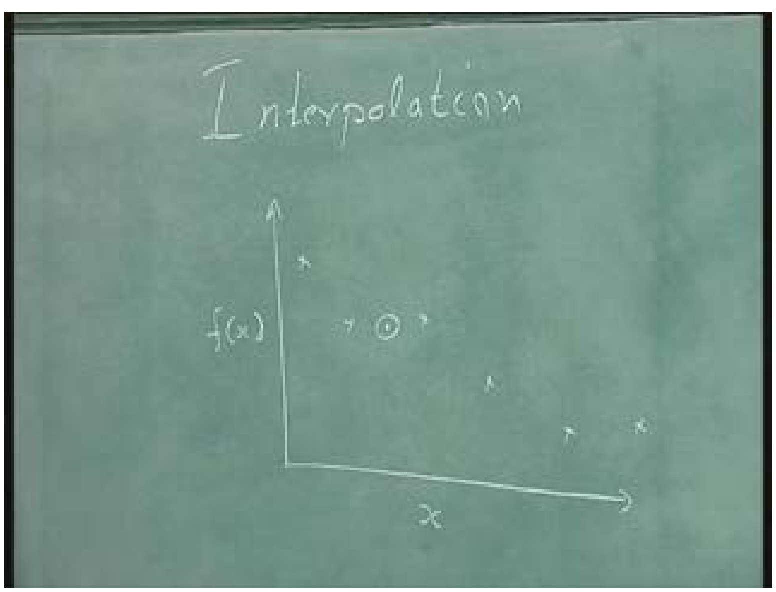
Numerical Methods and Programming
P. B. Sunil Kumar
Department of Physics
Indian Institute of Technology, Madras
Lecture - 9
Polynomial Interpolation

Today, we start looking at some of the numerical methods. So far, we have been looking at the programing part, and the representation of data on the computer floating point, and the errors due to that. So to summarize, we so far looked at some elements of C programing, and how do a floating point number represented by a finite number of digits in a computer, and what are the errors which come because of finiteness of the floating point, and also the round-off, and how those errors propagate in a calculation which involves a series of mathematical operations. And also, what are the, how to identify the sensitive steps in the series of operations.

These are the things we looked at. The next part of this course we would look at some of the techniques of numerical programing, the numerical methods, the methods which are commonly used in scientific programing. So, one of the topics is interpolation. That is what we start today. So we start today with interpolation. So interpolation of data. So that is what we will be looking at in today's lecture: when do we use interpolation and where do we use it. So, suppose we have a set of n discrete points, x of i and y of i, which I have represented here, and it is required to find the value y at a point between x of i and x of i plus 1.

Let us say we have, because of some numerical calculation or simulation or some experiment which is done, we have a set of data points as a function of some variable. So we had some set of data points which I have represented like that.

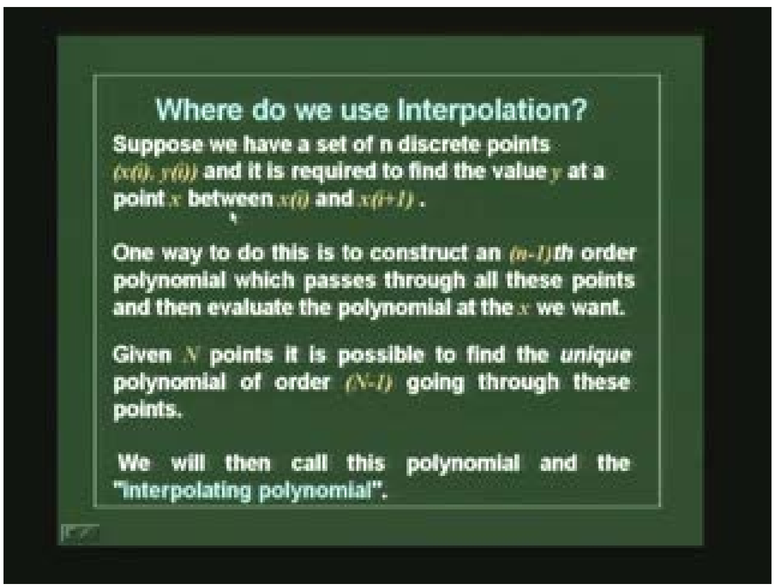
(Refer Slide Time: 04:01)



It could be a measurement of temperature as a function of time, or any other property, or it could be output from a simulation. In this case, we have a discrete set of data points, and you want to find out what is the value of this function, is at some point in between, which is here represent by this dot. So in those cases we need to have a function, a curve which goes through all these points, and it approximates the value here.

So this is, we are talking about a method of approximating some values, a function. So that is what I have represented here. These points are given by i noted here as x of i and y of i, i notes the set of discrete points. So y of i is my function f of x. So x of i are a set of discrete points, and y of i is a function value, and you want it between any two points, i and i plus 1, some value, x. And I want to find out what is the functional value of y at that point is?

(Refer Slide Time: 04:46)



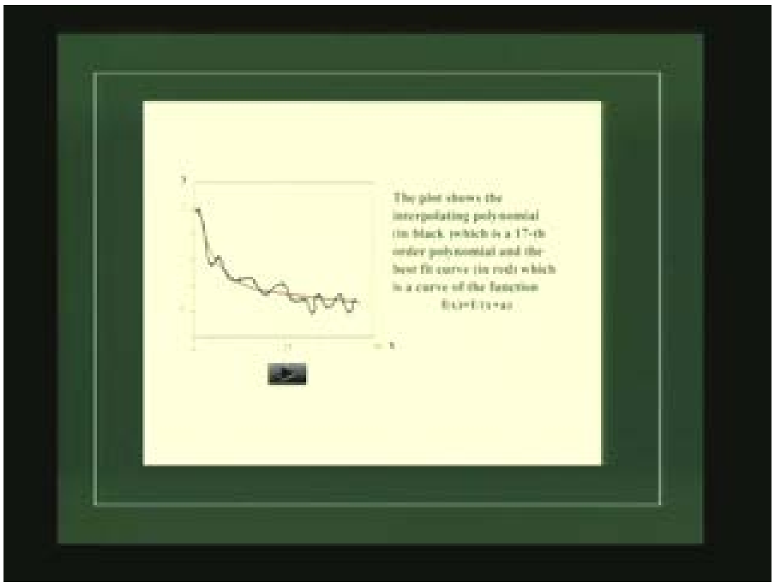
So one way to do that is to say that there is a polynomial p_n of x. So we can say a polynomial p_n of x which passes through all these points, it takes values pn of x_i is equal to y_i. So this is what we have put in here.

So we have this function y_i as f of x_i. So we have y_i as f of x_i, and if you can construct a polynomial which takes in values y_i at a set of points, x_i at the discrete points, x_i, and then that is a good approximation to this curve. That is one way of doing it, and then once we have the polynomial given to us of order n, given this n points, then we could, order n minus 1. So if you give n points, you have an order, n minus 1, and then we could evaluate the function value at any point, x, in between. That is one way of doing this, and we all know this given n points it is possible to find a unique polynomial of order n minus 1, which is going through these points.

So, given n points it is possible to find a unique polynomial of order n minus 1 which goes through all these points. So we call this polynomial as the interpolating polynomial.

So that is the notation we are going to use. We are going to use, say, that the points are the discrete points are given by x_i, and the functional value is given by y_i or f i, and the polynomial of order n would be given by p_n of x. That is a polynomial of order n. So this polynomial is called the interpolating polynomial, and in today's lecture we will try to find out how to construct this polynomial. This is a unique polynomial, i agree, but there are many representations of this. So how do we construct this polynomial and what are the approximations involved?

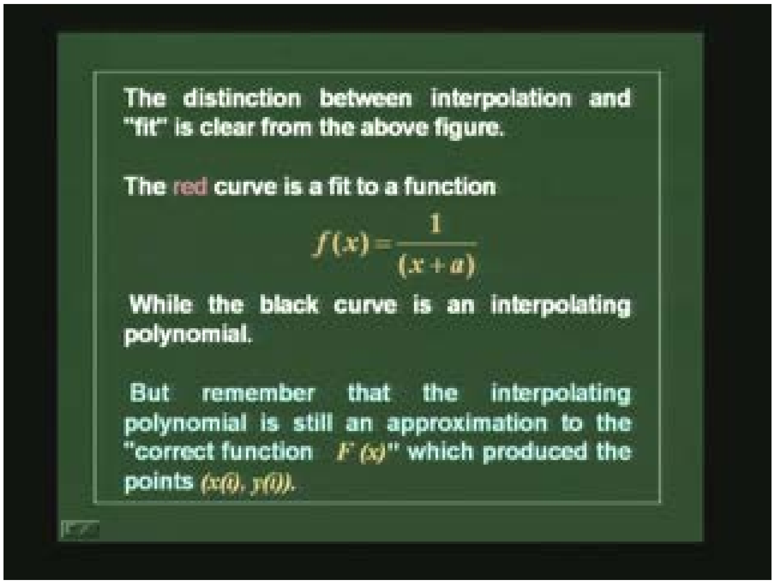
(Refer Slide Time: 06:59)



That is what we would try to find out today. So here is a plot which shows a set of data points, and then I am drawing two curves here just to show you the difference between what do we mean by interpolation, or interpolating polynomial, and what is the difference between an interpolation polynomial and a fit. So here are two curves. As we just saw, this is the set of data points which we obtained from simulation, or from experiments, and we have one curve which goes through the, a red curve which goes through, which goes in between these points, and then we have another curve which goes through all of these points. So the curve which goes through all of this here, this black one, is my interpolating polynomial, while this curve, the red curve, is a data fit. It is a fit to that thing.

