**Quantum Physics**

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**Lecture No. # 13**

Let’s go back to where we were at. We are in the middle of our discussion of coherent states of the oscillator. I have already mailed you problem set 3 which had large number of questions on coherent states and the like. So let me continue in this tenure a little bit. We will come back to coherent states as we go along, in particular the significance of coherent states. Student – using the Dirac’s operator method, where we decomposed $2^{nd}$ order into $1^{st}$ order differential equations, can we judge how any of these equations be broken down in that method? Professor – It’s a good question again. If I look at problems of a particle moving in a potential and for the moment let me look at particle moving in 1 dimension in a potential, then the time independent Schrodinger equation is the following.

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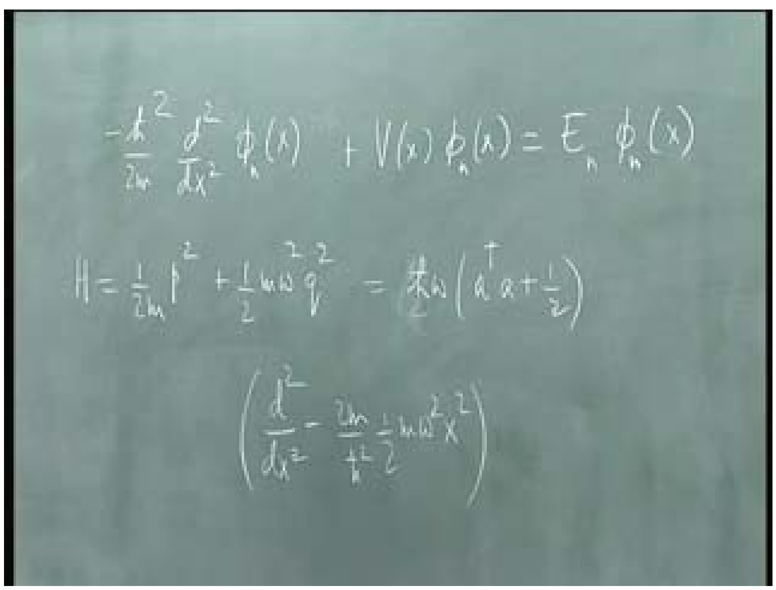

- h cross squared over 2 m d 2 over dx 2 phi$_n$ (x), for the moment assuming that there is a quantum number n which labels the eigenstates of this problem + V(x) phi$_n$ (x) = $E_n$ phi$_n$ (x). And the question asked is if we can solve this for the case when x = 1/2 m omega squared x squared. That is the harmonic oscillator and essentially, what the operator method did was to break this down. so if you recall this Hamiltonian was 1 over 2 m p squared + 1 half m omega squared q squared and then I defined operators a and a dagger.

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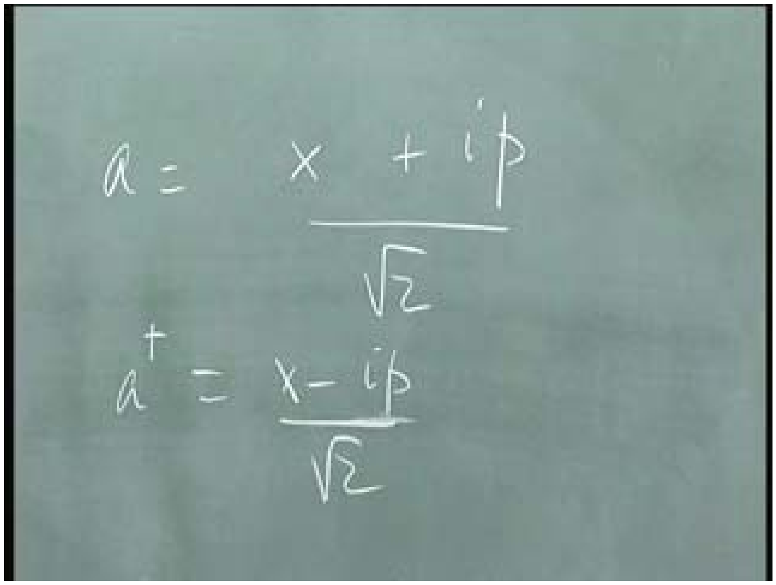

I defined a as = x + ip over square root of 2 in suitable units. x in units of h cross over m omega to the power 1/2 and p in units of 1 over square root of h cross m omega, then a became this and a dagger was x - ip over root 2 and this Hamiltonian went over into 1 half into h cross omega a dagger a + ½ . What this did was, the a dagger is x - ip and p in the position basis is - i d over dx. So this was like x - d over dx and this was x + d over dx and you multiplied those 2 together and you ended up with this (Refer Slide Time: 4:06) operator d 2 over dx to + x squared. so essentially the purpose of this exercise was to take this operator d 2 over dx 2 - 2 m over h cross squared 1/2 m omega squared x squared which acted on $phi_n$ to give you $E_n$ and factor this into 2 first order differential operators. This is what was done. And the question is for other forms of V (x) are it possible to do this or not. This is not always possible to do. So you can't always write this as a product of 2 operators. For certain kinds of V (x), you can do this. This method is called the Born-Infeld method or the factorization method or the method of intertwining operators.

There is a very extensive literature on this and people have carried out very extensive studies on precisely when you can do this and it has been systematized and it's been related to something called super symmetric quantum mechanics which again is specific to these kinds of potential problems and has other implications elsewhere. So there is a field of study which has actually worked out in the cases where you can solve it by this factorization method. The harmonic oscillators are the most notable examples of all. What is this special thing about the harmonic oscillator which nothing else has? Equal space, of course. I will show you that an infinite number of potentials for which you have equal spacing. They are called isospectral oscillators because they have exactly the same spectrum as the harmonic oscillator and you can systematically construct things and the harmonic oscillator is simplest of the lot.

Now we looked at coherent states as superpositions of the eigenstates of the number operator a dragger a, you can construct all kinds of super positions but the important thing is to find out is it an eigenstate of some operator or not. And we found the operator a even though it is not Hermitian as a set of normalizable eigen states. a dagger in contrast doesn't have this. Now it is a simple matter to actually write down the matrix representation of a a dagger x and p. So let's do that quickly.

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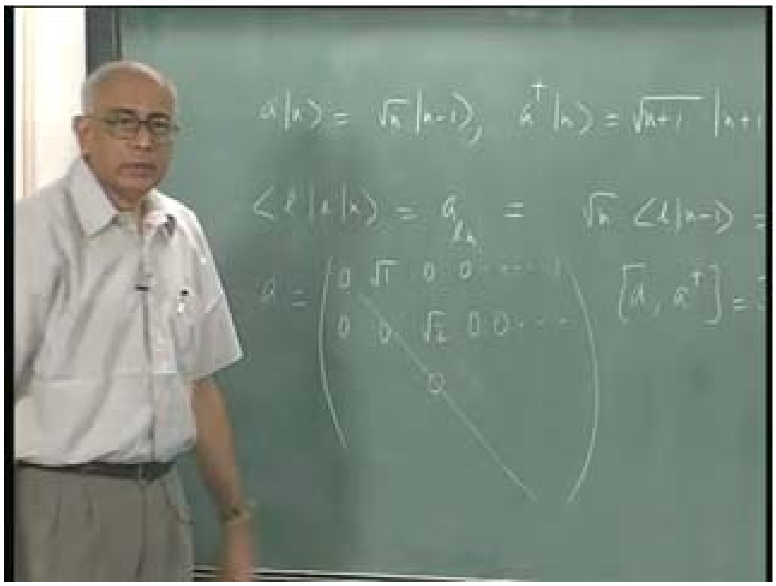

