Quantum Physics

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Lecture no. #15

Let's go on to problems of a particle moving in an electromagnetic field. This is what I would like to do next, namely; look at the motion of a quantum mechanical particle and applied dielectric and magnetic fields but before I do that, there was one little aspect which I wanted to mention. We looked at the problem of a particle in a potential well like an infinite well, like in a box and then I mentioned something about the finite well. We looked at the problem with the delta function well. We didn't look at barrier problems. So let me briefly mention scattering from a potential barrier and then go on to the problem of a charged particle in an electromagnetic field.

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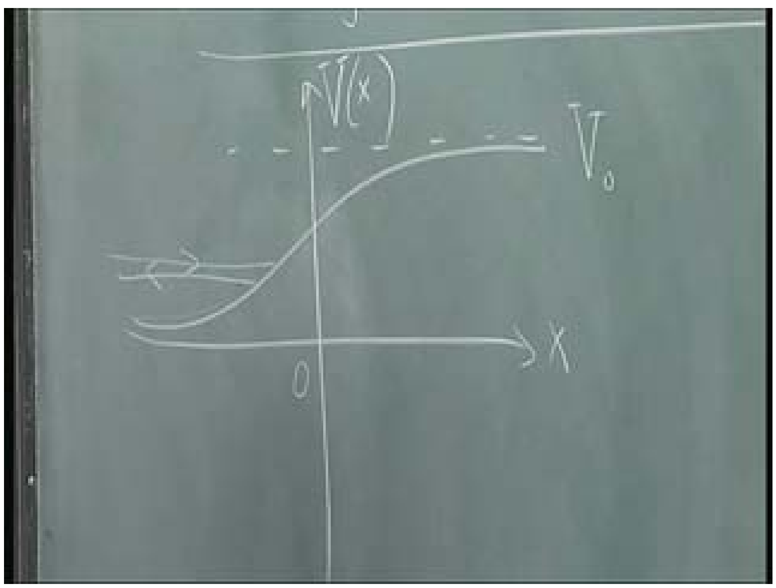

If time permits, we will talk about 3 dimensional scattering subsequently. Now, the kind of situation I have in mind is as follows. You have a potential which is really a well, a barrier in the sense which has V (x) along the y direction and x along the x direction. Let's assume that the potential is 0 as x tends to - infinity and when x tends to + infinity, it is some finite value and not 0. So it's a kind of a little hill and this is some asymptotic value V_0 and this is 0 (Refer Slide Time: 02:45).

Now the question is what does the quantum mechanical particle, put in such a potential do. Classically, of course you see that the energy can never be total energy. It can't be negative. you could have a total energy which is this (Refer Slide Time: 02:57) much for example, then a particle coming along with this kind of energy would climb up to this point and roll back. So it would essentially get reflected of this potential and go back. This is what would happen classically to a classical particle. Now quantum mechanically, the particle is described by a position space wave function. And then the question is what happens if you shoot a particle in this direction and what would it do.

Now what happens in the quantum case is that, if the total energy is less than V_0, then the particles comes along, a portion of this wave function actually exists in this region but dies down exponentially. It is then reflected back. But there would exactly as in total internal reflection. The incident wave could be reflected back and there is no transmission as such. Classically, if you have an energy greater than V_0, a particle would come along and move in this region with reduced velocity because there is some potential energy.

The total energy is conserved and therefore the kinetic energy would decrease and with reduced speed, it will move in this direction. It will overcome the potential barrier and go across to + infinity with a reduced velocity. Quantum mechanically, something interesting happens. A particle with energy greater than V_0 is an incident wave in which a portion of it is transmitted and a portion of it is reflected back on this side. The ratio of intensity will tell you that there is a transmission coefficient and there is a reflection coefficient. So let’s work that problem out.

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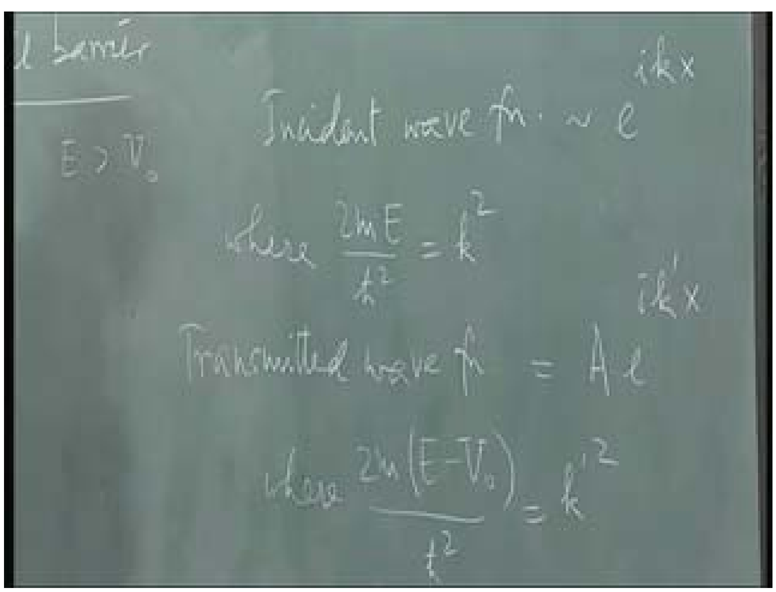

So we start by saying that i am going to look at E greater than V_0 because that is the case of interest. E less than V_0 is a trivial case. We can solve that once we do this. And then we ask what the wave function does. Well, asymptotically from this side, the incident wave is a plane wave of some fixed momentum. It’s a free particle. so the incident wave function is proportional to e to the power ikx, where as you can see 2 mE by h cross squared = k squared or E is h cross squared k squared over 2 m or p squared over 2 m. this is all that I have written.

So there is certain wave number associated with the incident wave and that's k. the wave function is proportional to e to ikx. It's a plane wave and therefore it is not normalizeable. The extent of this wave is - infinity to infinity. The energy is certainly greater than 0 and it's not a bound state. It's a free wave moving. By our convention, a space dependent part of a wave moving to the right would be E + ikx and moving to the left would be E to – ikx. so this is what the incident wave looks like. And now we ask what does it do very far out. on the right hand side, the transmitted wave function = some constant A e to power i k prime x where k prime squared is 2 m (E - V_0) over h cross squared. Because when we write the Schrodinger equation down at a point very far away there, you write the normal time independent Schrodinger equation where there is an E on the right hand side for h psi E psi but on the left hand side, there is also a potential V_0. You bring this over to the other side and then you have (E - V_0) as the available energy. E is the energy eigen value of this particle.

Therefore you can see k prime is reduced in magnitude from the original wave number k. this is like a light wave moving from one medium to another. Clearly there is a refractive index and then the effective speed decreases. What is the wave functional the side where some of it is reflected? This is some constant B times e to the power - ikx because it's moving in this medium (Refer Slide Time: 08:08), asymptotically very far away and there is no potential. So the wave number is once again k but it is moving in the other direction. Therefore, it's B e to the – ikx.

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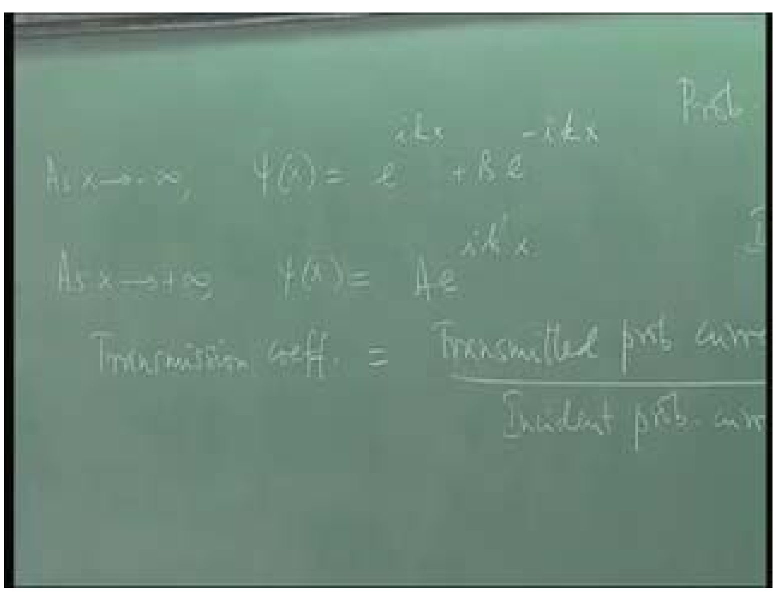

so this says that as x tends to – infinity, psi (x) = e to power ikx + B e to the power – ikx, apart from an overall normalization constant of some kind which you take that to be unity, for example. As x tends to + infinity, psi (x) = A e to the power i k prime x.