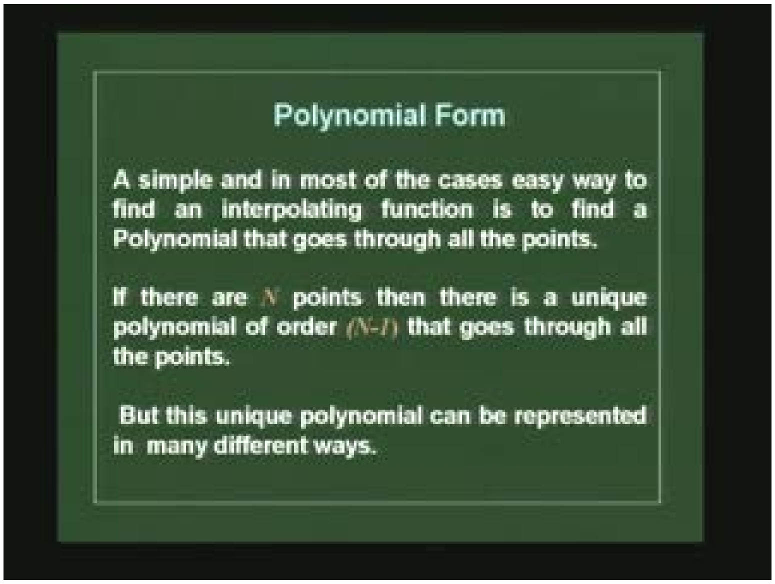
By this, I want to show you the difference between these two curves. So, here is a 17th order polynomial which goes through that, and here is a fit to this data by a function f of x equal to 1 by x plus a. That is the difference. So the difference between data fit and interpolating polynomial should be clear from this particular plot. Also keep in mind that even though this data goes through all these points it is still an approximation. It is not the exact function, f, which we are getting by doing this fit. It is still an approximation to the data. So that is what I showed you before, that is, the red curve which we showed before is an approximation, is a fit to this curve 1 over x plus a, but while the other curve was, which goes through all the points, was a polynomial. But remember that the interpolating polynomial is still an approximation to the actual function, f of x, which produced those data points.

(Refer Slide Time: 09:02)



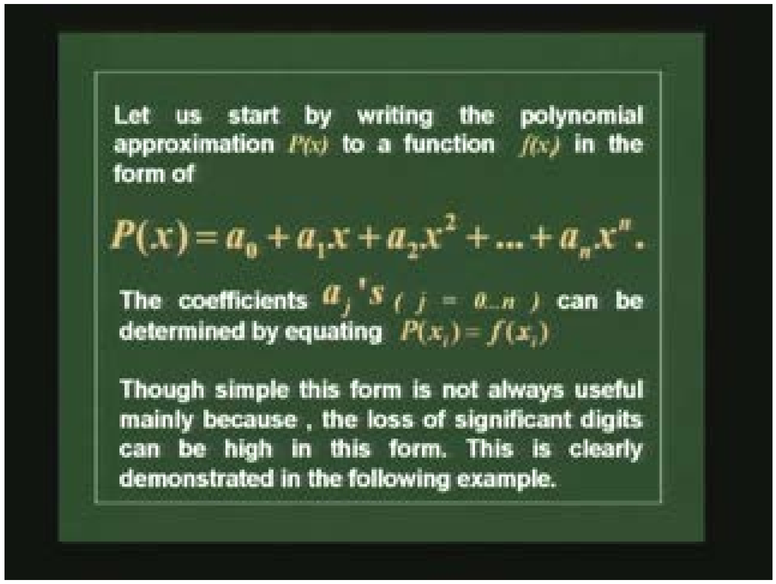
It is not the function; it is an approximation to the function. So as I said, there is a unique polynomial of order n minus 1 if you are given n points. If you know the function value at n points, we have a unique polynomial of order n minus 1, but there are many ways of getting this polynomial, or many ways of representing this polynomial.

(Refer Slide Time: 09:44)



So that is what you would go through now, and then we would look at some methods in which it is easiest to construct the polynomial. Instead of going through all the methods of constructing the polynomial we will just look at one or two methods where it is easy, programing-wise, to construct this polynomial.

(Refer Slide Time: 09:54)



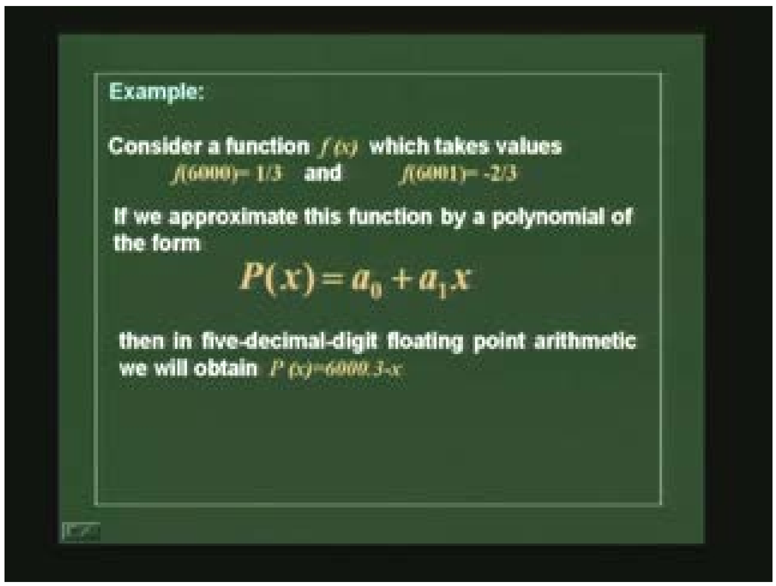
So here is an approximation to a polynomial of order n here. So we have said that a polynomial p of x which fits in the function values at f of "x_i" functional value of "x ''.

So here is a polynomial of that which I have written down here. So p of x is ao plus a_1 x plus a_2 x square up to a_n x to the power of n, so nth order polynomial. These coefficients then, we have a condition that this p of x should match the function value, or the measured function value, or the functional values. We have these functional values, this p_n of x should satisfy these functional values, or it should satisfy this relation that p_n of x_i is equal to y of i. We should satisfy this condition, and that will allow us to determine these coefficients, a_0, a_1, a_2, a $_3$, up to a_n. That is the method we would use. And so we could equate p of x_i to "y" of i and y of x_i or f of x_i, and then we could say, we could determine these coefficients, a_0 here.

So we will have, since we have n plus 1 function values, and we have n plus 1 equations. We are equating that, and we have n plus 1 coefficients to determine, so we can determine that. So the problem is that even though it looks simple when I say this, that you know you have n plus 1 equations, and you have n plus 1 unknowns, and you can simply determine this n plus 1 coefficients from this set of equations, but it is not always trivial, because we will see that we have tried to compute these coefficients. We have loss of significant digits, and because of that, these approximate, this polynomial can be very errorless. That is what we would see in the, we will try to see through an example.

So here is an example. So we have a function, and we have just 2 values. The functional values are given at x equal to 6000 as 1 by 3. I have chosen this thing because we know that this number is particularly to represent in a finite digit floating point representation, and f of 6001 as minus 2 by 3. So we have f of 6000 as 1 by 3, and f of 6001 as minus 2 by 3, and we want to approximate this function. So there are two points. So we have a polynomial of order 1 that it is a straight line, and you want to approximate this curve by an interpolating polynomial of the form a_0 plus a_1 x. So we have p of x as "a_0", plus "a_1" x as our polynomial.

