Quantum Physics
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Lecture no. #12

I am still to write the problem set which I will do today and send it you by email about Harmonics oscillator and all the eigenfunctions. I'll start with a couple of quick points which I had left out last time.

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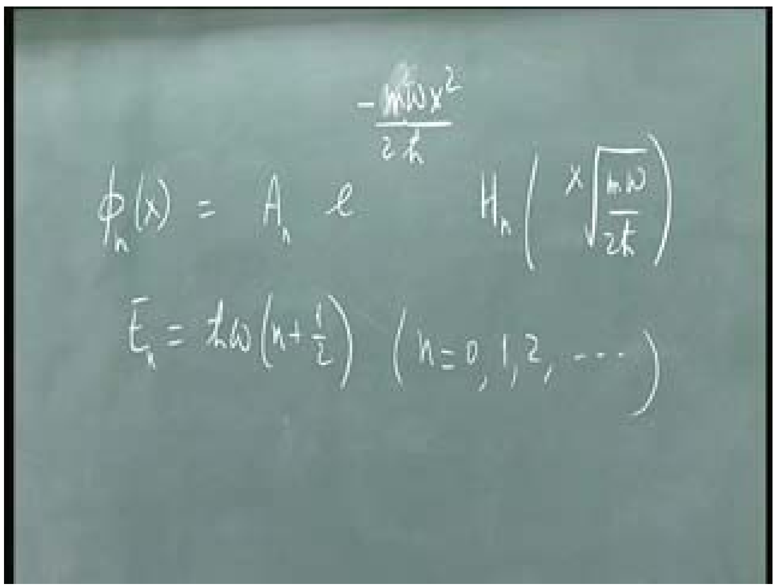

The wave functions themselves phi $_n$ (x) for the oscillator are of the form a normalization constant multiplied by e to the - x squared over 2 and this was measured in units of m omega or h cross. So 2 h cross and then m omega x squared. So Gaussian function of this kind and then H_n (x) root of m omega by 2 h cross which is a dimensionless quantity and these H_n's are the Hermite polynomials.

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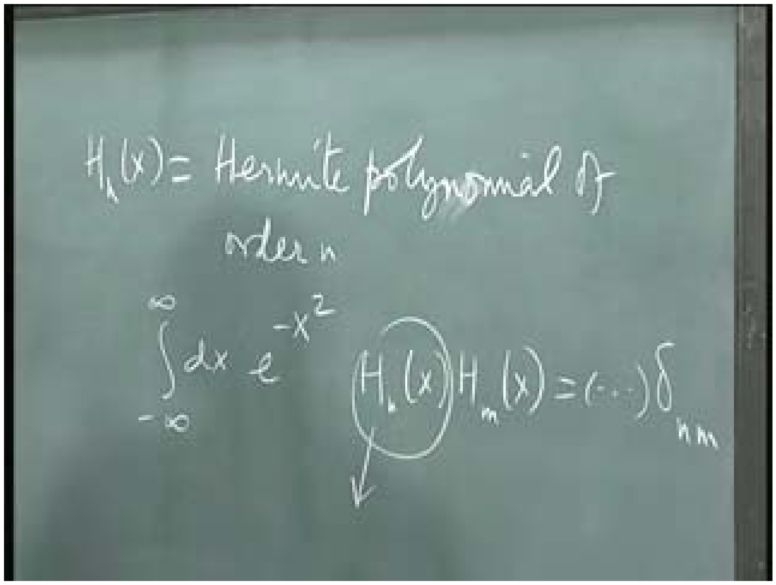

H_n (x) equal to polynomial of order n H $_0$ (x) is 1 and H $_1$(x) is 2 x and then the even ones are all even functions of every power of x and the odd H_n's have only odd powers of x and these functions here from a complete set of mutually orthogonal functions and it's a family of orthogonal polynomials. So you have a relation of the form integral - infinity to infinity, dx in the weight factor is x e to the - x squared. so that's a converse multiplied by H_n (x) H_n (x) equal to something times delta nm. i don't recall the exact factor here but this is the orthonormality relation between H_n's. there is also a generating function for this H_n. orthogonal polynomials, linear vector spaces, Legendre polynomial, Laguerre polynomials, etc. well, these are all like unit vectors in function space if you like and the weight factor here is this quantity Gaussian for the Hermite polynomials. You could have any range or weight factor here and similar orthonormality relations, and the Legendre functions polynomials run from - 1 to 1. You have integral P_l (x) P $_m$ (x) is delta l_m and so on apart from some constant.

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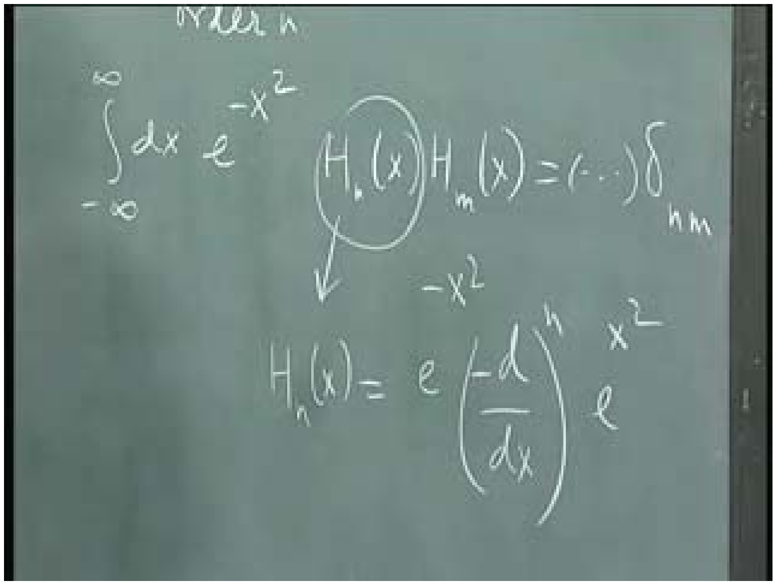

Now this thing here (Refer Slide Time: 04:20) is given by a Rodrigues formula as are all these orthogonal polynomials. so H_n (x) = e to the - x squared - d over dx to the power n e to the x squared. So you differentiate e to the x squared n times - 1 to n times that and then multiplied by e to the - x squared to get rid of the overall e to the x squared factor and you get a polynomial and they are Hermite polynomials. then there is a generating function and that relation says e to the power 2 t x - t squared where you could expand in powers of t because this is an entire function and you have only non negative powers of t. if you expand this exponential out and you collect powers of t together but remember there is a t here as well as a t squared there (Refer Slide Time: 05:10), so the coefficients would be some functions of x. and these coefficients are precisely the Hermite polynomials 0 to infinity H_n (x) over n factorial t to the power n. so this is a generating function this function here generates a power series in t whose coefficients are just the Hermite polynomials divided by n factorial. It's a very useful relation. And then there are these usual recursion relations and so on and so forth. The Hermite polynomials are the regular solutions of the following second order differential equation.

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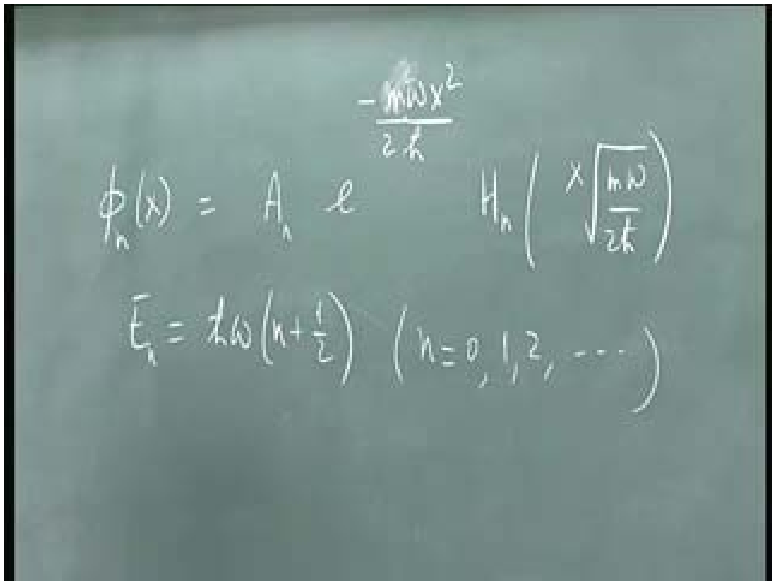

d 2 H_n over dx 2 - 2 x d H_n over dx + 2 n H_n =0. so they satisfy this second order differential equation. There are 2 solutions and they are linearly independent ones. One of them would have singularities and other one is polynomial like in the Legendre case. So I presume you are a little familiar with the theory of second order differential equations. So you can solve these equations by the power series method or by the Frobenius method and then the regular solution that you get is the H_n. of course, there is also a singular solution but that's not the one that occurs physically in the harmonic oscillator problem.

So this is a very useful piece of information to know. You can use all the properties of H_n in order to write down what the properties of the Eigen functions are. recall also that the energy Eigen values corresponding to these Eigen functions were these (Refer Slide Time: 07:04) things and n = 0, 1, 2, etc. so this finishes the harmonic oscillator everything you need to know is now known because you can take an arbitrary state of the system of the oscillator and you can expand it uniquely in terms of these Eigen functions of the Hamiltonian.

