Quantum Physics

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Lecture No. # 14

Today, I would like to go back a little to the Schrödinger equation for a particle moving in the 3 dimensions and show you that associated with the probability density; which is mod psi squared, there is also a probability current and whenever you have a density, you know in electricity & magnetism for instance, there is a current which is related to the density by the continuity equation; delta rho over delta t + delta j = 0.

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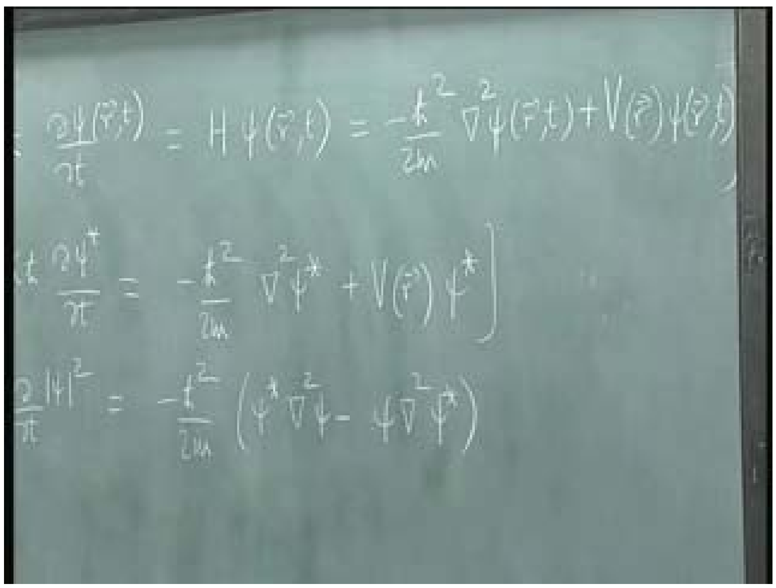

that is true for the charge density and the charge current. Similarly in fluids, when fluid is in motion between the current density and the fluid density, there is a continuity equation. So, the question you could ask is, for example, if mod psi squared in the position space representation has the connotation of probability density, is there a current associated with it. The answer is yes and it goes as follows. So we start by writing Schrödinger equation and now let me write it in the position representation for a particle moving in some potential V (r). So it says ih cross delta psi (r,t) over delta t = the Hamiltonian in the position basis on psi (r, t). and the Hamiltonian is the sum of the kinetic energy + the potential energy and in the position basis, this is - h cross squared over 2 m del squared psi (r, t) + V (r) which multiplies psi (r,t). So that is the time dependent Schrödinger equation in the position representation for a particle moving in some potential V(r). The way you derive the probability current is to multiply this equation by psi star; the complex conjugate of psi and the complex conjugate equation which is - ih cross delta psi star (r,t). Notice its h cross squared over 2 m and when you complex conjugate it, this becomes psi star here. So del squared psi star; the complex conjugate + the same potential because it's a real function, so no complex conjugation there with psi star. So all you have to do is to multiply this equation by psi star and this equation by psi and subtract one from the other (Refer Slide Time: 03:38 to 03:45). And if you did that, you end up with ih cross delta over delta t with psi star delta psi over delta t + psi delta psi over delta t which is just delta over delta t mod psi squared on the left. And on the right, it is = - h cross square over 2 m times the psi star del squared psi - psi del squared psi star. The potential term cancels out on both sides when you subtract. No matter what the potential is, this equation is true on this side.

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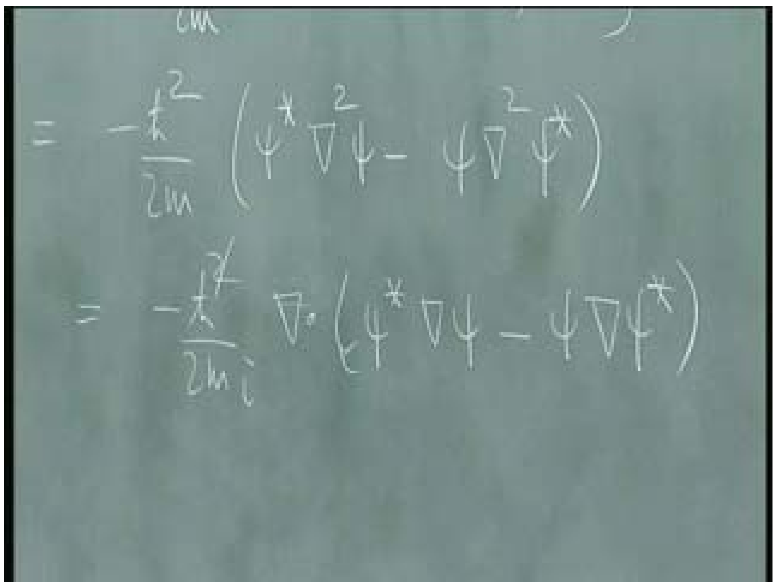

But this term here (Refer Slide Time: 04:45) is like the divergence of a current because notice that if you took that divergence (psi star grad psi - psi grad psi), star this is a divergence of a scalar times vector. You know that there is a formula for the divergence of u times v where u is a scalar and V vector. Its grad u dot V + u divergence of V. so that gives you this is = del psi star doted with del psi + psi star del dot del is del squared on psi and then - grad psi dot grad psi star - psi del squared psi star. The cross term cancels out. Therefore you could write this as = - h cross square over 2 m divergence of psi star grad psi - psi grad psi star. And lets cancel out and divide by ih cross. So move this out and then I have put an i here (Refer Slide Time: 06:10) and this becomes h cross.

