**Numerical Methods and Programming**
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**Lecture - 35**
# Explicit and Implicit Methods for Partial Differential Equations

We have been looking at partial differential equations and the methods of solving these equations numerically in which we saw one particular case of an elliptic partial differential equation of the form del square u by del x square plus del square u by del y square equal to 0 and we solve these equations by writing down the second derivatives as difference equations and we wrote del square u by del x square as u at x plus delta x minus 2u x plus u at x minus delta x divided by delta x square and similarly for delta y square and we found that these equations, we can solve iteratively by writing it in the following form that is if you substitute these equations for del, the second derivatives in the difference form in these equations.

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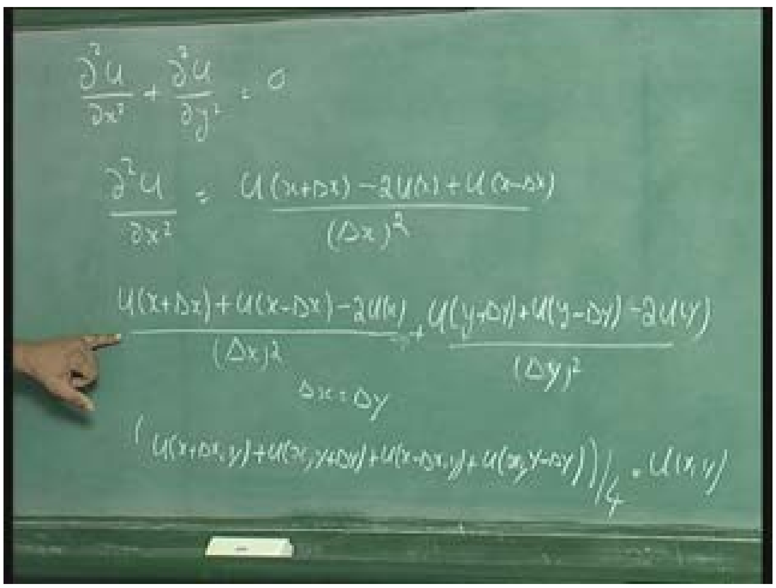

The equation which we have if you have if you substitute these terms in to that and this in the case where delta x and delta y are the same. So when delta x equal to delta y we can write these equations as, so that is what we saw in the last lecture that I can substitute this difference equation into this and then write this equation, this particular equation in this form that is the u at xy is written as ux plus delta x, y, u x y plus delta y u x minus delta x y and u x y minus delta y whole divided by 4 and so that is what we have written out and we will solve this iteratively that is to start with we divide the space into a grid.

So we have this we divide this into a square grid and then we had the values specified at each of these grid points to begin with and then at the next step, we would evaluate the values at each grid point by using this procedure, this equation and we will continue this procedure till we have the left 2 sides are the same that is when u x, y obtained in the next iteration loop is the same as what was in the previous step that was the way of solving the elliptic equations which we did in the last class.

So now we look at slightly different form that in which now we look at equations which are time dependent. So this particular equation we solved in the context of the distribution of temperature that is we said that if you have a slab in which a slab a conducting slab and which is kept at whose boundaries are kept at different temperatures for example, this is kept at let us say hundred degrees and this is 0 and 50 and 75 and the slab is insulated from the top and the bottom, so the whole heat flow is only inside the plane and then we wanted to know what the temperature distribution would be and that we saw that leads to an equation of this form where u is the temperature.

Now the same problem if you want to look at the case where it is time dependent okay that is we want to know how does now this is steady state equation. So now you want to know how starting from a given initial condition that is this is kept at 175, 50 and 0 and everywhere else also is 0 to begin with and you want to know how does it go in to the steady state and in that kind of cases we will have to solve a time dependent equation. So now in this course we will only look at one dimensional time dependent equations that is one space dimension and then one time dimension that we will look at only those particular cases, okay.

So that is equations of the form we will be looking at equations of the form del u where u is again temperature is equal to some coefficient times t is little t is the time here some coefficient times del square u by del x square. So we will look at equations of this form so here is one example where this arises in the case of a heat conduction through a thin rod.

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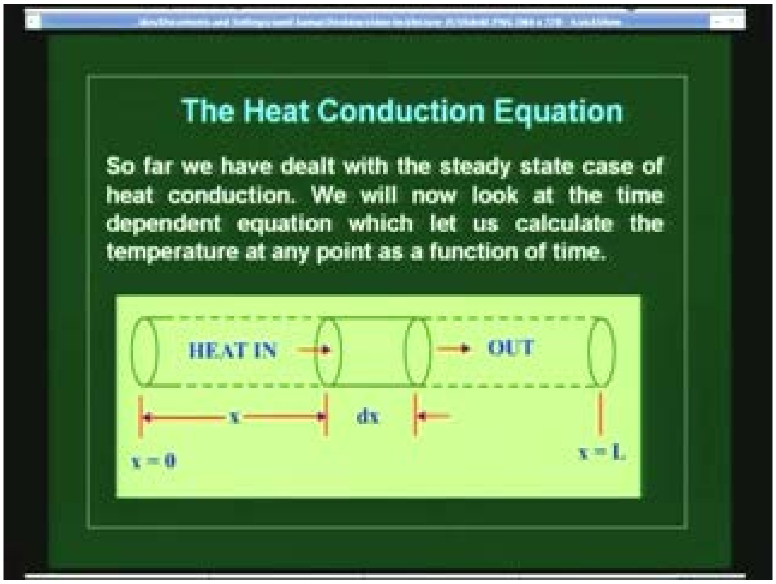

So I can assume that this to be a very thin rod, so that across the cross section we can assume the temperature to be uniform and let us say it is insulated all over its length. So what is exposed is only the two boundaries and the two boundaries are kept at two different temperatures and we want to know what will be the temperature distribution along the length of this rod as a function of time. So that problem would lead to an equation of the form which I just wrote.

So we are not looking at steady state here we are looking at this long thin rod thin, we assume it to be thin, so that we can we take it as thin. So that we can assume in the cross section across the cross section the temperature is uniform and it is insulated all along its length. So the heat enters here and goes out here so you take any element where the heat comes in here at is qx and what heat goes out here is q at x plus delta x. So that will be the heat flux in to the cross sectional area which is just take it is as a.

