum. This is the square of an angular momentum here (Refer Slide Time: 28:26), so it matches on this side. This is called the angular momentum algebra. This is going to be my definition and all the properties of angular momentum are going to come out of this. Of course we know at the back of minds that these angular momentum operators would generate rotations of the coordinate system just as linear momentum operator generates translations of the coordinate system. And the Hamiltonian generates time translations. It takes you from one instant of time to the state at a later instant of time.

So what can we say from here? Everything we derive is going to come from here. So now, i remove this scaffolding. I am not interested in a single particle. It's not of the form r cross p or anything like that. This is the definition and that is what one has to ask. This algebra is more complicated than the original algebra we started with, where we said x p = to ih cross times the unit operator. Because it was the unit operator, things became very simple. But now that's not true. $J_1$with J_2 is ih cross J_3 and cyclic permutations. Therefore this is a much more complicated algebra. It is also called the SO 3 algebra because it has to do with rotations in 3 dimensions. It is a Lie algebra, in the sense that if you recall what a Lie algebra is.

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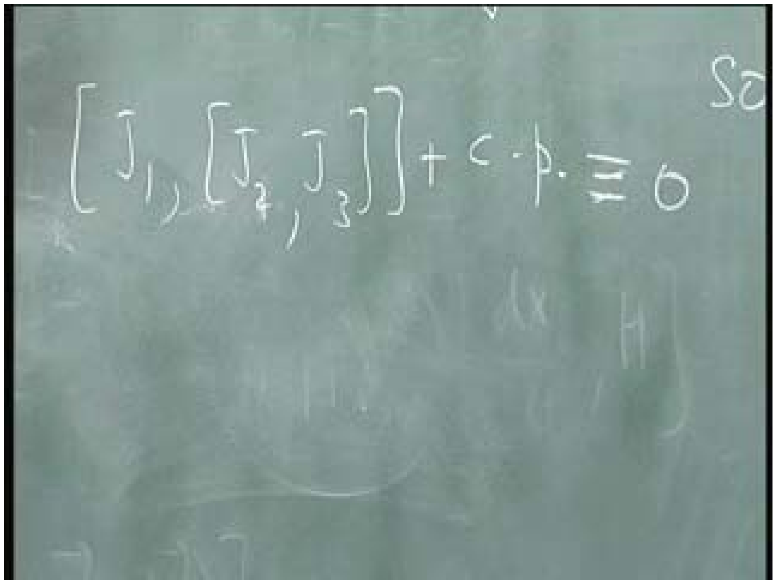

If you have a set of elements and you define a bilinear operation among them, in this case, the commutator which is anti-symmetric and which satisfies the Jacobi identify, which is also true here by the way. You can see that $[J_1,[J_2,J_3]]$ + cyclic permutations is identically zero. so they forma a Lie algebra in this case. But our immediate purpose is to ask given this information, what can we say about the states angular momentum states of any system whatsoever. Remarkably this question can be answered just on the basis of this algebra and nothing more. What we need is a few observations.

The first thing we need is the following:

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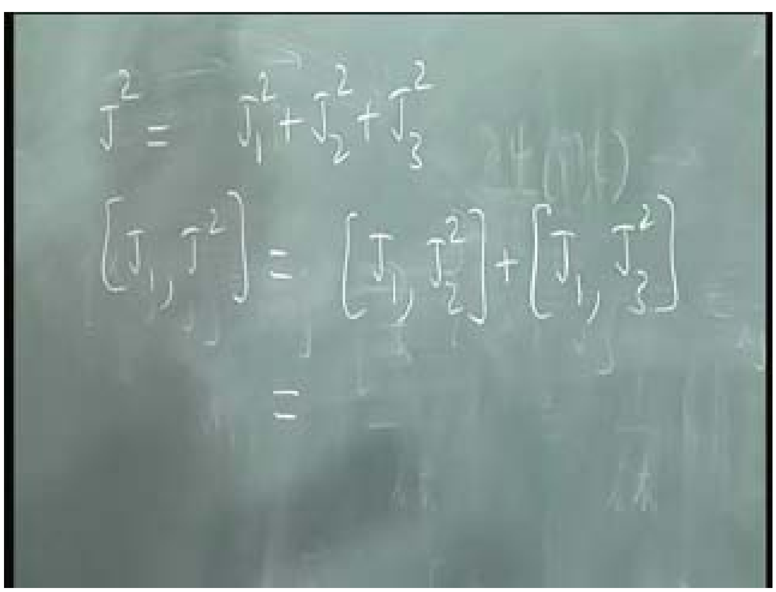

J squared, the square of the total angular momentum is by definition, J_1 squared + J_2 squared + J_3 squared and we could ask, what's the commutator of any of the components with J squared. And not surprisingly, as you know in classical mechanics, the square of the angular momentum, its Poisson bracket with any component is zero. They can be simultaneously written down and the same thing is true here. J squared will commute with each one of them and that's not hard to show. Look at [J_1, J squared] for example. J_1commutes with J_1 squared. So there is no problem and then you are left with [J1, J_2 squared] + [J_1,J_3 squared] in this fashion.