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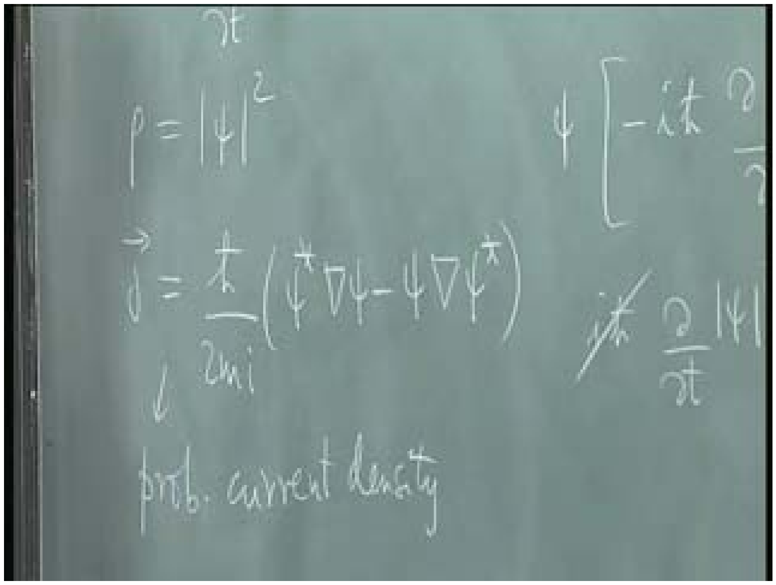

So indeed you have an equation of this kind with rho = mod psi squared because density in this case is just the probability density + the current. So j = h cross over 2 m i psi star grad psi - psi grad psi star and this quantity is called the probability current density. It's a real quantity because it's this complex number - its complex conjugate divided by 2 i. so in fact, it's a real quantity and it is the imaginary part of psi star grad psi essentially. This looks exactly like a continuity equation and just as the probability density had a bilinear psi and psi star which involves both of these, similarly the current density also involves both these quantities here. Now if you have a charged quantum mechanical particle moving, then if you multiply this by the charge of the particle, you could in fact associate with the real physical charge density and the electric current density. So this is something we keep in the back of our minds that for a quantum mechanical particle, we are going to look at the behavior of a charge particle in electromagnetic fields. This quantity here (Refer Slide Time: 07:25) multiplied by E will become the actual current density but right now let's look at this expression itself.

It's Hermitian or its real in this case in this representation and it obeys a continuity equation here. So although the Schrödinger equation was a linear equation and psi, notice that these (Refer Slide Time: 08:14) quantities are not linear. They are bilinear and they involve combinations of psi as well as psi star. When we do a little bit of scattering, i am going to come back to this quantity here (Refer Slide Time: 08:24) and talk about meaning of this quantity and we will see what how this can be related directly to a physical quantity in scattering processes. Now the other point i wanted to mention which had been left out was the spread of a wave packet. So let me spend sometime on that and it will also give us some practice with regard to the Heisenberg picture and the Schrödinger picture. in the problem set that I have sent you, I mentioned something called Ehrenfest theorem and i have also mentioned something about how Gaussian wave packet spreads.

Now let's look at this in the simplest of instances and see what happens in one dimension. So i have a free particle or a particle in a potential to start with, it's propagating in the x direction and now the under some potential possibly. the question is, if i prepare an initial state of this particle such that it is localized around some point x_0, so a little wave packet peaked at x_0 and let go, what happens to it. This thing here would propagate under the Schrödinger equation and you could ask the question what happens to the shape of this wave packet as time goes along. And let's look at what a general result is.

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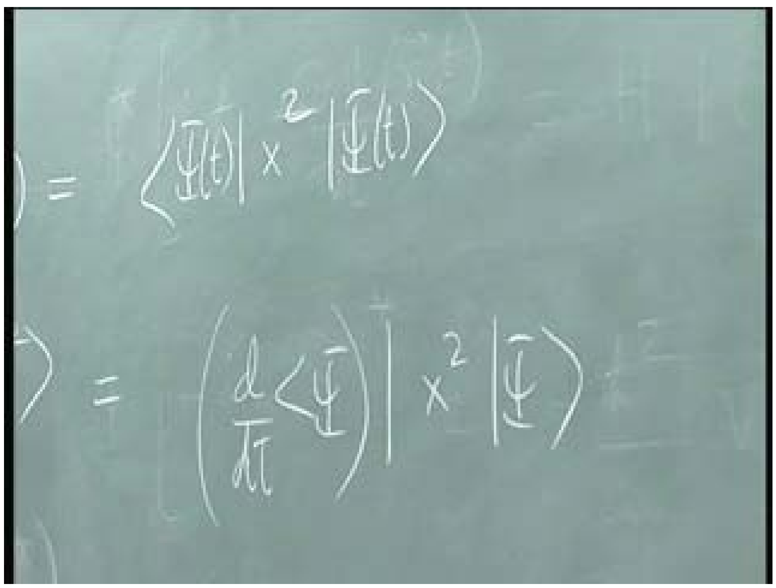

So I am going to look at a state psi (t) of a particle and i am going to ask what happens to the mean square value of the position and so on. Now you know that if you have a Gaussian wave packet, the peak position would actually give you the value of the mean and if it spreads out or if it contracts, you know that the wave packet is either dispersing or its getting tighter. So let's ask what this quantity <psi (t)|x squared|psi (t)> is. x itself is trivial to compute, so let's look at x squared and see what happens. The Hamiltonian of this particle is p squared over 2 m + some potential, in general.

A little later, we will look at a free particle. So what is this going to be and of course this is what i call x squared as a function of t and like I pointed out, you can compute this either in the Schrödinger picture or in the Heisenberg picture. In the former case, I would regard the dependence as coming from the wave function and the latter case i would say the operator itself depends on time.

Well let’s see what this quantity does. so d over dt x squared is got by differentiating this and this (Refer Slide Time: 11:25). so this is d over dt psi acting on x squared psi on this side + assuming i have normalized it, it is psi x squared d psi over d on this side and I use the Schrödinger equation for these 2 quantities. So i know the Schrödinger equation as ih cross d psi over dt = H psi and the adjoint equation here is - ih cross d over dt of the bra vector psi. but H is a hermitian operator. So there is no dagger here and it remains as it is and i pluck those in. so this is psi H x squared psi but that was with a - ih cross. So lets put the - ih cross downstairs in this fashion + psi x squared H psi and there is an ih cross down. so this is = 1 over ih cross expectation psi Poisson commutator of x squared with H psi because you have got a + x squared H and - H x squared and you have to find the expectation value of this commutator here. What is this commutator [x squared, H]? This is = 1 over 2 m commutator of x squared with p squared and then x squared with V (x).