So the heat flux let us take it as q here and then q of x and then what goes out will be q at delta x as we saw earlier so now we know that the q is the gradient of temperature that is del u by del x in this particular case.

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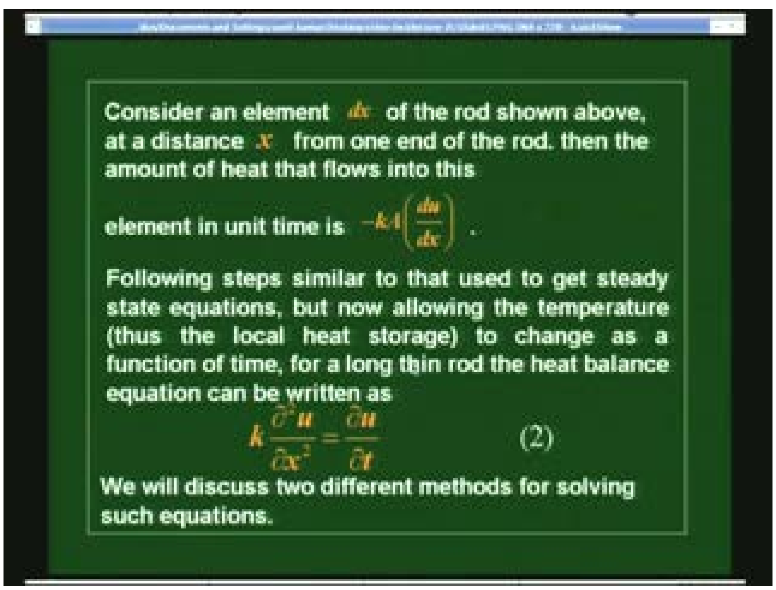


So these equations would lead to something like this we consider an element dx and then the heat enters at x and it goes out at x plus dx, now the amount of heat that flows into this thing is qx, this qx is k a in to del u by del x. We know that that is what we, this is the Fourier’s law. So what the heat enters is qx at this point here. So that is k times a into the gradient in the temperature, u is the temperature here. So gradient is the temperature “a” is the area and “k” is the heat conduction coefficient, so that what we have and then we can write and we know that this heat which goes out is qx plus delta x.

So the net change in that what comes in and goes out would be equal to del, let me write this in this form. So you have this rod, so we are considering this particular section, so what comes in here is qx and what goes out here is qx plus dx for this thickness and so the change would be qx plus dx can be written as q of x plus del q by del x into dx. So what goes out of what is retained here the heat which is retained here in this section is qx plus dx minus qx which is del q by del x del x del q by dx.

So qx plus dx minus q of x is del q by del x into dx that is what we have. Okay now this q itself, we know is minus k times a times del u by del x. So substitute that here so the heat retained here per unit length per unit area, so the heat which is retained in this area per unit length per unit cross sectional area would be given by the heat retained would be per unit length per unit area would be del q by del x right.

So that will be the heat which is retained here, okay per unit are per unit length would be so 1 over into 1 over a, that will us k times del square u by del x square, now that heat which is retained causes a change in temperature there.

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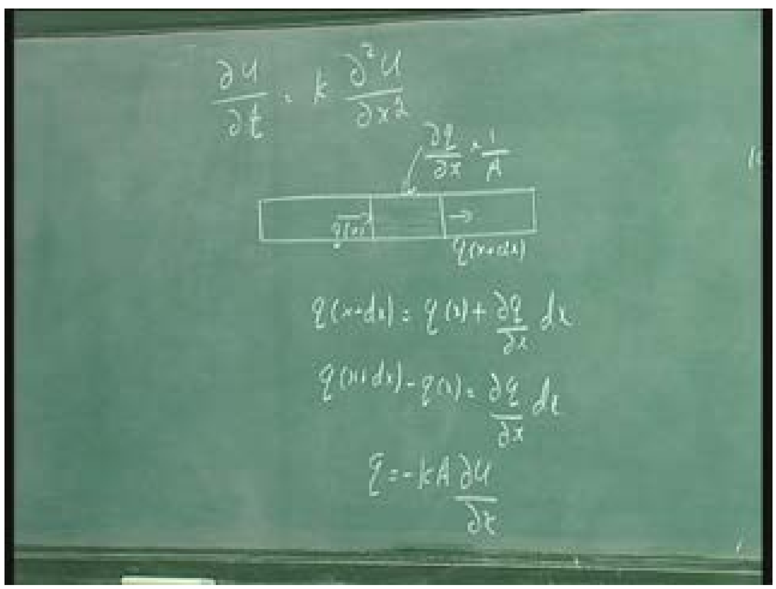


So that leads to the equation that del square u by del x square into k is del u by del t. So that is one case where you get this equation, that is take a long thin rod insulated on the sides, okay so heat comes in here goes out there and the cross section we assume the temperature to be uniform and then the heat comes in here if you take an element of length dx what comes in here is qx what goes out is qx plus delta x.

So retained is del q by del x into dx, so the heat which is retained per unit area, unit area of cross section, per unit length would be 1 over a del q by del x but we know q is KA into del u by del x. So we get the heat retained is actually del square u by del x square and that retained heat is actually used for increasing the temperature there. So the rate of change of temperature should be proportional to the rate of retaining of the heat. So that gives this equation del $x_u$ by del x that is 1 example, so we saw the steady state case in the Laplace's equation now we are looking at this case where the temperature changes as a function of time. So now to conclude what we want to look at is an equation of this form and we want to know how you can solve this numerically.

So again we could write del square u by del x square as u at x plus delta x minus 2 ux plus ux minus delta x divided by delta x square and we could write del u by del t as ux t plus delta t, okay now this is all, so now I can write like this and then minus ux, t divided by delta t. So that is the forward difference method of writing the derivatives del u by del t, now the question is so now we can do this.

Okay so we can there is a choice here that is when you write substitute this in to this equation, since this is only a derivative with respect to time I can write this as u x t plus delta t minus u x at t divided by delta t at the same position x, now the right hand side is only derivative with respect to space variable x then the question is whether we should take this derivative at time t or time t plus delta t, so there is a choice there.

So what I am saying is that you can actually evaluate this derivative at either t or t plus delta t. So that leads to two different methods if I evaluate this at t, I would call that as an explicit method and if I evaluate this derivative at time t plus delta t, I call that implicit method. So these are the two methods which you should be looking at today.