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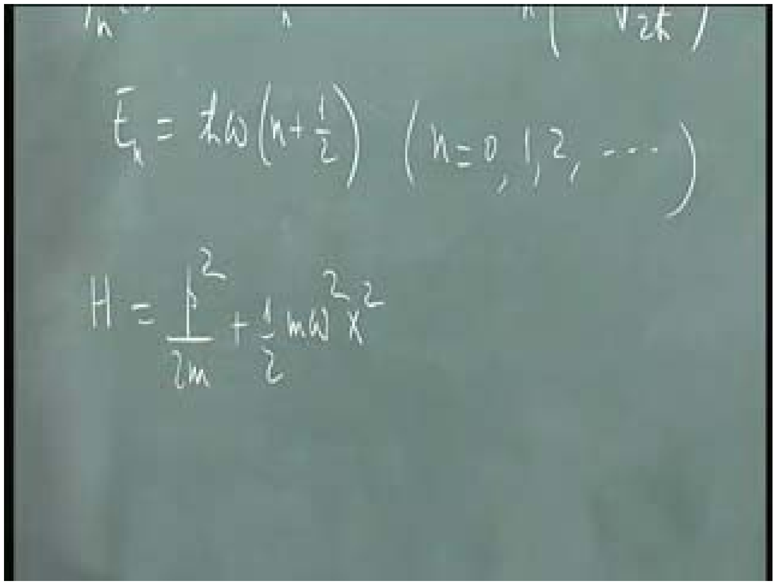

Now recall that the Hamilton for this problem was p squared over 2 m + 1 half m omega squared x squared where x and p are operators; the position momentum operators. apart from these coefficients its quadratic in x and its quadratic in p. so there is a complete symmetry between the position space wave functions and the momentum space wave functions in this case because what appears here is completely symmetric in x and p.

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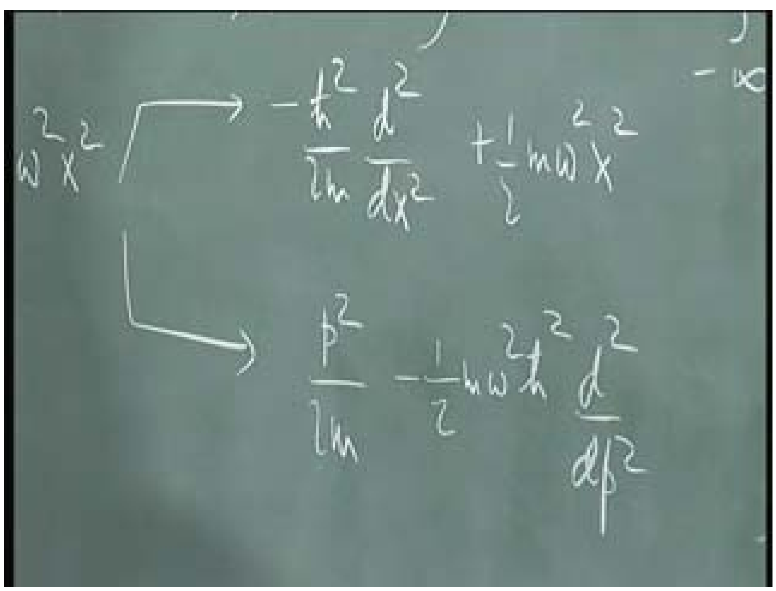

So if you were to write this out in the position basis namely; acting on wave functions $phi_n(x)$ in the position basis, then this is - h cross squared over 2 m d 2 over d x 2 + ½ m omega squared x squared in the position basis.

In the momentum basis, if you are acting on wave functions in momentum space, then p remains as it is. so this is p squared over 2 m, just multiplication, but x is replaced by + i h cross d over dp and when you square it, it becomes – 1/2 m omega squared h cross squared d 2 over dp 2. so you see apart from these constants which can be scaled away, you have an operator which is essentially x squared - d 2 over dx 2 in the position basis and the same operator in the momentum basis is p squared - d 2 over dp 2. so it shouldn't be surprising that the solutions you get for the momentum space wave functions would also be Hermite polynomial times e to the - p squared over 2 in proper units and it will satisfy exactly the same equations in the case of the x basis.

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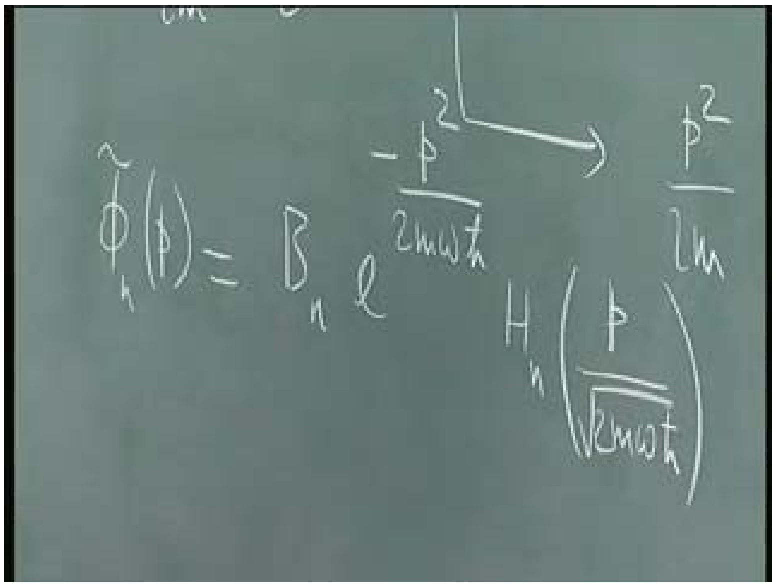

So the a momentum space wave functions would be phi $_n$ tilde(p) for the same energy eigen values, these would be some suitable normalization constants multiplied by some B_n e to the power -, not m omega x squared over 2 x but p squared over 2 m omega h cross because that's the dimensionless momentum, times H_n (p) over square root 2 m omega h cross. So even without calculation, you are absolutely guaranteed that this identifies what the wave functions would be and they look exactly the same in both bases.

the ground state wave function in each case is very special. The ground state wave function corresponds to n=0 and apart from some constants, it's e to the - x squared over 2 H_0 but H $_0$ is 1, so this goes away and you just get a Gaussian. So in the ground state of this harmonic oscillator, the wave function is a Gaussian both in x as well as in p. and that's the reflection of a well known fact that the Fourier transform of a Gaussian is also a Gaussian function because we know these two bases are related by Fourier transform.

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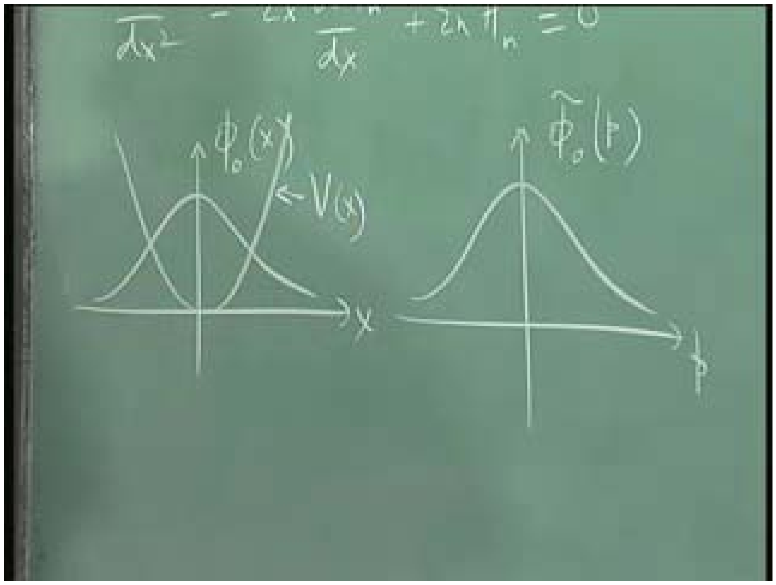

so the ground state wave function phi $_0$ (x) as a function of x is some kind of Gaussian and similarly, phi $_0$ tilde(p) as a function of p is also a Gaussian. so in the ground state, whether energy is half h cross omega, the wave function is extended and runs all the way from - infinity to infinity. Recall that the potential itself was just this (Refer Slide Time: 11:30), this is V (x) and the wave function itself is this (Refer Slide Time: 11:35). so even though classically, at energy 1/2 h cross Omega the particle can only run between 2 finite points quantum mechanically it can be found anywhere on the x axis from - infinity to infinity with ever decreasing probability and dies on extremely fast as you go out. But it's a nonzero probability that the particle in the ground state is actually in the region outside the classically allowed region. Now we generated all the higher order terms by the operator method. We generated the solutions in x and all higher order solutions. We would like to find out what the uncertainties in the position and the momentum are. Let's see if you can calculate this.