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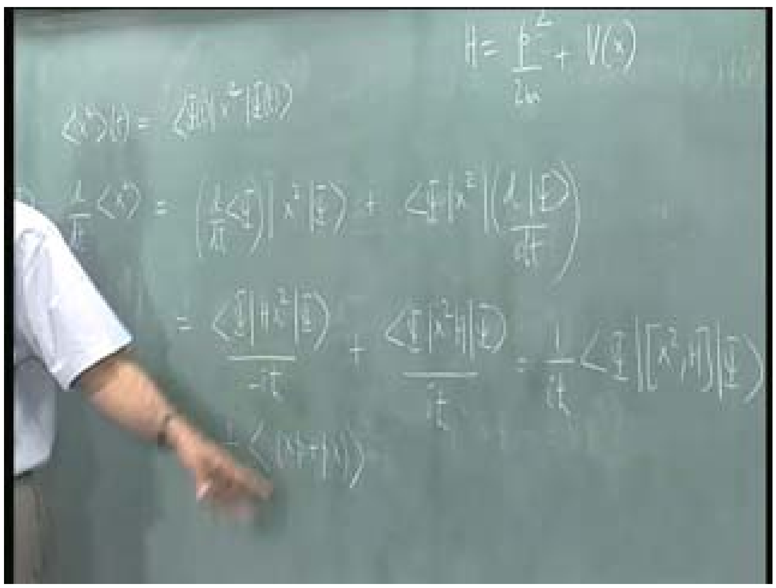

That it is 0 because it is an x operator and a function of the x operator on both of these terms and it commutes with itself. And this (Refer Slide Time: 13:56) is 1 over 2 m times x [x, p squared] + x [p squared, x] because that is the rule for confining these products for some commutators of products. So you got to be careful with the order here because x and p don’t commute with each other. So x squared is x times x which is the first factor the second factor i write it there. this is 1 over 2 m and x with p squared can be written as as x p [x, p] + x [x, p] p. so i do the same thing again x with p squared, I write it twice in this fashion + p [x , p] x.

please notice, if I write this as p times p, one factor p comes out and its x with p with an x (Refer Slide Time: 14:58) and the second time around its [x, p] px. so you have to be very careful with the order of these terms. So it’s 1over 2 m and [x,p] is ih cross throughout times the unit operator. So I pull that ih cross out. And then i have xp and px twice. So it becomes 2{xp + px}. So this is = ih cross over m (xp + px). So let’s put that in here (Refer Slide Time: 16:10). in general, the 1over ih cross goes away and you are left with 1 over m expectation value of the anti commutator <(xp + px)> .that’s a general result and its typical of these problems because you see it’s not as if this is going to reduce to something simple. The expectation value of this term (Refer Slide Time: 16:35) reduces to something. This is the second moment of x and it involves a symmetric combination of x and p here. And now we say xp is certainly a physical observable, xp + px is certainly a hermitian combination. So I now try to find its equation of motion which is d over dt of that and then you will discover pretty soon that this series never truncates. It just keeps going because you will get more and multiple commutators and even though you started with x commutator p which is just the unit operator, when you start taking all the moments of the x and p in general, it is an infinity hierarchy of equations.

So even in the simplest of problems you end up with an infinite hierarchy of these things. It is not surprising because after all what you can say about the system is in terms of its state vector or probability distribution. And you know that the probability distribution in general, is specified by infinite number of moments. That is exactly what will happen here. You need to know the first moment of x, second moment and the third moment and so on but, because of the non commutativity, you have other problems. You need to know the same thing for p and for the expectation values of various combinations of x and p like xp + px, xp squared, xp xp and so on. So this is our first result that the d over dt of this has turned out to be just this quantity here (Refer Slide Time: 18:06).

Let’s look at what happens if you find the second derivative of it. Now, of course you have to sit down and write what this expectation value is and its not going to be trivial because you have xp + p x. so let’s simplify this a little bit and see what happens.

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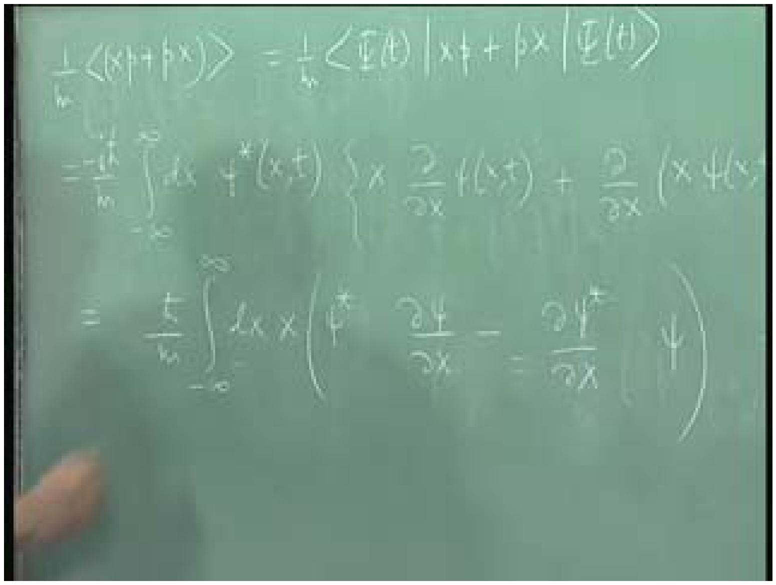

So 1 over m xp + px, this is expectation value in any state of the system psi. Let's do it in the position basis and see what happens. So in the position basis, in the abstract sense this is 1 over m psi (t) x operator p operator + p operator x operator on psi(t). But you could also write this in the position basis. I will write everything in terms of the wave function psi (x, t) and then this (Refer Slide Time: 19:08) becomes integral - infinity to infinity dx psi star (x, t). That's what this stands for. The bra here is replaced by psi star in this space of square integrable functions, times x times the p operator, but p operator is - ih cross d over dx. Let's put partial derivatives because you have t dependence as well. This acts on psi (x,t). That is the meaning of this (Refer Slide Time: 19:44) p in the position basis. Notice, it's like the derivative operator acting on whatever is on the right hand side + delta over delta x x times psi (x, t). And I have no choice because this (Refer Slide Time: 20:12 to 20:21) p is to the left of this x operator. so it acts on everything on its right hand side and there is d over dx gradient with respect to x of everything to its right hand side.