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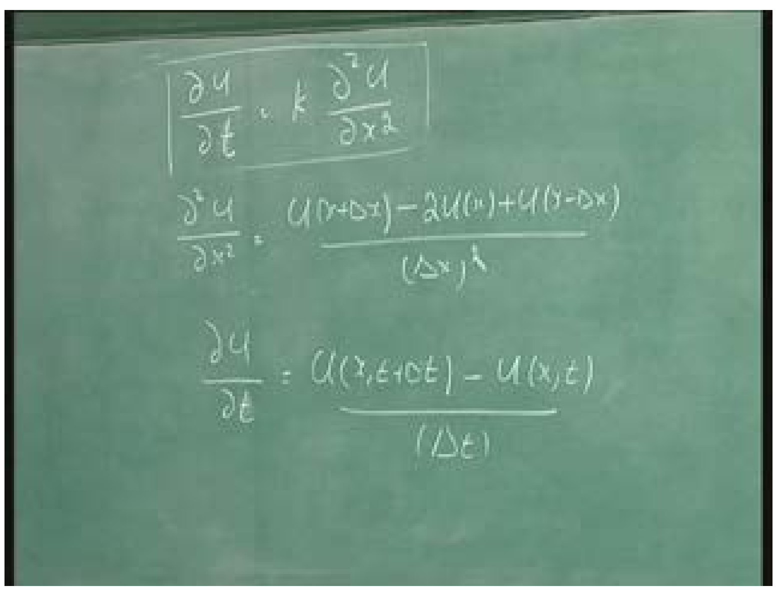


Today our aim is to solve this equation this particular equation using two different methods that is substituting for these derivatives, their difference equations but taking the space derivative in one method at time t that is called the explicit method and another method in which we would take this derivative at time t plus delta t and that would be called the implicit method we will look at these two methods, these two equations. So let me summarize this again.

So we want to look at equation of this form which arises in the case of thermal transport in a long thin rod and the time dependent equation for that and we want to solve this and we will discuss two different methods and one is the implicit method in which the second derivative here u is evaluated at time t plus delta t and there is another explicit method in which this is evaluated at time t.

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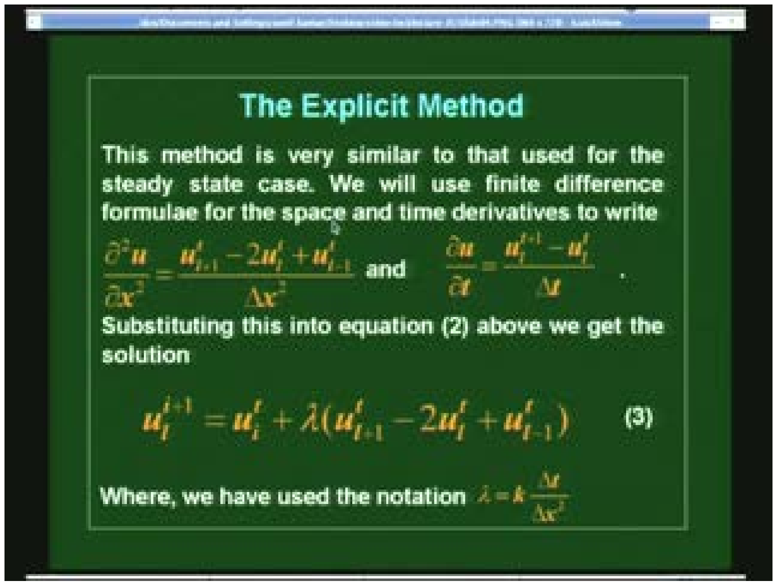


So here is a explicit method, so it is very similar to the one which we use for the steady state case, so we replace the second derivative here the del square u by del x square by this difference equation. So now the notation we use is the following that we descriptize space and with the space positions are known l, so u of x we call as $u_l$ and u x at time t. So u x at time t you would now denote by $u_l^t$, so u x plus delta x at time t will be written as $u_l$ plus 1 which is equally spaced points we are going to take we are going to take equally spaced points in the spatial variable, so it is u $_l$ plus 1 time t.

Similarly, t plus delta t will be t plus 1here.Okay so here we are taking all second derivatives at time t. So that will be $u^t_l$ plus 1 minus 2 $u_l^t$ plus $u_l$ minus 1 t divided by delta x square and now there is the time derivative del u by del t as $u_l^t$ plus 1 minus $_l^t$ by delta t. So now these 2 equations substituting into our time dependent partial differential equation, we get we can write that as this one that is u, now what we did was substitute these two into this equation. So we are going to write it in time t, so we are doing explicit method so all this is evaluated at time t, so what do we get if you do that.

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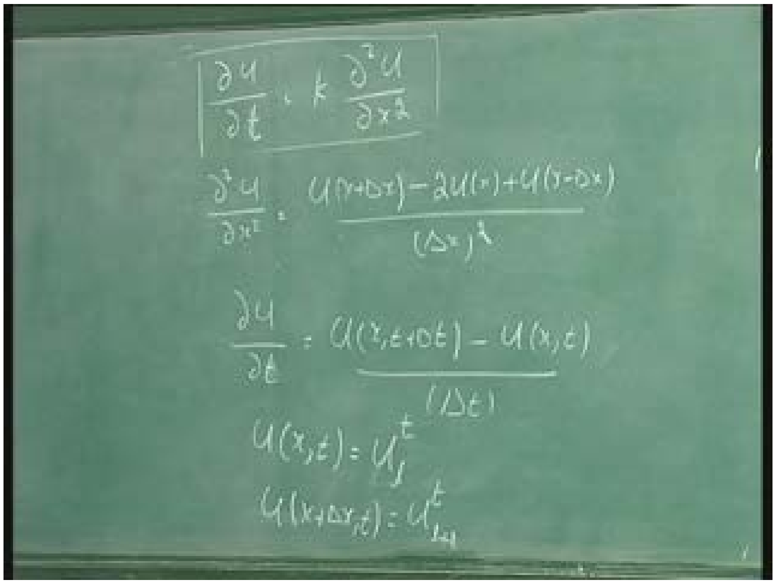

So we are going to get $u_l$ at t plus 1minus u $_l$ at t divided by delta t is equal to u $_l$ plus 1at t minus 2 $u_l^t$ plus $u_l$ minus 1 t divided by delta x square and there is a KA coefficient here it is just k. So that is the equation we have, so we can rewrite this in terms of, so what we want to find out is what is the temperature at t plus 1 at time t plus 1, so we can write $u_l$ at time t plus 1 is equal to k times delta t by delta x square into $u_l$ plus 1t plus 1 minus 2, minus 2 $u_l^t$ plus $u_l$ minus 1 t plus, so now we have so now plus $u_l$ at t.