In the harmonic oscillator problem, all we need to do is to notice is that a on n = square root n on n - 1 and a dagger on n is square root n + 1 on n +1, for n = 0,1,2, etc. because, the moment you have this, then you can write down the arbitrary matrix element here. So lets use l a n = a l n by definition. This is the matrix element because I am going to write these operators as matrices in a bases of these number operator states. So by definition this is the matrix element. and this gives you square root of n l with n – 1 which is square root of n, a kronecker delta of $_{l,\ n\ -\ 1}$ because these are orthonormal. Therefore what is the matrix representation of a? You have 0 0, 0 1, 0 2, 0 3, 1 0, 1 1 etc. so we start by labeling with 0 here. So what is the 0 0 element? l is 0 and n is 0 and that is not possible as you can see l must be = n - 1 or = l + 1. So the 1st element is 0 but the 2nd element exists and n is 1 in this case, so this is equal to square root of 1 and then 0 all the way with infinite number of them. and then you get a 0 here (Refer Slide Time: 08:27) and a 0 here and you get a square root of 2, 0 etc. so the matrix a is an upper triangular matrix in which you have 0's on the principle diagonal and then everywhere else you have the square roots of the natural numbers. That is the matrix representation of a. and a dagger is just the Hermitian conjugate of it. It's easy to verify that the commutator of a with a dagger is indeed equal to the unit matrix, the infinite dimensional unit matrix. [a, a dagger] is I which is just ones on the principle diagonal everywhere.

It should be obvious to you that you cannot represent a and a dagger by finite dimensional matrices. The reason is i just take the trace of both sides and the trace of the left hand side is 0 by definition, because trace ab = trace ba, even if a and b don't commute with each other. But if you have a finite number there, then of course you have a contradiction right away. For infinite dimension matrices, these things don't generally work if these are unbounded operators. These operators are not bounded because x and p are not bounded operators. Their eigenvalues go all the way up to infinity. you could also write down x and p because x is just a + a dragger over root 2 or something like that and therefore it will have a root 1 here, root 1 here, root 2 here, root 2 here, etc. and similarly for p, so while in the position basis, the operator x just means multiply by x and the operator p means differentiate with respective to x in the Fock basis. The operator x is a certain infinite dimension matrix and a dagger is its conjugate matrix and similarly x and p.

Now having said this, let's go back a little bit back track and talk about the uncertainty principle itself. We know that there is an uncertainty principle involved here because x with p, the commutator is ih cross that implies that delta x delta p must be greater than equal to 1/2 h cross. We would like to see what is the general uncertainty principle is for 2 operators a and b. so i start by saying the 2 physical operators a and b reference to a certaitn system are compatible with each other if they commute with each other. There are several ways of asking what the implication of this is. So let me write that down.

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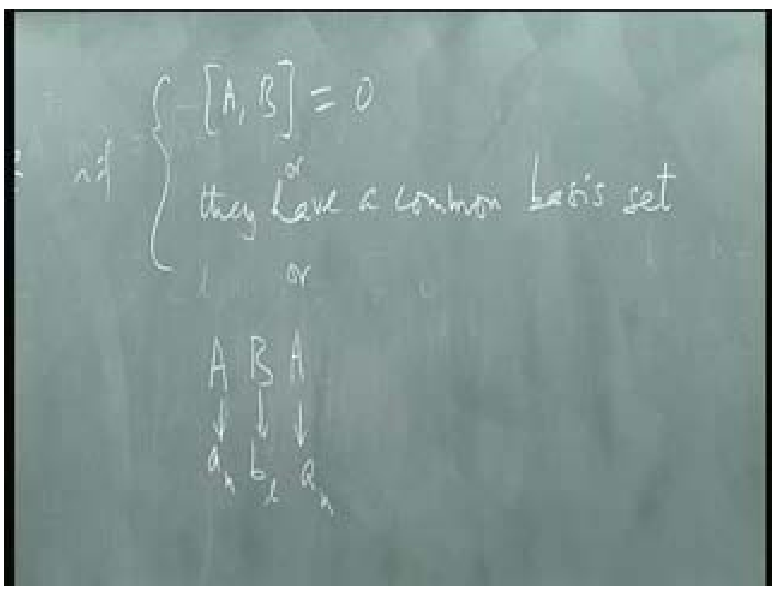


A and B are compatible if any one of the following things is true. Commutator of A with B is 0 or they have a common basis set. There is a common basis set of eigen vectors that you can find in the Hilbert space such that these are eigen states of A as well as B simultaneously. Or if the following is true, you measure A and you get a certain number which is one of the eigen values of A and you immediately measure B, you get one of the eigenvalues of B and you measure A once again in principle immediately before anything can happen and if you get the original number back, then it means that the state is unchanged at all and therefore A and B are compatible with each other. This is entirely equivalent to saying that A and B commute with each other. so its a measurement sequence A followed by B followed by A, if this leads to one of the eigen values of A, let's call it $a_n$, this will lead to the one of the eigen values of b, let's call it $b_l$. and you measure this once again and you get back $a_n$, then a and b are said to be compatible with each other. Now of course, once you have a quantum mechanical system, the trick is to find all the operators which are compatible with each other, the maximal set of mutually compatible operators and they would have a common eigen basis. Generally, you would include the Hamiltonian in this set simply because that governs the time evolution of the system and as you see, the eigen states of the Hamiltonian are stationary states and they particularly play an important role in quantum mechanics.

Therefore in any maximal set of mutually commuting observables or compatible observables, you would always include the Hamiltonian. Other than that, the choice can be non-unique. It very often depends on what your physical system is especially when we look at an example like angular momentum; you would use the Hamiltonian, the total angular momentum and any one component of the angular momentum because they don't mutually commute with each other.  so let's suppose you have a quantum system in which you have 2 operators, A and B which need not be compatible with each other, could be but need not be compatible with each other. Then we can say something about the uncertainty in A and B. I am going to assume that A and B are self adjoints so that they are real eigen values.

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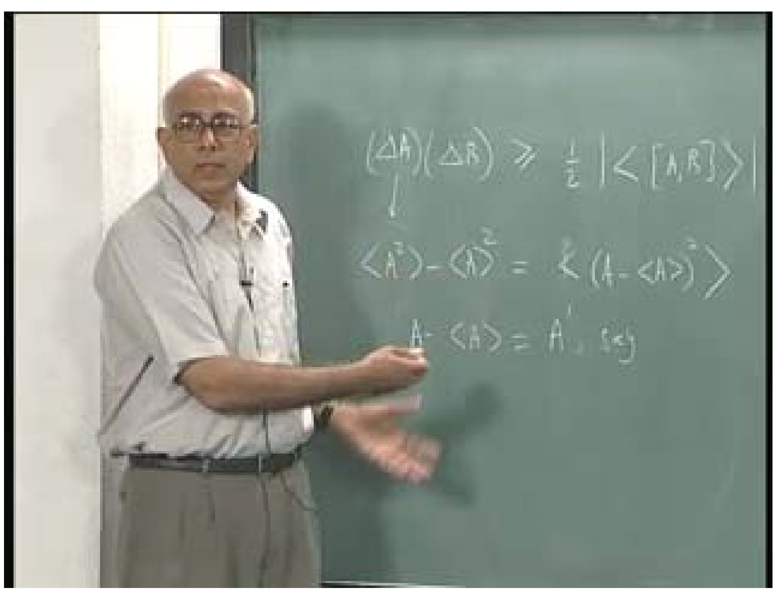

It can be shown that in any state of the system, delta A delta B, which is the standard deviation in A in the state concerned and for B similarly, is greater than equal to 1 half the modulus of the expectation value of the commutator of A with B in that state. So this is the generalized uncertainty principle. We will establish it in a minute. How do we establish this? We will simplify a matters a little bit. What is meant by delta A? This stands for A squared - A average whole squared in the state concerned which is the same as saying it is the expectation value of A - A average squared. This of course we know with the variance is the mean square - the square of the mean but it is also the mean value of the square of the deviation from the mean. so for simplicity, lets remove this A average and B average and subtract those 2 and lets call this something else. Let's call A - A equal to A prime, say and similarly for B. What is the expectation value of the commutator of A with B here? It is the same as the expectation value of the commuator of A prime with B prime because A prime and B prime are just A and B with some number subtracted. These are C numbers, classical numbers not operators because this is an expectation value.