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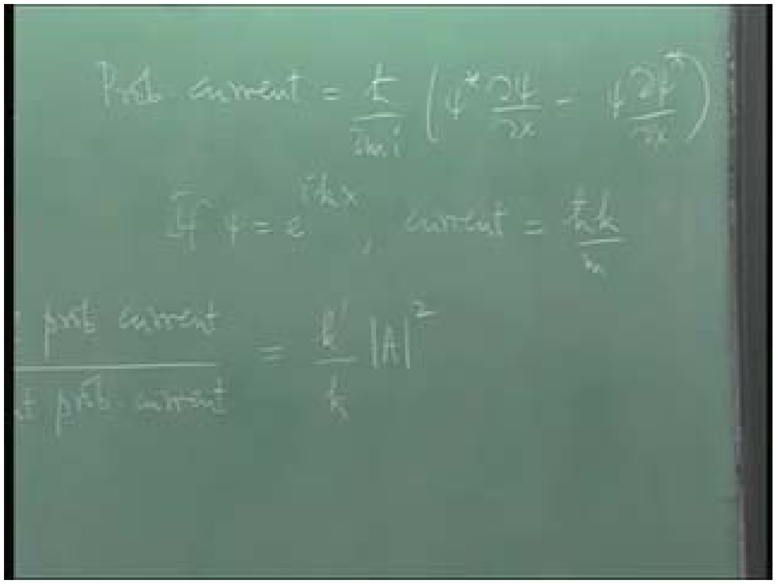

So probability current everywhere is h cross over 2 mi and if you recall the expression we wrote last time, its psi star delta psi over delta x - psi delta psi star over delta x. that was our expression for the probability current. For a plane if psi is e to ikx, apart some constant then this current = there is an h cross over 2 mi and this is going to give you e to – ikx. When you differentiate it, this is going to give you an ik and the exponential cancels and there is an extra - here for the psi star. So it gives you 2 ik and the 2 i cancels and you get h cross k over m. that's the momentum divided by the mass. It's like the velocity. This is what you would expect the current to be for a particle of unit charge for instance. So I define the transmission coefficient as the ratio of the transmitted probability current to the incident probability current. so one takes this (Refer Slide Time: 11:14) portion of the wave function, finds the current and divides it by the incident current which is h cross k over m.

And when you take this function and find the transmitted probably current, there is a psi and a psi star. so there is going to be an A and an A star as well, multiplying it and therefore this (Refer Slide Time: 11:34) is = k prime over k mod a squared because this is going to come by differentiating this plane wave and then there is a mod A squared and there is h cross m factor which cancels out on both sides.

So the physical meaning of this arbitrary constant A multiplying the transmitted wave is at the modulus of A squared is essentially the transmission coefficient multiplied of course by the ratio of wave numbers in the 2 media. Similarly the reflection coefficient the reflection coefficient s = reflected probability current divided by incident probability current. And that is given by the reflected wave which is this (refer Slide Time: 12:38) and the incident wave here (Refer Slide Time: 12:39). This is just mod B the whole squared.

Now of course probability must be conserved. Initially, if you normalized the wave function such that the total probability is 1, when you add the reflected and transmitted waves, you should again get 1. A part of it goes through there and a part of it comes back here. So I call this T and I call this R (refer Slide Time: 13:10). So it follows that T + R must be 1 always. So that relates mod A squared to mod B squared. And then to actually find out what these terms are, it requires you to solve the Schrodinger equation with that specific form of the potential and then find out what happens asymptotically.

We are given the asymptotic values but out of those coefficients A and B, one of them would be determined in terms of the other up to a phase and that can be found only if you actually tell me the shape of the potential. So let's look at an extremely simple case where the potential is very easy to write down. Let me explain when we have absorption, a perfect reflection and an internal reflection. We will not have absorption normally in this problem at all, unless there is something causing the particles to disappear.

So if you shoot for example, a beam of electrons or neutrons and I had a nucleus, the nucleus absorbs some of it, then the total incident flux of neutrons disappears and it goes into the nucleus. Then you indeed have a real absorption coefficient and part of it is scattered. This is going to be taken into account by making the potential complex so that some probability disappears probably mass. Otherwise you will have a conservation of probability. So you need a complex potential and this is called the optical potential method where you actually pertain the potential is complex but its imaginary part is related to the absorption coefficient because something another reaction takes place. Otherwise it's pure scattering.

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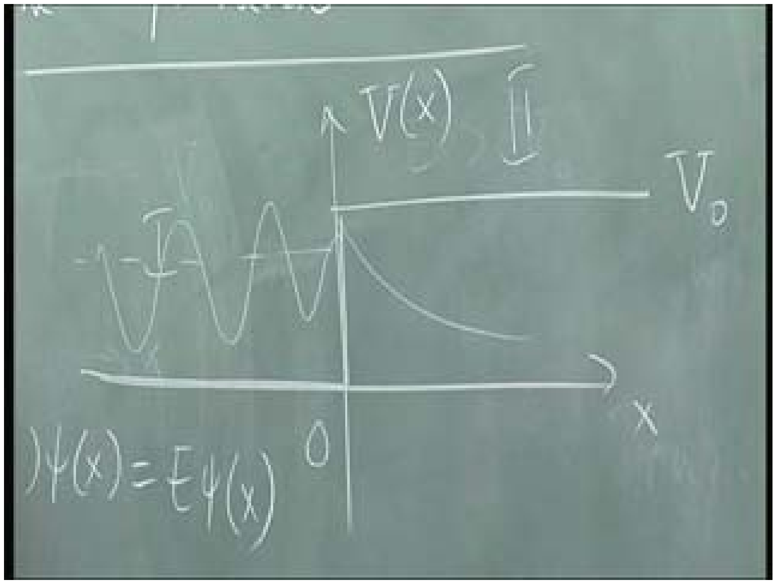

So in this problem you have pure scattering which is why R + T is exactly = one. There is no leakage or loss anywhere. so the potential i have in mind is the simplest of all so here is x, here is V (x) and now lets just say its 0 all the way up to 0 and then it's a barrier and you have a V_0 (Refer Slide Time: 15:14 to 15:20). Now we can write down the solution to the Schrodinger equation trivially because there is no potential anywhere here. Therefore this region I and this is region II (Refer Slide Time: 15:34). In region I, the wave function psi (x) is exactly = e to ikx + B e to - ikx.

There is no approximation involved here because there is no potential here and it's just a free particle. Some of it is getting reflected back and that's the B portion. And then in II, psi (x) is exactly = A e to the power i k prime x. now how are we going to fix A and B? k and k prime are known to you because you know what R and V_0 are. Every E greater than V_0 is an allowed eigenvalue and you have a continuous spectrum. So E and V_0 are given numbers. Well you also have the normalization condition that T + R must be = 1.