So that is the equation which we will be writing, so this is easy to solve because at given time t, that is at time t equal to 0, for example or at any starting time we are given the temperature at all the points. So we have this particular rod long thin rod, so we have been given the conditions at the boundary and then we write we descriptize this into equal intervals in space with interval delta x and at time t equal to 0 we have given the temperature at all the points.

We know what the temperature at all the points at time t equal to 0 is and then at time t plus 1 that is t plus delta t what is the temperature is given by this equation. so the right hand side is completely at time in terms of temperature at t and the left hand side gives us the temperature at t plus 1. So using this equation we can easily solve this, so now this term is called the parameter lambda so that is called lambda this equation this particular parameter is called lambda k times delta t by delta x Whole Square.

So now note that one problem with this equation is that, it is second order in delta x in space variable and in first order in time so because of this particular reason it is not equally its sensitivity is not the same in terms of the spatial descriptization and time descriptization that is reflected in this particular parameter.

So that is when we change delta x what is the effect which it has on this equation is not the same as when we change delta t the time descriptization .So we have descriptized both time and space here but the effect of the descriptization on the stability of these equations are not the same for time and space and that is because it is first order in time and second order in space derivative.

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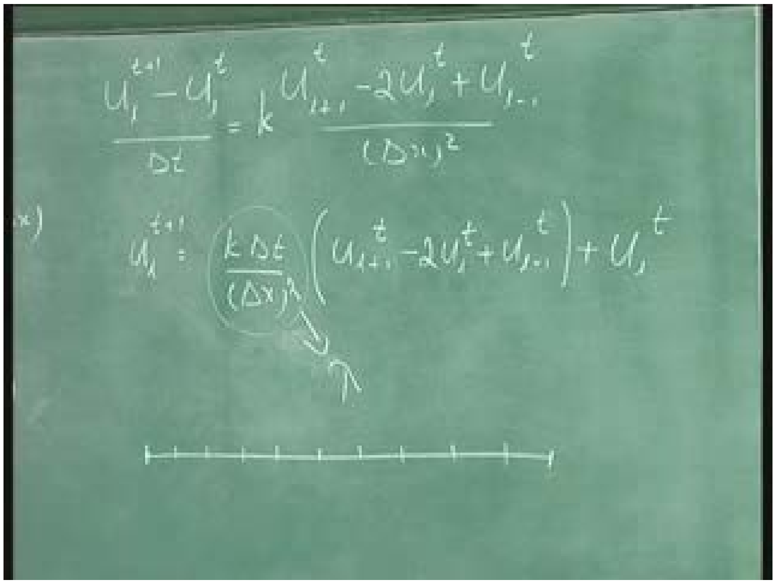

So this simple equation can be used to solve the heat equation but the only problem is that this particular equation is found to be extremely sensitive to the value of lambda which we use that is this if you use a very large value of lambda this is unstable.

We have to use small value of lambda that is we have to use lambda typically less than or equal to “.5” so which is less than equal to half to get good results on this equation that is the biggest problem with this equation otherwise, it is extremely simple that we know at all time starting time the initial time we know the temperature at all points on this rod we can use this equation to get the temperature at the all later times using this once we have fixed this quantity lambda.

So that is what is written summarized here. So we have this spatial derivative written as this discrete equation and this time derivative written using the again forward derivative forward difference equation and we substitute that into the heat equation and we obtain a simple equation of this form, where lambda given by k times delta t by delta x square. So now we can use given a time the temperature at t minus 1 we can compute the temperature at time t using that equation.

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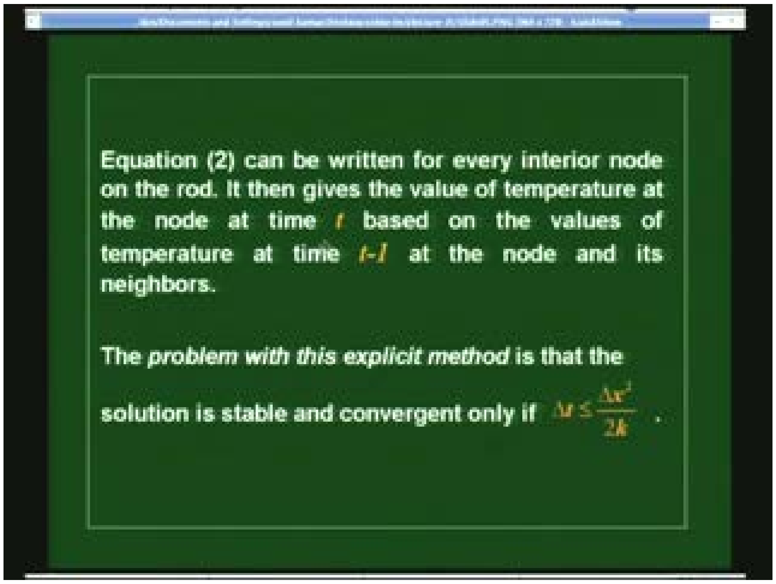


So then as I said the problem with this method is that this method is convergent that is stable only when delta t is less than or equal to delta t square by 2k or lambda has to be less than or equal to half lambda being k times delta t by delta x square has to be less than half and that is what we see. So now that is the problem with this explicit method.