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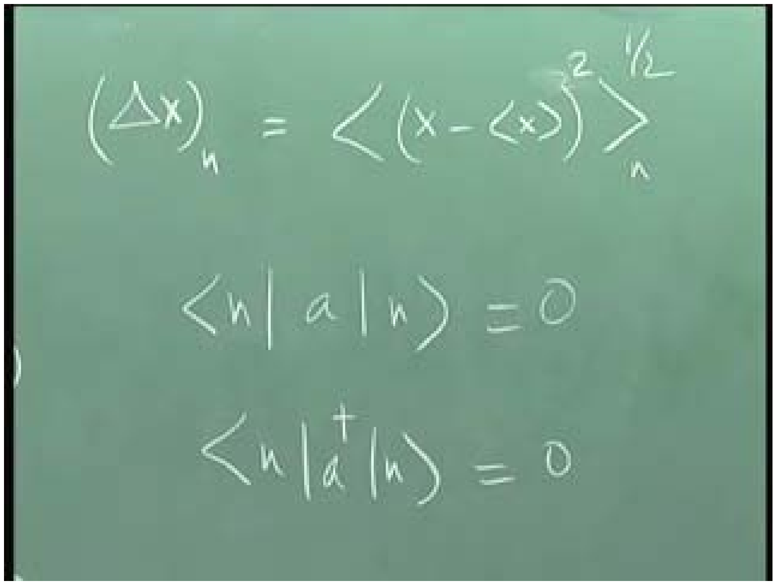

So what was need to calculate is delta x in the n’th state, so let me call it n, this by definition is x - x average in the n’th state always. So we would like to calculate the variance and then take the square root of this quantity. so this would be delta x and similarly delta p. and you shouldn’t be surprise that delta x and delta p would be exactly the same in both cases because each of the n’th wave function in the position basis is a Fourier transform of the wave function in the momentum basis. And they are exactly the same in functional form. So I expect a great symmetry between delta x and delta p in this case. So, in fact calculate just delta x and then i write down delta p by just changing units. x is measured in units of h cross over m omega square root and p is measured in units of square root of m omega h cross. So that’s all that you need but there is an easier way to do this and the way to do this is to ask this quantity here. We would like to calculate in the n’th basis. so let me just call this the ket vector corresponding to the n’th energy level En and let me first find out what is this quantity (Refer Slide Time: 13:50). A was essentially x + ip over square root of 2 x in appropriate units and a dagger was x – ip. So what’s a with n on both sides?

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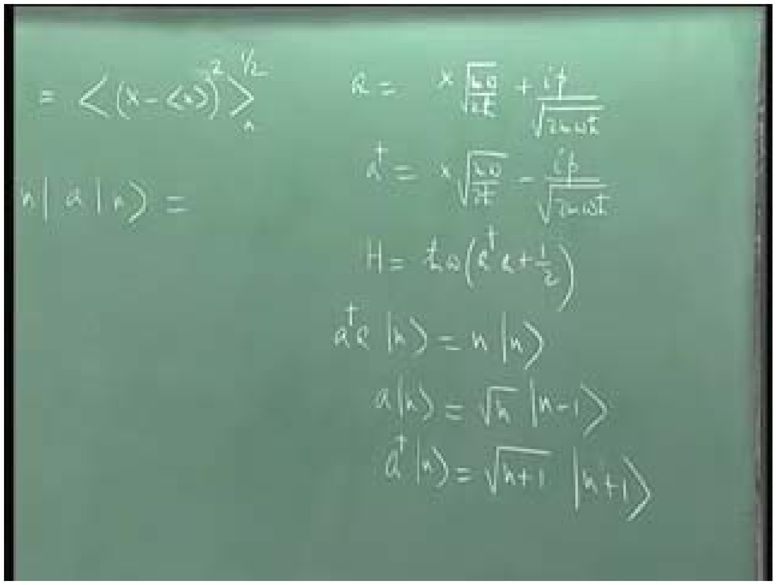

If you recall a was = x square root m omega by 2 h cross + ip over root 2 m omega h cross and a dagger will be = x root m omega by 2 h cross – ip over 2 m omega h cross, where x and p were the position and momentum operators and if you recall the Hamiltonian was h cross omega a dagger a + ½. And we know that in the n'th eigen state, a dagger a acting on it is just n on n. this was the number operator acting on the ket n and we also found out what a acting on n does. a acting on n gives you square root of n times n – 1. It lowers and a dagger acting on n takes you up. Notice this factor is not n – 1, its n. so when n is 0, a acting on the vacuum on a ground state just annihilates a vacuum. It gives you 0. And then, there are no levels below that. So given that, what's the diagonal element? Well these levels n are all orthonormal. The moment a acts on n, it lowers you to n - 1 and then n – 1, overlap with n is 0 because they are perpendicular to each other in the Hilbert space. So this straight away gives you 0 immediately and what is this? (Refer Slide Time: 16:24)? Well of course its Hermitian conjugate would also be zero that's the number complex number so its complex conjugate is also zero immediately.

So if you represent a and a dagger as matrices which you can, you need a infinite dimensional matrix in each case because n goes all the way to infinity. It starts at zero and the label goes all the way to infinity. So you can represent the position and momentum operators as infinite dimensional matrices. So what looks like a d over dx operator acting on function space, the same operator in the energy basis looks like an infinite dimension matrix. So these are abstract operators and they wear many clothes at different times. So you shouldn't be surprised that what looks like a derivative here; d over dx, really ends up looking like you know another basis which is just multiplication by p and then yet another basis. It's really an infinite dimensional matrix. Now this tells you that neither a nor a dagger is a diagonal matrix because all diagonal elements are 0. So a + a dagger and a - a dagger which would be essentially x and p also don't have diagonal elements at all. They only have off diagonal elements and now we can compute these uncertainties. So for the moment, let me set m= 1, h cross =1 and omega = 1. I can always do this. I choose a scale of mass such that m = 1, I choose a scale of time such that omega = 1 and then I choose a scale of length such that h cross = 1. I will put them back later on by dimensionless arguments.

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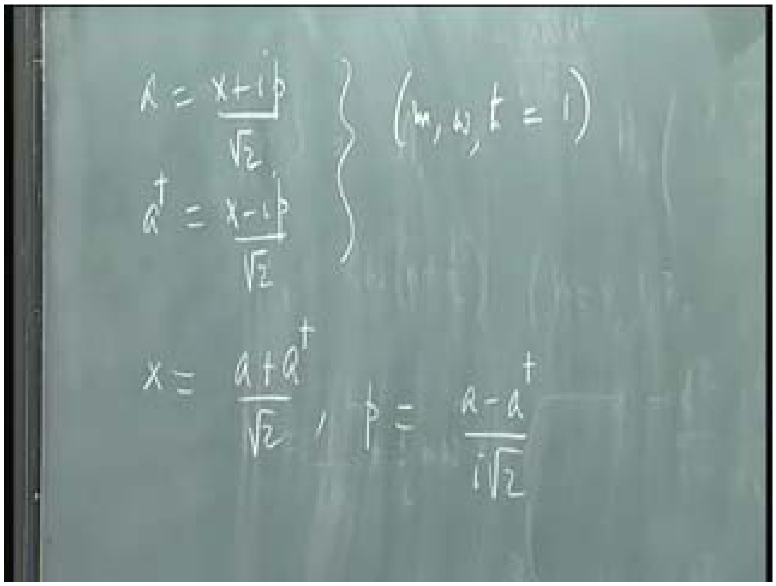

So a is x + ip over root 2 and a dagger = x - ip over root 2. M, omega and h cross are equal to 1. We will restore those units later on. So what's x equal to? If i add these two, x is a + a over root 2 and p = a - a dagger over i root 2. They have to be Hermitian x and p. so a and a dagger are not Hermitian. a and a dagger are not Harmitian operators but x and p are physical operators. They are Hermitian. They must have real eigenvalues. So I am in business.

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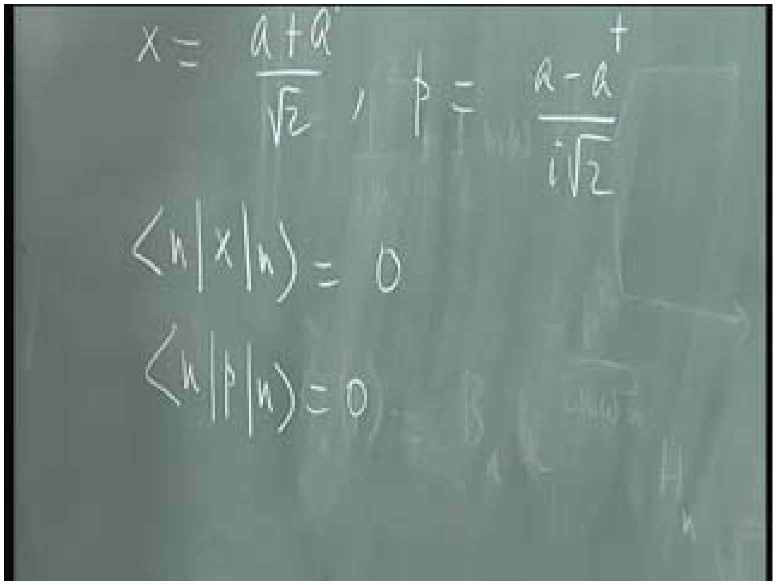

What's n x n? That is zero straight away and n p n is zero. That's sort of obvious because you have this oscillator classically going back and forth. Therefore its mean position is at the centre of oscillation. it should be zero. And it's not going anywhere. It's simply bouncing back and forth in the potential. So the mean momentum is also zero but the squares are not zero. we need to compute that and you can either do this painfully by doing it on the basis on messing around by differentiating Hermite polynomials or you can read neatly by finding out what's n x squared n?