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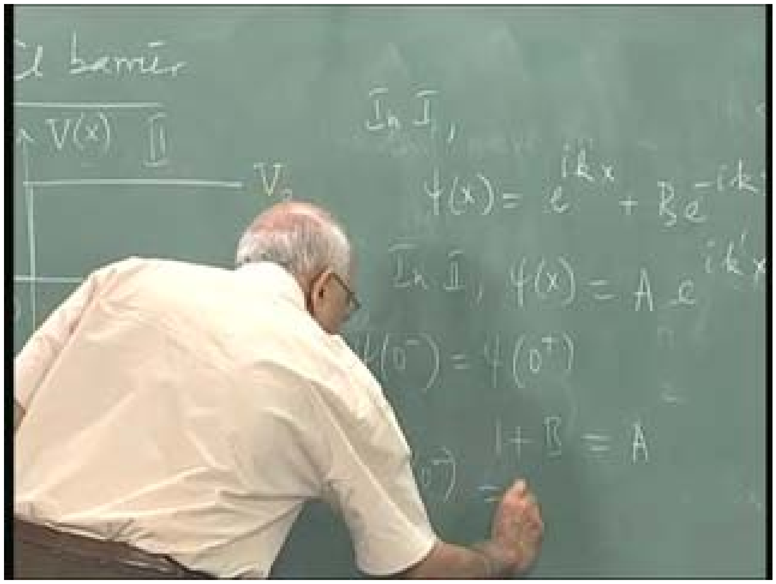

So we can write that down. Mod A squared times k prime over k + mod B squared has got to be one. But how are you going to find A? You have to use the continuity of the wave function at this point 0. So you have to use the fact that psi (0-) = psi(0 +). So coming from one side and the other side must be equal which implies that 1 + B is = A. so we have that relation now. You also need the relation for the complex conjugates and that you find by saying that the derivatives of the potential on both sides have to be continuous. Because after all what does Schrodinger equation tell you? It says - h cross squared 2 m psi double prime (x) + V (x) psi (x) is = E psi(x). And this function is continuous. V (x) has a finite discontinuity. So V double prime has a finite discontinuity to compensate for this term here (Refer Slide Time: 18:02).

So the wave function’s slope must also match. The slope should have a cusp so that the curvature; psi double prime would be discontinuous. So you also need to impose psi prime (0 -) = psi prime (0 +). When you do that, you have enough conditions to find A and B simultaneously. And now you can check that this relation T + R = 1 is automatically satisfied. So I will leave you to work the rest of the algebra out. These 2 conditions are enough to determine A and B and then verify that the reflection + transmission coefficient is indeed one. What happens if E is less than V_0 in this problem? What would happen in this (Refer Slide Time: 19:07) region? This would still be the form of the wave function but look at what k prime is.

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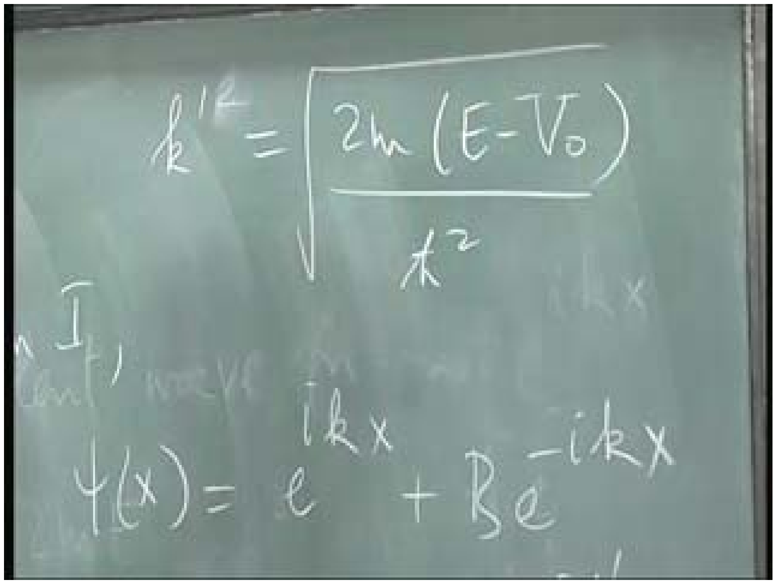

k prime squared is 2 m (E - V_0) divided by its h cross squared. So k prime is a square root of that and what happens if E is less than V_0? It becomes imaginary. And therefore ik prime becomes a negative number and so you have the wave here but once it comes here (Refer Slide Time: 19:45), it dies down exponentially. You can check this out that the transmission coefficient is 0. Nothing goes through asymptotically as x goes to + infinity, there is no wave function. however there is a penetration into this (Refer Slide Time: 20:05) barrier and it dies down exponentially fast which is essentially square root of 2 m V_0 - E over h cross.

Student – So what does it mean? Professor - you can put this whole thing in a huge box and normalize it in that box. It's not normalizable because its e to the ikx is not a member of L_2, from - infinity to infinity but it's a member of L 2 from - L to + L if you like. And then of course you would argue by saying if i put it in a huge box from - L to + L, then the momentum gets quantized. The wave number gets quantized because it has got to satisfy the boundary conditions. So this is called box normalization and I would like the box go to infinity. Then the levels become essentially continuous. That's one way of doing it. A more rigorous way of doing it would be to say I cannot find the sharp momentum eigenstate. A plane wave is a friction.

So I find a wave packet localized about some wave number k_0, may be a Gaussian wave packet where I shoot that and it comes back. I could do the same thing completely for superposition of waves. The whole thing is linear. So that's the second way to do it. When we do everything with wave packets, that's the rigorous way to do it and there is lot of complication. So, one chooses box normalization generally. So the physical point is not the important thing is not the overall factor but the relative factors here; 1 is to B and then is to A. those are the relative strength or intensities that we are worried about. So I leave you to work out the rest of this problem including the penetration here. This is actually even in classical electromagnetic theory when you have total internal reflection of a light ray which is inside a denser medium where the incident angle greater than critical angle. It goes up, it gets reflected in the denser medium but there A is an evanescent wave in the other medium. So there is something which exponential dies down. Only in ray optics is it strictly internal reflection. Otherwise in wave optics there is a little bit of penetration into the other medium. That's the property of waves and the same thing goes on here. As E is smaller, the wave dies down faster. In fact, you could go now to the limit of asking what if I took this barrier and cut it off after some time?

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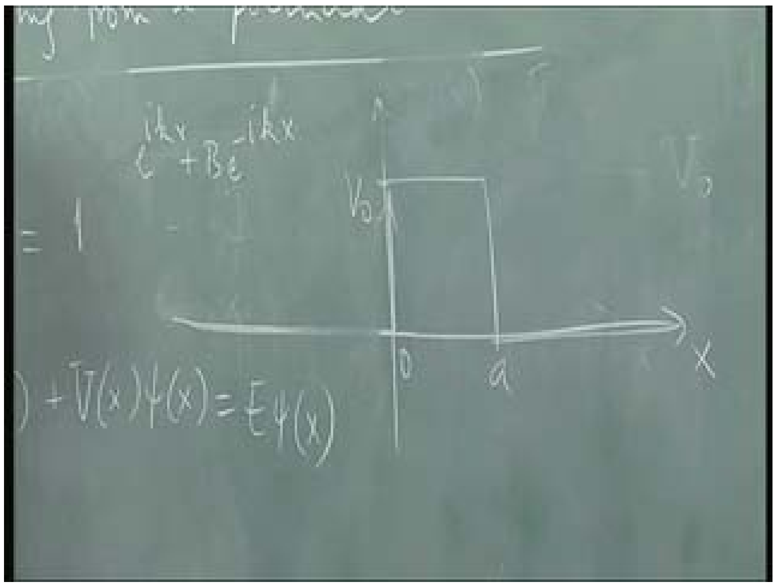

So what if this (Refer Slide Time: 22:56) was my barrier? So here 0 and a. what is going to happen now to an incident particle which is coming in within energy like that? There is going to be a transmission there but in this (Refer Slide Time: 23:07) region, where there is potential V_0, the effective wave number is different from what it is in the free case. So it's like a light ray passing in air through a slab of glass and back into air. That's exactly what is going to happen. And now to solve this problem in this region, you would say the wave function is of the form e to ikx + B e to - ik x once again.