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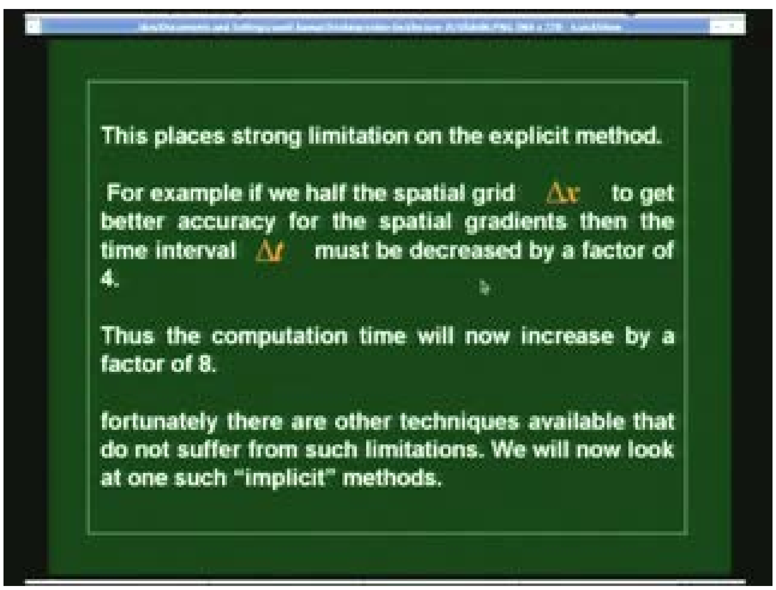


So if you want the implication of this is the following that if you actually change the grid size delta x in spatial grid, we also have to change the delta t correspondingly. So that is time we want to keep lambda a constant that is, so that we choose a lambda for which the equation is stable and we give stable solution and we want a little more, let us say spatial resolution. So you want to know so we choose a particular delta x and we solved the problem using this solve this equation using this particular explicit method and then we want to increase the resolution in space.

We wanted a temperature little more to each other that means you want to reduce the delta x. So what happens is if you change delta x here to keep the delta, to keep the lambda the same we also have to change the delta t so you can see that if I actually halve the delta x so to get the same stability and to see that this equation is stable I have to go 1 by 4 in delta t because this is delta x square this is delta t.

So if I want to increase the spatial resolution by going to half the space distance the descriptization half my descriptization in space then I have to go 1 by fourth in time. So my computational expenditure will go up by 8 times. So to solve that for a given time t we have descriptized both time in terms, in intervals of delta t and space in terms of delta x if you want to go to higher accuracy in the spatial variable and if you halve the delta x keep the same lambda, so that the equation is stable we have to go 1 by fourth in delta t. So that means we will increase our computational expenditure by 8 times so this is one of the drawbacks of this particular explicit method.

We will see this method implemented in a program before we go and see a little more accurate method which are the implicit methods. So we look at this program which uses the explicit method for solving the heat equation. O kay so we have we are using a lambda which is equal to ".5" and what we do is we descriptize the space into 5 points and you use the explicit equation which is saying that s at j it is actually saying that s at j plus 1 plus s at j minus 1 and 2 s j minus 2 s j multiplied by lambda added to s j is the next one which I had to store as d of j here.

So d is time t plus 1 and s is time t that is what you are going to use. So before we start that we need some boundary condition that is we have these equations and I am1 as d the temperature and then we need to implement this equation at every point on this descriptized space descriptized space, the problem is again as we saw in the Laplace's equation that we cannot use these equations at these 2 initial, 2 end points here, we do not know what l plus, l minus 1 is and here we do not know what l plus 1 is.

So either we have to use we have to use some boundary conditions, the boundary condition could be that this the two end points the temperature is specified. So we could say that this is equal to 100 and u is 100 here and u is 0 here or we could use the derivatives at the 2 boundaries fix the derivatives at the 2 boundaries and get the temperature corresponding to 2 fictitious points at that side. So that the derivative can be fixed and then we would have to solve for all the points including the boundary points.

So in the case where the boundary condition is specified in terms of the variable at the 2 boundaries that is what the program is for we have to solve only for the interior points. So these two boundaries the value of u or the temperature is fixed all time for all time it is fixed as 100 and 0 and we solve for all interior points using a lambda equal to ".5". So that is what we do so we have time going from 1 to 50 and we solve this for every time step so to start with everywhere else is put as 0, so one end is kept at 100 and every other point is kept at 0 to begin with and the last point is always maintained at 0.

So that is the two boundary conditions which we have, so now we solve this using this this is the time t plus 1 and this is the time t, so time t plus 1 the temperature is equal to time t plus lambda times this particular second derivative of the function and then after that calculation is over I transfer the time t plus 1 to time t and then goes back and do the next time step etcetera. So every five time step I write that temperature profile into a file named as explicit one explicit 2 etcetera now this is the method of writing the files like that.

So I can use this function which you have not seen before may be it is juts called s printf. So I have a character string here and into this character string I am putting a name for a file the name of the file is explicit and a number here and the number is given by ib the number is percentage d is given to put the number here. So now this character string will get this name explicit and whatever the number which ib takes.

So ib takes t by 5, so when t is 5 it is 1 and t is 10 it is 2 etcetera, now this function can be used to automatically generate file names this generates file names explicit 1, explicit 2, explicit 3 etcetera as we run this program. So every 5 time step it will store the temperature into a file whose name is generated as explicit 1, explicit 2 etcetera, so now it is storing the temperature and the l value or the i value in this case which corresponds to discrete points on the along the lot.

So we run this program and so we compile that and we run this program and then now we have all these data files which I have written as explicit 1, 2, 3, 4, 5 etcetera this corresponds to temperature at time 5, 10, 15 up to 50, so 0 to 50 is what we have so now we plot this one of the things that means we get a time t equal to 5 that is to it the temperature goes from 100 to 0 as we go from 0 to 5 in space.

So what we see is that the temperature drops and so that is what we expect and it goes to 0 in the continuous fashion. Remember, we have only 5 points in the interior and now we could plot this time two also so we write the next time step that is explicit 2 or explicit 4 may be now that is after few more time steps.

So you can see that by the 4 that is 20 time steps we have actually reached the value, the steady state value where this linearly drops that is what we the steady state solution would be that the temperature drops linearly from one end to the other end and that is what we see at the time 20 time steps. So this method definitely works for lambda equal to .5 and gives us the evolution of the temperature profile as a function of time.

So now let us look at the same program, let us look at the same program when we change the lambda to “2.7” so we increase to, we change this to “.71” that is what I have written in the comment here then you will see that it does not actually work very well for lambda equal to “.71”.

So what we have done is we change the lambda we increase the lambda that is we either increase delta t or we decrease delta x to increase our spatial resolution probably we increase delta x or decrease delta x and that is increasing lambda and then we try to run this again. So let us see what we get so we run this program and then we again plot this as a function for the first time step so and that is what we see.