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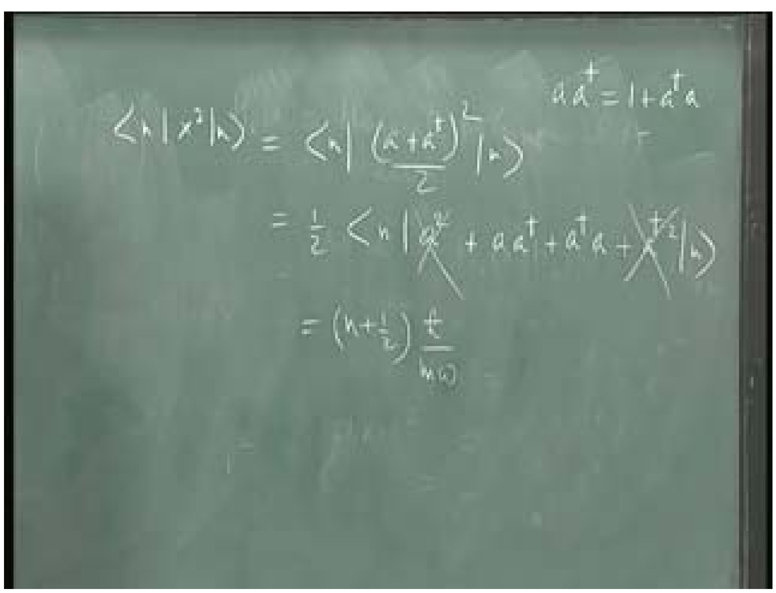

This is equal to n x squared, all you we have to do is square that, a + a dagger squared over 2 acting on n and that's equal to ½ n and what are these terms inside? So we will get an a squared + a dagger + a dagger a + a dagger squared. That's very important to remember. since a and a dagger don't commute, you have to keep both terms + a dagger squared on n but a squared when it hits n is going to lower to n – 1 and to n – 2. And that's orthogonal to this. The first and the last terms don't contribute. We want only the diagonal elements. This is equal to half and now the matter is exceedingly simple. a dagger a acting on n is just n. it comes out but what's a dagger? We use the commutation relation. So a dagger equal to 1 + a dagger a. so this gives you n + n + 1. That's 2 n + 1 over ½. That's equal to n + ½. So you right away have these answers for what the mean square value of the position is in the n'th normalized eigen state. If you restore all the other dimensional factors, then you get n + 1/2 times h cross over m omega. It's got to have dimensions of length squared.

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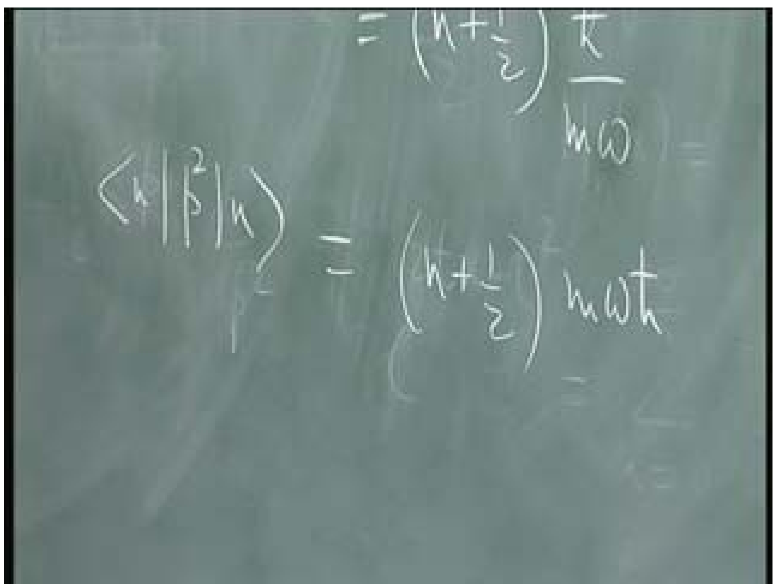

Similarly, n p squared n but that's equal to the same because the i squared and the minus cancel out with each other. So you see how symmetric it is and what's the ground state answer? In the ground state, so we have this very interesting answer which says the following.

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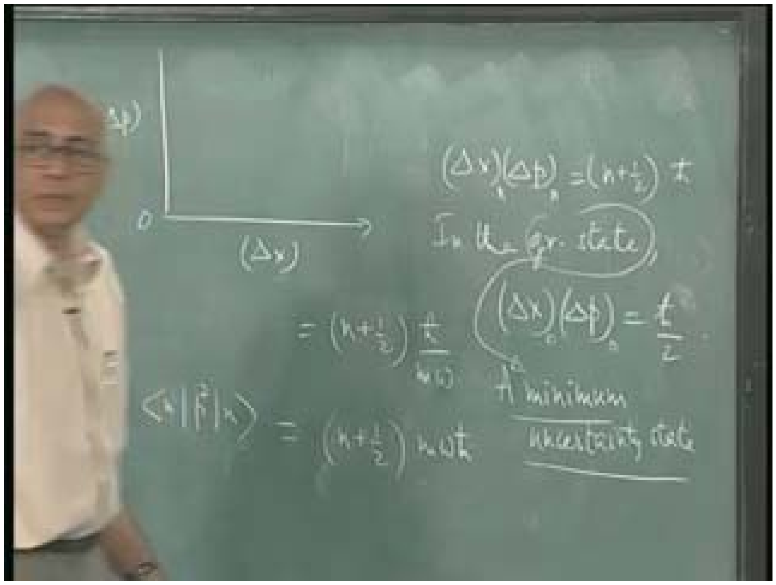

If I plot delta x here versus delta p, these are standard deviations. So they can't be negative. I am going to do this for each of the states delta p delta x.

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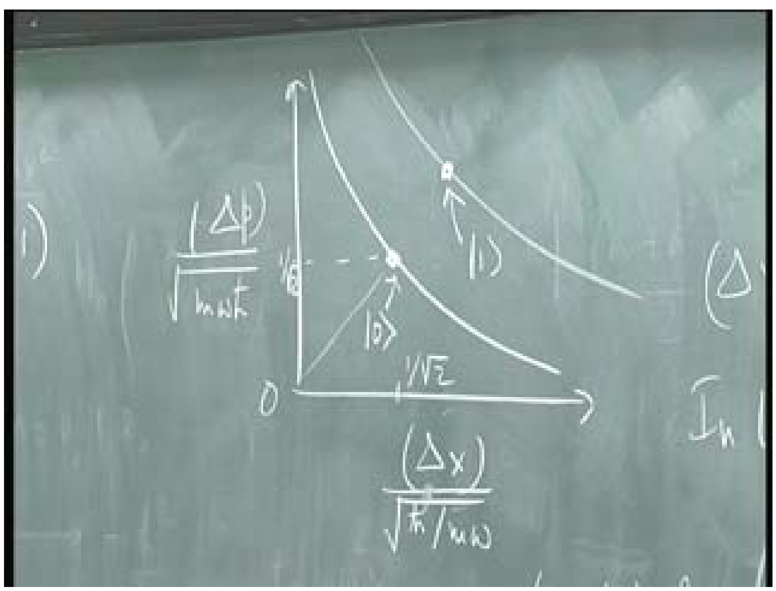

Delta x delta p in the n'th state = n + 1/2 h cross. Now delta in the ground state is exactly h cross over 2 but the uncertainty principle between x and p says this is the least value could ever have. So the ground state of the harmonic oscillator is a minimum uncertainty state. Actually if we plot delta x delta p = h cross over 2. That's a rectangular hyperbola in this fashion. So delta p over m omega h cross and delta x over root h cross by m omega. Let me measure it in these units. Then the ground states, its right here (Refer

Slide Time: 25: 13) at the value (1/ root 2, 1/ root 2) at the mid. That's the nearest point on this rectangular hyperbola from origin and is symmetric. And what do the excited states do? They are just n + ½. so this becomes 3 ½, 5 ½, etc. they also sit on these hyperbolas. This (Refer Slide Time: 25: 45) is the first excited state. We are guaranteed by the uncertainty principle that no matter what state of the harmonic oscillator you look at, you cannot come down below the curve or outside this curve.

Every point on this corresponds to delta x delta p exactly equal to h cross over 2 and you can't go below that. However, it's possible that there are states of this oscillator which you don't know to construct so easily at the moment but you could be here (Refer Slide Time: 26:16). You are below the 1 over root 2, the ground state uncertainty in the momentum but you are much bigger than that in the position. So this is possible and in quantum optics, this kind of state has been realized and they are called squeeze states because what you are doing is squeezing the uncertainty in one of the variables down to practically zero but you are in the expense of increasing the uncertainty in the other variable. So it's possible to have those states.