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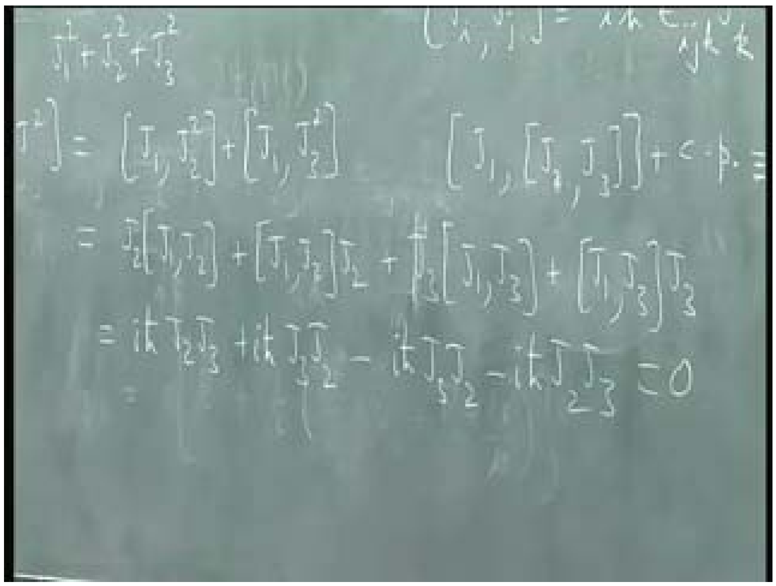

By our rule we must be careful about the orders of these things. It's J_2 times [J_1,J_2], i put one of these factors on the left and the next one i have put on the right. So its [J_1,J_2] J_2 in this fashion + similarly J_3 [J_1,J_3] + [J_1,J_3] J_3 in this fashion. That's equal ih cross J_2 J_3 + ih cross J_3 J_2 in this fashion. And this is equal to - ih cross J_3 J_2 - ih cross J_2 J_3 and of course this is 0. Since I could have chosen any one of the components to start with, this thing goes true immediately. In fact it implies something more. It implies that any linear combination of J_1, J_2 and J_3 commutes with J squared since each of them commutes.

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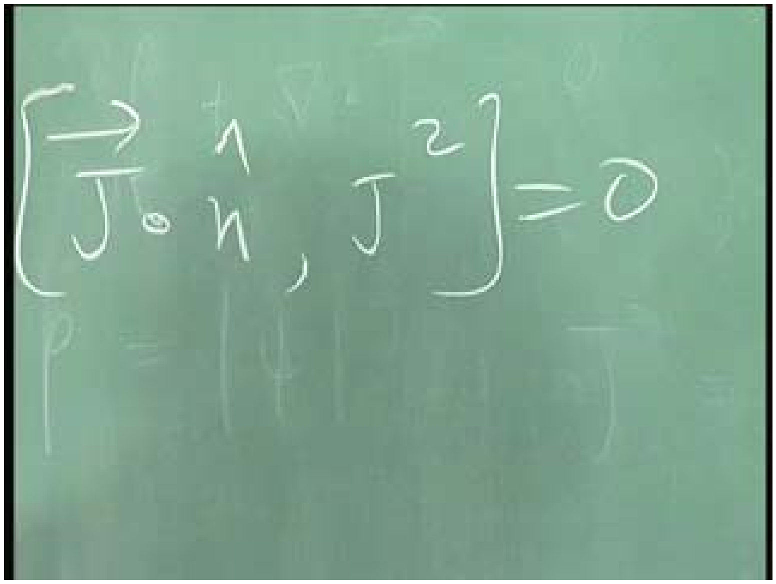

So it immediately implies that J dot n with J squared is zero for any unit vector n. so the component of the angular momentum in any direction in space commutes with the square of the total angular momentum. Therefore you can simultaneously find common eigen states. I emphasize this in particular because one always uses some particular axis of quantization as its called, usually J 3 and you say, let's find simultaneous eigen states of J_z and J squared. But there is nothing special about the z direction or any component. But no 2 components commute with each other. That's immediately clear.