Then that becomes - ih cross over m, integral - infinity to infinity, dx psi star x delta psi over delta x is the first term and then in the second term here (Refer Slide Time: 20:47), lets integrate by parts. So i have to do an integral psi star delta over delta x x psi. I would like to throw the derivative on the psi star now. And what has that become when I integrate by parts? It is a - delta psi star over delta x times x psi. because if i integrate this term by parts, you have an integral and then the surface term and then another term which is just the derivative acting on this guy with a - sign (Refer Slide Time: 21:16 to 21:26) and - ih cross over m psi star (x, t) x psi(x, t) at x = - infinity to infinity. That was a surface term i got by integrating this term here (Refer Slide Time: 21:47).

now if these wave functions are square integrable and you are in L 2 and the mean value of x is finite and so on, then this (Refer Slide Time: 22:04) term vanishes at +/- infinity sufficiently rapidly because a wave function is supposed to vanish at +/- infinity sufficiently rapidly and you are left with just this (Refer Slide Time: 22:11) term but there is nothing to stop you from pulling the x out in here because that's not getting differentiated anyway in this fashion. You also pull the i down so this (Refer Slide Time: 22:32) becomes h cross over mi . But you recognize the probability current was h cross over 2 mi psi star grad psi - psi grad psi star. So you could in fact write this as twice integral - infinity to infinity, dx x j(x,t) but j was the probability current density. So you see this current density is related to measurables because when you multiplied by x and you integrate, you in fact get the time derivative of the expectation value of the square of the position of particle. it is directly connected to this current density.

Now lets try to evaluate this (Refer Slide Time: 23:29) by taking the second derivative and see what happens and let's do it in the Heisenberg picture. We worked it out in the position representation in the Schrödinger picture. Let's see what happened in the Heisenberg picture. So lets write the result down and then I evaluate it.

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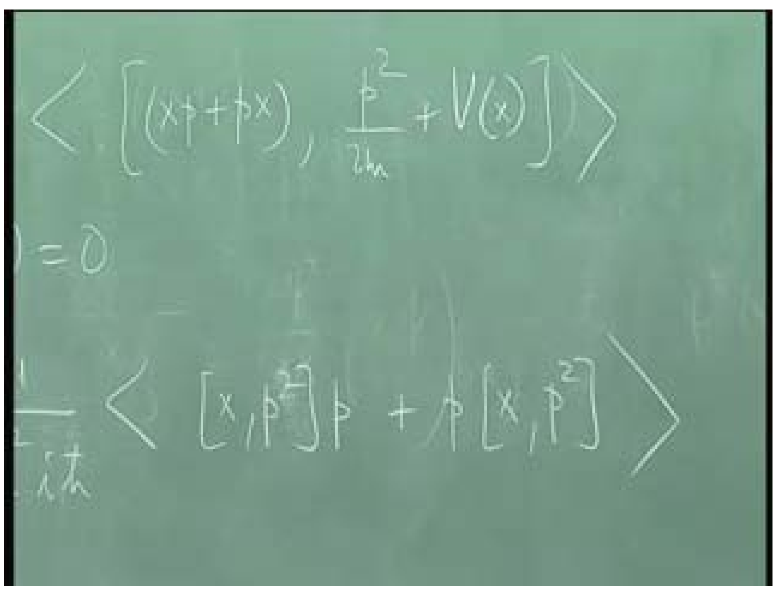

So we have d over dt of x squared = 1 over m expectation xp + px. So d 2 over dt 2 of x square the second derivative of this is 1 over m and now, let's work in the Heisenberg picture. But d over dt of any operator is the commutator of this operator with the Hamiltonian divided by ih cross. That's the Heisenberg equation of motion. so let's write this as 1 over mi h cross times the expectation value of the commutator of xp + px with the Hamiltonian which is p squared over 2 m + V(x). In general, you can't go very far because now you need the commutator of p with V (x) and there will be a big mess. It will be generally be V prime (x) and so on but look at what happens to a free particle.

So for a free particle, V(x) = 0. the Hamiltonian is just p squared over 2 m and then you expect that in the case of a free particle, the Hamiltonian commutes with the momentum operator and therefore you expect that momentum eigenstates are also energy eigenstates of the Hamiltonian so for a free particle since V(x) = 0, we can compute this quantity xp + px with p squared over 2 m. and so it says d of d 2 over dt 2 x squared = 1over mi h cross commutator of xp with p square over 2 m, so lets pull out another 2 m, so its 1 over 2 m square ih cross, expectation value of x with p squared and that’s you all you need.

So x with p squared with a p outside because p with p squared of course commutes and there is a V(x). So this term doesn’t contribute here. And p times again x with p squared expectation value. What is x with p squared? It is 2 ih cross times the derivative of p squared with respect to p which is 2 p. so it’s twice ih cross p and the twice ih cross cancels and this gives you 2 over m squared p squared. So actually you have a simple result now. The second derivative of x squared of the position is just 2 over im times the expectation value of p squared. But the expectation value of p squared over 2 m is just the energy for free particle. So let us suppose that it’s in a momentum eigenstate with some value of p.

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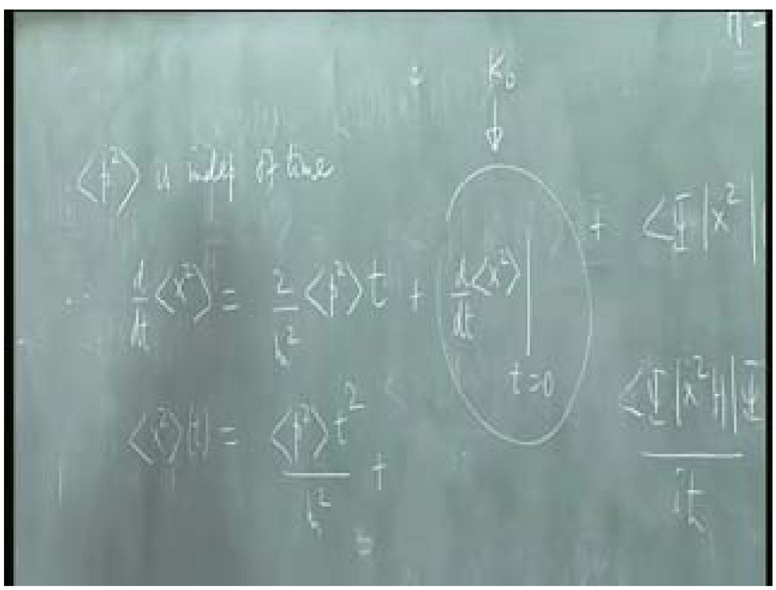

So let it be some value of p or may be a wave packet, in either case this quantity p squared is independent of time because this quantity commutes with the Hamiltonian. Therefore its expectation value is independent of time. It’s a constant and you have to prescribe to me what it is a t = 0. This is a constant of the motion. p squared is independent of time because this is just the function of the Hamiltonian itself. No quantity which depends on Hamiltonian without explicit t dependence has its expectation values independent of time. It depends on what the initial momentum spread was. We don’t care but it is some number.