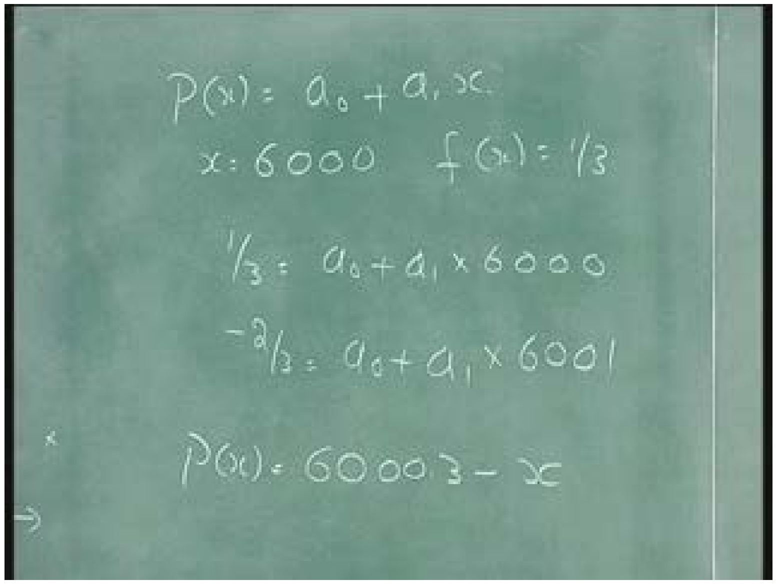
(Refer Slide Time: 12:58)



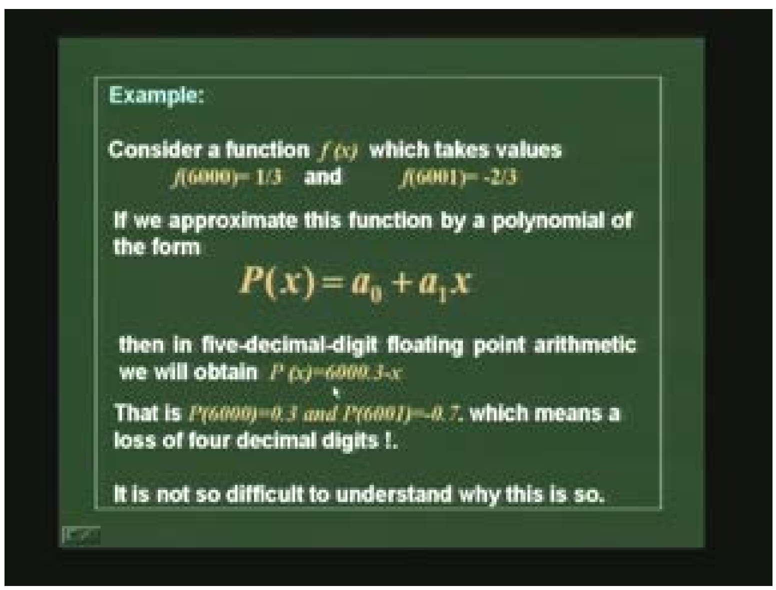
Now we try to determine the values of that. So how do we do this? We say, that p of 6000. So we would say we have this polynomial now as p of x is equal to ao plus a1 x. And we have the value as x equal to 6000. We had f of x equal to 1 by 3.We know that, and then we will substitute that here. So we will say 1 by 3 is equal to "a_0" plus "a_1" into 6000, and then we have f of 6001 as minus 2 by 3 here. We have f of 6001 as minus 2 by 3.So we will put that in here and we will say, f minus 2 by 3 is "a_0" plus "a_1" into 6001.That is what the 2 equations which we would have.

So now, we have to compute "a_0" and "a_1" from this which is trivial, and we will do that, and then we would get the approximation of the polynomial as written there as p of x is equal to 6000.3 minus x. That is the value which you would get. "a0" 6000.3 minus x. Now let us see what happens if you, I said we have used the condition that p of x is equal to f of x, and then computed "a_0" and "a_1", and we got the polynomial, and we would expect p of x would be equal to f of x when we substitute any of the x values into this polynomial. That is what we try, we expect.

(Refer Slide Time: 14:22)



(Refer Slide Time: 14:46)

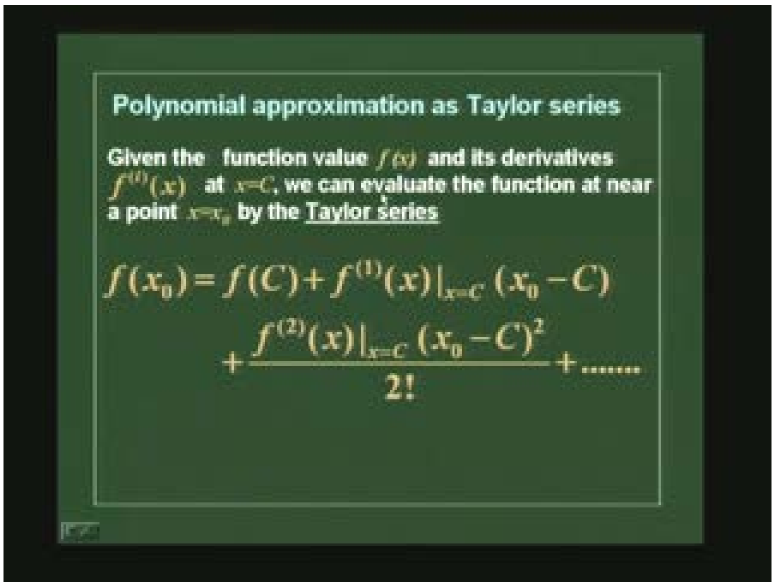


When we do that, what we find is that in a 5 decimal digit floating point arithmetic we would get p of 6000 as 0.3, and p of 6001 as minus “0.7”.So, which means the loss of4 decimal digits which is not something which you would be happy with.

So that is, so this is an example to show that even though such a method of equating a polynomial straight away into the measured function values, and evaluating the coefficients from that, will not always lead to correct polynomial, because of the error in the representation of the floating point numbers, or in the representation of real numbers in a finite number of digit floating point representation. So that is what we see. So there are better methods of doing this. So one method to do that is, to use a Taylor series evaluation. As we go on the course, we will see many times, we will use this particular series called Taylor series, so I thought may be a good thing to spend a few minutes. I am just trying to understand what the Taylor series is.

So given a function f of x and it is derivative, we set that point, and if the function is a smooth differentiable function, and then given that function and it is derivatives, or it is differentiable at many points, and then at a particular point x equal to c, if you have all the derivatives, then evaluate. We can evaluate the function near the point x equal to x_0, or x equal to c, or say any point x equal to x_0, or x equal to c by expansion around that. So here, I am showing that this function f of x is being expanded around c to another point x_0 to get its value at x_0.

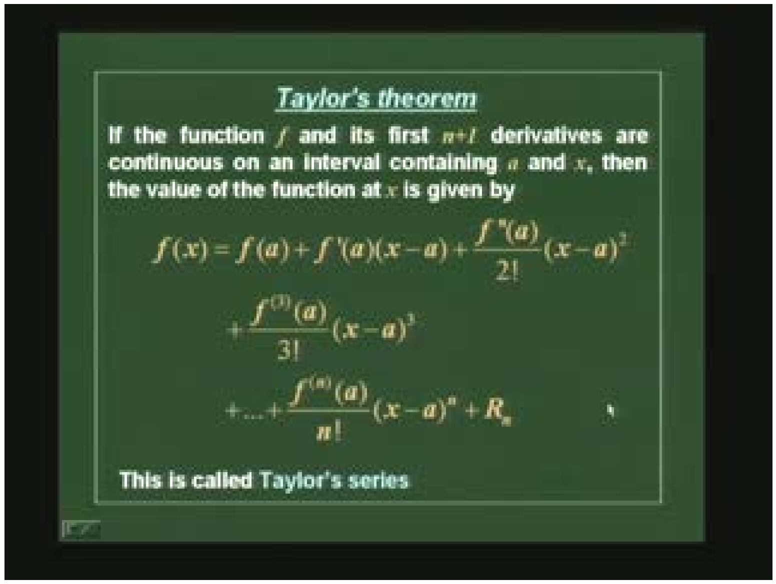
(Refer Slide Time: 16:49)



Let us look at this Taylor series a little more in detail. That is what a Taylor theorem says. If a function, f, and it is first n plus 1 derivatives are continuous, and on an interval containing “a” and “x”, then the value of the function at x can be determined by, from the value of the function at “a”, by this series.

That is what we have used there. What we are using here is the value of the function at a. And f prime of a, as we saw in earlier lectures, is a derivative of a with respect to x evaluated at x equal to a, multiplied by x minus a, and in the second derivative divided by 2 factorial times x minus “a” whole square, and all the way up to the nth order derivative with respect to x evaluated at “a” divided by n factorial into x minus “a” to the power n, and then there is some remainder in this series.