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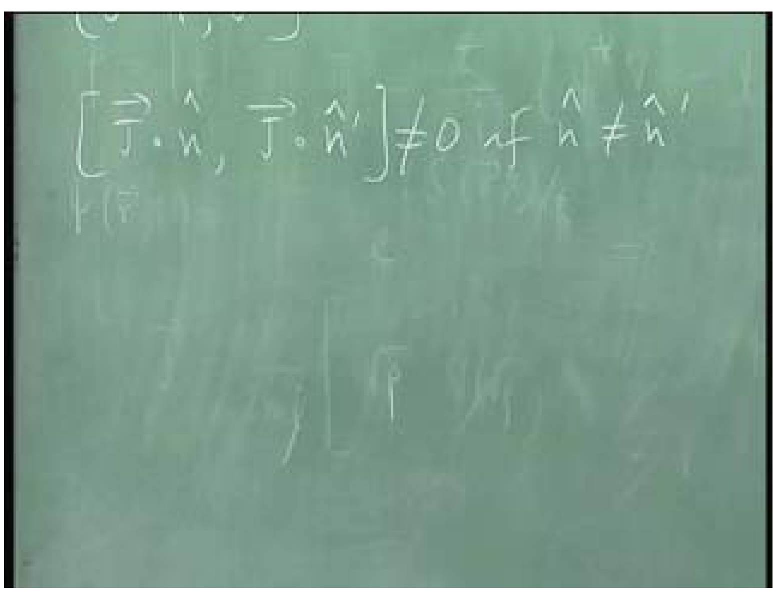

Because [J dot n, J dot n prime] is not equal to zero if n is not equal to n. If these are 2 different unit vectors, then these 2 don't commute with each other. Only 1component any 1component in any direction whatsoever and the square of the angular momentum commute with each other and therefore they form the maximal set of commuting operators in this problem. There maybe other physical operators connected with the system but if you restrict our attention to the angular momentum of the system, you could choose any component you like. So that's the first lesson. What else can we say? Well, we would like to really find out what's the spectrum of the system. What are the eigen values possible for J squared and J dot n?

What possible eigen values can you have based on just the fact that you have these commutation relations? we must do exactly what we did for the harmonic oscillator where i started by writing the Hamiltonian as a dagger a + ½ and i said if a commutator a dagger is 1,it implies that a dagger a in the space of square integrable functions has eigen values 0, 1,2,3 and so on. i did this purely algebraically. So one has to do the same thing here (Refer Slide Time: 35:53). There are several ways of doing this. Let me show you one way of doing it. It's a trick called Bosonization from the fact that everything can be made into Bosons. So, one can actually use the harmonic oscillator solution, which you already know in order to solve this problem. There are other ways of doing it but there are little more involved.

The oscillator I can work everything out explicitly and we will see how simple it is. So let's do the following. in the case of the harmonic oscillator we found that these operators a and a dagger were like ladder operators they raised you from 1state to another and brought you down from higher state to a lower state. So the question is can i find similar ladder operators here and the answer is yes, in fact you can. So lets do the following.

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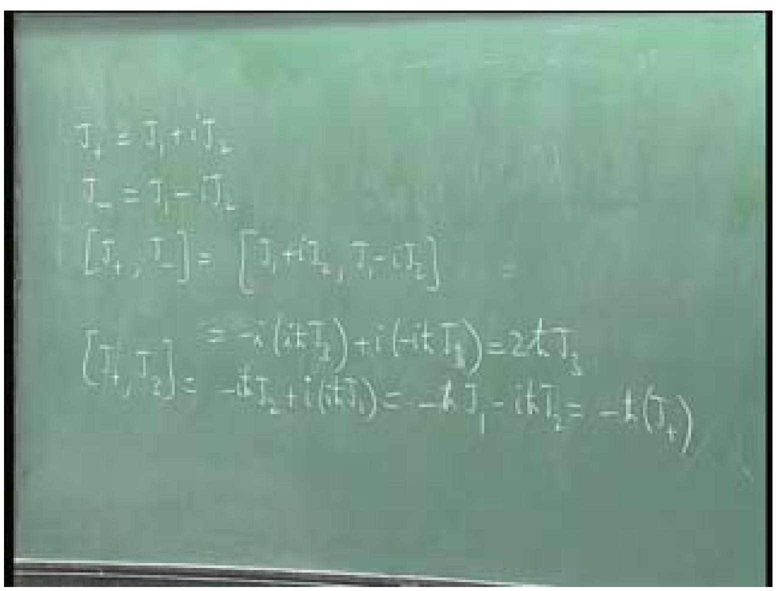

Remember J_1, J_2, J_3 are Hermitian. So let's define a $J_+ = J_1 + iJ_2$. This is very reminiscent of writing a as x + ip. We wrote a dagger as x – ip. In exactly the same way let's write J as $J_1 + iJ_2$ and J_- as $J_1 - iJ_2$. Then the commutator of J_+ with J_- equal to $[J_1 + iJ_2, J_1 - iJ_2]$ and what does this work out to? Well, it's equal to - i $J_1$with J_2 but that's i times J_3. So this is equal to - i ih cross J_3. That's $J_1$with J_2 and then this commutator here, that's + i times - ih cross J_3 and that's equal to 2 h cross J_3.