In general, it is a real number. so It plays no role in the commutation and you could just as well write this but that is equal to in whatever state your in, it is equal to psi A prime B prime psi - psi B prime A prime psi, assuming the state is normalized otherwise I carry that in the denominator all the time. Let's assume this state is normalized to unity. But A is Hermiatian and A prime is Hermiatian and so is B prime. Therefore this is just the inner product. Let me revert to the inner product notations. It's A prime psi with B prime psi - B prime psi A prime psi. So you make this a bra vector and that a ket vector and these (Refer Slide Time: 18:00) are Hermitian. That is why I have written B here rather then B dagger. But what is this (Refer Slide Time: 18:06) number it is a complex conjugate of this (Refer Slide Time: 18:08) number. When I take the 2 in a bra and a ket, when I do a complex conjugation, it just changes the order in which the inner product appears.

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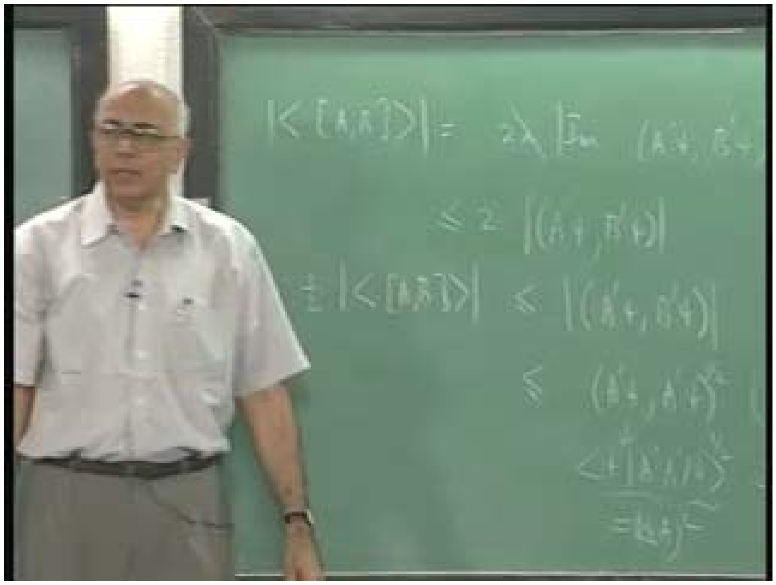

So expectation value of A B is equal to 2 i, imaginary part of the expectation value of this quantity (Refer Slide Time: 18:37) because this is some complex number and that - it complex conjugate is 2 i times the imaginary part. As the key step is this. This (Refer Slide Time: 18:56) is equal to A prime psi B prime psi star. Therefore, if I take modulus i goes away. And then it becomes twice modulus of quantity. But that (Refer Slide Time: 19:20) is less than or equal to twice the magnitude of A prime psi B prime psi. This is because the magnitude of the imaginary part is certainly less than or equal to the modulus of the complex number itself. therefore half expectation value of A B is less than equal to the magnitude of the inner product of this (Refer Slide Time: 19:53) quantity. This is some ket vector that is some other ket vector and you are saying this is the magnitude of the scalar product. Now what does the Cauchy inequality tell you? It says if you have any 2 vectors of this (Refer Slide Time: 20:22) kind, the magnitude of this is less than equal to the norm of chi the norm of phi. This is the Cauchy inequality. There is another way of saying that a dot b in 3 dimensional space is ab cos theta. therefore magnitude of a dot b is less than equal to magnitude of a magnitude of b therefore this then less than equal to (A prime psi, A prime psi) to the power half ,(B prime psi B prime psi) to the power half but what is this equal to?  This (Refer Slide Time: 21:54) is equal psi A prime A prime psi. And you want that to the power a half. Similarly, this (Refer Slide Time: 22: 10) is equal to psi B prime B prime psi to the power half. But A prime is A - A average. And when you square it and take expectation values, you get precisely delta A whole squared. So that establishes this (Refer Slide Time: 22:36).

This here (Refer Slide Time: 22:45) by definition is delta A whole square. it is A - A average whole squared, expectation value. so it is precisely this (Refer Slide Time: 23:00) quantity which is delta A whole squared and you take square root you get delta A. so delta A is the standard deviation, delta A squared is the variance. So this proves this generalized uncertainty principle. It is certainly the most general one but sometimes, you can actually do a little better because what we did was to replace this (Refer Slide Time: 23:46) by this. So you could actually put in how much less it is and so on and get a more general expression. There is another origin of the uncertainty principle called the Schrodinger Robertson way of writing the uncertainty principle which adds another term to this but this is true in all cases. As we have seen, there are cases like the ground state of the harmonic oscillator or every coherent state where the uncertainty principle is saturated, in the sense that you do get the minimum uncertainty. Now this (Refer Slide Time: 24:24) is the very useful form of result. We are going use this result here but notice even if A and B don't commute with each other, so that you have an uncertainty in a and an uncertainty in B which is non-zero.

I mean the commutator is not identically 0. It's conceivable that there exist a particular state in which this expectation value vanishes. And then in that particular state, the uncertainty product does go down to 0, even if A and B don't commute with each other. But if A and B commute with each other, this is identically 0. And then there is nothing here on the right hand side. It just says you can put a common eigen basis. You can measure A and B simultaneously with arbitrary precision. Otherwise you have this uncertainty principle and you can't get away with it.What happens in some cases is that the commutator of A with B becomes just a multiple of the unit operator and then you get a pure number times a unit operator on right hand side.

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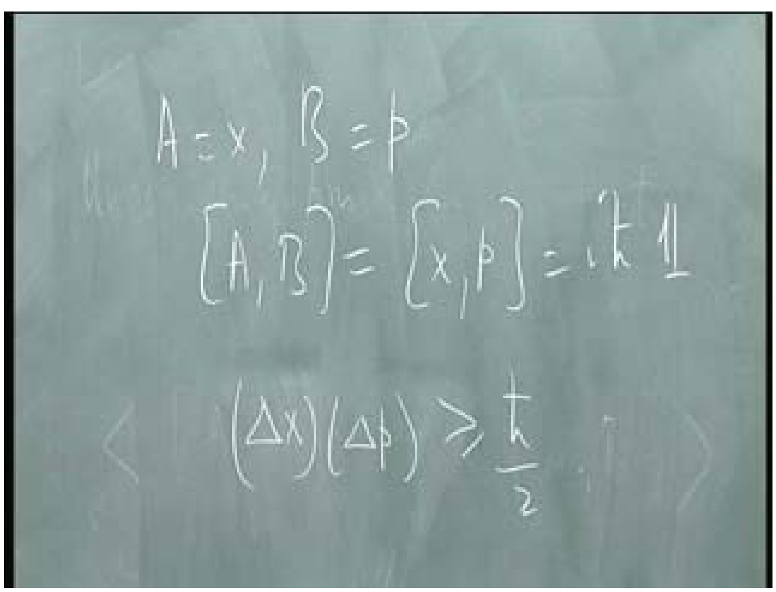

So for instance, the most common case is A is x, B is p, then [A, B] = [x,p] = ih cross times the unit operator. And then the matter becomes very simple. This goes away and then you get delta x delta p greater than equal to h cross over 2 because the expectation value of the unit operator is 1 in every state. But this is not always the case. When you have angular momentum for instance, then commutator of 1 component, say the x component of the y component will give you the z component. And then what appears on the right hand side is expectation value to z component. That is the most general form. Now having said all this, now I want to go to the following. We would like to ask now that we have solved the Schrödinger equation in a few cases, we have some feel for what the state vectors look like in position basis. They are represented for bound states by square integrable functions and so on. The question is, is there a way of making contact with classical Hamiltonian mechanics where you had physical observables which obeyed the Hamiltonians equations of motion? Is there a way of looking at quantum mechanics in that frame work? The answer is yes and this leads us to two ways of looking at quantum mechanics which are equivalent. So let me do that now.