We will see that there is actually an infinite number of states of the harmonic oscillator all of which are sitting here (Refer Slide Time: 26:46), not just the ground state. We don't have infinite number of states all of which would sit on that and you will see how to generate these states. In a nut shell, after all, once I have a harmonic oscillator, I can move its center of oscillation to any point. i can shift the potential and nothing would happen. Those states also sit right in that point and we will see where they come in and they are called coherent states. We will talk about that next but you see the operator way of doing this is much faster. So you can compute anything extremely fast once you know how to use this number operator basis. This number operator states are also called Fock states after Vladimir Fock who first derived these in Russia long ago. he solved a large number of problems and the number operator states he introduced are called Fock states in his honor and this space itself is called Fock space. I will explain what Fock space means a little later. One of the most important things you have to note about the harmonic oscillator is that the energy levels are equally spaced. Therefore when you quantize, the fields and look at actual fields, including the electromagnetic field, it turns out that the quanta, every time you add one more quantum of the field, you are adding the rest energy. Therefore you are adding the constant amount of energy if the quantum is at rest.

Therefore the harmonic oscillator plays a fundamental role in quantum field theory and the space in which those states exist are called Fock space. We will comeback to this but now I would like to ask the following question. We found the uncertainty in the position and the momentum for the ground state, what about eigenstates of a and a dagger themselves? They are not Hermitian but after all, they are related to x and p which are physical operators. So the question is can I find Eigen states of these operators. So let's try to do this.

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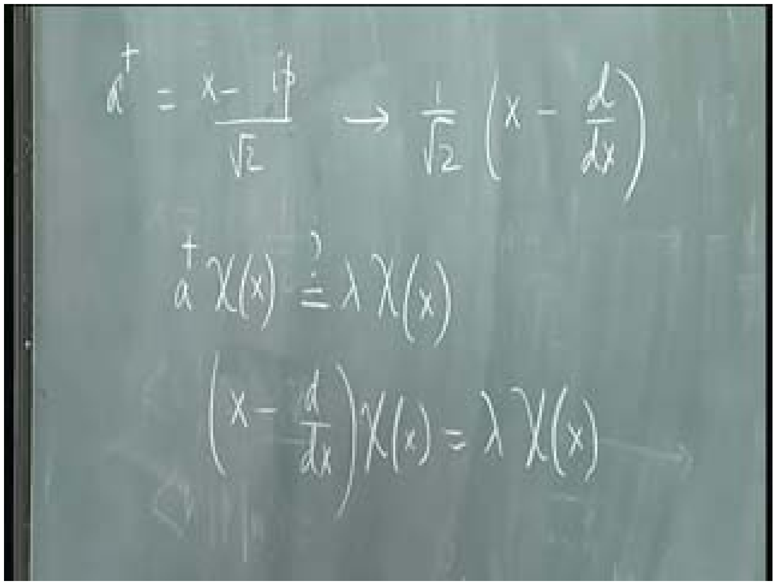

The first thing you must understand is that these are not Hermitian operators. So I don't expect the eigenvalues to be real. It may be complex. it doesn't matter after all if l over x + ip where x and p are physical and have real eigen values, the number x + ip is complex in general. So i can live with that but the question is can I find eigen functions or not? Now a dagger = x – ip. Let's use the units in which m, omega and h crosses are equal to 1. we put them back later on. And in the position basis, this is the same as saying 1 over root 2 (x - d over dx). And I said h cross equal to 1. So if this quantity has an eigen state and let's call it some chi (x) in the position basis, then the idea is a dagger on chi(x) should be equal to lambda on chi(x). if such a chi(x) exist, lambda could be complex in general. so it says that you must have x - d over dx on chi(x) = lambda on chi(x). That's not a hard differential equation to solve. It's a first order differential equation that can be trivially solved. We now find the solution and now see if you can normalize this or not. So I leave that to you as an exercise to find out but again we can do this problem also in the operator basis. So let's see if that works in the operator notation.

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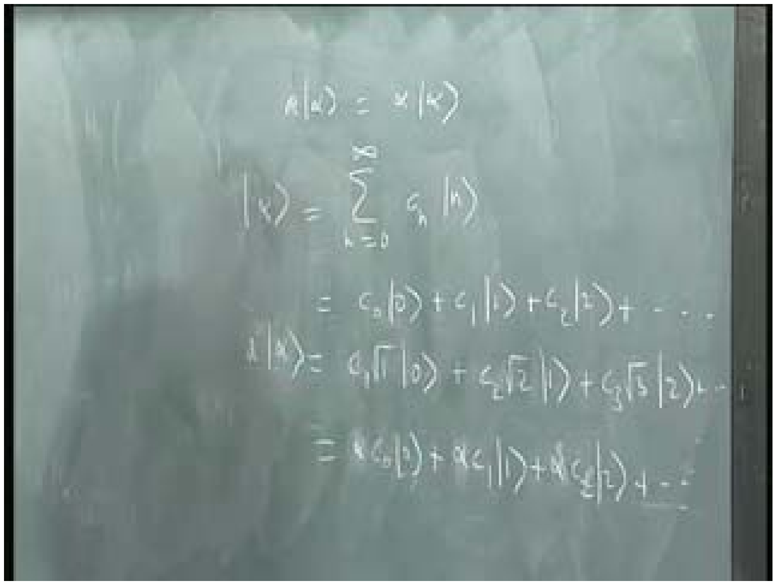

So I have a dagger on chi = lambda on chi. that’s the statement here that i have made except i am working in the energy basis. i can always expand this chi as summation n = 0 to infinity, some coefficients c_n times n. you know you can expand it in this form uniquely. Every state in the square integral normalizable states can be expanded uniquely in the form which forms the number operator basis. So what does this do? Because this implies chi = c_0 0 + c_1 1 + c_2 2 + … etc. Now what is a dagger on this? It’s a square root of n + 1 times n + 1. So square root n here is 0. so this becomes c_0 square root of 1 and 1 + c_1 square root of 2 on 2 + …. This is all the way to infinity but this must be equal to lambda times chi. so it must be equal to lambda c_0 1 0 + lambda c_1 n 1 + lambda c_2 on 2 +… etc. therefore each coefficient must be equal on both sides. There is no other choice because you require 2 vectors.

Therefore they must be equal component by component. There is no zero here. So the conclusion is c_0 must be ${}_0$. So that gets killed and so on down the line. What is that telling you? There are no eigen functions. But I can solve this equation. i can find the solution which is not trivial. You are telling me that chi(x) must be zero there. Chi must be the null vector but that’s not true. This can be solved. They will not be normalizable. They won’t be in the space of square integrable states. So that’s the crucial point. Remember that this expansion here has given you all the states in the space of normalizable states. And there are no normalizable eigen states of a dagger. There are eigen states in this space but they are not normalizable. so they will not be the total probability and will not be conserved. Now we do the same thing with a and let’s ask the same thing. So a is x + ip.

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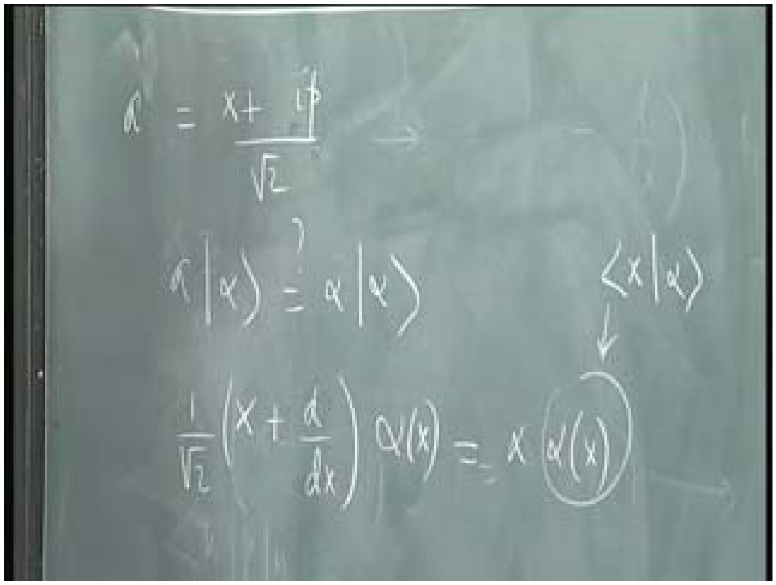

Can we have a on some eigen state, alpha is alpha on alpha. Alpha is some complex number. i play the same game as before and now i have x upon i have a 1 over root 2 x and this is an i and this is - i from there. So this is x + d over dx acting on this wave function. Let's call it alpha(x) for want of a better name, is equal to alpha times alpha(x). This alpha(x) is nothing but in the position basis, the representative of this state alpha. you can solve this equation and you can ask if they are normalizable eigen functions in this case because it is a + sign here. So it will immediately turn out that you get things which are normalizable because this is e to the - x squared sitting up there immediately.