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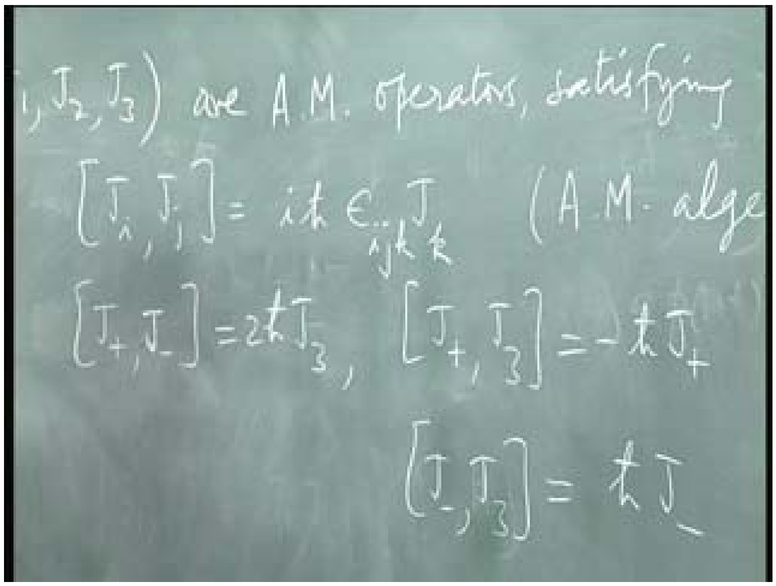

So this angular momentum algebra could also be written in the alternative way and that's $[J_+, J_-]$ = 2 h cross J_3 and we need to know what's J + with J 3 equal to. That's J 1with J 3 that's - ih cross J_2 + i times J_2 with J_3; that's ih cross J_1 = - h cross J_1- ih cross J_2 which is equal to - h cross J_+. so we have $[J_+, J_3]$ = - h cross J_+. Similarly you could ask what's J_-? J_-, J_3 is equal to $J_1$with J_3, that's - ih cross J_2 - i times J_2 with J_3. That's equal to ih cross J_1. That's equal to h cross J_1- ih cross J_2.

So this term gives me J_- with J_3 equal to h cross J_-. So i can rewrite the algebra in terms of J_+, J_- and J_3 and they look a little more complicated. The advantage is the i goes away and you have just these quantities here. Just another way of writing this whole thing. And this J_+, J_- will play the role of taking you from one angular momentum eigen state to the next. J_+ will raise it and J - will lower rate.

So that's why i have used + and - they play the role of a dagger and a. but of course this algebra is more complicated and it doesn't its not a trivial algebra because if you take further commutators, it keeps going. for example if you try to exponentiate e to the J 1+ iJ 2 for instance since J 1and J 2 don't commute they have there commutators J 3 and that doesn't commute with J 1and J 2 and so on, it will keeping going for ever. So it's a very complicated if you exponentiate these operators it's not trivial because of the structure. But because of this relatively simple algebra, its problem is solvable in the following way and is solved in the an ingenious way of doing it.

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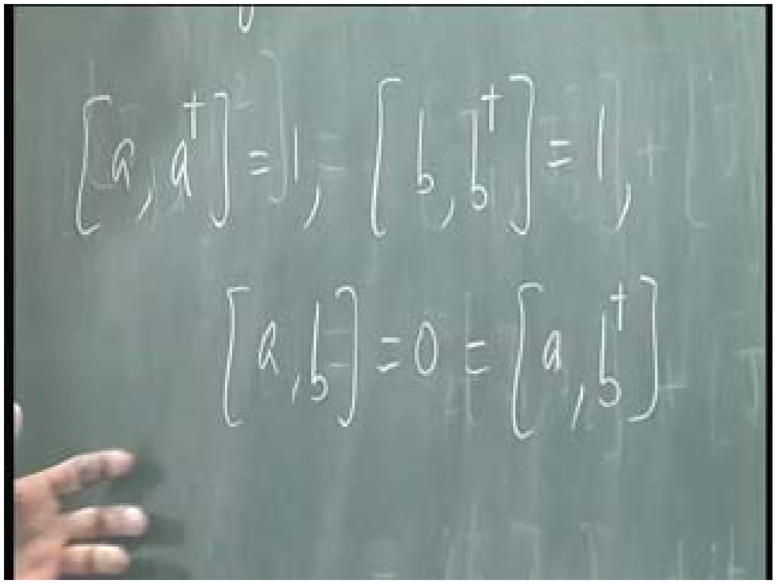

In this method, it’s perhaps due to Julian Schwinger and his idea was the following. Consider 2 different harmonic oscillators all together. One of them I call a and a dagger and the other i call b and b dagger. Consider that for a minute. So lets have [a, a dagger]= 1, [b, b dagger]= 1 and a and b don’t talk to each other. It’s just 2 independent harmonic oscillators. So [a, b] =0 = [a, b dagger]. So any a operator with any b operator is a zero. It commutes in this fashion. They are two independent harmonic oscillators.