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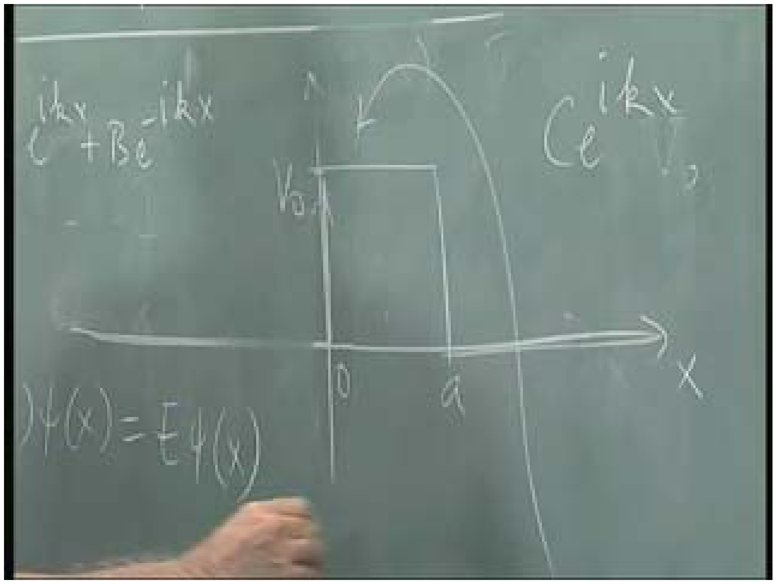

What would you say if it’s here (Refer Slide Time: 23:43)? It will be some C e to the ikx moving asymptotically in this direction. And what would it be here (Refer Slide Time: 23:54) in this region? In region II, it would of the form some D e to i k prime x + E e to the - i k prime x. there is no reason why you can’t have both. They are superposition and then you would equate the wave function and its derivative here and the wave function and its derivative there on that side. That’s how you determine these constants. What if I made A as 0 and increased the height V_0 of this barrier such that the product remains finite? You have a delta function in this case.

What would happen then? We did the problem of a bound state in a negative delta function but now I got a delta function barrier. So V (x) = + lambda delta(x). What would it the answer be now? Would there be a transmission coefficient? you can solve this by first solving the problem of V_0 and A with a finite V_0 A and then taking the limit in which V_0 tends to infinity and A tends to 0 such that the product is finite and relate it to lambda. And indeed there a transmission coefficient in this case. So any incident wave which comes along will be partially transmitted and partially reflected. You can compute the transmission and reflection coefficient.

What kind of boundary condition would you put there? If instead of looking at finite barrier of finite width and then letting the width go to 0 and the height go to infinity, I start with the delta function. What kind of boundary conditions would you put there? The wave function would be continuous but if you go back to Schrodinger equation this (Refer Slide Time: 25:53) has got an infinite discontinuity at the origin because it’s a delta function. Therefore, this has an infinite discontinuity because that itself is continuous and this has got to compensate for that. If psi double prime has an infinite discontinuity, it means psi prime has a finite jump. Therefore psi has a cusp at that point.

You cannot equate derivatives on both sides in this problem. You can equate the wave function on both sides but then you must integrate this equation from - epsilon to + epsilon and then the discontinuity in the first derivative would be related to the strength of the delta function here.

Therefore you can again check independently whether the finite barrier problem correctly goes to the delta function limit or not. Again that's going to be given to you as an exercise. Now these barrier penetration problems are crucial in many areas and have played a very big role in applications of quantum mechanics.

Conversation between Student and Professor: No because transmission is what i would have asymptotically as x tends to + infinity. Then I expect the potential to go to a constant and the wave function to become just a plane wave with some effective wave number. But if its barrier is finite and has a finite height, you see it does get transmitted. It will get through even if the energy was less then V_0. It's only if that potential barrier goes all the way to + infinity that you have an exponential cut off. If it's finite, then there is plane wave back once again and this particle can go through. It's exactly like a light wave traveling through a denser medium where it does get through and get transmitted.

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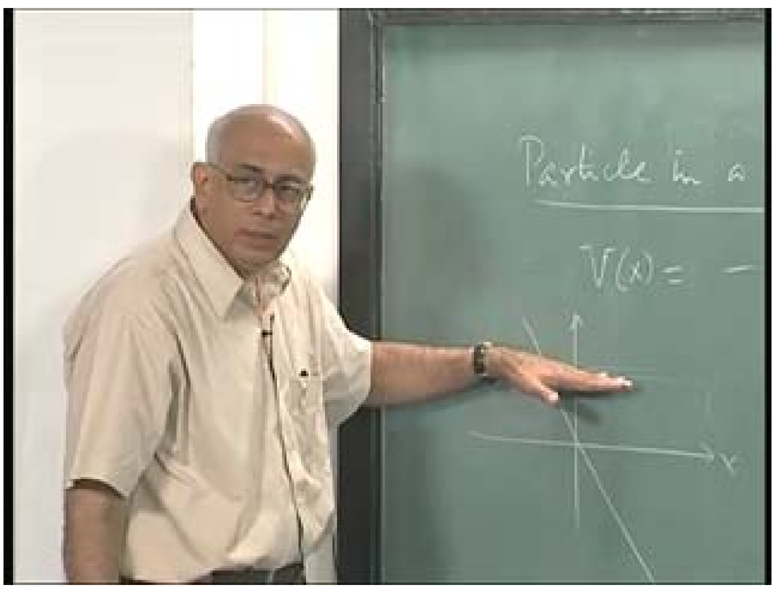

Now let us look at the problem of a particle in an electric field and in a magnetic field which is little more complicated. But as you know, the simplest electric field is a uniform electric field completely in which case the potential is just a linear function of x because a force is constant charge multiplied by the magnitude of the electric field. So let's look at a particle in a constant force field. Now what I have in mind is something where the force F is constant for example. So V (x) again in one dimension is = - Fx. F is a constant. Therefore - dv over dx is the force F. what kind of potential is this? It is just a straight line if i take F to be a positive number it is something like this (Refer Slide Time: 29:14). now it is clear that a classical particle in such a potential would simply role down to x = + infinity and whatever energy you give, if you it an energy of this much, it can go up to this point on the x axis, it can climb up this barrier and go up to this point but it can’t go to its left (Refer Slide Time: 29:26 to 29:40). It would then get reflected back. That’s what a classical particle would do.

And the spectrum of the energy in this problem in the Hamiltonian would be really - infinity to + infinity. Any energy is allowed from - infinity to infinity. So classically it is a complete a trivial problem. Now quantum mechanically, what do you except? E must run from - infinity to infinity in this problem.

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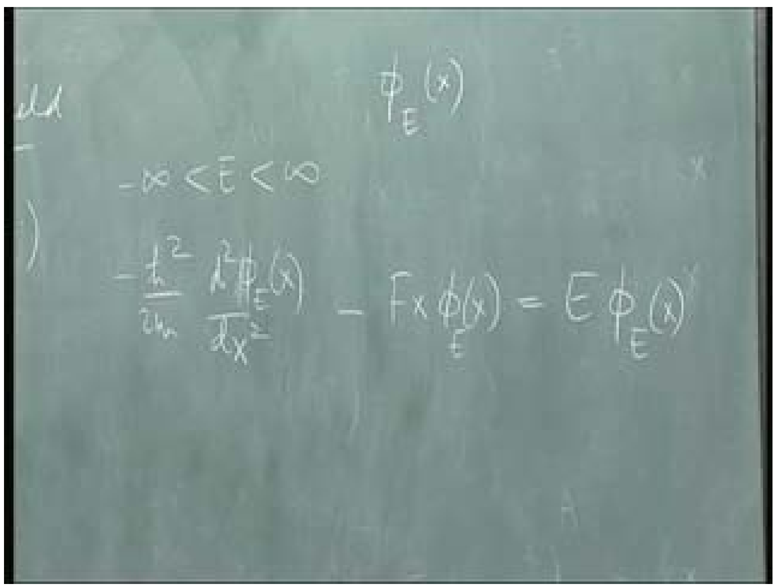

There is a confining potential and so, there are no bound states in this problem. The solution is not normalizable but what kind of wave would it be. On this region, if this (Refer Slide Time: 30:26) were your energy, what would you expect exactly as in a potential barrier problem? I expect there will be a little leakage into this but since this goes all the way to infinity, I expected it to exponentially die down on this side (Refer Slide Time: 30:36 to 30:41). And on this side, may be it would do some wave motion. This is what I would expect. This is certainly not bound states and so they are not normalizable. Now let’s see if you can solve this problem.