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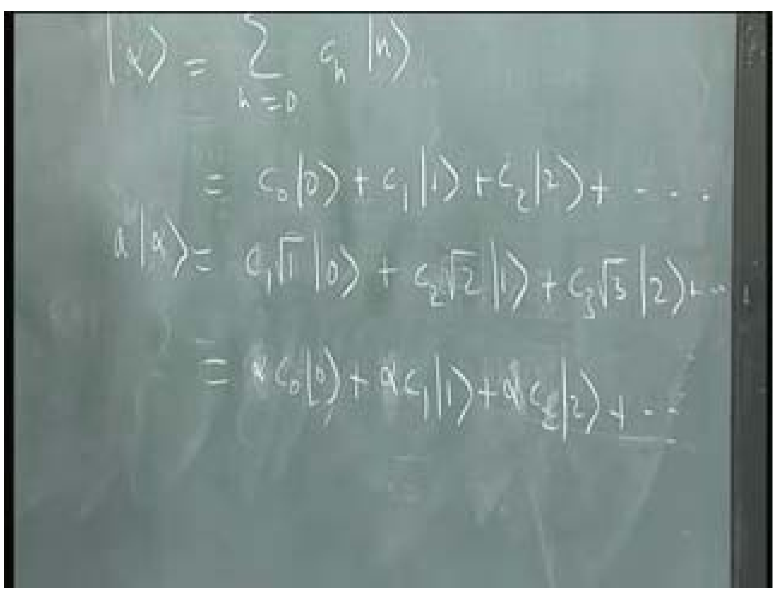

So we put alpha a on alpha to get alpha on alpha. I put alpha equal to these coefficients equal to c_0 etc and I put a on this alpha and now we are in business because this is + alpha here and that gives 0. It annihilates it and pushes it down but then acts on one and brings you down to zero and it acts on 2 and brings you down to 1. So you can match the series since this series is not bounded from upper side, you can equate these 2 series because this spectrum is bounded from below. a dagger doesn't have an eigen state. Had this spectrum and bounded above but not from below then you would have had eigen states of a a dagger and not of a.

You must have something unbounded on one side or the other, then the magic works. so this quantity here gives you c_1 acting on 1 that is root 1 times 0+ c_2 root 2 on 1 + c_3 root 3 on 2 + etc.

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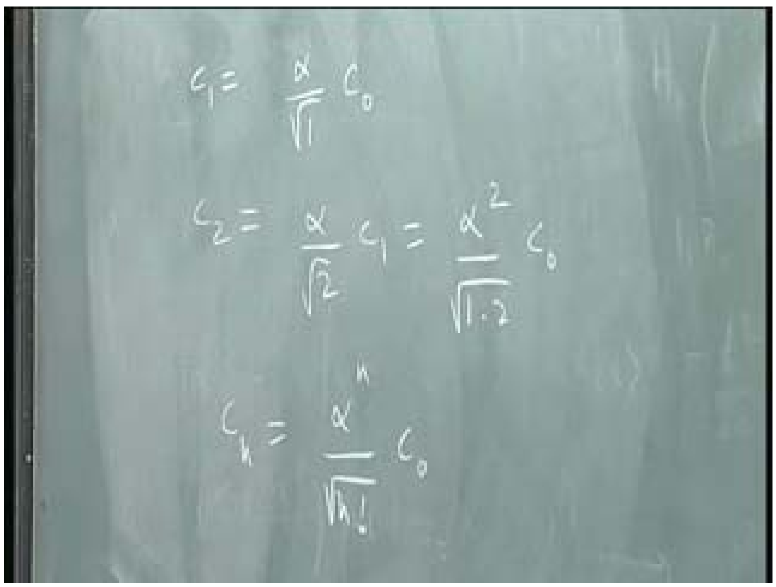

So we start of by saying c_1 = alpha over root 1 c_0, c_2 will be, you equate this term with this (Refer Slide Time: 39:08) quantity, that's equal to alpha over root 2 on c_1 which is equal to alpha square over root 1 acting on c_0 and so on. And the n'th coefficient is alpha to be n over square root of n factorial on c_0. So this fixes all the coefficients and in fact, it tells you that this is perfectly reasonable.

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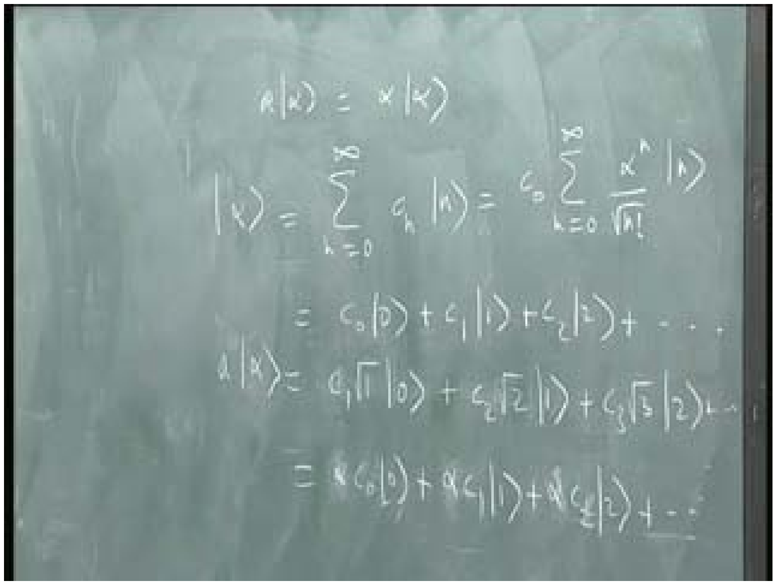

This is equal to a summation from n = 0 to infinity, there is a coefficient c_0 that comes out and then there is an alpha to the n acting on n over square root of n factorial. So we have a state. What are the allowed values of alpha? It can be any complex number. So it has a double continuous infinity of eigenvalues. All numbers in the complex plane are eigenvalues of the annihilation operator a. and they have corresponding normalizable eigen states. Let's normalize them.

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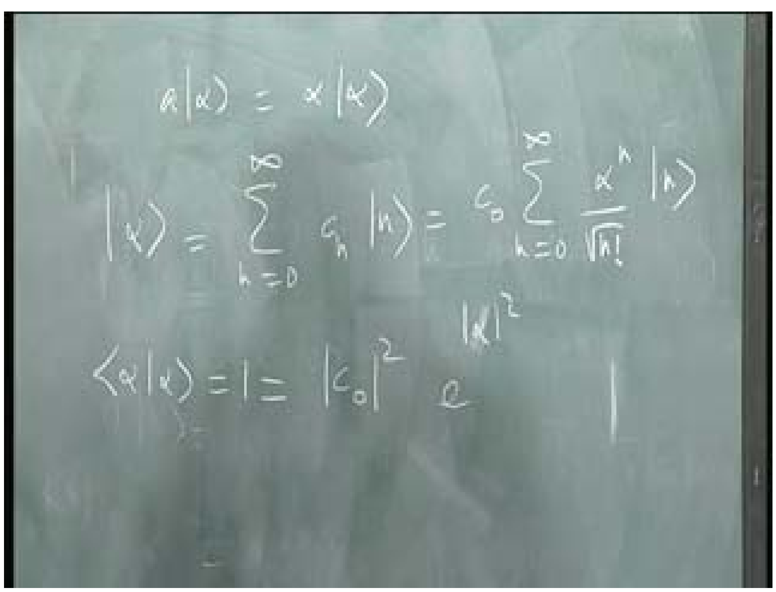

So alpha alpha = 1. i am going to impose that. so that gives you mod c $_0$ squared and then i have an alpha to the n on n and the other one would be alpha star to the m with root m factorial. But when you have an n and an m, you have a delta nm. so this whole thing becomes summation n=0 to infinity mod alpha to the power 2 n, that's alpha star to the power n over n factorial. But what's this (Refer Slide Time: 41:21) series? It is e to the power mod alpha squared. Therefore that tells you that c_0 = e to the – 1/2 mod alpha squared. i take the square root of it and I have set the phase of c_0 to be 0. The overall phase of wave function doesn't matter.

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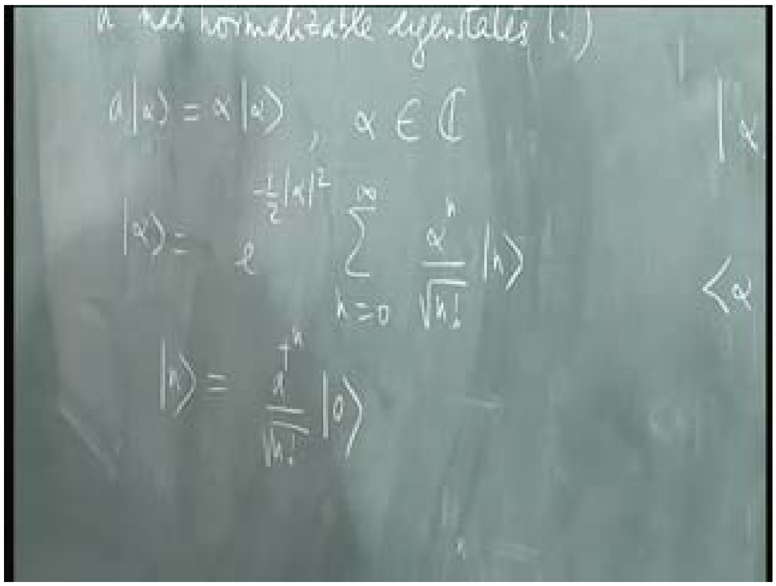

Therefore, we have a very important statement which says a has normalizable eigenstates. This is a little surprising, non Hermitian operator has normalizable eigenstates and these eigenstates are labeled by a complex number, alpha. And this alpha, the eigenstate is e to the – 1/2 mod alpha squared summation n=0 to infinity, alpha to the n over square root of n factorial on n. what does alpha= 0 correspond to? If we put 0 here (Refer Slide Time: 43:06), you get the ground state because a on 0 is 0 actually and you do get that because I put alpha= 0, only the n=0 term contributes.