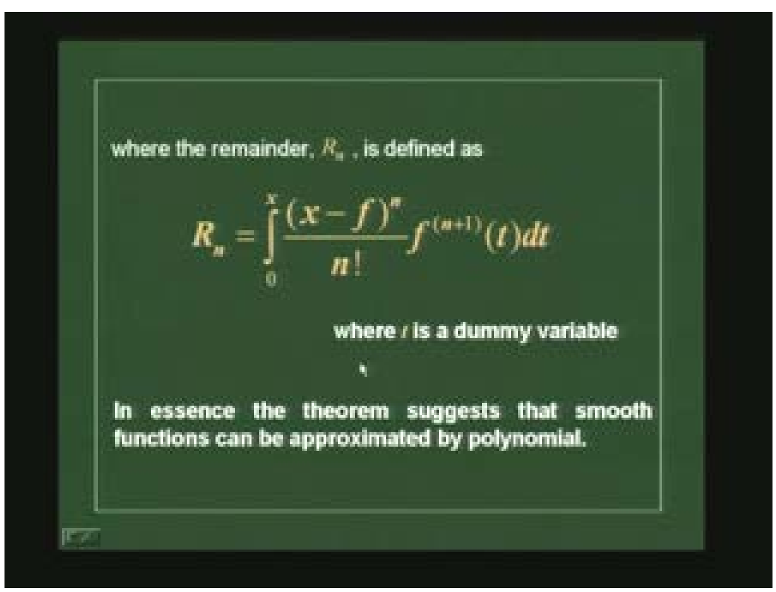
(Refer Slide Time: 17:46)



So we can see that the remainder of this series is given by a function of this form. That is basically if you do a Taylor series expansion, and we always, because we cut it off at some order n, we have some error, and that is what is given by this x minus f power n divided by n factorial, into the n plus 1th derivative.

So, the error in the Taylor series is always n plus 1. It is proportional to the n plus 1th derivative of that function. Here that particular function is a function of t, some variable t. That is, what is the essence of the Taylor approximation?

(Refer Slide Time: 18:40)



We will use this Taylor series to expand and get our data fit, to get our polynomial approximation. Instead of using a polynomial approximation of this form, what we will do is, we know, we will try to approximate the polynomial around these points by expanding the around this point. So we know the functional value at this point. So we will write a Taylor series expansion around that particular point.

So that will be a better representation, a better polynomial for region around this. So that is for short intervals. This is, we have better method than doing this kind of an expansion. So here, actually, expanded this polynomial around x is equal to 6000. We would have found a much better approximation to this polynomial than 6000 this, what you obtained here. That is what we are trying to see. So we will try to see how to minimize this error by using this kind of an expansion. So we had chosen x equal to c.

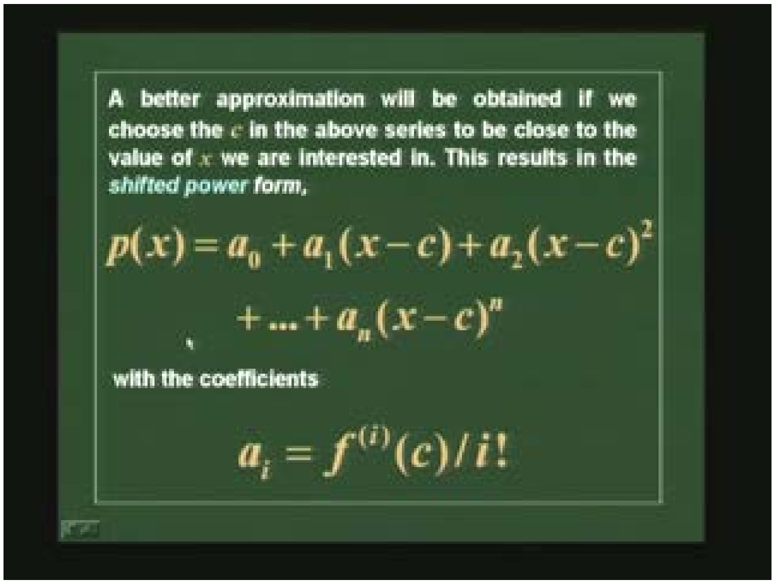
So we would take, see this particular example as 6000 and expand it around that region. So that is what you would see. Now, if I use c is equal to 0, then we have our polynomial which is obtained here. So basically, we want to write this polynomial as something minus c here. x minus c, where c, I would like to take as 6000.

I could take that as 6000 and then compute all those quantities again. That is what you should be trying to do. That will be an expansion of the function around x equal to 6000. So we will see that now. Now then, you will write the polynomial of many order as in this fashion now. Instead of writing a1 x we will write it as p of x as a_0 plus "a_1" into x minus c plus "a_2" into x minus c, whole square, etcetera, up to x minus c to the power n. So that is an nth order polynomial. So this is an nth order polynomial with x minus c to the power n as the highest term. Again, we see these coefficients, a_0, a_1, a_2, now we know are actually the derivatives of the function p of x, original function p of, function f of x, in this case, the function p of x, but we do not know the function.

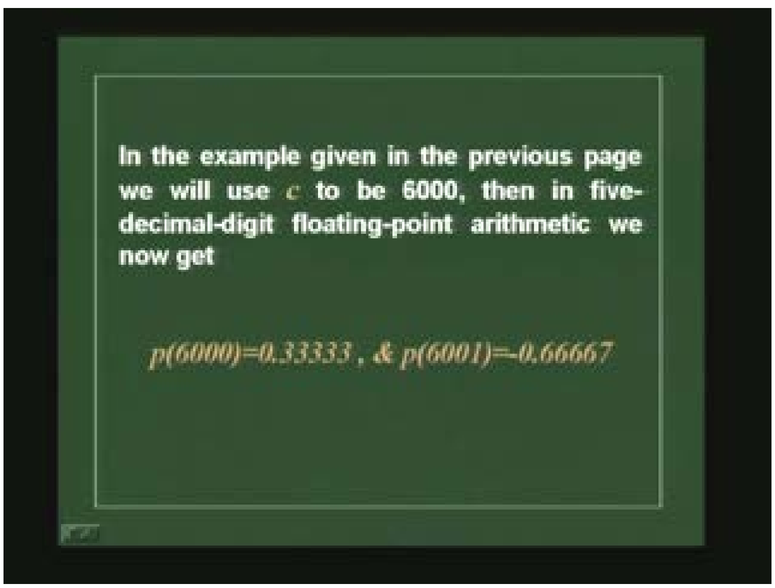
So we have to evaluate this instead in another way, but that we will do in similar fashion as we done earlier. This is, we will equate this function to various values. So again, we know what the c is here, in this case. Because we know where we are, expanding it around and then we would determine the values of a_0, a_1, a_2 by again equating this polynomial to the known values of the function.

Let us take x equal to 6000 here. That is c equal to 6000, and then we are getting p of 6000 now as 5 digit floating point representation. We are getting as "0.333333" and p of 6001as "minus 0.66667". So, we have a much better representation now, of this polynomial than what we obtained here. That is basically the essence of this calculation, this idea.

(Refer Slide Time: 21:25)



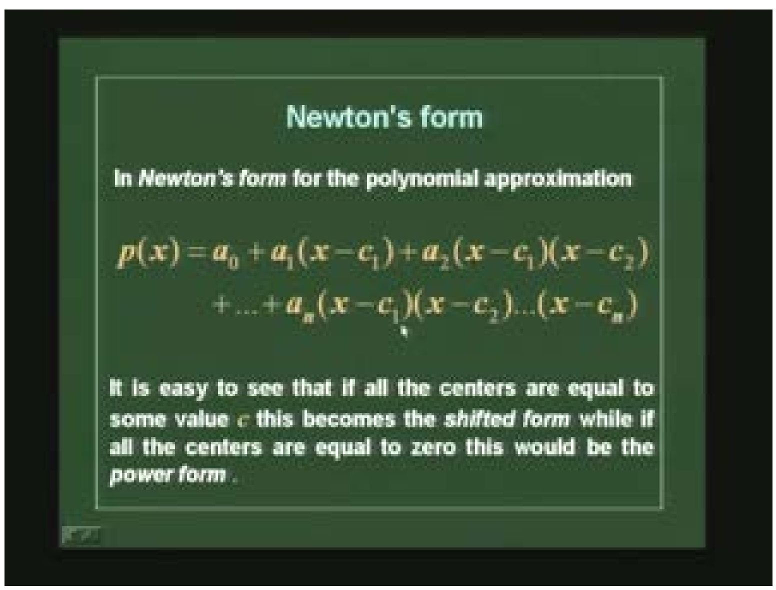
(Refer Slide Time: 22:00)



That is, we expand it in a series. So now, the same thing, as I said here. So what we were doing was writing f of x as f of x_0 plus, we said that x minus xo into derivative of x at x_0. So actually, it is what we are writing is del f by del x at x equal to x equal to x_0, and then we wrote it as x minus x_0 square by 2 factorial into del square f by del x square at x equal to x_0. We said that it is x minus x_0 to the power n divided by n factorial into del n. So that is what we had written in short notation as fn this derivative, and f_n of x_0, we said, that is basically the derivative the nth derivative of "f" with respect to x_0. This is the series which we have written. So this is what you were writing before as f of x being the polynomial, now as p of x as we are writing this as ao, and then we are writing this as a1 into x minus x_0 plus a_2 into x minus x_0 to the square, etcetera. So that is what we were writing. So now, this can be written in a slightly different fashion. So we could say that, in this particular case, we have always expanded the whole function around x equal to x_0. Now there is a slightly different way of writing this polynomial in which we will not use for different, you will not use x equal to x_0 as expansion point all the time but we would use something like this.