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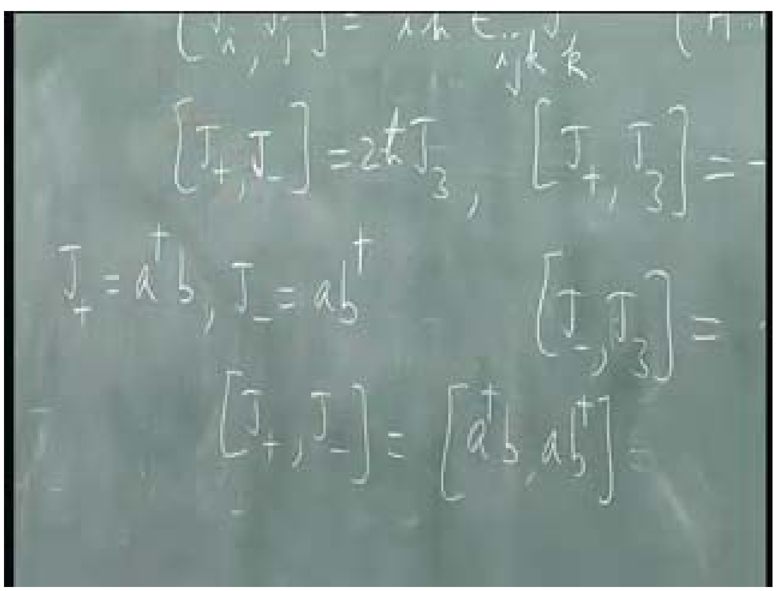

Then if you put J_+ = a dagger b, notice J - is the Hermitian conjugate of J + because J 1and J 2 themselves are Hermitian. So J - is J 1- iJ 2. So what does J - give you? What's the Hermitian conjugate of this (Refer Slide Time: 43:11)? It is b dagger a but a and b commute with each. So this could be written as a b dagger. b dagger a is the same as a b dagger. What's the commutator of J_+ with J_- equal to? It's the commutator of a dagger b with ab dagger. What does this give you? But one has to work this out carefully. Otherwise you will make a mistake.

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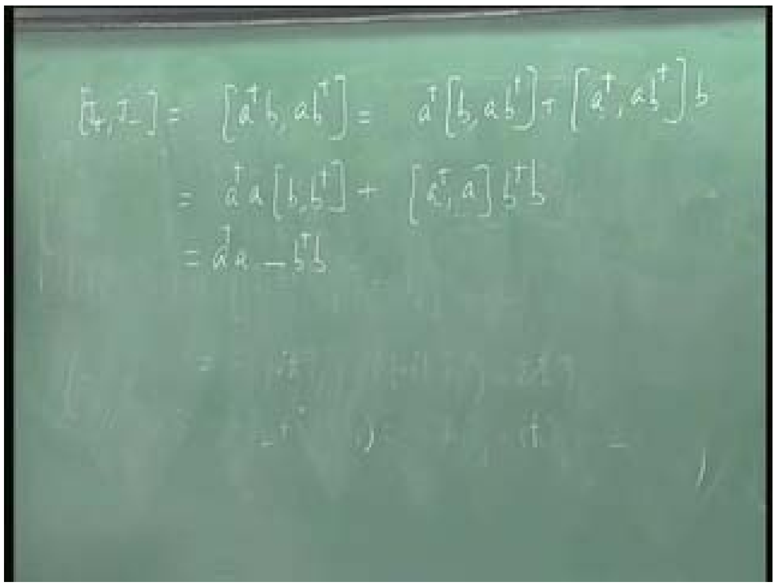

So [J +, J -] = [a dagger b ,b a dagger] = a dagger [b , ab dagger] + [a dagger, ab dagger]b. Remember that b with a commutes. So there is no problem at all. So a dagger a [b, b dagger] and the other term is zero. Similarly a dagger with b dagger is zero. So this is just [a dagger, a] b dagger b. that's equal to a dagger a - b dagger b because this (Refer Slide Tmie: 45:21) is – 1. b with b dagger is + 1,a with a dagger is + 1. So a dagger with a – 1 and that's equal to twice J 3 here.