All other n equal to zero alpha to be n is 1 so everything else goes away and you get the vacuum and this becomes unity. So these states which are eigen states of a, one state coincides with an eigen state of a dagger a and not surprisingly that's the state such that a on that state gives you 0. So alpha = 0 is the vacuum. It is a number operator state 0 but all other complex alphas are different states all together. You also know what is this (Refer Slide Time: 44:00) equal to in terms of a ground state? I keep raising this (Refer Slide Time: 44:05). So I act a dagger on the ground state which gives you a state 1 with the 1 over square root of 1 here and then i act once again on it, a dagger squared, and that's again going to give you a 2 here but it's going to give you a square root of 1 times square root of 2 here and so on. So it is clear that you can also write this n as a dagger to the power n over square root of n factorial acting on the ground state. You can also write it in that form.

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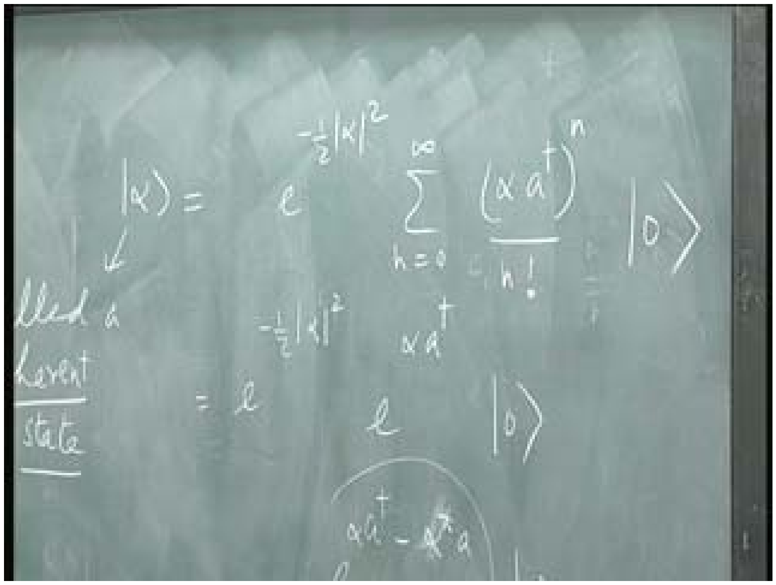

So this implies that this eigen state alpha can be written as e to the – 1/2 mod alpha squared summation n = 0 to infinity alpha a dagger to the power n over n factorial acting on this ground state; acting on the vacuum. But now you can do the summation formally. It's again an exponential series. So this is equal to e to the – 1/2 mod alpha squared e to the power alpha a dagger acting on the ground state. It's a very compact form of writing the eigen state of a. these eigen states are called coherent states. They play a fundamental role in quantum optics. When you apply this to the electromagnetic field and you quantize it, the coherent states play a fundamental role. In fact, if you take ideal single mode laser light with a given polarization, this is in fact the state of the radiation field is a coherent state. This has mathematically identical properties. You don't have a position and a momentum there for the photon or anything like that. This is quantizing the electromagnetic field itself and then you end up with a state which has exactly the properties of a coherent state of this kind.

It's not an accident that the exponential of this operator acts here of this operator a. because, notice that i said a and a dagger form a Lie algebra. A dagger in the unit operator that commentator of a with the unit operator 0. A dagger with that is 0 and a with a dagger is 1, the unit operator itself. They form a Lie algebra. This is called the Heisenberg algebra and from this Lie algebra, you can form a group by exponentiating just as we did in the case of rotations. So you exponentiate the generators with parameters and you end up with the group elements. So that's exactly what these quantities are and when it acts on this 0, you get each of the coherent states. In fact, you can even get rid of this (Refer Slide Time: 47:30). You can show it's essentially e to the power alpha star a with the - sign + alpha a dagger acting on 0. So with a little reduction i will give this is an exercise. You can write this as alpha a dagger - alpha star a acting on 0. It is this (Refer Slide Time: 47:55) operator, please remember that if a and a dagger don't commute, then e to the a dagger + a is not equal to the product of the individuals terms. So this can be reduced to that form and then you end up with this operator acting on 0 on the ground state which gives you the coherent states. Physically what does coherent state mean? What does the wave function look like and what are normalization conditions and so on? Well we normalize these states to 1. Remember that's how we found this factor here (Refer Slide Time: 48:25) but you could ask would you expect them to be orthonormal. The question is the following. In the space of the harmonic oscillator, all normalizable states could be expanded uniquely in terms of number operator states. They form an orthonormal basis. So there is no question that we can expand.

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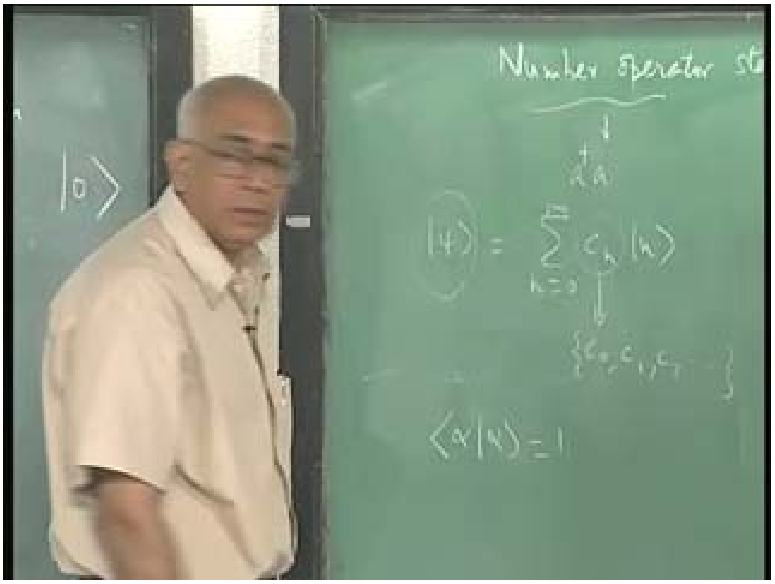

Everything in terms of this number operator which incidentally is a dagger a + ½, that is just 1/2 times unit operator multiplied by h cross omega. Essentially in terms of the eigen states of a dagger a. a dagger a acting on n was n times the same state and n runs from 0, 1, 2, 3, etc. so all states in this space could be expanded in this form. So every normalizable state psi could be written in the form c_n n, n = 0 to infinity. Since this is a unique expansion, specification of this state is the same as specifying that infinite set of numbers c_0, c_1, c_2 etc. these are the components. So if you give me an infinite number of coefficients, i give you the state uniquely. In the case of alpha, alpha is a doubly continuous variable, both the real and imaginary parts.

So you would expect that alpha also forms the basis and they are normalized. the question is, if you have a different alpha, is the inner product of this alpha with this beta equal to zero or not, if the alpha is not beta what could you expect? Then I would say this is orthonormalized. This is what we would like to find out. Which would you expect? What would your intuitive expectation be? You seem to have many more doubly infinite number of labels, both the real and imaginary parts of alpha. So if you can uniquely expand the state in terms of this doubly infinite labels, how come it suddenly get compressed to just the nonnegative integers? So it doesn't look right. So let's compute this overlap.

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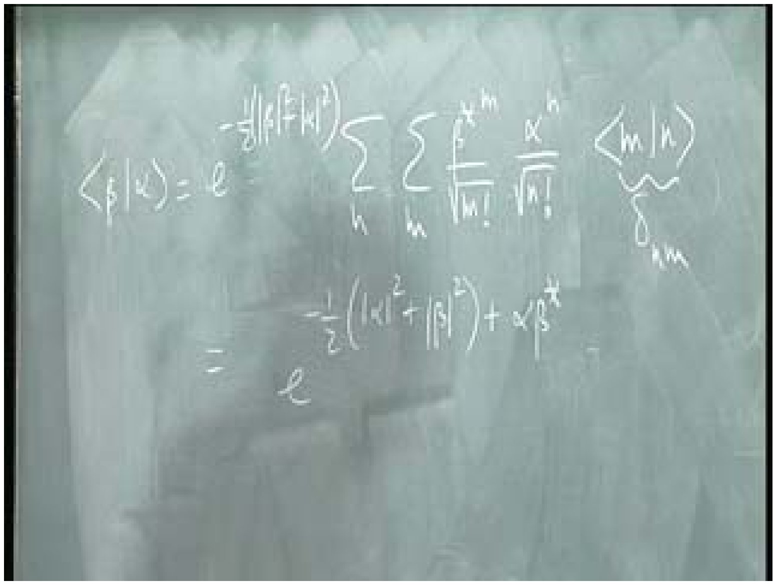

Let’s do beta with alpha = summation over n summation over m and this state alpha was = alpha to the n over root n factorial, there was an n on right side and there is a beta star to the m over root m factorial. you shouldn’t forget beta with alpha equal to e to the – 1/2 mod beta squared + mod alpha squared those who are the 2 normalization factors for the state and i took the blob data so you got a put a beta star here over m over this and then there was an m n but this is delta $_{nm}$, that’s orthonormal. So therefore the summation collapse is to a single sum and you get moderate + square + mod beta squared and then you get alpha beta star to the power n over n factorial. This is what you get. if alpha= beta, you get a 1 of course because you want alpha with alpha to be 1, you we normalize it.