We write down the Schrodinger equation for energy eigen values. It is - h cross squared over 2 m d 2 phi over dx 2. so i am looking at stationary states of the system with some energy E. so I should really write phi $_{E}$(x) as the position space representative of a stationary state corresponding to energy E. E is a continuous spectrum - infinity to infinity. So for any E i have d 2 phi E of x over dx 2 + the potential - Fx phi (x) and this is = E phi $_E$(x). The quantum number n in this case is replaced by E. that is a continuous variable that labels the eigen value of the Hamiltonian. This is the equation we have to solve. It is not a trivial equation because of this term (Refer Slide Time: 32:06). So lets get rid of the - signs and so on and see what happens.

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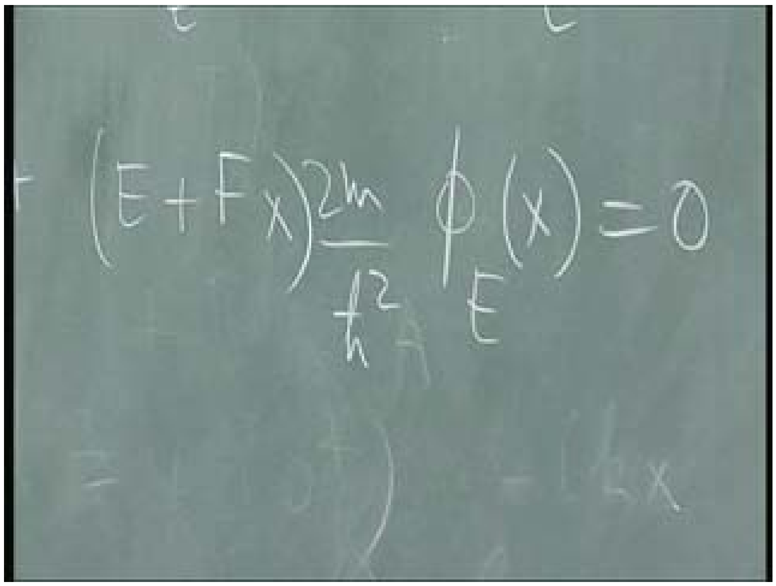

We have phi $_E$ double prime (x) + (E + Fx) 2 m over h cross square phi $_E$(x) = 0. Now it is obvious that I should change variables to this quantity (Refer Slide Time: 32:38). It is clear I should put a new variable there.

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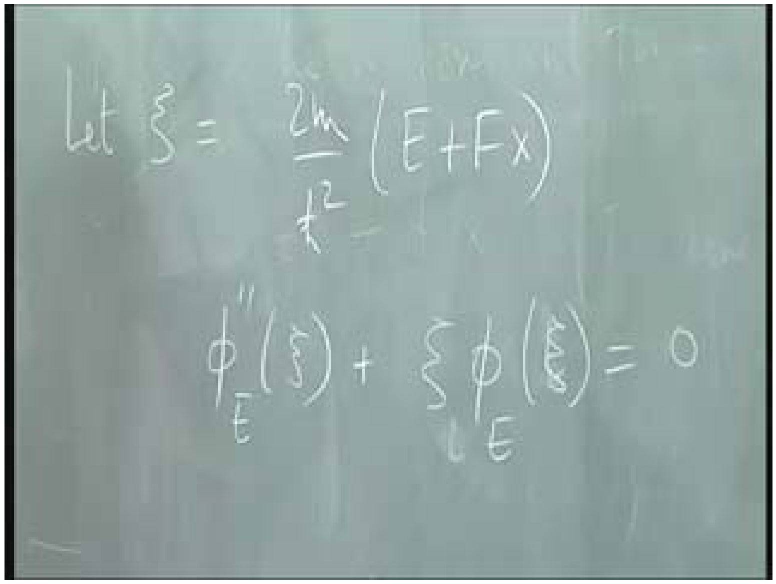

So let xi = 2 m over h cross squared (E + Fx). And then what does this wave function do? so phi double prime (xi) + xi phi_E (x) = 0.

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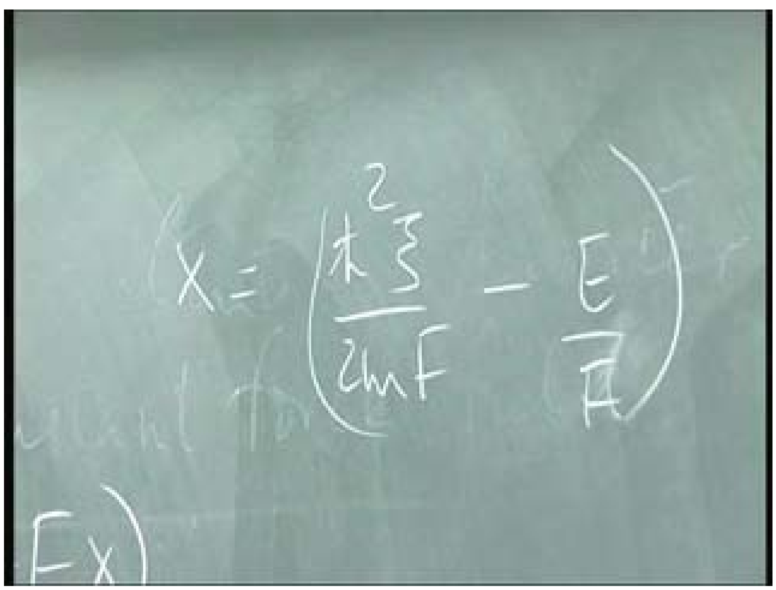

So I differentiate twice and I get some F dependent constants which I would move here. That is the derivative of this term (Refer Slide Time: 34:16) if I differentiate twice. But this constant doesn't involve E. therefore there is no reference to E here at all. No matter what the E is, you have exactly the same functional from of the solution. So independent of E. now what kind of solutions you have for this equation? It is a nontrivial equation and this is called Airy's equation. And the solution is a very strange object.

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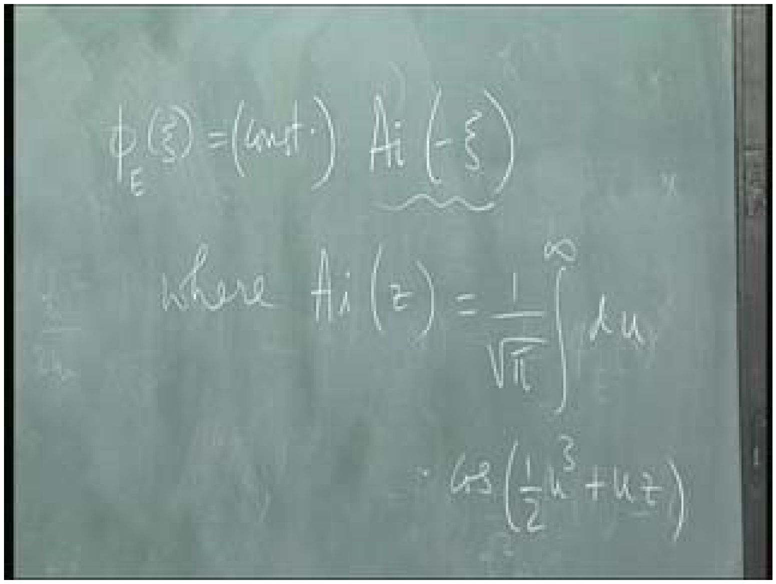

They are of the form phi $_E$ (xi) = some normalization constant times the Ai(– xi). This is called the Airy function of argument - xi by convention where the Ai(z) = 1 over root pi integral up to infinity, du times cosine (1/2 u cubed + u z). It is a special function and it is tabulated. But it's a solution of y double prime + xy = 0. So this is not very transparent although this is well studied and people have found the asymptotic form and so on. The reason is I have chosen a very bad representation to work in. in this problem it would have been easier to work in the momentum representation because you see the Schrodinger equation become much simpler in the momentum representation.