So then we see that when we increase lambda instead of getting a smooth profile as we obtained for smaller lambda, now we have oscillations in the temperature and we are not supposed to be getting these oscillations. We know that if I keep a rod at 1 at 100 degrees and other end at 0, I expect the temperature to drop in a monotonic fashion not oscillatory in space and actually these oscillations would blow up and we would get by next time step you would see that we cannot actually get the solution correctly. So you can see that the temperature now has gone completely crazy, that is now we have temperature going to minus 2000 plus 6000 etcetera, so which is obviously wrong.

So this particular method does not work for lambda larger than ".5" I just use a larger which is higher to get probably a better spatial resolution and we see that the method does not actually work.

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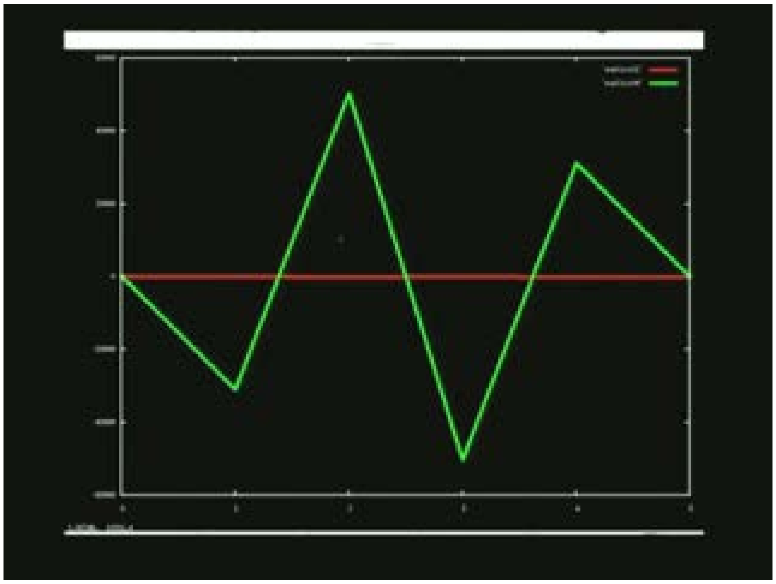

So this is an example to show that even though very simple this method is not very efficient in solving equations of this form. So we go back to our next description and look at a little better method called an implicit method.

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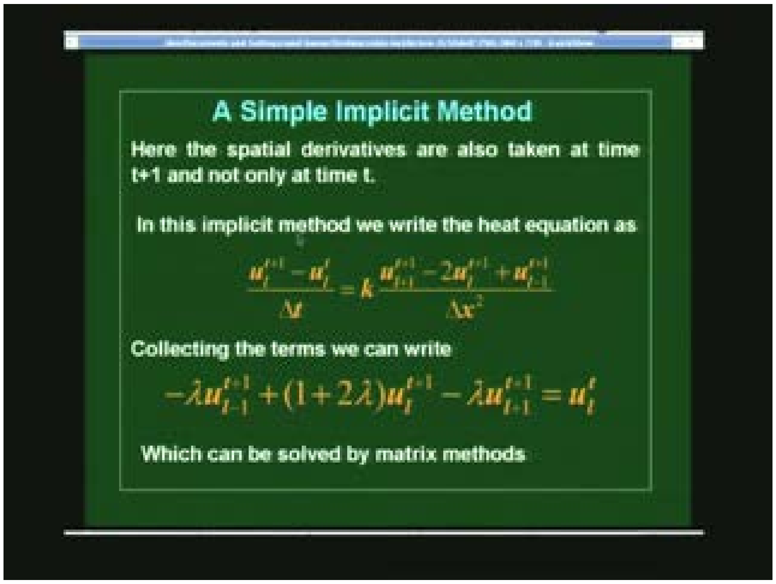


So as I said earlier, the difference is that in this method the spatial derivative in a simple implicit method, the spatial derivative on the right hand side of this equation which we have here, this spatial derivative is now evaluated at time t plus 1. So that is we will evaluate all these derivatives at time t plus 1. So this derivative would be evaluated at time t plus 1, so that means that when you write this equation. So that is the discrete form of the heat equation and all these derivatives are taken at time t plus1. So now obviously this is, this will not work anymore, so we have to write a different equation for that.

So when this is all evaluated at time t we have a very simple equation for the temperature at time t plus 1, okay but now we have the right hand side evaluated at time t plus 1 and because of that now we have to when we write this equation for time, when you write this equation separating t plus 1 and t time variables, so you will have $u_l$ minus 1 t plus 1 and plus 1 plus 2 lambda minus lambda. So we have lambda is remember k delta t by x square, so we have minus lambda $u_l$ minus 1 t plus 1 and then you would have 1 plus 2 lambda $u_l$ t plus 1 and minus lambda $u_l$ plus 1 at t plus 1 is equal to $u_l$ at t.

So now we have the equation in this form and we have to solve for their time at t plus 1. So we write this equation for every spatial point l and then we will get a set of linear equations. So if you write this equation for every spatial point, we get a set of linear equations and that set and then has to be solved using matrix methods to get the correct answer. So again if you write this for example, l equal to 0 we do not need, so we have again $u_0$ let us say is 100 and we say u some n is equal to 0 that is fixed at all times for any t this is fixed and then we write this equation starting from l equal to 1 to n minus 1.

So for l equal to 1 is a special case because then this is 0, so then for l equal to 1we have it as 1 plus 2 lambda $u_1$ at t plus 1 minus lambda $u_2$ at time t plus 1 is equal to $u_0$ at time t plus lambda times u, $u_1$ at time t and $u_0$ at time t that is l equal to 1, for all other l that is l greater than 1 and less than n minus, n less than n, n minus 1 we have this equation that is minus lambda $u_l$ minus 1t plus 1 plus 1 plus 2 lambda $u_l$ at t plus 1minus lambda $u_l$ plus 1 at t plus 1 equal to $u_l$ at t.

So that works for all l between 1 and n minus 1 for l equal to n minus 1 again, we have problem on this end. So we can write that now as minus lambda $u_n$ minus 2 at time t plus 1 plus 1 plus 2 lambda times $u_n$ minus 1 at time t plus 1 is equal to $u_l$ $u_n$ minus 1 at time t plus lambda times $u_n$ at time t. So lambda times lambda $u_0$ lambda $u_n$ are fixed by this, so these 2 equations are slightly different from the other point equations.