So we have to use here, we have to use the first term as x minus c_1, and the second term is, instead of x minus c_1 square we are using x minus c_1 into x minus c_2 and continue up to n, and the nth term would be a n, x minus c_1, x minus c_2, x minus c_n. So notice that where c_1 equal to equal to c_2, c_3, etcetera, up to c_n are equal to c_0, or x_0, in this particular case, then we would get this polynomial which I have written here. So this would follow from this if I use all the coefficients, c_1, c_2, c_n equal to x_0. That is what we call the shifted form. This is what called the Newton's form. So this is called the Newton's form for the polynomial, and what we would get if I put all these coefficients equal to the same thing is the shifted form which were all centers are equal to, all centers are equal, and if I put all coefficients equal to 0, then we get the normal power form which we obtained in the beginning.

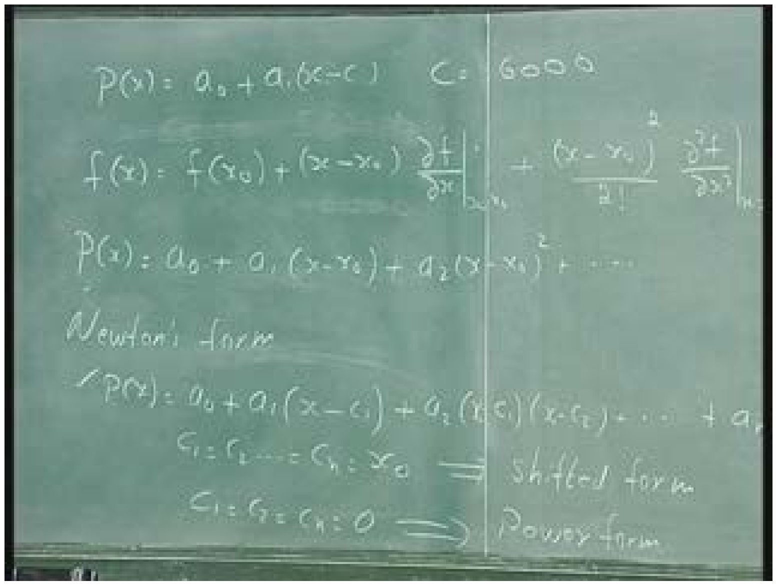
(Refer Slide Time: 25:13)



So to summarize, we have the Newton's form. So the Newton's form, we had p of x as a_o plus a_1 into, in this language, x minus c_1, plus z_2 into x minus c_1 into x minus c_2 up to a_n, which goes as x minus c_1, into x minus c_2 all the way to x minus c_n. We said that if I put c1 equal to c_2 equal to c_n as xo, then we go back to that which is called the shifted form.

So that is, shifted form or we could have c_1 equal to c_2 equal to c_n equal to 0. So that is the power form. They are all the same polynomial. They are just represented in different ways of, in different ways, and because of the finiteness of our representation, the real number representation has been represented as a using a floating point of finite number of digits. We have different answers, numerical answers coming from this, but mathematically, they are all the same polynomial, because the polynomial which passes through, the nth order polynomial which passes through the n plus 1 points is a unique polynomial. So now, so we will write this particular polynomial in a slightly different format. That is what we would look at.

(Refer Slide Time: 27:09)

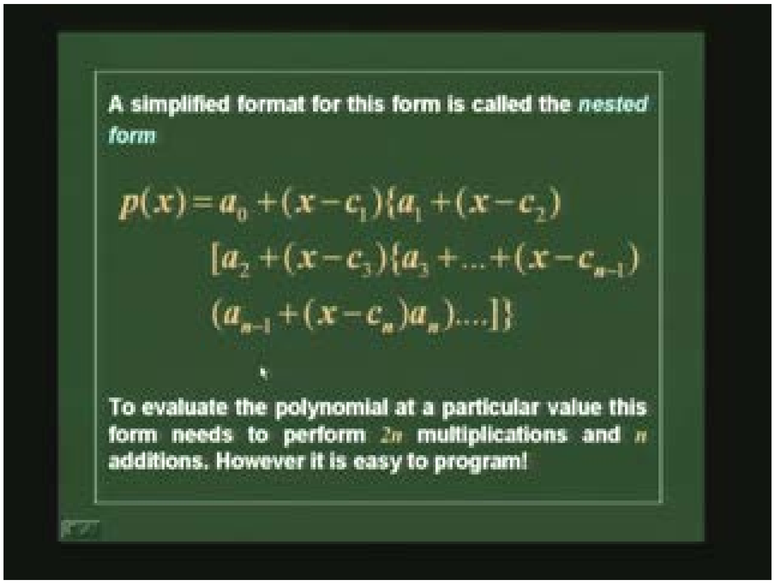


For computational purpose, it is better to write this in a kind of a nested form, and that is what I am trying to show here. So the same thing which I have written there is now written in a slightly different form. That is, we write it in what is called a nested form. In this form, we write it as a_o plus x minus c_1 into a1 plus x minus $_{c2}$ into a_2, etcetera. So you see, x minus c_1 is common for all terms starting from the second one. So I could take out x minus c_1 outside, and write it as a_1 plus a_2, then x minus c_2 is common for all the terms down this. So, I could take that out and then continue writing that in this fashion, and then you would get a function, a form of this type, and this is called the nested form, and this is what you would see in many text books when you talk about Newton's form in the nested form.

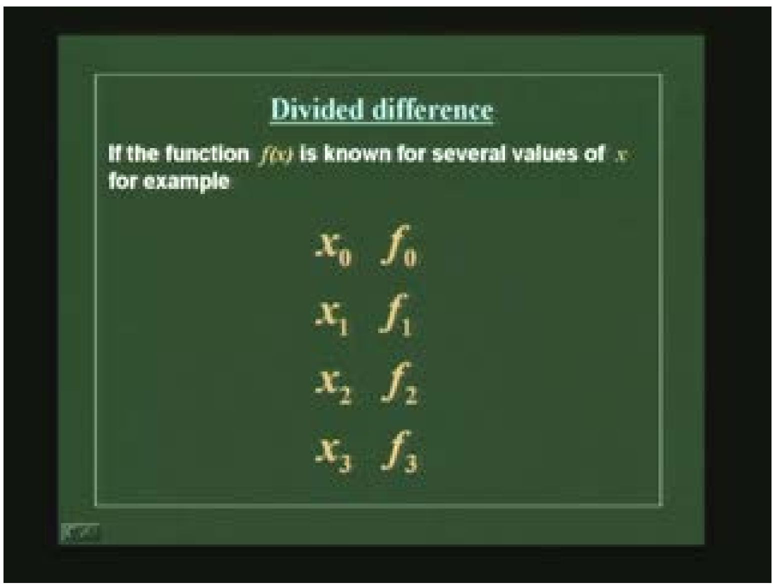
We need to do, as I said, if you want to do, if you want to evaluate after obtaining this polynomial, so we equate this polynomial to n plus 1 function values which we know. We can obtain these coefficients a_o, a_2 up to a_n minus 1. We will take c_1, c_2, c_3, c_4 as the functional values which we know. The x values at which the function values are known are c_1, c_2, c_3, and then from that we can actually compute this whole polynomial, all the coefficients in this polynomial, and then we have to evaluate. Since, we know the polynomial we can evaluate this polynomial at any given value x. So when we do that, we have to do two n multiplications and n additions. There are n terms here, so we have to do two n multiplications and n terms.

Each bracket here has one multiplication to do, so we have two n multiplications, and we have n additions to do, but it is, however it is a very easy thing to program. That is a very, it is still a very popular form to use.