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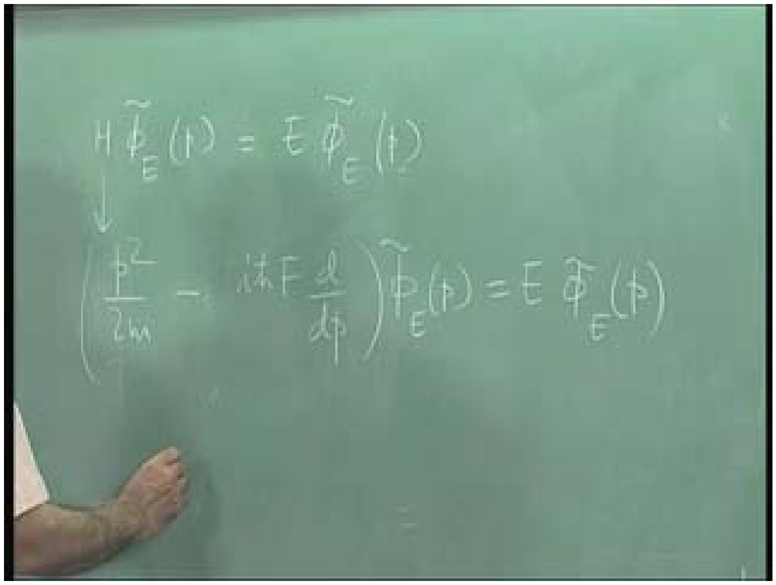

You have H phi$_E$(p) = E phi $_E$ (p) in the momentum basis. And the Hamiltonian is p square over 2 m - F times x. but what is x in the momentum representation when acting on phi $_E$ tilde(p)? It is + ih cross d over dp. So this would become very easy. That (Refer Slide Time: 37:40) is the momentum space equation. This is much easier to solve because it is completely a trivial equation. You don't have to write the solution down. It has got constant coefficients and it is immediately separable and integrable. Moreover it has a great advantage that it's of first order differential equation. So simply move this term to the other side and get rid of the – sign (Refer Slide Time: 38:04). So we get a trivial equation to integrate (Refer Slide Time: 38:47). You just have to do integral p squared with respective p and that gives you p cubed and the other term gives you an Ep.

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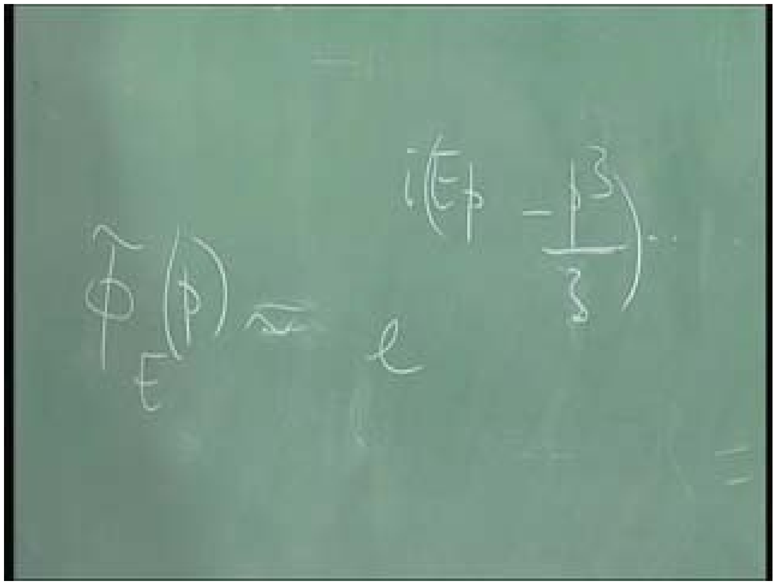

And therefore you are going to get a solution which is going to look like phi $_{E}$ ~ the exponential of i(Ep - p cubed over 3).so as for as the p dependence is concerned it is going to have a linear term and a cubic term multiplied by various constants. So the Fourier transform of the Airy functions is quite straight forward to right down. Incidentally, that is why the u cubed turned up here which is related to the fact that you got a p cubed there. What does this function look like is the question and it is possible to do an asymptotic analysis. it will turn out that as i tends to – infinity, i expect the wave function to actually die down. Remember i was proportional to x here. So I expect the wave function to die down on the left hand side.

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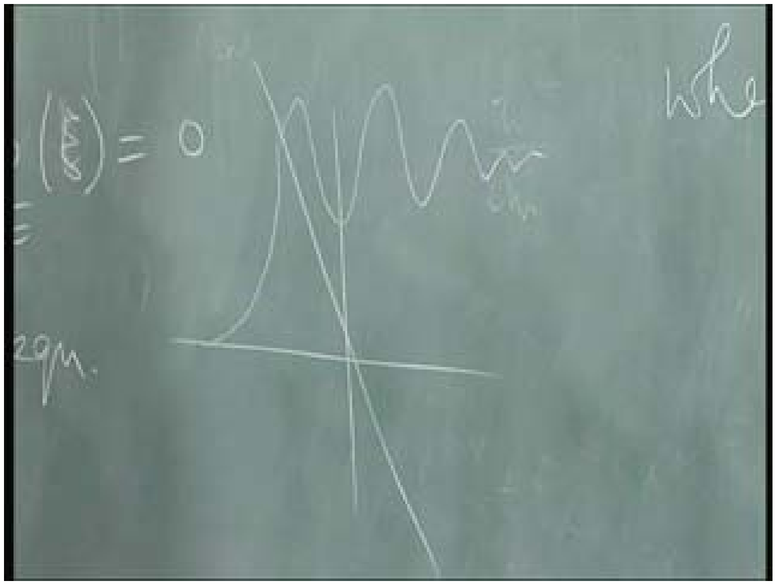

So i expect on this side something to die down but on this side, I expect some wave object like this (Refer Slide Time: 40:28). And this is what the airy function does as I tend to – infinity, the wave function phi tends to e to the power - mod xi to the 3 half. It dies down like e to the - mod xi to the 3 half on the left hand side faster than exponential.

On the right hand side, it remains wavy but the amplitude dies down very slowly. So when xi tends to + infinity, phi goes like xi to the - quarter times a sin function of something of the other involved inside. So there is something that oscillates but with very low amplitude and you can find this constant that multiplies it by normalizing it. Now since these are not normalizable wave functions because there is no discrete spectrum, you normalize it momentum space. You do an energy normalization. In other words, it is reasonable to expect that both in position and momentum space, I have the following normalization.

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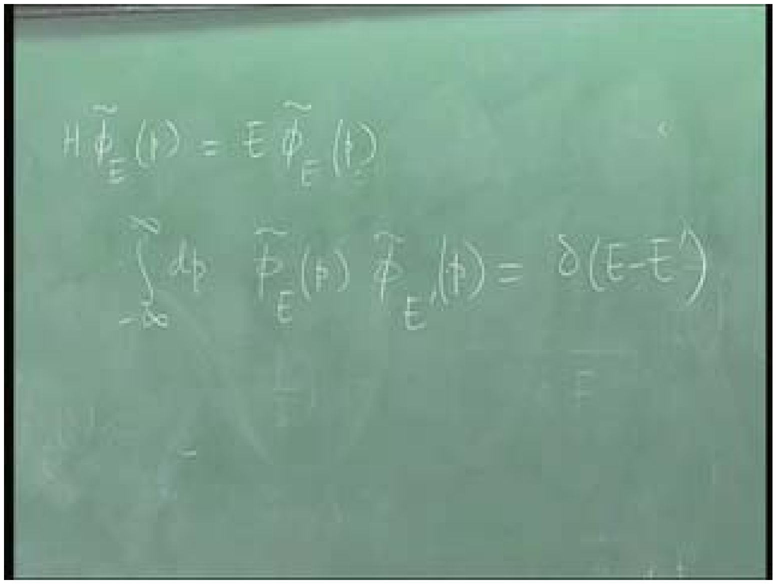

So a wave function of energy E and the wave function of energy E prime should have an inner product which is proportional to the Kronecker delta (E - E prime). So the normalization condition would be dp - infinity to infinity phi $_E$ star (p tilde) phi $_{E\ prime}$ tilde (p) should be = delta of (E - E prime). This is the normalization condition that I will put on wave functions which are not normalizable in position space. I simply say that the eigen functions of the Hamiltonian form an orthonormal set and it is a continuous spectrum. So it is reasonable that it is the normalization condition, a similar one in the position space that will fix our constants.