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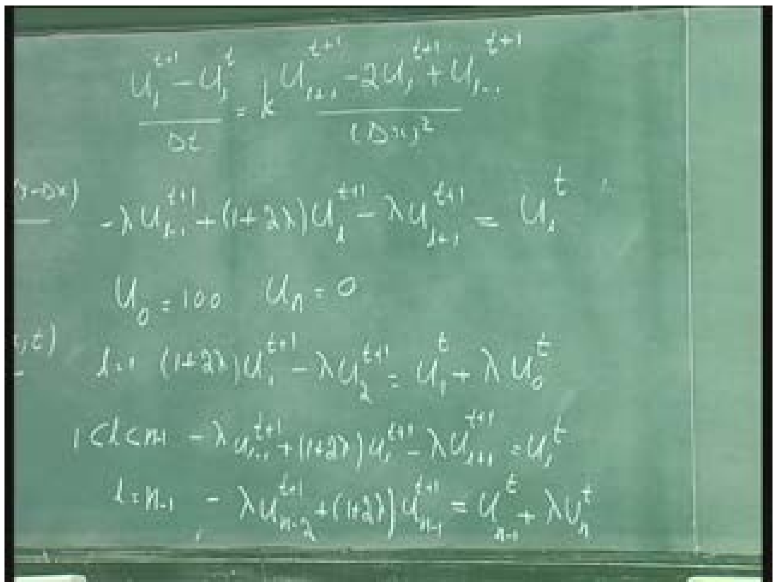

So that is, so that will be this will give us a set of, so we have a set of linear equations and the variables are the unknowns are like $u_l$ minus 1, $u_l$ and $u_l$ plus 1. So you have a tri diagonal matrix to solve. So we can write this in a matrix form and you get a tri diagonal matrix with elements which will be of this form. So I write that here, so you will have a equation of this form 1 plus 2 lambda minus lambda and then you have 0 0 0 etcetera.

So let us write for 5 points so you have 1 plus 2 lambda minus lambda 0 0 0 and the next 1 would be minus lambda 1 plus 2 lambda minus lambda 0 0 and then you will have 0 minus lambda 1 plus 2 lambda minus lambda 0 and etcetera. So you will have until you have the last point would be okay I will write of it, so 0 0 minus lambda 1 plus 2 lambda minus lambda and then 0 0 0 minus lambda 1 plus 2 lambda that is our matrix and this multiplied by so this matrix will have to be multiplied by we will have then $u_1$, $u_2$, $u_3$, $u_4$ and $u_5$. So that is what you will have to get, so we have to get these 2, this matrix multiplied by this column vector and the right hand side of that will be given by z, t plus 1.

So this is all at t plus 1 which are our unknowns, so we u z time t plus 1 is our unknowns and the right hand side of this equation would then be this, the first one will be $u_1$ at time t lambda $u_0$ and the rest would be $u_2$ $u_3$ $u_4$ at time t and the last one would be $u_4$ at time t plus lambda $u_5$ that will be right hand side of this matrix, this is the matrix which we have to solve, so we solve this. So we have the equation in this form that is the first one as the right hand side as, so matrix equation now is $u_1$ at time t plus, $u_0$ at time t plus lambda $u_0$ at time t, $u_1$ at time t and then $u_2$ at time t, $u_3$ at time t, $u_4$ at time t and $u_5$ is $u_4$ at time t, u plus lambda times u, actually $u_5$ at time t and $u_6$ at time t, now this point is solved by this and this are fixed these two values and other and these also and these are values at time t. So we know all these the right hand side, so we can solve for these equations at time t plus 1.

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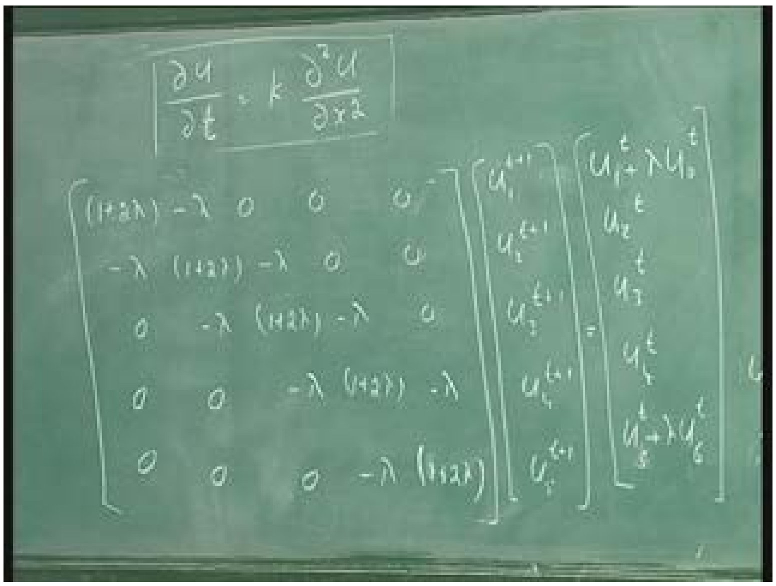

So that is the method and we will see an implementation of this method in this particular program. So here is another program which does that we call it implicit method. So it is simple implicit method, so here I am using a lambda which is large which is ".7" just to show you that it works for this large lambda and other things are same as the temperature is 2, boundaries are fixed at $s_0$ and $s_n$ are the boundary temperatures that is 100 and 0 and all temperature at time t equal to 0, everywhere else is 0. So now I construct I have to construct this matrix so here is the matrix which is constructed.

So I have to start with all matrix elements are put equal to 0 and then I give the matrix elements these values as lambda 1 plus 2 lambda and minus lambda along the diagonal, that is along the diagonal, along this is a tri diagonal. So along the diagonal I have 1plus 2 lambda minus lambda 1 plus 2 lambda and minus lambda as the matrix elements.

So that is what is been given here for each of these columns I have i minus 1, I, i minus 1 I, i and i, i plus 1 as minus lambda 1 plus 2 lambda and minus lambda and the right hand side for the matrix which is given of this equation is given by d i equal to s i that s remember is the temperature at s time t and d in this particular case s is the temperature at time t.