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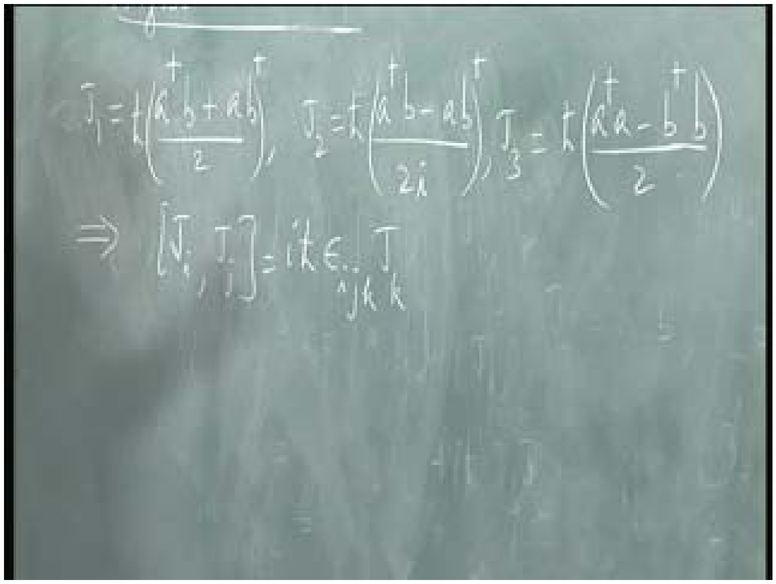

So its clear and you can check out the rest of it that if i define J_1= a dagger b + ab dagger over 2, J_2 = a dagger b - ab dagger over 2 i. this (Refer Slide Time: 46:11) is guaranteed to be Hermitian because this term becomes ba dagger which is this term here and there is a - sign there. That is taken care of by the 2 i. so this Hermitian (Refer Slide Time: 46:11 to 46:19). J 3 =a dagger a - b dagger b over 2. So with these 3 combinations you guaranteed that the angular momentum algebra is satisfied. This automatically implies that $[J_i, J_j]$ = ih cross epsilon $_{ijk}$ J_k. I put h cross equal to 1. i will fix it up. i will put it at the end and we need it. i set h cross equal to 1 in defining this. Otherwise you need to put h cross multiplying each of these terms.

We choose a and b to be dimensionless in our problem. So there must be h cross everywhere. So in the angular momentum algebra, J_i, J_j is ih cross epsilon $_{ijk}$ J_k. so its nice that, you see in nature, we have the fundamental constant of nature; Planck's constant which has got dimensions of angular momentum. So we have been provided with a quantum of angular momentum already if you like. So that's why it appears everywhere. So my statement is that these 3 operators here satisfy the angular momentum algebra and therefore to study the angular momentum algebra, i may as well study the properties of a's and b's which i already know. Now lets work out what J squared is. We verified that J squared commutes with J1, J2 and J3.

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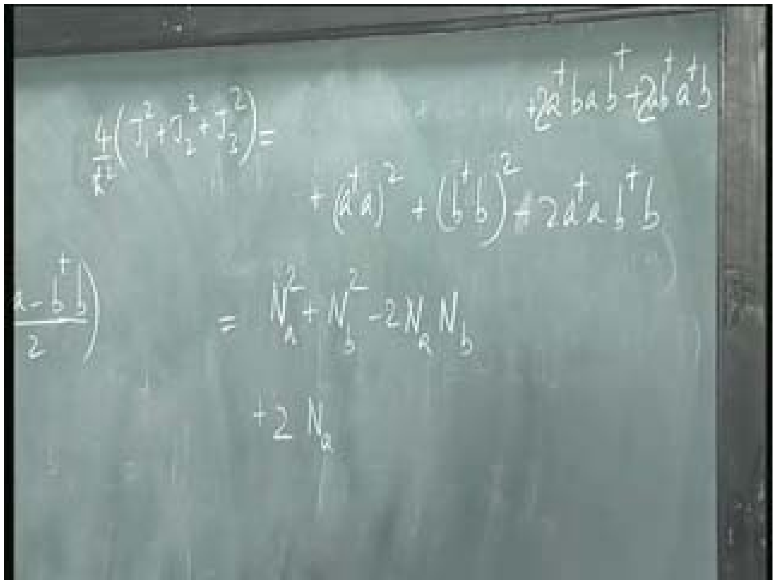

So it's 4 over h cross squared (J_1squared + J_2 squared + J_3 squared) and this is going to be painful. It's a dagger b a dagger b + ab dagger ab dagger + a dagger b ab dagger + ab dagger a dagger b. i have to be careful about the orders here. This (Refer Slide Time: 49:48) part doesn't commute with that part. So we are extremely precautious in writing it out. And then the square of J_2. So there is an i there. So this is - a dagger b a dagger b - ab dagger ab dagger. So you permit me to write twice this and twice this and cancel these terms out (Refer Slide Time: 50:30 to 50:37). And then you have the square of J_3. We have an (a dagger a) squared + (b dagger b) squared – 2(a dagger a b dagger b), because a's and b's commute with each other. So i just write it in this fashion. Let's call a dagger = N_a, b dagger b= N_b.