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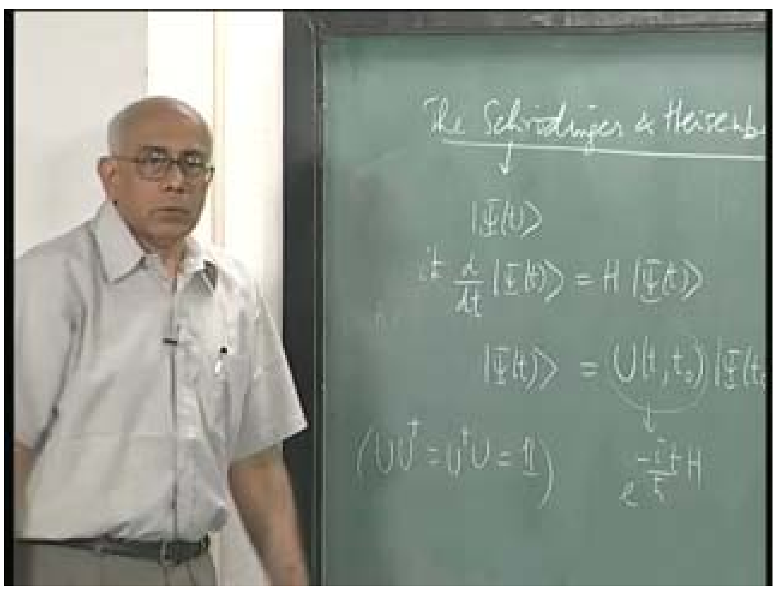


These are called the Schrodinger and Heisenberg pictures or representations, but I like to use word picture because it is more graphic. What we did so far was the Schrödinger picture. So we started by same that a quantum mechanical system is described at any instant of time by a state vector in some Hilbert space. So we assume the existence of a state vector which was explicitly time dependent and the time dependents was specified by the Schrödinger equation. So we had this input and we had ih cross d over dt psi (t) = the Hamiltonian times psi (t). what this meant was that if you specify the state of the system for me somehow at t = 0 or some instant $t_0$, I specify what the state is at every later instant of time and the later state is a unitary transformation acting on the earlier state. so the formal solution to this, in the case when H was explicitly time dependent, we showed this but then even and more general cases this is still true, this is equal to some unitary operator which takes you to time t from $t_0$ acting on psi ($t_0$). It's unitary because the norm was preserved. so the fact that psi (t) psi (t) norm which is the square of the norm at time t = psi (0) psi (0), this implies that the operator is unitary in between and in the case when the Hamiltonian was explicitly time independent, this U actually turned out to be e to the - i over h cross t times H. this was a Hermitian operator and e to the i times Hermitian operator is unitary operator. So we were guaranteed that U U dagger is U dagger U is equal to the identity operator.

So the picture is, you have a state vector which describes everything you know about the system and the state vector is undergoing in the Hilbert space, a unitary evolution. This is going from 1 state to another by a continuous set of infinitesimal transformations which build up to unitary transformation. Now as you pointed out, the norm is preserved which is like saying the magnitude of this vector is preserved. The picture one has in mind is that this vector, if you look at in ordinary 3 dimensional space and you ask when the norm of a vector is changing, you would say it is moving on the surface of a sphere. If you put the origin at the tail of this vector, then this vector is meandering around on the surface of a sphere so that its length doesn't change. So it's something like that except we got to be a little careful. If I multiply this state vector by a complex number of unit modulus, I still don't change this state vector. Remember this meandering is happening in Hilbert space.

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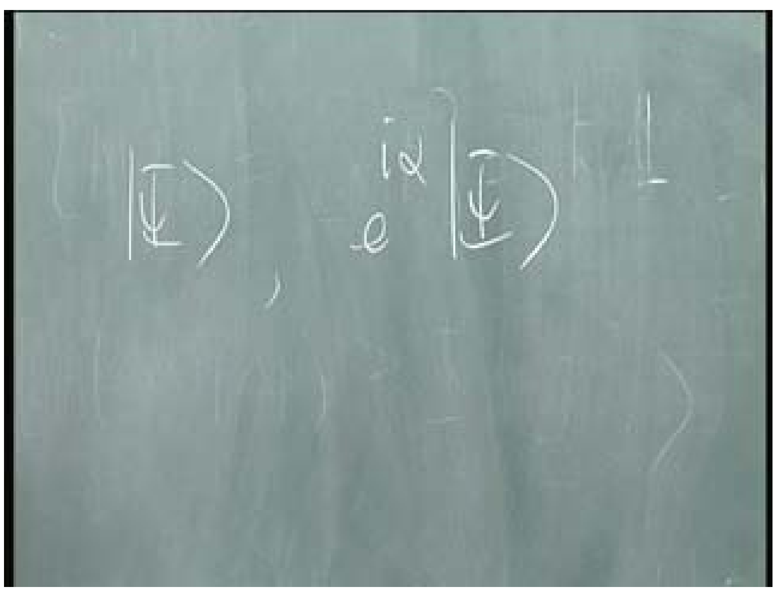

So in Hilbert space, if I have a state vector psi and I have a state vector e to the i alpha on psi, where alpha is the real number, these have exactly the same norm because a bra is e to the - i alpha and that cancels out. So multiplying by a complex number of unit modulus doesn't change the norm of a vector in the vector space. So all those vectors are equivalent. It is as if in ordinary 3 dimensional space; imagine that all vectors along a given direction have all been projected to just the unit vector. And you go to another direction that is a different unit vector. But everything has been projected down to this unit vector. In ordinary 3 dimensional space, if I say that multiplying a vector by a real constant doesn't change its direction.

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So it's as if i start by saying if you have a vector here (Refer Slide Time: 31:54), all vectors in this direction is equivalent to just this vector (Refer Slide Time: 32:00). They have all been projected on to this. This (Refer Slide Time: 32:06) direction is another story. Then you have a unit vector here and these are different from each other. They are related to each other by a rotation in the Hilbert space or a unitary transformation. This is called the ray representation because is Hilbert space is a projective space. What is physically important is not the state vector itself but the entire set of state vectors which differ from each other by just multiple of complex number of unit modulus. They all correspond to the same state physically. They all correspond to the same state. So it's exactly like saying I am looking at this space of ordinary 3 dimensional vectors but I will agree to say that everything along this direction is equivalent to this (Refer Slide Time: 32:58) unit vector.