So that immediately says therefore, d over dt of x squared = 2 over m squared p square multiplied by time, i am just integrating with respect to time + d over dt of x squared evaluated a t = 0. That's a constant of integration. These expectation values are real numbers and they are functions of time. They are not operators or anything like that. They are not wave functions. in general, these would be some complex numbers but these are Hermitian operators and the expectation values are some real numbers. And they are functions of time. So we get our first result which says the second moment of the position is related to time by a linear function + a constant of this kind. You can integrate it once again. so it says x squared as a function of time is p squared t squared over m squared + let's call this k_0 (Refer Slide Time: 30:10). This is the value of the spread in the wave packet, the second moment of the position at t = 0. Its some constant + k_0 t + x squared at t = 0.

Now notice that for a free particle classically, moving at a constant speed, its position as a function of time is linear in time and the square of its position is a quadratic function of time. in general, these are of the form at squared + bt + c. you are almost there except these are for expectation values and looks like this (Refer Slide Time: 30:52). And it's a trivial to ask what is x itself.

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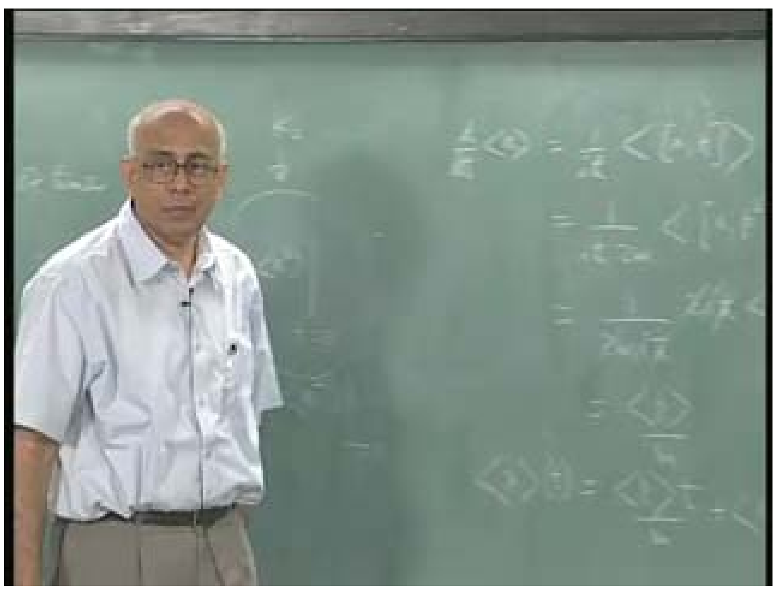

So let's do d over dt of x average itself for a free particle. Once again this is completely trivial. This = 1over ih cross expectation value of x with the Hamiltonian, H. once again i use the Heisenberg equation of motion and that is = 1 over ih cross over 2m, the expectation value of [x,p squared] because there is no potential and even if there were, this would cancel out in this particular case. This = 1 over 2 m ih cross 2i h cross, expectation value of p in this case. The 2ih cross cancels and it gives you result which you expect is p over m. So the peak position, if you start with a wave packet which is centered at the mean value, that works expectation value of x (t) is therefore = a constant which is independent of time because it commutes with Hamiltonian. This is <p>t over m + <x> at t = 0. What does this remind you of? this is how a classical particle would moveif you tell me its speed, its p over m and the speed is constant and momentum is conserved in the problem. Then its position at time t is a position at time 0 + speed multiplied by time or velocity multiplied by time.

And now the same equation is being obeyed by expectation values in general. Even for a non free particle this, is true because this quantity here (Refer Slide Time: 33:17) would have an extra V (x) but x would commute with that V(x). The expectation value of this physical quantity behaves like a classical particle. This is true for a free particle because I assumed this (Refer Slide Time: 33:41) quantity was independent of time and that’s only true if you don’t have the potential. So while this first step goes through, this (Refer Slide Time: 33:41) commutator is correct and this (Refer Slide Time: 33:51) quantity you can’t integrate now to say this is pt unless you have a free particle.

So we are only working with a free particle. And incidentally this was also true for a free particle as we saw a little earlier. So quantum mechanical expectation values obey classical equations of motion. That is the content of the Ehrenfest theorem. This is not always true. it is not true for instance if the classical dynamics is chaotic. Then this theorem breaks down and that took a while to realize but this is a very simple case because there are no other problems like ordering problems and so on. This theorem is true and this was first pointed out by Ehrenfest and i have written a little about it in the problem set.

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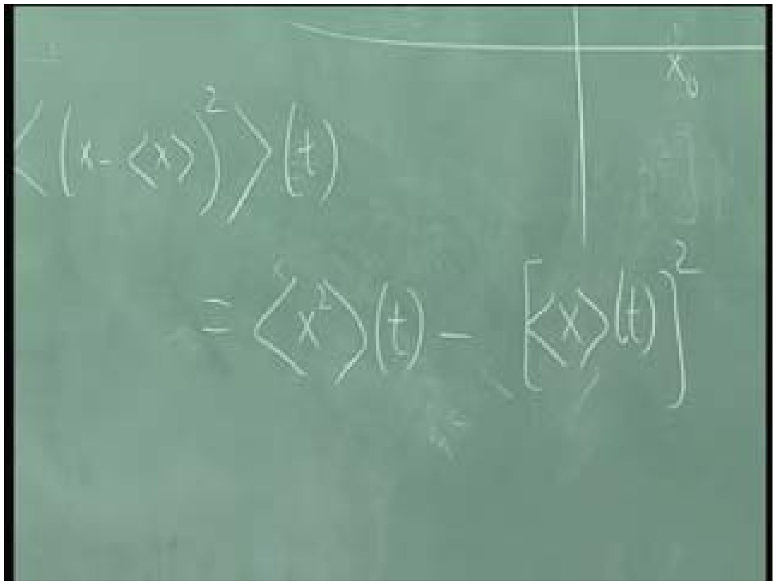

Now you could ask what does the uncertainty do and what does the wave packet do? And you will see immediately that i start off with the wave packet at t = 0, suppose along with x you had wave packet preferred like this. This was my mod psi squared, the initial probability density. So mod psi (x, 0) whole squared looked like this. So there we some initial position x_0 and there was some width here which is related to delta x, which is = x - average x whole squared to the power half at t = 0. That was my initial uncertainty in the position and the initial wave packet centered about some x_0 c (Refer Slide Time: 35:00 to 35:47). Now I let it go and let's suppose the expectation p is in the forward direction. Then you are guaranteed that the expectation value of x will change with time linearly in this fashion. This (Refer Slide Time: 36:07) number was x_0 out there for that symmetric distribution and the mean squared would increase with time also like t square here (Refer Slide Time: 36:15). Now the question is, what does the uncertainty do? So what you have to compute is, the expectation value of x - x average whole squared as a function of t, but that is the same as x squared as a function of t - x average as a function of t whole squared. So all you have to do is to square this term and subtract from this (Refer Slide Time: 36:45). What would the uncertainty do?