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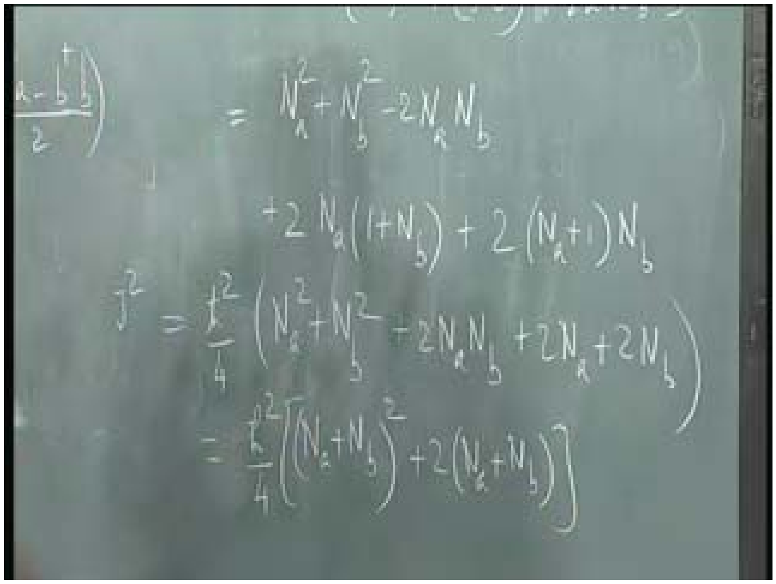

So this is N_a squared + N_b squared - 2 N_a N_b and these commute with each other, + 2 N_a, but i have a bb dagger here. I know bb dagger - b dagger b is 1. so bb dagger is 1+ b dagger b or 1+ N_b. + 2(Na + 1) Nb. so this is equal to N_a squared + N_b squared + 2(N_a N_b) + 2 N_a + 2 N_b. so J squared is equal to h cross squared over four times this (Refer Slide Time: 53:24). and N_a and N_b commute with each other. So this is h cross squared over four (N_a + N_b) whole squared + 2(N_a + N_b).

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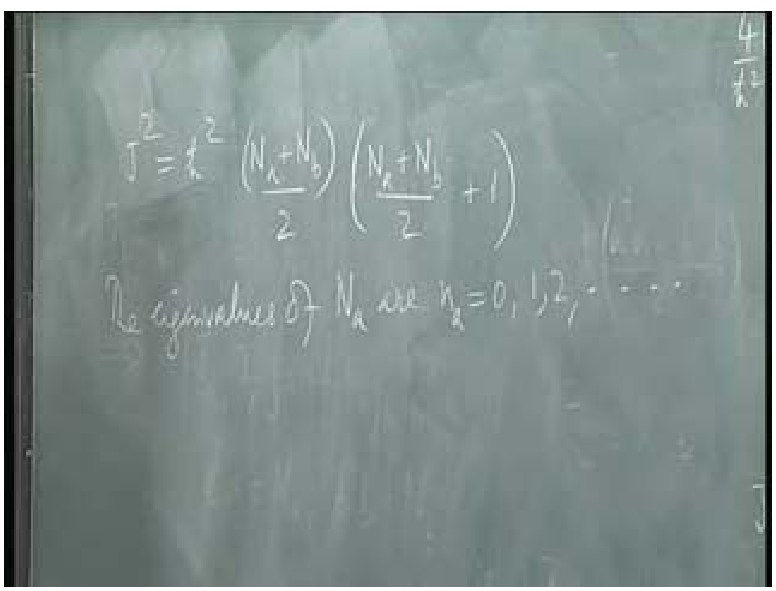

Therefore, we have our result which says J squared = h cross squared over 2 ($N_a + N_b$) ($N_a + N_b$ /2 + 1). That’s my final result. What are the eigen values of N_a? We still have to find the states of the system. That’s a different story all together but and this is nothing to do with the harmonic oscillator. but purely at an algebraic level, the fact that these commutators have been handled properly, it says that solving the problem of finding the possible eigen values of J squared, physical system being anything, is exactly the same as solving the problem of 2 harmonic oscillators independent of each other which we have actually solved.

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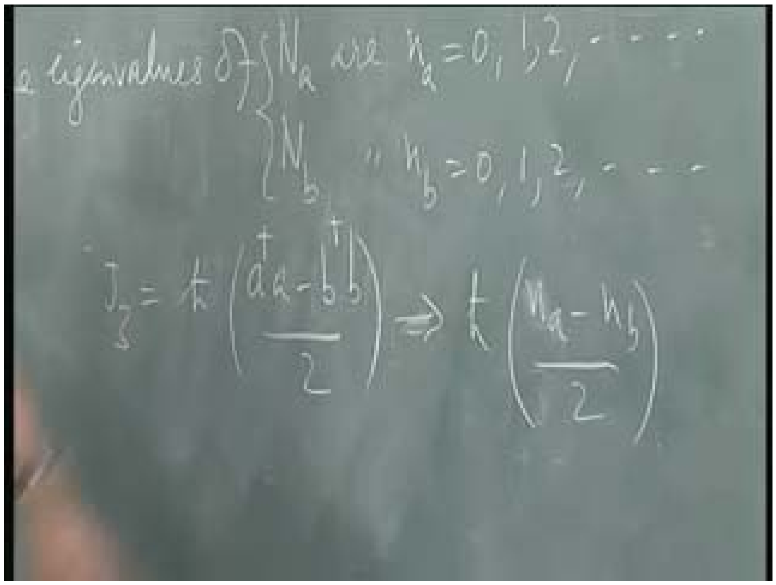

So the eigenvalues of N_a are n_a = 0,1,2 …, whatever be the states, we will write the states down and then we will try to construct angular momentum states. Similarly the eigenvalues of N_b are n_b = 0,1,2,…. So we know the eigenvalues of the total angular momentum, whatever be the system, states, the physical nature of the system or the angular momentum, we don’t care.