So I leave to work out the rest of it. It is easy to find phi $_E$ (p) subject to this normalization condition. This now solves the problem of a particle in a uniform field of force. Unlike bound states where the wave function could be made real, that is no longer true here. There are these E to the ikx's and so on. So the wave function is a complex quantity and this is called energy normalization. So that fixes another problem in a uniform field.

Now if you put a hydrogen atom in a uniform electric field, you have what is called the Stark shift but that's the 3 dimensional problems. But again, you would have the similar thing. You have a potential which is a 1 over r. that is the coulomb potential due to the nucleus. On top of it if you put a field in the z direction, you would have a potential proportional to z. this actually makes the problem much more difficult to solve because on one hand, 1 over z doesn't have spherical symmetries. So you lose this spherical symmetry. And then it is not clear what would happen to the energy levels formally because this potential actually becomes unbounded. It is not a confining potential any more. We will try and mention this when I talk about the central force problem. I will come back to it. Now that we see what happens in a constant electric field, let us see what happens in a magnetic field. I put in magnetic field and look at a particle moving in a 3 dimensional space. This is a more nontrivial problem.

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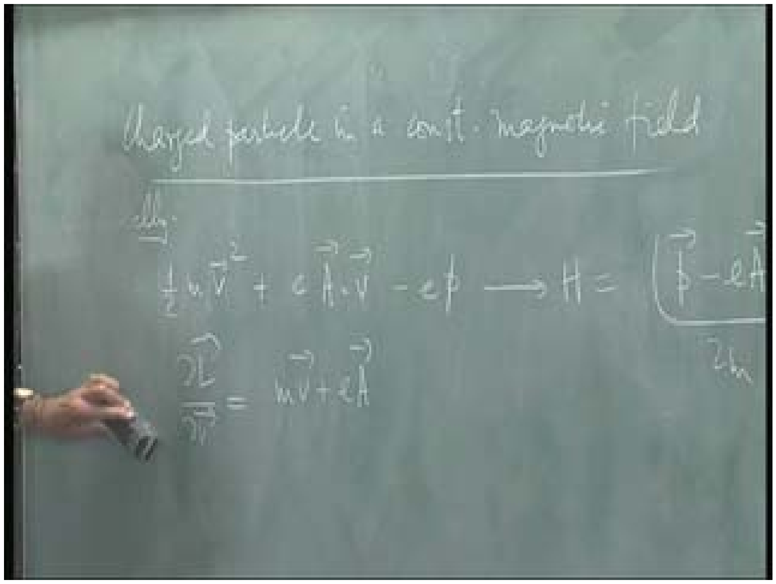

Let's look at a charged particle in a constant magnetic field. This is going to take some time. So I just set the preliminaries up today and then we will resume this. Let's go right back and ask classically what happens? Now when we do quantum mechanics you need to be able to write the Hamiltonian down. So first let's ask what happens classically i put a charged particle in a magnetic field and I need to know the Hamiltonian formalism. Now to remind you of what went on, the Lagrangian of the particle classically was one half mV squared. That is the free particle. Then you put it in the magnetic field and then you have e times; e is the charge of the particle A dot V; the vector potential. And then if there is a scalar potential, that term contributes too and you have an e phi. They are functions of r and t in general but if you have a static magnetic field, then you can switch off phi, the scalar potential and you have some A which is a function of r alone. But this is true in general.

From here, you would go to the Hamiltonian by going through first a momentum. The canonical momentum that you define is p = delta L over delta V, the derivative of the Lagrangian with respect to the generalized velocity. And this is = mV + eA. So the canonical momentum is not the mechanical momentum. It is not mass times velocity but involves the vector potential as well. therefore when you put that in, you go to Hamiltonian which, for the free particle was p squared over 2 m. but the moment you put in a magnetic field, this potential is replaced by p - eA whole squared over 2 m + e phi. So this is all that happens where p is the canonical momentum. And now we are working in the constant magnetic field. So we don't have a scalar potential. We have this term here (Refer Slide Time: 47:40) and what we have to do is to take this as the quantum mechanical Hamiltonian where p is the momentum operator. So now our quantum Hamiltonian is given as follows.

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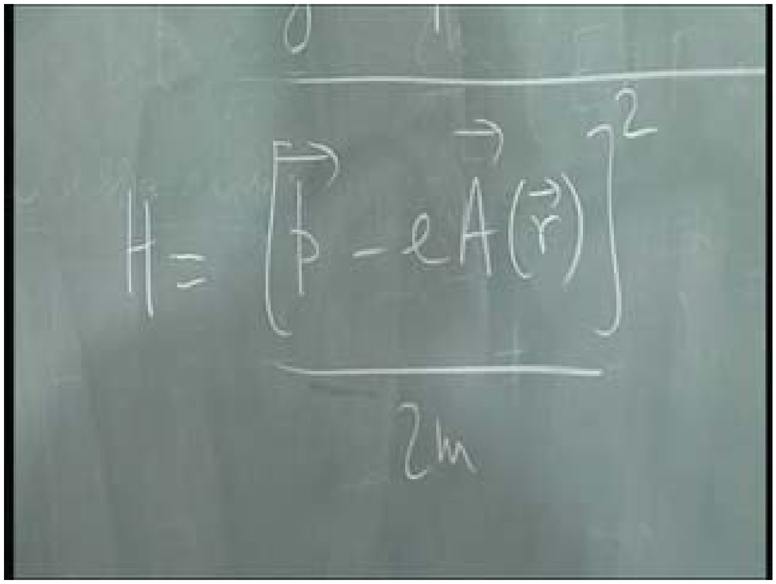

It is H = p operator - e operator. but that could be a function of position, in general A (r) p squared over 2 m. now we got to be careful because we know that p and A don't commute with each other. they may not commute with each other in general because A is a function of r and therefore when you square this term (refer Slide Time: 48:33), you are going to have terms which depend on p squared over 2 m but there is going to be a p dot A as well as an A dot p term and they don't commute with each other. Classically, you don't have this problem but quantum mechanically you do.

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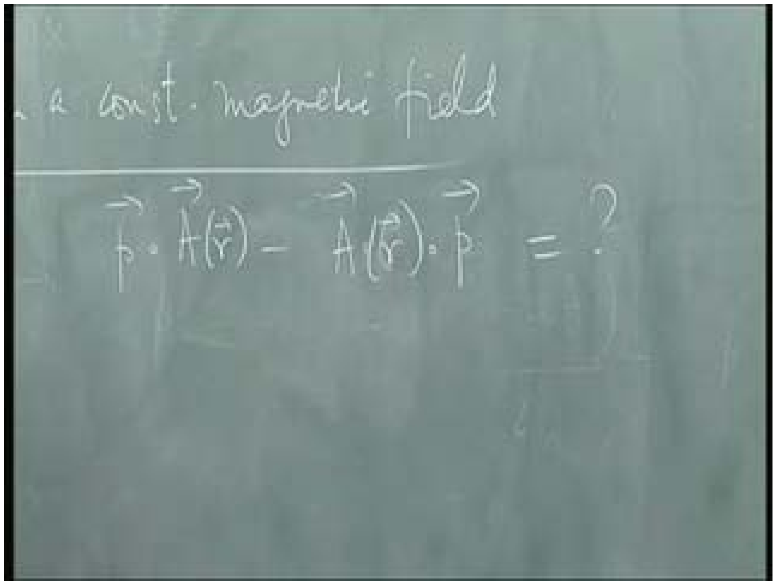

So finally you will have to compute p dot A and then you want the commutator of p dot A with A dot p. and therefore we have to compute the number p dot A (r) - A (r) dot p. the cartesian components of r and p don't commute with each other. So we really need to address the question what is this commutator because if I know that, then I can write everything in terms of p dot A. A dot p is given in terms of p dot A.