(Refer Slide Time: 29:14)



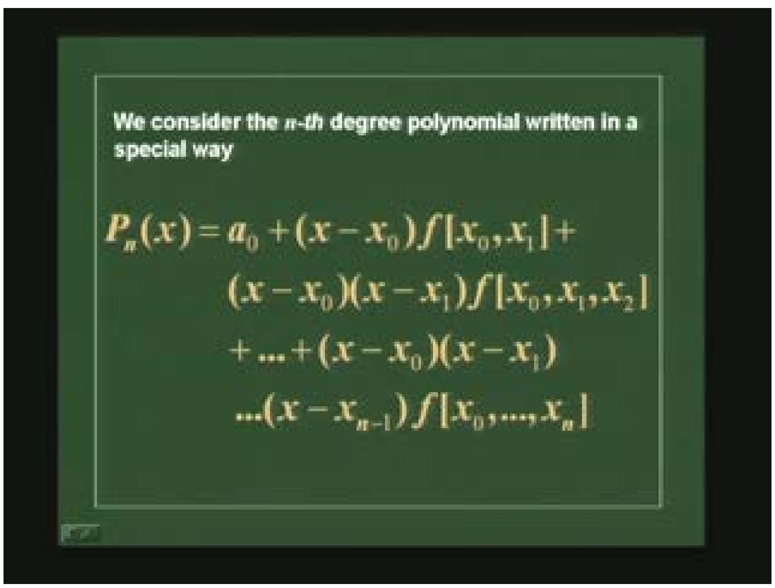
(Refer Slide Time: 29:32)



So now, there is, once we have the nested form and we saw this quantity, there is when you program it we would use what is called a divided difference method to compute the values, the coefficients. So we just saw that there are, while equating this polynomial to the function values at x_i or x equal to c_1, c_2, c_3, etcetera, I can actually compute all these coefficients, a_o, a_1, a_2. So that is easy to see. We can see that I could just take p or at x equal to c1, then I would get it as a_o. So "p" at x equal to c_1 from this polynomial, it is that all other terms would vanish and that is just a_o and "p" at x equal to c_2 would be then a_o plus "a_1" into "c_2" minus c_1. All other terms will be 0. So by doing in this fashion we can actually compute all these coefficients, a_o, a_1, etcetera. But that is not a very elegant or a very good form of computing it. So we can use what is called a divided difference to actually achieve this quantity. So we will see how this divided difference is executed. So first let us fix a notation. So we would call the function values at x_o as f_0, and the functional value at x_1 as f_1, x_2 as f_2, and x_3 as f_3, etcetera. This is clear. So this is the notation we are going to use.

So we have the points at which the function is known, and these are the function values which we have got up to x_n f_n. So basically, f of xo will be called as f_0, f of x_1 is called x_1, etcetera, and then we would write this using a slightly different notation, exactly the same thing. We are going to use a slightly different notation to make things more clear there instead of writing a_o, a_1, a_2, a_3, etcetera. We are going to use this in this notation. So we are going to use, change the notation a little bit. We are not going to use a_o, a_1, a_2, etcetera. We are going to use f of square bracket, x_o, x_1, f of square bracket, x_o, x_1, x_2, etcetera. So that tells us what the coefficients are. so the coefficients are now not called a_o, a_1, a_2,it is called f square bracket x_o, x_1, f square bracket x_o, x_1, x_2, etcetera. So this is just to make programing easy.so now we have this form.

(Refer Slide Time: 32:07)

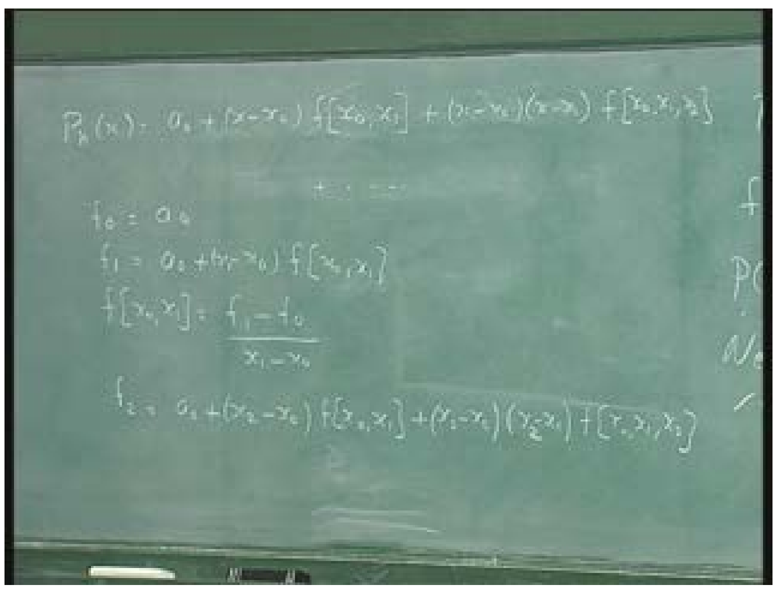


So we are going to write our polynomial as p_n of x equal to f. Remember, this is a square bracket. So it is x_o, x_1, and then the next term would be, so that is the form which you are going to write, and then as I said we can now take p_n at x_o. That will be a_o, and p_n at x_1 would be a_o plus x minus x_1 into f of x_o x_1, and then we would get a series, we would get a set of equations of this form, and that is what then, we would make up a difference table.

So let us compute this quantity here. So we can say that the function values are now called f_1. So, f of pn at xo is now called f0. So we have equations which say f_0 equal to ao. And then we have equation called f1 which is f of pn of x_1, this will now be ao plus x minus x_1 into f x_o x_1. So we can easily that f "x_o x_1" is nothing but f_1, and ao was f_0, so it is f 1minus f_0 divided by, this is x_1 minus x_o. I substitute x_1 there, so it is x_1 minus x_o. So it is f_1 f_0 by x_1 minus x_o. So that is the value. That is the coefficient. That is what we have written here. So then we can compute the next coefficient in a similar way.

As I said, we could now write down f_2, so that would be ao plus x_2 now minus x_o. I am substituting it here into f of x_o x_1 plus x_2 minus x_o into x minus x_1 x_2 minus x_1, into f of x_o x_1 x_2. Now I can see that I can read off, I know what ao is .I knows what f of x_o x_1 is. That is from this equation. So I have this here, and this one here, and then I know x_2 x_1, etcetera, so I can compute this quantity. From that I know f_2, so we can compute this quantity from that. But this is not the way we are going to do. We are going to do, use, by using what is known as the divided difference method. So I will show you what that is, and we will compute that using an example.

(Refer Slide Time: 35:20)



So what we will do is, we will first construct these functions, the two-point functions. We will do that here. We have written that x_o as f_0, x_1 as f_1, x_2 function value f_2, x_3 function value "f_3", and "x_4" function value "f_4". We have written that, and having done that now what we will do is we will construct the differences.

We did one here right? We did between "x_o" and "x_1". So I will do that here. So I will write here as this quantity called here f_1 minus f_0 divided by x_1 minus x_o, and I know that this is the coefficient of the first term in the polynomial, and I know that this is the coefficient of the second term in my polynomial. The polynomial, the second term is given by that. So now I have to construct the third term, the coefficient of the third term, etcetera. So to do that I will first make what is called the first order differences. So that is the, so we would make the first divided difference, the first divided differences, the second order divided differences, etcetera, and then we will construct all of them in a series. That is what we will try to do. So we had the first order, first divided difference given by x_o, x_1, or in general, given by f_t minus f s by “x_t” minus “x_s”. That is the first divided differences.

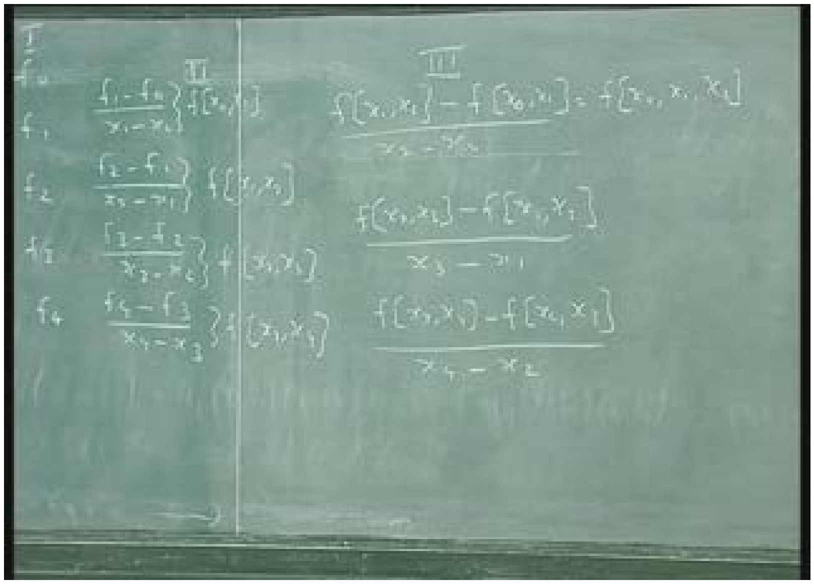
So we will construct that for all of them. We will write them here as “f_2” minus “f_1” by x_2 minus x_1, and then between these two as “f_3” minus “f_2” divided by “x_3” minus “x_2”, and then between these two as “f_4” minus “f_3” divided by “x_4” minus “x_3”.So right now, this thing has no consequence to the polynomial here. It looks like because we have constructed the coefficient one here and the second one here, and then we will do the second divided difference. So the second divided difference would be the divided difference between these quantities. So these two will give us 1, and these two will give us 1, and these two will give us 1, so the second divided difference is then given by this quantity.