Would it increase with time? In general this (Refer Slide Time: 37:07) doesn't cancel. This is guaranteed to be a positive number and will increase with time. There is dispersion and this wave packet actually increases with time. So I leave it you to find out what the increase is. its very much like the solution of the diffusion equation in this case because this equation is more or less the same for a free particle. and this dispersion occurs because you can regard this pulse as being made of different Fourier components and the phase velocity and the group velocity would not be the same because for a free particle, remember that the energy momentum relationship is quadratic. E is p squared over 2 m or His p squared over 2 m. so if I write the energy as h cross omega, the angular frequency omega in the wave picture and the momentum as h cross k, where k is wave number, then i get a relation which says omega (k) = h cross k squared over 2 m. so there is quadratic relationship. this immediately means that d omega over dk is not = omega over k. therefore the group velocity is not = the wave velocity or the phase velocity and different wave numbers would propagate with different speeds. Therefore you get a dispersion.

That's being reflected here (Refer Slide Time: 38:46) in saying that if i start with an initial wave packet, its going to spread as a function of time even if there is no potential because the relationship between the energy and the momentum is not linear one or the wave frequency and the wave number is not linear. Give me an example of dispersionless motion. Well, again if i put light through a medium, it certainly disperses. If you put it through prism or water or air, it will disperse because passage of light through a medium involves the interaction of photons with the atoms of this medium. And there is no guarantee that this interaction is the same for all frequencies. It is certainly not true. so this is what is responsible for dispersion. You know the refractive index depends on the wavelength but when does light not suffer any dispersion? Certainly in a vacuum or free space, it doesn't suffer any dispersion at all.

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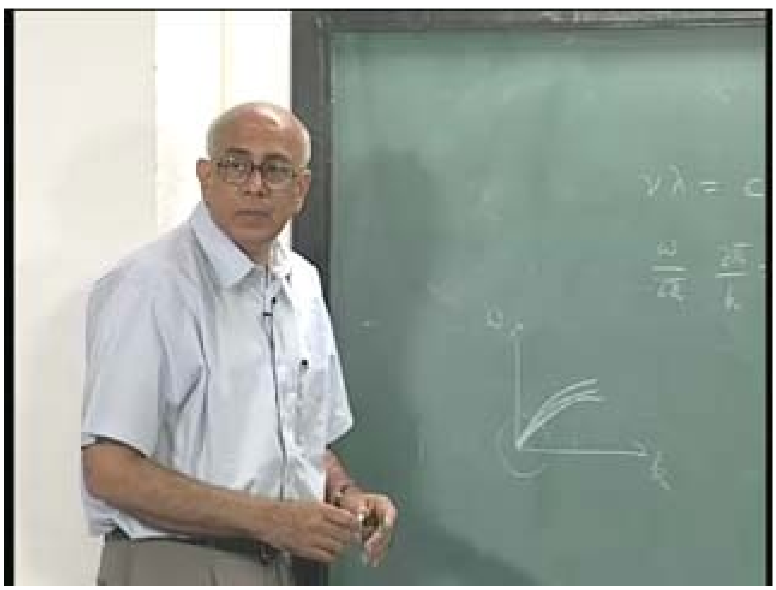

So we start by saying nu times lambda = c, where nu is the frequency and lambda is the wave length and c this is = the speed of light in vacuum. mu is omega over 2 pi, the angular frequency and lambda in terms of the wave number is 2 pi over k. Hence omega = ck. so this implies that d omega by the k = c = omega over k and you have no dispersion. So electromagnetic waves of all wave lengths travel with exactly the same speed in a vacuum and the reason is of course that the relationship is linear in this case.

Similarly if you look at sound, omega is = ck where c is the speed of sound. If you look at sound propagating in a solid then for acoustic phonons, omega still = ck. but in a crystal this is not what happens. If you plot the wave frequency versus wave number for acoustic phonons inside a crystal because in a crystal, sound is propagated by all these phonons moving cooperatively in a crystal. While the relation starts linearly, it tapers off to what is called the Brillouin edge and in this region, you have what are called acoustic phonons (Refer Slide Time: 42:06). If this crystal has anisotropy, in other words, along the principle directions, the crystal structure is different or the lattice constant is different. You have a simple cubic crystal and all 3 directions have exactly the same lattice constant. then you have the tetragonal crystal where the 3 principle axis are still perpendicular to each other but 2 of them have the same size, same length a and a and the third side is c. then sound propagation in the 3 directions splits up and you have branches one of which is doubly degenerate and the other splits off in this fashion (Refer Slide Time: 42:56).

The slopes are different. so the speed of sound is different in the 2 directions and if you have an orthorhombic system where you have 3 lattice constants a, b, c in 3 perpendicular directions, then you really would have 3 branches (Refer Slide Time: 43:19); and the propagation is very different but they are all linear to start with or at least very close to this edge. What would happen if the photon had a mass? After all, that’s what electromagnetic waves are. When you quantize them, you get photons. And this relation here (Refer Slide Time: 43:38) is equivalent to saying that the energy of a photon is linearly dependent on its momentum because all you have to use is the Einstein De Broglie relation. Epsilon is h nu and so, nu is epsilon over h, Planck's constant and lambda is h over p. that’s = c or epsilon = cp. that is the linear relation between the energy of a photon and its magnitude of its momentum and you have no dispersion at all. suppose the photon had a rest mass, it doesn’t as far as our theories tell us, but suppose it had a rest mass what would happen? What would happen to this (Refer Slide Time: 44:32) relation? It would go over into the relation between the energy and momentum for a relativistic particle and what is that?