But this number has a modulus less than unity but it’s not zero. This is the complex number here but you can take its phase and its real part and so on and take out modulus times e to the power whatever it is and you can compute what the modulus of this quantity is. So calculate mod beta alpha. Mob alpha beta squared will turn out to be proportional to e to the - alpha - beta whole squared times some number. So in the complex plane of these eigen values, mod alpha - beta is the distance between those 2 points. Now the 2 points coincide then of course you get unity as normalization but in the difference you get some number less than 1, e to the - some positive number.

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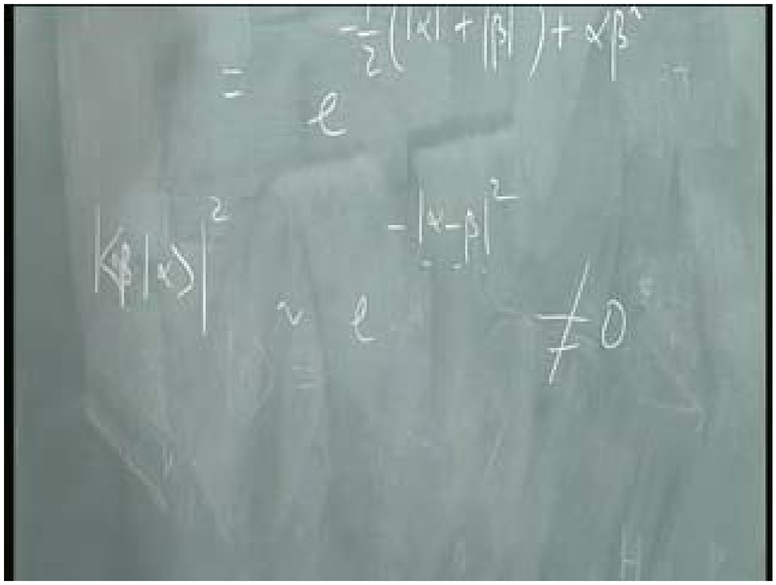

So it is not zero in general. Its exponential doesn't vanish. So this is not an orthonormal set of states. What to do you expect regarding completeness? Once again it's not a complete set of states but it's over complete. It is actually doing much more than that. And this is another thing. I am gone a leave it as an exercise.

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You can show that integral and now you have to sum overall possible alphas and alpha is a complex number so you have to integrate over the real part of alpha and the imaginary part from - infinity to infinity. so that's over the entire complex planar, let me call it d 2 alpha to show it's a 2 dimensional integral. Alpha alpha over pi = the identity operator.

Again I'm leaving that you as an exercise to show. It's slightly harder in this case but this should be fairly straight forward and I will indicate how to do that in the problem sector. These states are said to be over complete. So the set alpha is an over complete set. But now comes a key question. What does the wave function look like for alpha? Again I will go back to the definition.

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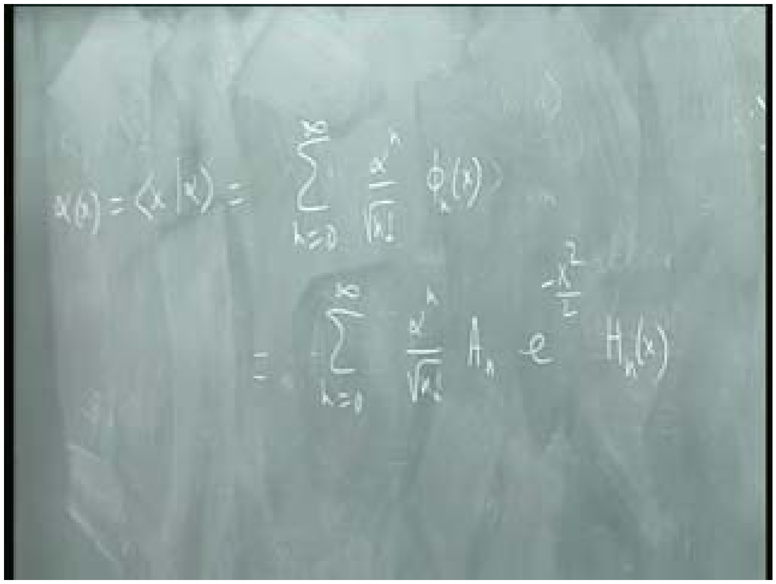

I have alpha = summation n = 0 to infinity, alpha to the n over root n factorial on n and i would like to look at it in the position basis. This is the wave function in the position basis corresponding to this coherent state alpha. So this is what i call alpha(x) labeled by this x. the wave function must be some square integrable function and that would be equal to this (Refer Slide Time: 56:42) but what's x with the n? This is our famous phi $_n$ (x). This is phi $_n$ (x) but that = apart from some normalization constant, summation n = 0 to infinity, alpha to the n over root n factorial and then this normalization constant A_n multiplied by e to the - x squared over 2 in those units that we chose, $H_n(x)$. Now you are stuck with this. You have to do this summation but you are saved by the fact that this A_n has a 1 over square root n factorial and that will kill this (Refer Slide Time: 57:42) and then the rest would be an exponential which you can actually sum.

So I am going to leave this to you as an exercise to do. And what could you expect once you sum it? You get an exponential which involves this (Refer Slide Time: 57:55), remember the expression we have for the generating function. That's where it comes in use and it will turn out this is a displaced Gaussian. So it's not e to the x squared but it's e to the x - alpha$_1$ squared where alpha$_1$ is a real part of alpha multiplied by some phase factor. Similarly the momentum space eigenfunction would be e to the p - alpha $_2$ but alpha $_2$ is the imaginary part of alpha. so all that's happening by applying this operator e to the alpha a dagger - alpha star a on 0 is that the center of oscillation which was 0 for the ground state and for all other states as well and the momentum eigen value was also 0 that gets shifted.

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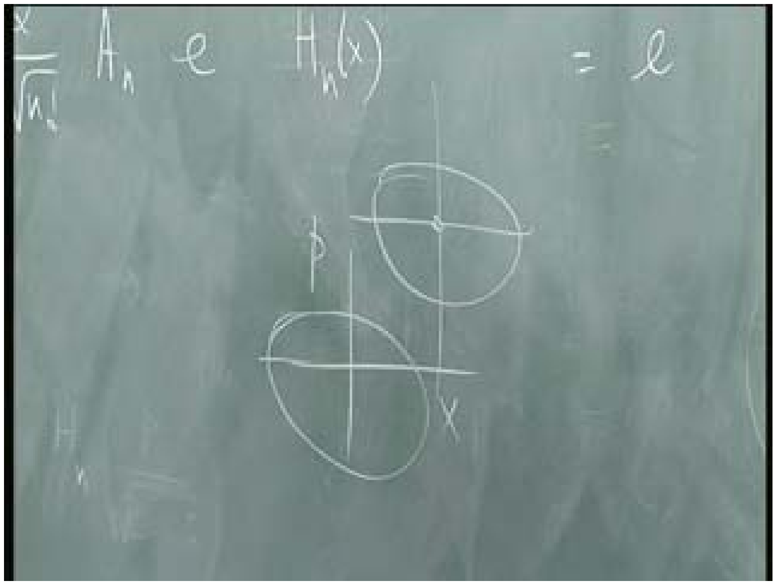

So classically what it means is that in the x-p phase plane, the original oscillator corresponded to an oscillation like this (Refer Slide Time: 59:02). Once you apply this operator, the phase trajectory is somewhere else at this point shifted by $alpha_1$ on this side and $alpha_2$ on that side. It's just a displaced oscillator. So you would expect nothing new has happened by doing this. So in fact you can start with now a state alpha, treat that like the vacuum and apply a dagger on it or the equivalent of the dagger and create another set of excited states everywhere. So all these statements are deep implications in quantum optics and I will explain a few of them as we go along. let me stop here today and next time we will discover some more properties of this and what I should like to do is to give you some insight into what does this (Refer Slide Time: 59:49) quantity do. It's a unitary operator and it's called the displacement operator for a few reasons. We will study a few properties of these. This will also help us understand the algebra a little better. So we will stop here. Thank you!