So this is what we call, so this is what we call as f of x, we call that as x_o, x_1, and then by same notation we would call this quantity as f of “x_1 x_2”, and this quantity here as f of “x_2 x_3”, and this one as f of “x_3 x_4”, and then we will construct the next divided difference as f of “x_1 x_2” minus f of “x_o x_1” divided by x_2 minus xo, and here it will be f of “x_2 x_3”. We have been writing it as, the notation we write, f_1 minus f_2, f_2 minus f_1, f_3 minus f_2. So we should write it as similar way here.

So we would write it as x_2. We call it as x_2, x_3 minus f of x_2, “x_1 x_2” divided by x_3 minus x_1, and here it will be f of “x_3 x_4” minus f of “x_2 x_3” divided by x_4 minus x_2. So that is the second difference. And now what I am trying to show is that. So this is the coefficient of the first term, and that is the coefficient of the second term. So we will call this the first term, and this is the coefficient of the second term, and this will be the coefficient of the third term.

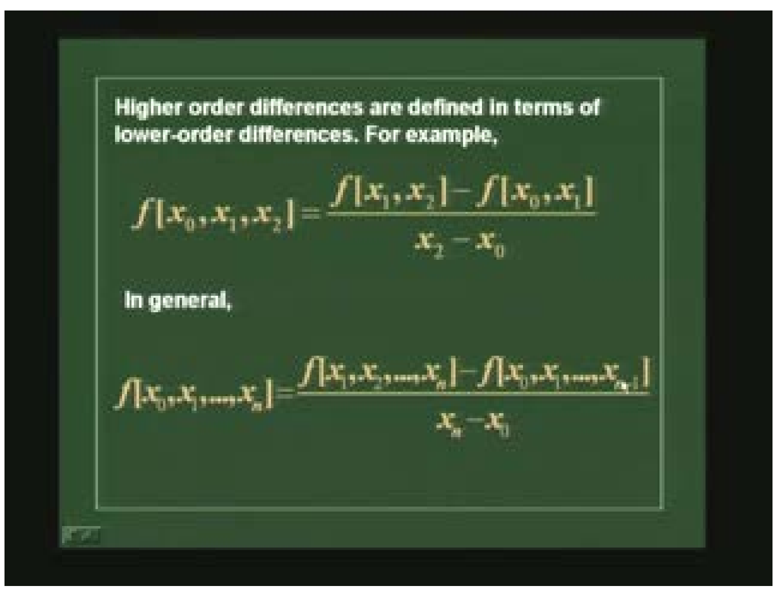
Basically, that is what f of, so this is f of “x_o x_1 x_2” is. So you can just rearrange terms here and write everything in terms of these functions which are known, and then we can show that this term, what we get from here, is just exactly the same as this quantity. That is f of “x_1 x_2” minus f of “x_o x_1” divided by x_2 minus x_o. So what I am trying to say is that I can, here I can substitute for this. This I know, and I can rearrange terms here, and do a little bit of algebra, and then I can show that f of x_o x_1 x_2 is f of x_1 x_2 minus f of “x_o x_1” divided by x_2 minus xo, so that it can be shown from this, and that I leave that as an exercise for you to do.

(Refer Slide Time: 41:08)



In general, we would compute the higher order differences in this form. That is the functions with third order term. The coefficient of the third order term which had x_o, x_1,x_2, I showed you, is the difference between the second order coefficients, second order divided differences, the first order divided differences, divided by x_2 minus x_o. We look at the intervals here. This also keeps increasing as we go down or go up on the order.

(Refer Slide Time: 42:00)



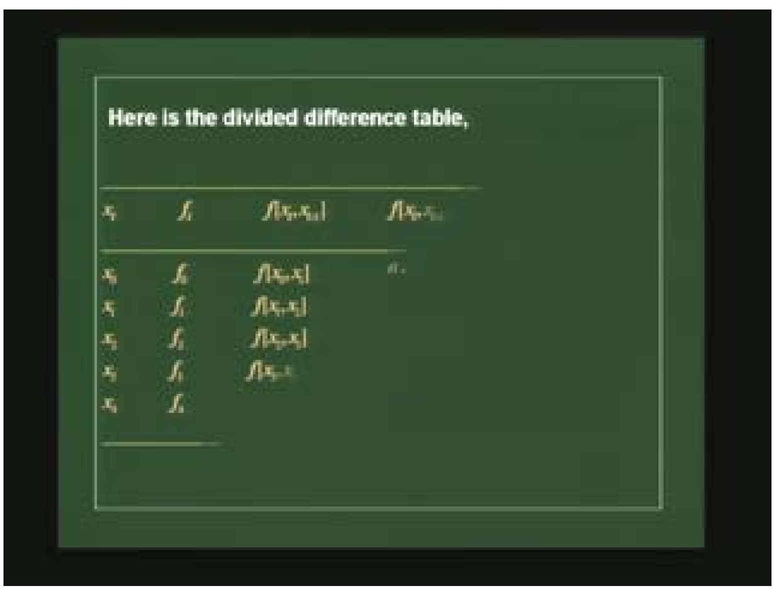
So in general, we can write f of x_o x_1 up to x_n, as the divided difference of just the level before that is f of x_o f of x_1 x_2 up to x_n minus f of x_o, x_1 up to f of x_n minus 1. So that is the way, you are going to write.

So you are going to write everything one order before. So here, notice that here x_o x_1 x_2 is the difference between x_1 x_2 and x_o, x_1 divided by x_2 minus x_1 x_o. So now, when I take f of x_o x_1 x_2 x_3, that is the next term here would be, we know that it is x_3, the next term which we would write would be f_3, would be equal to ao plus x_3 minus x_o into f of x_o x_1 plus x_3 minus x_o into x_3 minus x_1 into f of x_o x_1 x_2, which we now know from that and the next term, would be x_3 minus xo into x_3 minus x_1 into x_3 minus x_2 into f of x_o x_1 x_2 x_3. That will be the next coefficient. So this equation should determine that.

So now again, instead of going through all this to determine f of x_o x_1 x_2 x_3, we could use a series from here. So then, we will have this term now, would be called f of x_o, x_1 x_2 x_3, and this would be called f of x_2 x_3 x_4, and this term here, that is, f of x_o x_1 x_2 x_3, this coefficient of the term which is the third order, or the fourth term, the third order coefficient, would be then given by f of x_1 x_2 x_3 minus f of, minus f of x_o x_1 x_2 divided by x_3 minus x_o. So that is the way we would write. So that is what is there in general nth, so when you go to the nth term we will take 1 to n minus 0 to n minus 1 divided by x_n minus x_o. That sets the coefficients. By this way we can divide, we can compute all the coefficients. So here is the table which we are going to try to construct here.

So the table is here. So we have all the function values, all the points, and all the function values, and then we have the first order differences, divided difference, and then we have the second order divided difference, and then the third order divided difference, and then we will have one more here which will be the fourth order divided difference.

(Refer Slide Time: 45:44)

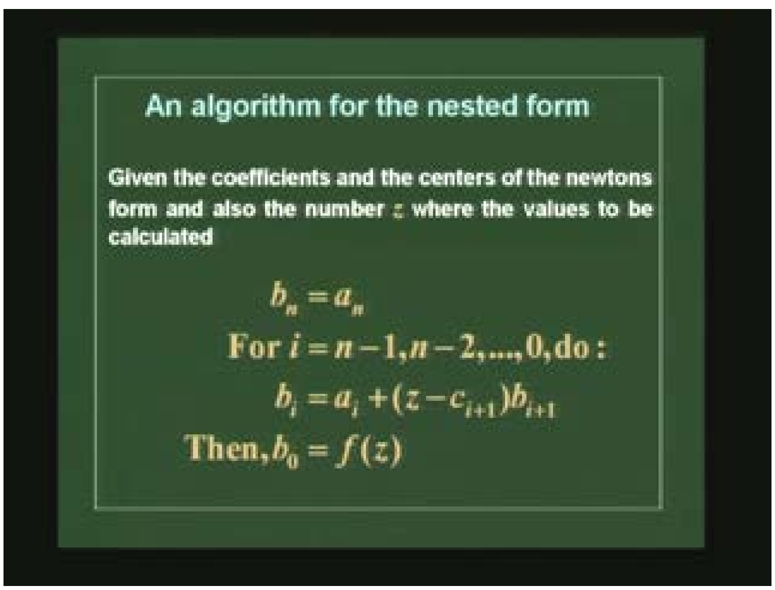


So in this way we can compute all of them in a more systematically, and this is easier to compute, easier to program, than actually doing each of these terms individually when you go to large order in the polynomial. So when you go to large order of the polynomial, it is easier to do it using the divided difference method. So here is an algorithm. Once you have this divided difference method and then to evaluate the polynomial again we will not use this. We will use the nested form which we mentioned earlier. So remember what was the nested form? We will not write the polynomial in this fashion.