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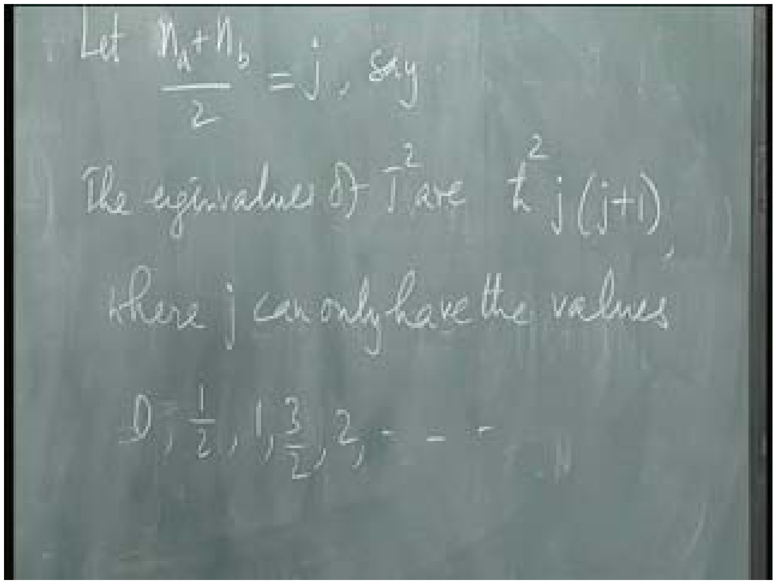

Let $n_a + n_b = j$, say. It's a nonnegative number it's a real number na and nb are nonnegative integers. So the eigen values of J squared are h cross squared j times (j + 1). So you see this famous formulary l(l + 1). So the angular momentum is appeared here but with a little twist. It's even more general than we expected where j can only have the values 0, 1/2, 1, 3/2, 2, 5/2, and so on. So we see half integer appears naturally in almost a magical way. Orbital angular momentum is quantized and only integer values can appear. So there is some more input into it. Orbital angular momentum is a special kind of angular momentum but not the most general one possible. We need an additional input to rule out things like 1/2 and 3/2 and so on. But you know already that things like spin do have half integer values and this is where it comes from, at least mathematically. This is not a very transparent way of saying where this half comes from. It just came out by the algebra here. So the physical interpretation of it is still missing completely and we have to see why that happens. I will explain that in some detail because it's very important.

But this is our first result. it says the total angular momentum squared is quantized for any system as a consequence of the commutation relations and its only possible eigenvalues are h cross squared j(j + 1), where j is either a nonnegative integer or a half odd integer. These are the only possibilities. We worked in 3 dimensions, we can show that this is actually a little more general but we have used 3 dimensional commutation relations and so on and this is a consequence immediately. Now the question is, i have another operator along with j which must commute with it, what are its eigen values? What would be the answer? Well, in our representation, its clear i already chose 3 as the quantization axis and that operator, if you recall J_3 = h cross times a dagger a - b dagger b over 2. But this is equal to h cross times n_a - n_b over 2, in terms of eigen values.

And now, the question to ask is given a value of j, given $n_a + n_b$ over 2, what are the possible values of na - nb over 2? So it's sort of clear and i will explain this tomorrow in detail. It's clear that if you are given a value of $n_a + n_b$ over 2, you can form it from n_a and n_b in many different ways. One of them would be to set this 0 and have this as large as possible. Then this would be the largest possible here. And the other would be to set this term 0 and have that as large as possible and then this would be as negative as you can get (Refer Slide Time: 10:00:15 to 10:00:30). It's not hard to see that the allowed values here would run from - j to + j in steps of 1. This is how angular momentum gets quantized.

So we will take it up from this point tomorrow morning and we still have to prove. And after that, we have the massive task of finding out what do $j_{+ \text{ and}}$ j_{-} do, what do these states look like, how can we write a representation for the states and so on. And finally we have to comeback to orbital angular momentum and i ask what happened what happened to the 1/2's, 3/2's and so on. And last of all, what are these terms (Refer Slide Time: 10:01:07) and what do these represent. We will see that these terms are called spinner representations of the rotation group. They would correspond to particles with half integer spin.