They differ from each other only by some normalization constant. So it's like a projective space. This has deep implications. We will see some them later on. But for the movement I want it to appreciate the fact that it is as if this vector is rotating on the surface of the sphere expect it is happening in infinite dimensional space and this U is not a rotation but a unitary transformation. A unitary transformation which preserves a norm is the counterpart in quantum mechanics in Hamiltonian mechanics what you call a canonical transformation because that preserves phase space volume. The Poisson brackets structure was preserved and in fact time evolution under the Hamiltonians equations of motion was nothing but the gradual unfolding of a sequence of infinitesimal canonical transformations.

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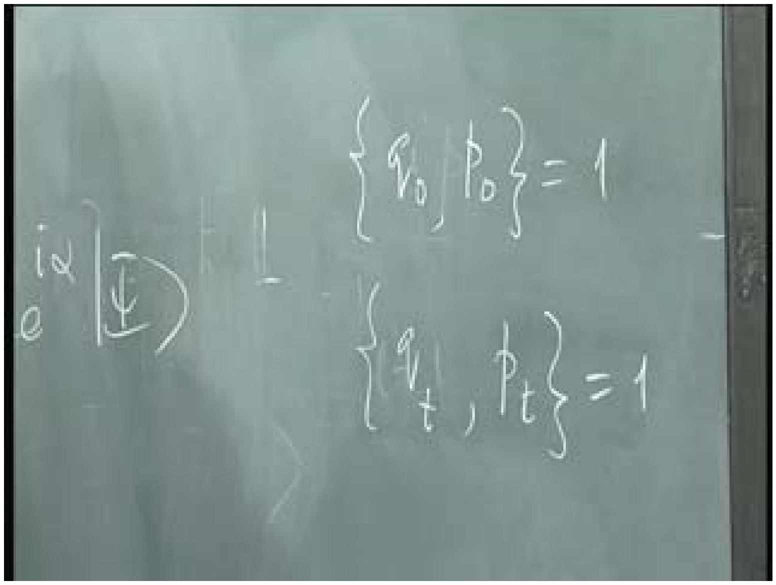

We saw classically that if you had at time t = 0, $\{q_0, p_0\}= 1$ and $q_t$ was the solution of Hamilton's equation for a given Hamiltonian with initial condition prescribed so that this is the function of $q_0$, $p_0$ and time, then we know that $\{q_t , p_t\}=1$ was still true. And this leads to the fact that volume elements in phase space were preserved. So the point in phase space occurred like that of an incompressible fluid in real space. That was the significance of these infinitesimal canonical transformations. So in the same way, just as classical Hamiltonian dynamics evolution is the gradual unfolding of a sequence of infinitesimal canonical transformations. In exactly the same way quantum evolution is the gradual unfolding of a sequence of infinitesimal unitary transformations on the state vector. So that was our picture.

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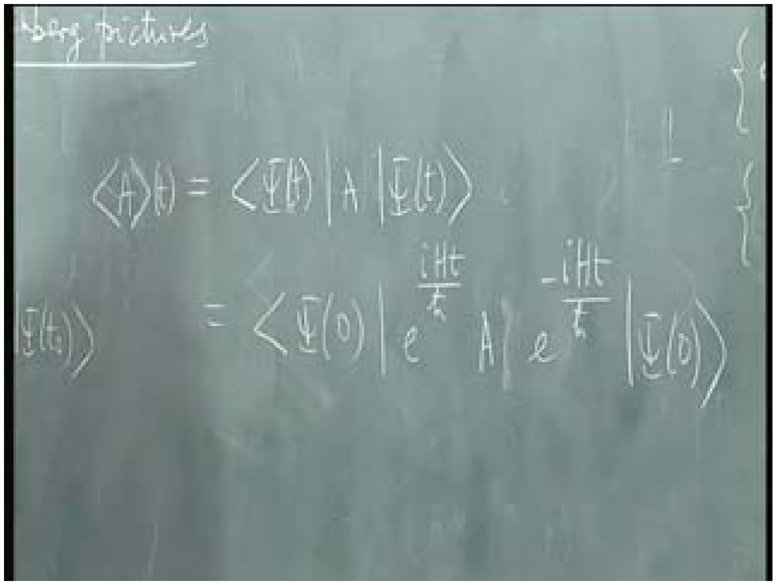


Now what about observables? Well, any observable A had an expectation value which was given by psi (t) A psi (t). So this observable itself could be position, momentum, angular momentum, etc and because you took expectation value with respect to a state that depended on time, the observable acquires a time dependent. I have assumed that the denominator remains 1 at all times. but if I rewrite this (Refer Slide Time: 35:53), and if i use this formula here (Refer Slide Time: 35:56) on this side, then this becomes psi(0) e to the power i Ht over h cross A e to the - i Ht over h cross psi(0), by definition this was equal to that( Refer Slide Time: 36:25). Please notice that there was a - here and therefore when i take the bra vector it becomes a + (Refer Slide Time: from 36:32 to 36:34) and that is very crucial. Now Heisenberg comes along and says that you have another way of looking at the whole thing. Suppose, I say that the operator itself has time evolution, just like a classical dynamical variable like position or momentum has an equation of motion, I say the operator itself has time dependence and the state vector doesn't have time dependence. So forgetting the fact that this t here came from this term here (Refer Slide Time: 37:14 to 37:23) and this thing came from this state vector here, suppose this is the same as writing some phi and then an operator A which has t dependence, and then a phi, I could as well do this. I start by saying my state doesn't change at all in any quantum mechanical system, the operators change in such a way that if you give me the operator at t = 0, which was my original operator, then the operator at time t is given by this formula (Refer Slide Time: 37:57).

No physics is changed because finally all the information we get is via expectation values. This is the Heisenberg way of looking at it. So let me now formalize this. the idea is to say if I now start putting subscripts just to tell you what Schrodinger and Heisenberg are, Schrödinger's idea was to say that a system is described by a state vector and let me say $psi_s$ for Schrödinger, the operator A is now in the Schrödinger picture $A_s$ and there is no explicit time dependence.

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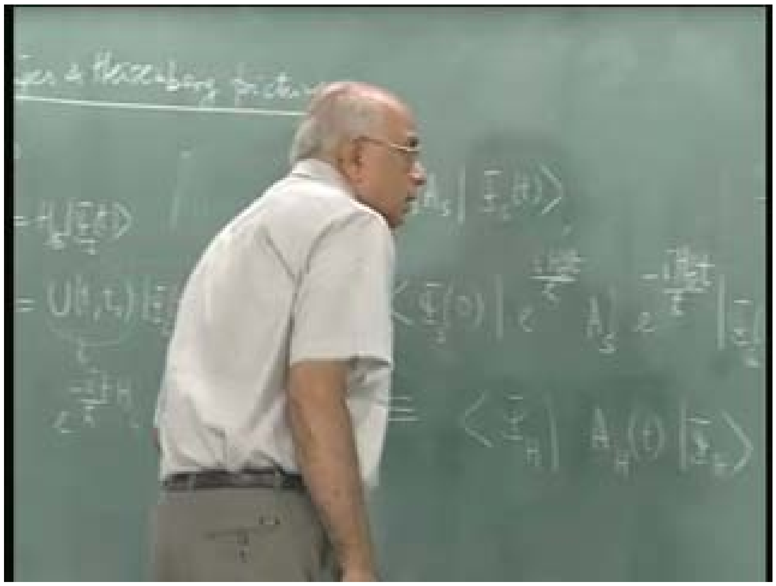

So you have psi $_s$ (t) A $_s$ psi $_s$( t) that was equal to (Refer Slide Time: 39:15 to 39:30) and that we insist is equal to, by definition a state vector which has no time dependence, what so ever. So psi $_H$ $A_H$ $psi_H$, there is no time dependence.  I insist these 2 are equivalent completely and it must be true for every observable and whatever state vector you start with.