We will write the polynomial now as pn of x, then the nested form as ao plus a_2. So we have x minus xo into a1 plus x minus xo into x minus xo into a_2 plus x minus x_2 into a $_3$, etcetera. So that is the nested form which we had. So we will have many things to close there. So that is the nested form which we wrote and that is what you would use to evaluate the polynomial at any value x. So we will use divided difference method to compute the coefficients by writing this 1 2 3, these coefficients, by doing this divided difference, and then we will use these coefficients and put them back here.

So, let me substitute this, "a_o" as f of "x_o x_1" and this will be then f of "x_o x_1 x_2" plus x minus x_2 into, etcetera. So that is what you will have. So we can compute it in this fashion much easier because it is again nested form and is much easier to program. That is what I have shown here. We start from the last term in the nested form and then we go up on the, in this case, so we will keep going up by doing this. So first we compute the last term. So that is the a_n. We know this coefficient, and then we go up. These are the coefficients which I have represented here as f of "f" square bracket x_o x_1, etcetera, and then we will go up from there. That is why I go from loop n minus 1 to n minus 2.

(Refer Slide Time: 48:21)



We will actually do this using an example. We will actually do this computation then, it becomes more clear. The idea is that if I have only 3 terms, so I have only my polynomial was, x minus x_o into f of a_1 and x minus x_o x minus x_1 a_2, that is, if that is only a second order polynomial. And then I could stop here and I say that my function is, you see now it is only a second order polynomial. I know the function value at x_o x_1 and x_2, 3 points. Then, I have a second order polynomial. And then the way I would use it in the nested form would be just to say that my function is actually equal to f of, my function is actually equal to, I start from this and I will say f is f of x_0 x_1 $x_{2.}$

So that is what I would say. That is what I call here by saying b_1 equal to a_n, and then I go to the next point. That is n in this. In my particular case n was just 2. So I will go to the next one and I will say that I will multiply that n. i is n minus 1 here. This is actually i plus 1 is n. That is my function here. I multiply that by z minus the value which I knew this and then add the next coefficient to it. That is saying that is the next step. This is the first step, and the next step I will do, I will have to compute, then I will have to multiply this by that. So I will compute, multiply x minus x1 with this coefficient, f. That is what I did, and then I add the next one to this, that is, f of x_o x_1. That is what I get, and the next step I would take this f, this whole coefficient, and multiply that by x minus xo and then add ao to it.

So that is what is here. So the term before that, you multiply that by x minus x_o or x minus x_1, whatever that number is. So this is now my new f, and then I take the next f. I will take x minus x_o and multiply f and then add ao. So that is the step1, step2, step3.That is what it is here. This is exactly what is being given here. We can write this down and try to do this by substituting numbers here. You can see that this is exactly what I have written on this board here. So that is, now I give a small problem here which you can work out. We will just do that here.

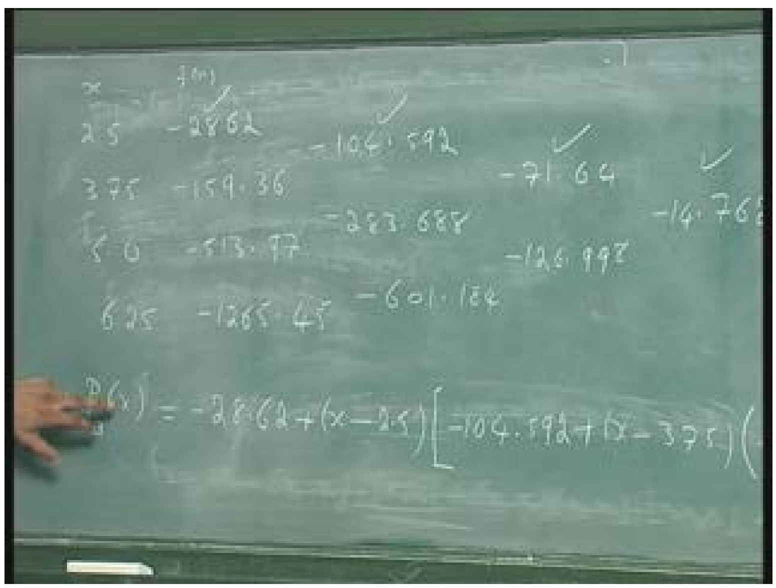
So we have, let us say, a function value function given at x "2.5" "3.75", "five" and "6.25", and the function values are given as minus "28.62", "159.5131265". So we have 1, 2, 3, 4 points at which the function is known, and then we want to construct a polynomial which goes through this Newton using Newton divided difference method. So we will get a polynomial of order 3. So what we get is a polynomial of order 3 because we have 4 points here. So we will just, we will use this form to actually we will use the form which is given here, this function of order 3. So we will substitute the actual numbers for this, and we will just go on with this calculation. I am listing the values here. So the values of x and f of x, so they are "2.5" and minus "28.62". And then I have "3.75", and that is minus "159.36", and then I have "5.0", and that is minus "513.97". And then I have "6.25", this is minus "1265.45", and I have this and then, I compute the first differences, that is, minus "104.592".

I am doing this computation. I am doing exactly this, that is f_1 minus f_0. I subtract this from this and divide it by the difference between these two. That is what I have done here. So I get this number, and I do the difference between these two, and divide it by the difference between these two, and then I will get minus "283.688" and I take the difference between these two numbers and divide it by the difference between these two numbers, and then I get it as minus "601.184", and then I would take the difference between these two numbers, these two numbers, and divide it by the difference between these two "2.5" and 5, and similarly, I take the difference between these two and then I divide it by the difference between "6.25" and "3.75".

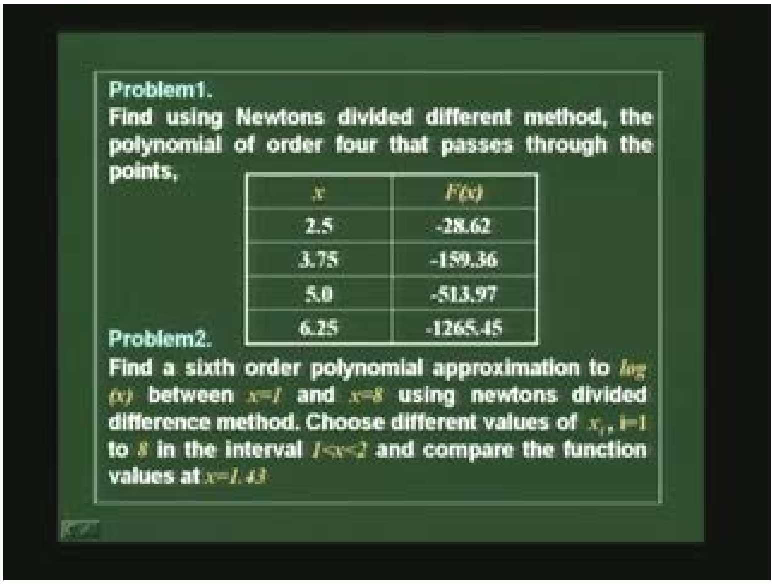
So that will give me minus "71.64" and minus "126.998".Then I take the difference between these two and divide it by the difference between "6.25" and "2.5". So that gives me minus "14.76224".So we have all the coefficients of the polynomial 1, 2, 3, 4. The polynomial now is p of x, that is 3n is 3, would be equal to minus "28.62", plus x minus x_o is "2.5" in the nested form, then I will write that the next point as minus "104.592", plus the next one is x minus "3.75" into the coefficient will be minus "71.64".Then I will write it as, so I have the next coefficient, and then I will have minus "671.64" plus x minus "6.25", and then I will multiply that again by minus "14.76224".

So that is the way, I would compute this. So I can close all the brackets. So I computed all the coefficients that is starting with minus "28.62" minus "104.592" minus "71.64" minus "14.76224", and then I can write that in a nested form, and that gives me my polynomial. So you can see that x minus 2 x minus "3.75" and x minus "6.25". So that is the, that determines the order of the polynomial. It is order of 3, the polynomial. This is my final polynomial, and then I can substitute any value of x and evaluate any value of x in between these numbers, and evaluate this polynomial. So we will see other forms of writing this, and some more problems in the coming class.

(Refer Slide Time: 55:40)



(Refer Slide Time: 55:54)



You could actually use this method to evaluate, to get a set of discrete points of log x between 1 and 8, and then you construct using this divided difference method, a polynomial, which goes through all these points, and then compare that with the function log x. So that gives you a good idea about how accurate your polynomial would be as you increase the number of points in between. With this, we will stop here. We will go to other forms of representing the polynomial in the next class.