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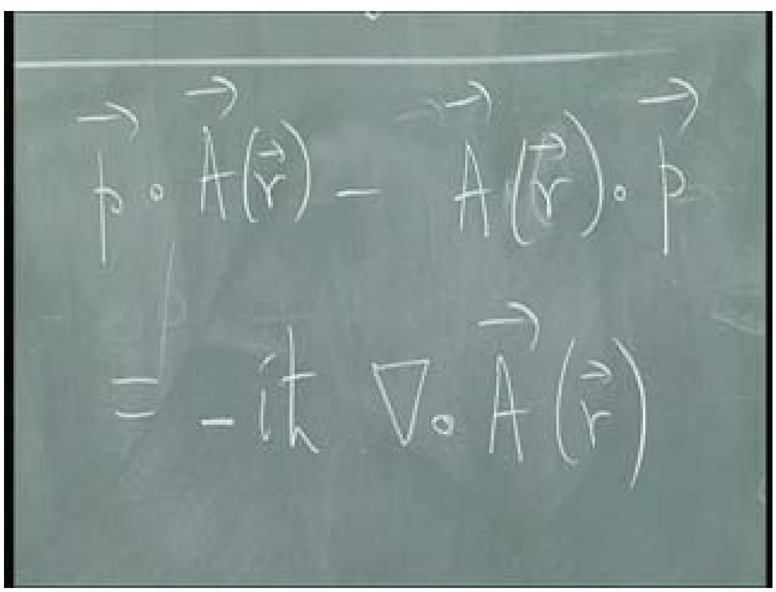

This is like asking what is [x, p]? This ih cross times the unit operator. Then the question was what is [f (x), p_x]. It is ih cross times f prime(x). Therefore one can write this down as – ih cross divergence of A(r). So as along as del dot A is not 0, you are in trouble but now of course you remember, from classical electromagnetism that you can always work in a gauge in which divergence of A is 0. It is called the coulomb gauge. We can always work in a coulomb gauge. Now let's look at our problem.

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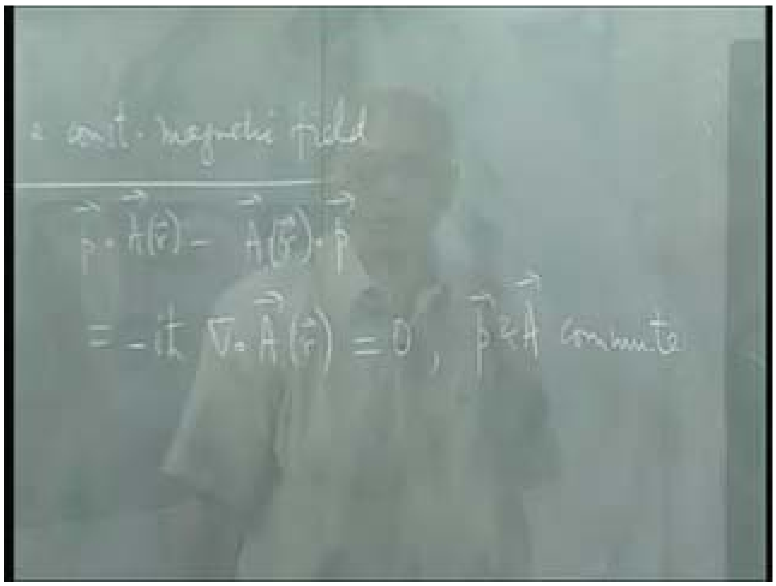

Our problem is A (r) for a constant magnetic field B in some arbitrary direction. Not surprisingly, since the curl of be A must be B, a linear function of the coordinate, so that when you differentiate it you get a constant. What is the solution? Does anyone what the famous representation for the vector potential is curl A = B? But B must be a constant. So what is A in terms of B such that when you differentiate you end up with ½(B cross r). So I like you to work this out.

You will find that the curl of A is exactly B. this is the inversion of the equation curl A = B. so A = half B cross r if B is constant. Is there divergence of this 0? It is trivially 0 because del dot A = del $_i$ A_i in tensor notation. This is 1/2 del i epsilon $_{ijk}$ B_j x_k. And B is a constant. So B_j comes out. I will leave you to check out that this in fact goes to 0. So the divergence is 0 and therefore you are in shape. Thus p and A commute. So that complication is off our hands. The next thing we got worry about is, how do we make sure that the entire thing is independent of arbitrary gauge transformations on A? This is because there is nothing sacred about A.

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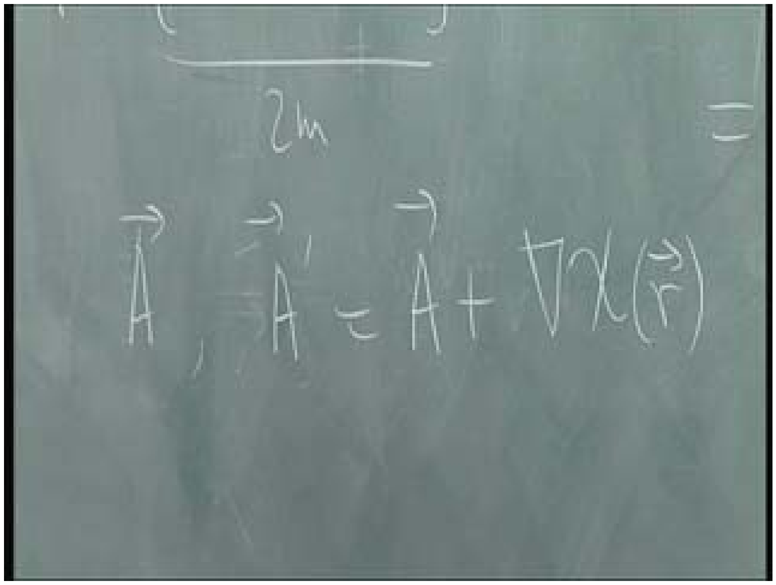

I could choose A prime as A + the gradient of any scalar function of the position and it will not change the curl of A which is B. so no physics is changed if I add a gradient of a scalar function to the vector potential. What will happen to the Hamiltonian? It will change but it will be a canonical transformation. Nothing will get affected but I must make sure that the Schrödinger equation doesn't change. Next time I will tell you how to do this. You need to introduce a phase factor into the wave function. So along with the gauge transformation, for the field you also have to make a transformation on the wave function. And it is a package deal. Then you are assured that the system is gauge invariant. This phase will not be a measurable quantity. It will be arbitrary and not a measurable quantity because physical quantities will depend on phi star phi and this phase factor will cancel out except in some very interesting situations where the space you are working in is not simply connected.

If there is a flux line passing through for example. And then you do have a quantum non integrable phase and this is the start of the so called non integrable phase in quantum mechanics. The very famous effect called the Bohm-Aharonov's effect which I will talk about and tell you how it arises from the phase factor which you need to put in on phi in order to preserve gauge invariance. So we will do this. The next thing we will look at is the velocity operator because we also have to ask what happens physically to the velocity of this particle. And it will turn out that the different Cartesian components of the velocity do not commute with each other, although in momentum it would. So that's one more indication why the momentum is far easier to handle than the velocity. Let me do that next time. Thank you!