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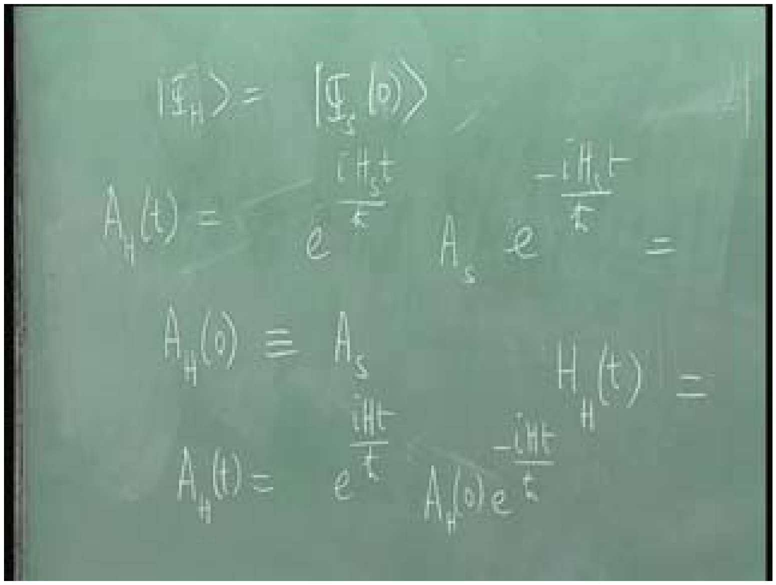


So these 2 are completely consistent with each other provided we define $psi_H$ = $psi_S$ at t= 0 or at some fiducial instant of time. it could be some $t_0$ once and for all i decide, so this state vector is moving around and at some instant of time, we know the Heisenberg state vector. Then in the Heisenberg picture, it is the operators that move around and evolve but the state vector remains where it is. moreover $A_H$( t) is by definition equal to e to the $iH_S$ t over h cross $A_S$ e to the - $iH_S$ over h cross. Therefore, not surprisingly $A_H$ at time 0 by definition is equal to $A_S$. Please notice in the Heisenberg picture, the state vectors have no time dependence, the operators do in the Schrödinger picture. The operators have no time dependence for physical quantities which don't explicitly depend on time but the state vector does carry all the time dependence. Therefore, i can write this immediately as equal to e to the i $H_S$ t over h cross $A_H$ at time 0 e to the - i $H_S$ t over h cross.

This is still good not enough for me because i would like to know what is the Heisenberg operator at time t. i shouldn't to go back to the Schrödinger picture at all. But I could do the same thing for the Hamiltonian itself and would ask what is H Heisenberg at time t. what is the Hamiltonian operator itself at time t? i put that back in here (Refer Slide Time: 43:06) and I get e to the i $H_S$ t over h cross $H_S$ e to the - i $H_S$ t over h cross by definition by definition. But this commutes with itself. Every operator commutes with the function of itself. Therefore this is identically equal to $H_S$.

The Hamiltonian itself is an autonomous Hamiltonian. It doesn't change in either picture. It is common to both. so I can forget about the subscript as far as the Hamiltonian is concerned and just call it H. therefore, going back here (Refer Slide Time: 44:29), I have this definition of the Heisenberg picture operator at any time t = e to the i Ht over h cross $A_H$ (0) e to the - iH t over h cross by definition. This is my definition of the Heisenberg picture operator. It says, unlike the state vector, where these (Refer Slide Time: 45:10) 2 terms were connected by unitary transformation, because this is a unitary transformation, psi $_H$ = psi$_S$ at 0. Psi $_S$ at time any time t is a unitary transformation U acting on psi $_H$. in the case of the operator, we have an e to the - on the right, e to the + on the left or U and U dagger on this side. Now, having written this, the next step is to write down the differential equation. So what we doing is to work backwards to find the Heisenberg equation of motion. We could have started with that but we will see what happens. So what is the d $A_H$ over dt?

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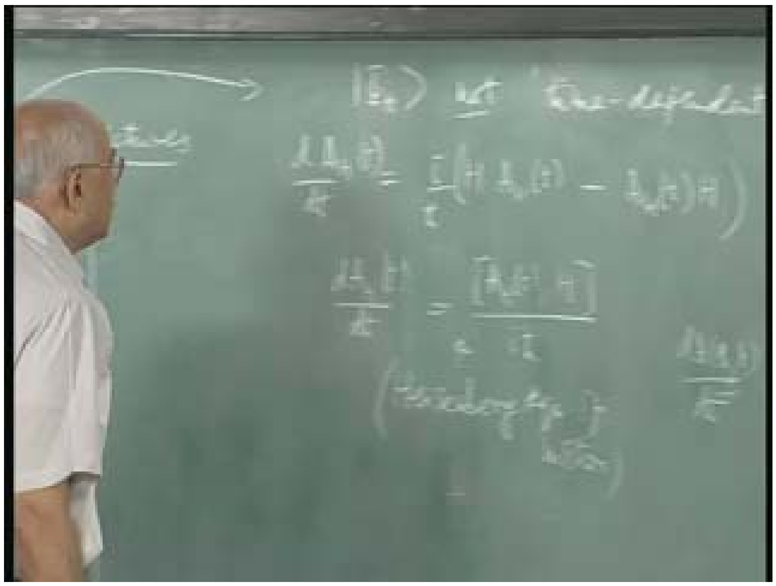


The only time dependence is coming from here and here (Refer Slide Time: 46:06). So what is that equal to? Student – The Heisenberg H is the same as the Schrödinger H. But we know that the Schrödinger H is independent of time where as the Heisenberg's H is dependant. Professor- in the case of H alone, it is no longer dependent on time. That is the whole point. The Hamiltonian is a very special operator and for autonomous systems, whichever picture you're in, it doesn't acquire any time dependence because it commutes with itself.

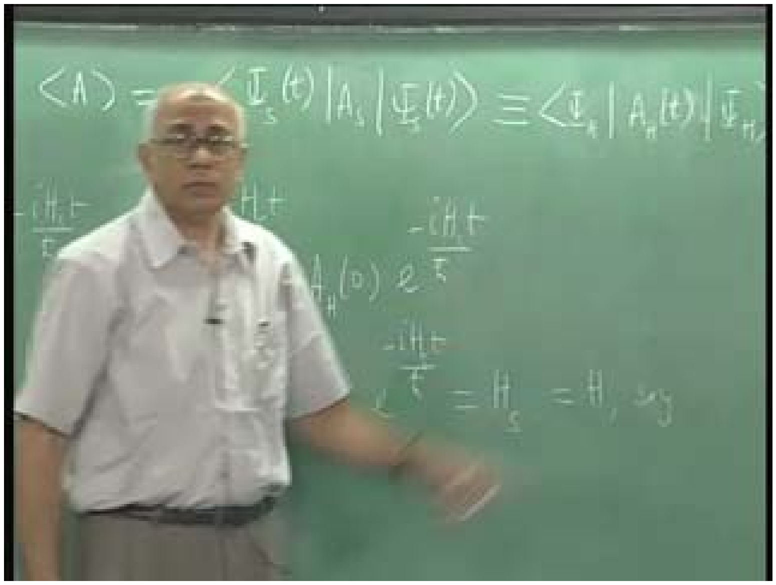
So the Hamiltonian is a link between the 2 pictures. That is the fulcrum. So you have this relation and all we have to do is to differentiate it. We haven't put in explicit time dependence. i should put in H (t) and then of course, this is not the solution. There is a unitary operator here that is a separate formalism. I am going to claim it goes though there too but what we are doing here is to work backwards to see what the differential equation is. And then I will establish that it is true in general.