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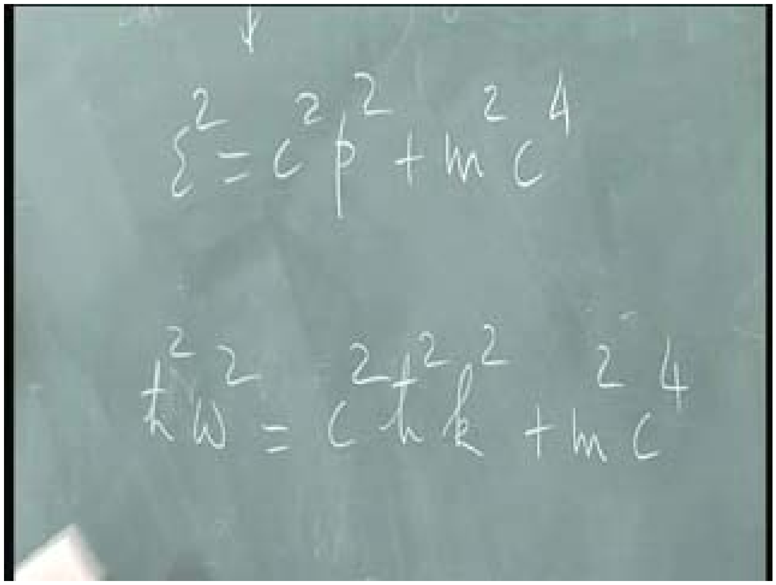
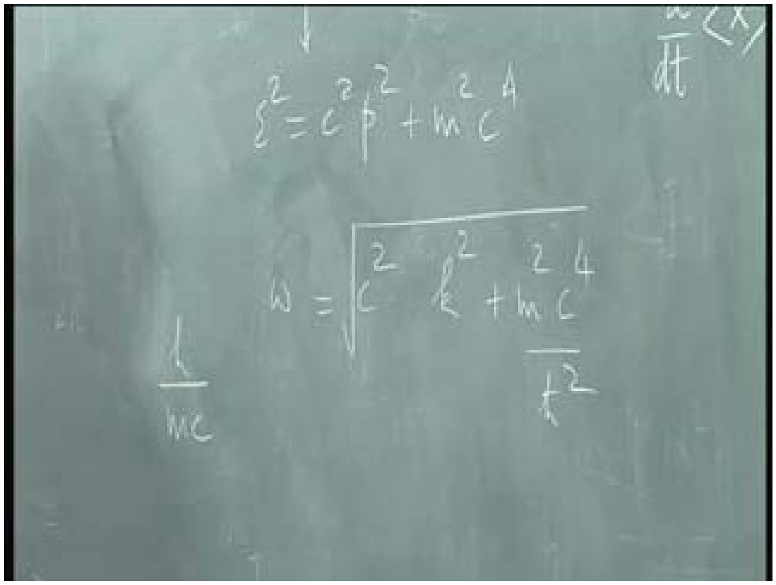


Instead of this (Refer Slide Time: 44:48) you have a different relation and that's not a linear relation. that's epsilon squared = c squared p squared + m squared c four where m is a rest mass of the particle that is the correct relativistic relation between the energy and momentum of a relativistic particle of rest mass m. in the moment you have this (Refer Slide Time: 45:18) and i put this is = h cross omega. so you have got h cross squared omega squared as c squared and p is h cross k, so h cross squared k squared + m squared c 4 and I divide by h cross squared. This is no longer a linear relation. and therefore d omega over dk is not same as omega over k, which means you would be able to detect a non zero m by looking at an event from very far away going through vacuum and finding out if there is a time delay between the arrival of protons of one wave length and another wave length. And you can actually compute what that time delay is.

And there are these very intense gamma ray bursts which happen in the universe very far away and you have to now make many corrections. You have to ask whether its going through interstellar gas and is getting dispersed on the way because of the medium and so on but by and large, most of space is empty. You look at very distance intense gamma ray burst and there are many of them happened. A notable one happened in 1998, for example, they are called The Giant Gamma Ray Burts, GRBC. You then wait for the gamma rays to come and for the optical photons and the radio photons to come and so on. And sometimes there could be a delay of days or seconds between them depending on the wave length. By measuring the time delay between say optical and radio waves from the same event, gamma would come in practically instantaneously and you can compute what m is.

(Refer Slide Time: 00:47:11 min)

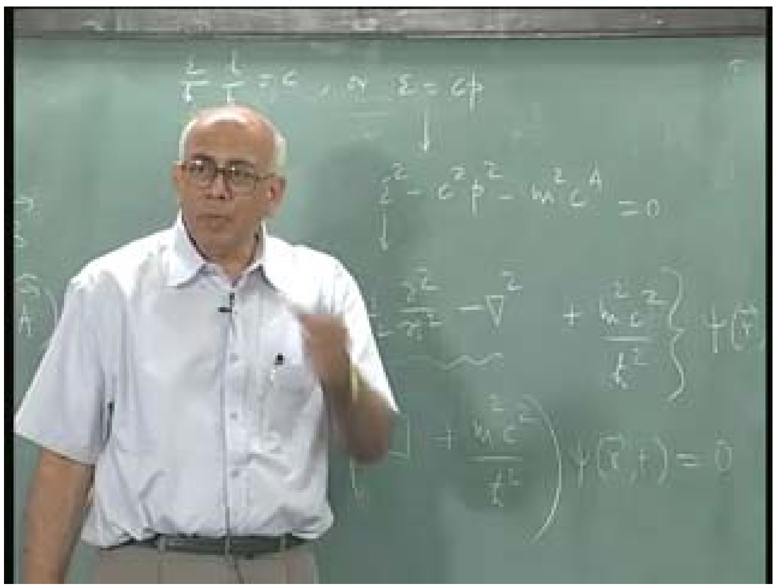


By the way what is the significance of mc? You notice that mc is a typical momentum. It's called the Compton momentum and of course h over mc is the Compton wavelength of a particle. And its like the size of the particle. if it all you can attribute a size to these, quantum objects h over mc is a good candidate, it's like the extent of this particle. What happens if m is 0? It is infinity. So a photon is completely delocalized. A single monochromatic wave is delocalized and it extends for - infinity to + infinity. On the other hand, you get it more and more compact, it's almost 0. And then, there is a big pulse here and then 0, you localize it more and more. It's clear that in wavelength, you really spreading it more and more. The more you localize in the position, the more you are spreading in the momentum or wavenumber. So this is used in fact, to put up bounds on this m. we never say something in 0 in physics. That's very clear. All our theories require that the photon have an exact 0 rest mass but then of course experimentally, we can never say something is 0.