So what is dA H over dt? Well, you got to be very careful because this operator may not commute with the Hamiltonian and that is the whole point. Otherwise this just goes through and you have no difficulty at all. Any operator which commutes with the Hamiltonian this thing goes across there (Refer Slide Time: 47:42) and cancels and then the Heisenberg and the Schrödinger picture operators are exactly the same. So what happens if I differentiate? I have to be very careful. Differentiating with respect to t produces iH over h cross, I can put that either here or there (Refer Slide Time: 48:00) but i can't put it across that. So I must write this as equal to i over h cross H and then this entire bracket (Refer Slide Time: 48:17) which is $A_H$ (t) itself. And then when I differentiate this term here, I get an iH over h cross with the – sign. I can put it here but I can't bring it here and i can also put it on that side (Refer Slide Time: 48:40). You end up with – $A_H$ (t) H and what is that quantity? It is the commutator. So let me write it as $dA_H$ over dt is [$A_H$ (t), H] over ih cross. What we have there is a solution to this equation. Since these are operator equations, you have a linear operator here acting on this. So this is a super operator acting on A. it says take AH– HA and that is the right hand side here. That is equal to dA over dt as an operator, dependent on a parameter t. and the question what's the solution to such an equation solution.

The solution is this (Refer Slide Time: 49: 57). You have this factor on the left and that factor on the with the specified initial condition. After all the solution is unique only if you specify the initial condition. So for that initial condition, for this equation with that initial condition, that is the unique solution. But what does this remind you of classically? if A was the function of all the q's and p's, d A over dt is equal to the Poisson bracket of A with the Hamiltonian and indeed all that has happened in quantum mechanics is that the Poisson bracket has been replaced by the commutator divided by ih cross which is what we said right in the beginning. That is the way you go from classical to quantum physics.

This (Refer Slide Time: 51:19) is the Heisenberg equation of motion for any physical operator. It replaces the Schrödinger equation because there is no time evolution of the state. So you got a fixed psi $_H$ not t dependent. And the operators are time dependent. In the Schrödinger picture, state vector is time dependent but the operators are time independent. And this little derivation we gave here (Refer Slide Time: 52:078) is not really a derivation. it actually ensured that you could find the expectation value of an operator in either picture and you got exactly the same answer.

(Refer Slide Time: 00:52:20 min)

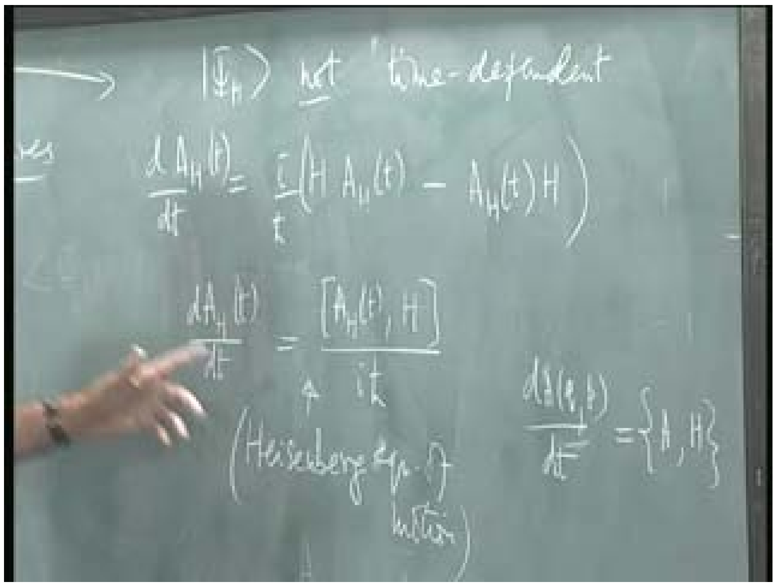


So therefore I can now go ahead and write, any A is either = psi $_S$(t) A $_S$ psi$_S$(t) which is the same as psi $_H$ A $_H$ (t )psi $_H$. so you put the blame for the time dependence either on the state vector or on the operator itself. that is like saying when an average value changes either I say that the average changes because the probability distribution changes and the variable itself is not explicitly time dependent or I say, no the distribution is fixed once and for all, but the variable has got dynamical evolution and they lead to exactly the same answer. So this consistency condition here takes you form one picture to another and the Heisenberg and Schrödinger pictures are said to be unitarily equivalent to each other because they are connected to each other by a unitary transformation. And which one you use it depends.

Obviously when you have 2 ways of doing something, you have a great advantage. You can do it either this way or that depending on what is easier under the conditions that you are interested in. it will turn out that in real problems, you use something in between the 2 called the interaction picture or the intermediate picture. And the reason is that, if you have a Hamiltonian which you can’t fully solve but you can solve most of it and there is a small perturbation, then you get into Heisenberg picture in which the evolution due to the portion you know is taken care of. its just like if you have a problem of 2 fellows sitting on a turn table and a throwing a ball between themselves or there is a particle moving in a rotating frame of reference, the obvious thing to do is go to that coordinate system in which this turn table is at rest.

So you go to the core rotating frame of reference, you get rid of that extra evolution which you already know and then look at the dynamical revolution of whatever you are interested in. so this is the strategy used in quantum mechanics to go to the Heisenberg picture to remove the portion of time evolution which is known to you already. It is the exact analog of looking at dynamics which is occurring in a rotating frame of reference. There is yet another way of saying this and I leave that you as an exercise to do this and that is the following.

(Refer Slide Time: 00:55:11 min)

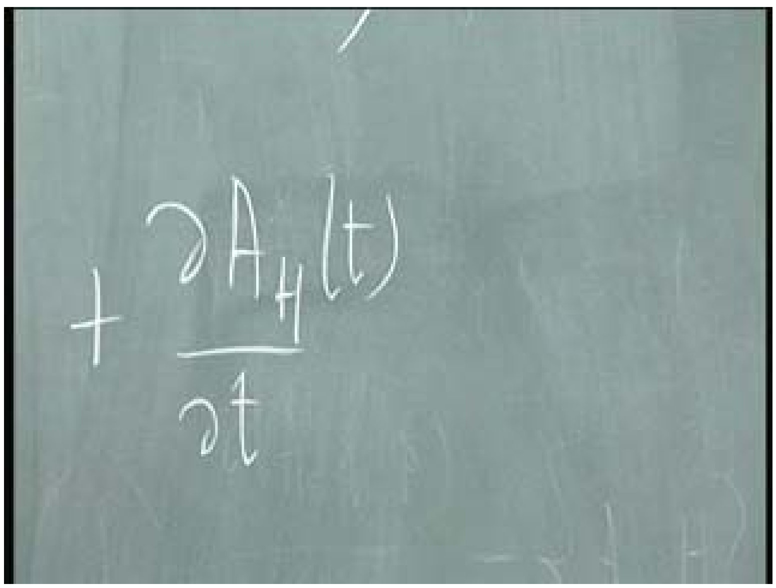



After all, if I have an expectation value A of an operator, I would write this in the schrodinger picture as psi(t) A psi(t) and you don't even have normalize it, so divided by psi(t) psi(t). And then i could ask what is dA over dt? So what is the meaning of dA over dt where A is an operator? It doesn't have any meaning. What you can do however is to say i find A, that is dependent on time. So let me compute d over dt of A which you can do because as time dependents all over there (Refer Slide Time: 56:06) and you can write down what d over dt is and insist that this be identically equal to expectation value of dA over dt. In other words, insist that the meaning of the time derivative of an operator is that its expectation value be identically equal to the time derivative of the expectation value of the operator.

Clearly, this is the Schrödinger picture and that is the Heisenberg picture. If you equate these 2 quantities, you get exactly what I said without writing the solution down. because what would you do, you would take dA over dt and this is equal to d over dt psi(t), this is some bra vector which acts on A psi(t) + if the bottom is normalized, then the other term is psi(t) A d over dt psi(t), assuming this is normalized to 1. Then what you do for this (Refer Slide Time: 57:23)? You are in the Schrödinger picture. So use the Schrödinger equation. This (Refer Slide Time: 57:42) is equal to H psi (t) over ih cross. Therefore this thing here (Refer Slide Time: 57:45) is - bra psi with H, Hermitian divided by ih cross.

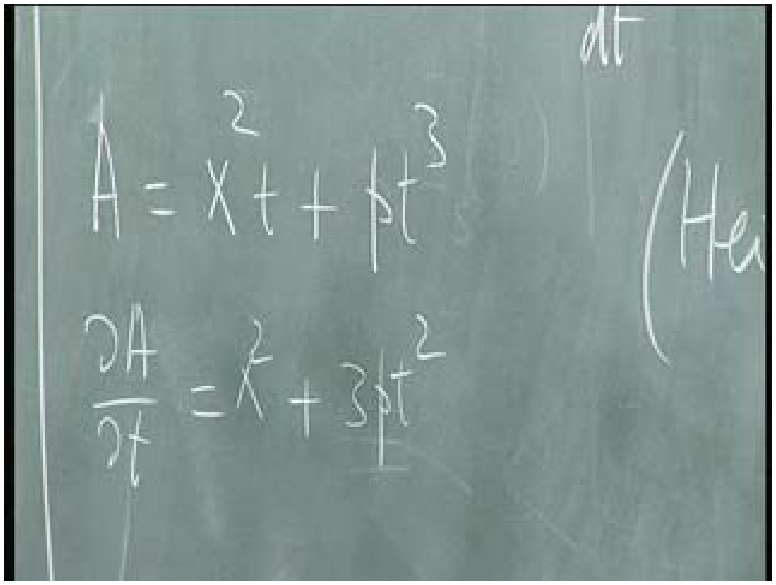
And when you put the two together, you get precisely the statement that this is the expectation value of the commutator of A with H divided by ih cross. That is the Heisenberg equation of motion. What I did earlier was to write the solution down explicitly and I differentiate this exponent and so on. But there I don't write that down at all. i just differentiate and use the Schrödinger equation which is true even if the Hamiltonian is explicitly time dependent. And you get exactly the same result. So actually this result is true even if the Hamiltonian is time dependent but if there is explicit time dependence in A itself, what do you think will happen? Suppose there is explicit time dependence on A even in the Schrödinger picture, I don't want to look at position; I want to look at this square of the position multiplied by the cube of the time. What will happen? This (Refer Slide Time: 59:28) will be have t dependence. That's a partial derivative of A with respect to t. there is one more term. So what would happen to this (Refer Slide Time: 59:36)?

(Refer Slide Time: 00:59:42 min)



It is just the partial derivative because imagine if this A has t dependence, then there is a third a third term in this (Refer Slide Time: 59:55) equation and when you differentiate, you have a psi (t) psi (t) with delta A over delta t in between. Suppose i want the position of a particle that is my A. Let's look at a case.

(Refer Slide Time: 1:00:10 min)




Lets look at A equal to= x squared t + pt cubed. That is the physical operator. x and p are operators but t is not. When delta A over delta t in the Heisenberg picture = x squared + 3 p t squared. This can happen in ordinary Hamiltonian mechanics also because there is an extra term.

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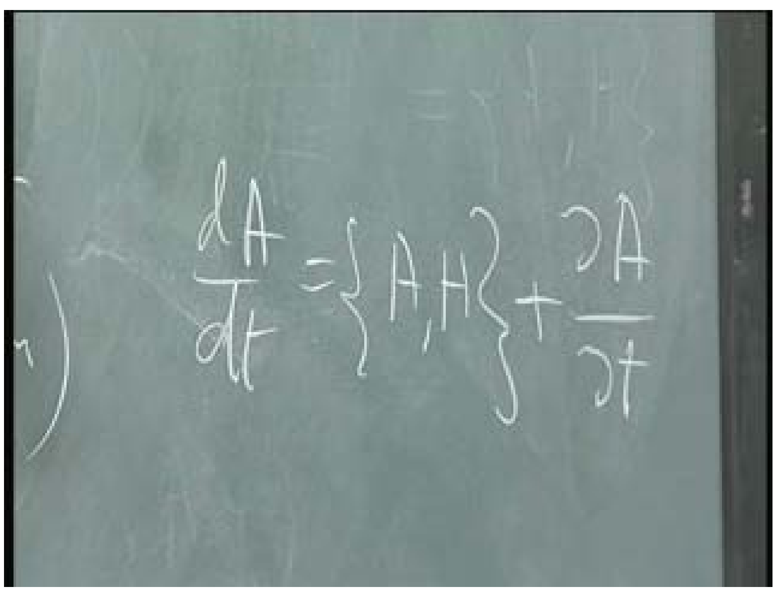

Whenever you write dA over dt is Poisson bracket of A with H remember, if this A had an explicit dependence, then remember there was a delta A over delta t term. So it's exactly the same thing. So the take home lesson is that in general in the Heisenberg picture this (Refer Slide Time: 01:01:16) is the equation whenever you have a no an autonomous Hamiltonian, this is the equation of motion. But this remains true even when you have time dependents. Then this would be H Heisenberg at time t because there is no guarantee that the Hamiltonian commute with itself at different instance of time and it doesn't have to. So that is the most general way of writing this equation down.

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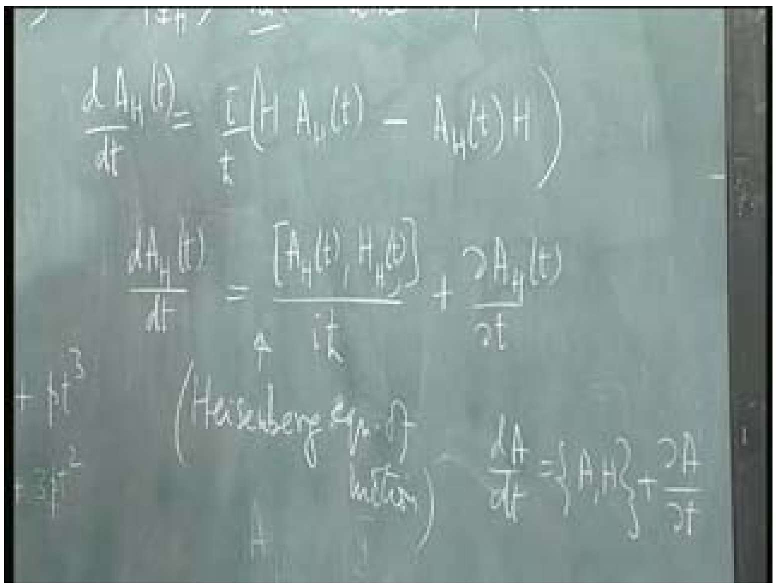

It replaces the Schrödinger equation. Very often it's convenient to use this because once you use an operator method, then it is convenient to use every thing algebraically. Because eventually, you just trying to find expectation values for physical quantities and then it may be more convenient to use this. What I need to do is to show you what happens if you got H t dependence here (Refer Slide Time: 01:02:30), explicitly non autonomous. What happens to this (Refer Slide Time: 01:02:34) unitary operator? It is no longer this exponent. It's a little more complicated formula but this we will do subsequently. So we should stop here today. Thank you!