It turns out that the best bounds on the photon would correspond to m of the order of less than 10 to the - 50 kilograms or 52 kilograms. There are estimates which come from the galactic magnetic fields of the order of 10 to the – 60 and more kilograms but they are not as reliable. What is the other consequence of the photon not having 0 rest mass? You would have to go back and re-examine Maxwells equations. And they have written down for 0 rest mass because if you if you recall, you had the wave operator acting on the electric and magnetic fields in free space. You had electromagnetic radiation in which the electric and magnetic fields obeyed the wave equation and that gets modified by precisely this (Refer Slide Time: 49:45) term here because this is again quantum. Let me write this equation quantum mechanically for a particle and use the old Einstein De Broglie way of going to quantum mechanics. So bringing all the terms to one side, we get epsilon squared – c squared p squared – m squared c 4 = 0 and now this is this is like an eigenvalue equation. So really this comes from the Hamiltonian and the Hamiltonian acting on a wave function is ih delta over delta t acting on the wave function. So this is - h cross squared d 2 over dt 2 acting on the wave function psi - c squared and p in the position basis is - ih cross gradient acting on the wave function.

So c squared - ih cross squared del squared acting on the wave function - m squared c four acting on the wave function in position space = 0 if the particle is described by a single component wave function like psi (r, t). Simplifying further, you get this (Refer Slide Time: 52:16).

(Refer Slide Time: 00:51:59 min)



If you didn't have this (Refer Slide Time: 52:18) term, then it just the wave equation which is what the electric and magnetic fields and so on would obey. and in fact, in the case of the electromagnetic field, you know that you can replace the electric field E and the magnetic field B by a scalar potential phi and a vector potential A. together these form the components of a four dimensional vector in four space and this quantity here(Refer Slide Time: 52:56) is the four dimensional analog of the Laplacian operator in 3 dimensions with the time metric taken care of; namely that is c squared t squared - r squared which is the square of the interval and is called the D'Alembertian operator or the box operator box + m squared c squared over h cross squared on psi (r, t) = 0 would be the equation for a particle which is described by a single scalar wave function, psi(r,t) this equation is called the Klein Gordon equation.

It replaces the Schrödinger equation in relativistic quantum mechanics. It's an interesting historical side light that Schrödinger actually wrote this equation down first even before he wrote down the original equation for the nonrelativistic case. He wrote this down first and then gave it up as unphysical. and the reason he gave it up was because if you calculate from this quantity, the probability density in this problem, then instead of the probability density being mod psi squared which is the probabilistic interpretation of quantum mechanics, if you compute from this quantity the probability density from this equation it turns out to be psi star delta psi over delta t - delta psi star delta t psi. That turns out to be the probability density rho. Then he was puzzled and he thought this was wrong because that quantity on the right hand side take either sign divided by i.

But it can take either sign and he wondered how probability density can be negative and so he gave it up. And then Klein and Gordon wrote this down and they got the correct equation. And then once he wrote the non relativistic equation, mod psi squared was a probability density and everything came out consistently and there was no difficulty. But in fact, he was not wrong. it turns out that this equation is applicable to charged particles And then if you multiply by e, it is the charge density and there is nothing which says those charge densities have to be positive or negative. So he missed out on discovering relativistic quantum mechanics although he discovered it first. But in any case this equation goes by the name of the Klein Gordon equation.

Coming to the photon, you have four quantities and together, you know that phi over c A is = the four vector potential A mu and Maxwell's equations in their original form is represented by del A mu = 0 or the current density on the right hand side if you have sources. That's what a free proton does. but now you replace it by this + that (Refer Slide Time: 56:23) acting on A mu as a function of r and t = 0. So Maxwell's equation, the D'Alembertian on A mu = 0 is replaced by this wave equation. You must know regard this as the wave function of the photon. This equation has a name it's called Proca's equation.

So even the basic Schrödinger equation for the photon which is really relativistic always and therefore its a version of the Klein Gordon equation in this case, is something else once you go to a massive photon. now please recall that if you go to electrostatics and this (Refer Slide Time: 57:20) term goes away and you work in what is called the gauge the Coloumb gauge, then you know that this leads in electrostatics to del square phi = rho over epsilon $_0$ and if you got a single charge at the origin then this is = a delta function at r over epsilon $_0$ with - sign multiplied by the charge density. What is the solution? it is the electrostatic potential due to a point charge at the origin and that will tell you that its phi = e over 4 pi epsilon $_0$ r. the moment you have this (Refer Slide Time: 58:17) extra term in this static limit, this potential gets modified. Notice that the scalar potential going like1 over r leads to the inverse square law force.

It leads to the fact that Gauss's law in electrostatics is invalid. You put a charge inside some region and the flux over a surface or of the electric field around this surface is = the charge divided by epsilon $_0$. All those things go out of the window the moment you have correction to the photon mass. This (Refer Slide Time: 58:52) gets modified and instead goes like e to the power - mu r over r and this potential is not a 1 over r potential. It dies down and it will for much greater than that and this mu must have the inverse dimensions of length. It is the inverse of the Compton wavelength. so this mu = mc over h. therefore, the inverse square law is not valid, that Cavendish experiment to test the inverse square law will undergo modification, Gauss's law in electrostatics will undergo modification and you can actually test and find out if the inverse square law is valid or not and get a limit on the photon mass. This is our best experiment yet and it shows that the photon mass, if it exists, is certainly less than 10 to the - 50 kilograms and constantly people try to improve it. So you see there is a very intricate chain of argument but not very difficult to understand. it takes you to these things and now in this course, I am also going to discuss, not the electromagnetic field but i am going to talk about spin and then we will make some connection between why you get more than one component here (Refer Slide Time: 01:00:02) and what the role of this spin is in making sure that you don’t have the scalar wave function here.

So that’s one of the topics we will look at very shortly but coming back to what we did about wave packets, i thought you should appreciate the fact that going the nonlinear relationship between energy and momentum even for a free non relativistic particle, any initial wave packet disperses in space. Thank you!