So here they are given that and the two special cases, so the first row the first row I have to this is for the interior points and the first row I have only 2elements that is 0 0 is 1 plus 2 lambda and 0 1 is minus lambda and the right hand side is $s_0$ plus s of 0. So that is 1 more it is actually for the first row and for the last row again we have only 2 points that is n minus 1, i minus 1 and n minus 1 i. Okay now this is the last part that is actually i is n here.

So n minus 1, n minus 1 and n minus 1 n, i is n minus 1. So n minus 1, n minus 2, n minus one n minus 1. So these are the two last points the last row, so I am just writing these 2 elements and these 2 rows especially that is here I have n minus 2 and n minus 1 and here I have 1 and 2. So a special case and similarly this right hand side equation this is $u_5$ plus lambda $u_6$ which I call u n in the program and this is $u_1$ plus lambda $u_0$.

So that is the equation, so then I call this function which finds the inverse of this matrix d, so n is the dimension of this matrix. So the idea is that I would just find the inverse of this matrix right and multiply with this then I will get that temperature at time t plus 1. So the advantage of that is see this is a constant matrix this does not depend on the time the matrix here does not depend on the time.

So what depends on time is only these 2column vectors, so once I have constructed this matrix for the given descriptization I have and I found the inverse of this matrix and then I can just go through a loop to find the t plus 1 as a function of t and that is what I will be doing in this program. To find the inverse of the matrix, we know we have this is a function here which finds the inverse of this matrix and so we look at their functions.

So this is the function to which that matrix d as pass as an array of pointers which you have learnt earlier and the method this function uses is to find, do the l u decomposition and so we find the inverse by using the l u decomposition which again we have found earlier. So we have the l u decomposed form of this matrix d which we have seen earlier in our earlier lectures and then use that l u decomposed form of this matrix and supply the right hand as 1000100 etcetera.

We give the d matrix the d column vector in this form and find the inverse we have gone through this earlier. So we have the inverse here this inverse is written back into this program. So once when I call this inverse function here in this program I call the inverse function and I pass the matrix d and when it comes back the d is replaced by its inverse.

So because we do not need the d matrix what we need is the d inverse and then I do it for 10 time steps here. So I am doing for time equal to 1 to 10 and each time step what I do is I multiply that matrix, since this is a matrix multiplication I multiply now the temperature at time t plus 1 is temperature at time t plus the matrix multiplied by the d and then I replace d by that particular time t plus 1 and again treat the 2 end points separately and goes back into this loop again.

So the method is this, I find the inverse of this matrix and multiply this vector column vector here by the inverse of this matrix and I find at time t plus 1 what the values are.

Then I transfer the time t plus 1 values in to time t in to this vector and again multiply it by the inverse of the matrix to get the next time step.

So that is what we are doing here. So to start with you have the temperature 0 every where except at the 2 end points and then I multiply the matrix the right hand side, the d column vector by the inverse of the matrix d. This is now replaced by its inverse and then I transfer what I get into the d column vector and I treat the 2 end points specially because they are not s m but they are they are not u at time t plus 1 but they are $s_0$ times lambda plus the u at s values and similarly the last point and then I write those temperature as a function of time into 2 different files here again I use the same technique to generate different file names and then I store them all into different file names called implicit 1, implicit 2 etcetera for every time step and remember I am going to run this for lambda equal to ".7".

So we will now run the implicit program, so instead of explicit we will run it now implicit program. So that is, so now we will now plot the implicit method. So okay and you see that there is no oscillations.

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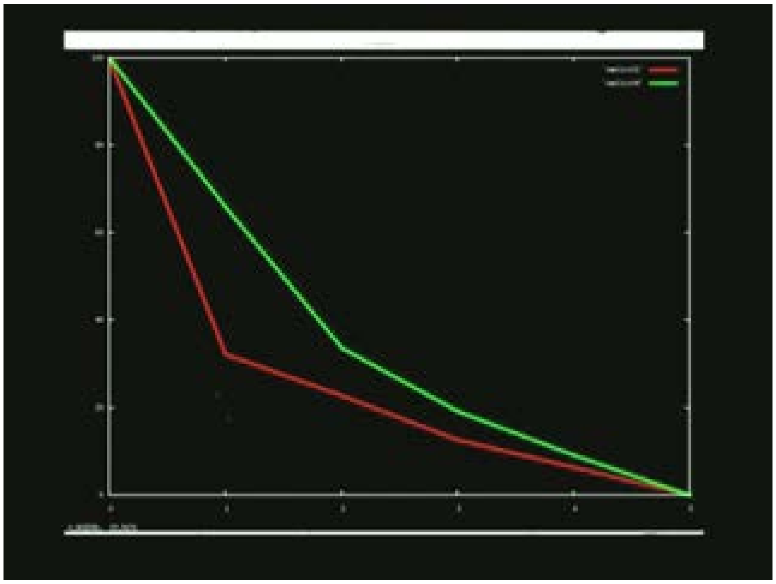

So the temperature this is at time 1 and this is at time 4 etcetera. So there is no oscillations here and it slowly keeps improving. Okay so the temperature at any point in between the 2 ends is increasing as a function of time and it goes towards the straight line. So what I wanted to show you here is that again I used only 5 points, so I used 5 points between the 2 limits and I use lambda which is larger than ".5" that is why I use lambda ".7" for which our explicit method did not work and you saw that the implicit method works for this particular value of lambda.

So what is the difference, in the implicit method we wrote the right hand side evaluated the right hand side that is the spatial derivative at time t plus 1 and while in the explicit method the right hand side was purely a function of a temperature at time t. So that is the difference between the implicit and the explicit method. Okay there are better schemes than this.

So if you look at this this particular method is again an Euler scheme itself but the only thing is that we have evaluated the right hand side at time t plus 1. So we could do even better than this by actually taking a mixture of the type derivatives at time t and time t plus 1, so that is also that will be what you will be looking at next.

So to summarize here, so we had evaluated the derivatives at time t plus 1 and then we wrote down the equations for all the interior points and the 2 boundary points were specified as boundary conditions for all time and then we write down the equations and we write down the equations for all interior points and then solve them using matrix methods.

So we find the inverse of this matrix and multiply the right hand column vector by the inverse of that matrix to find the temperature at time t plus 1. So the other method in which we can use, other implicit method which uses a mixture of these two derivatives that is we could use the derivative on the right hand side the second derivative that is del square u by del x square evaluated at both time t and t plus 1 and take the mean of that.

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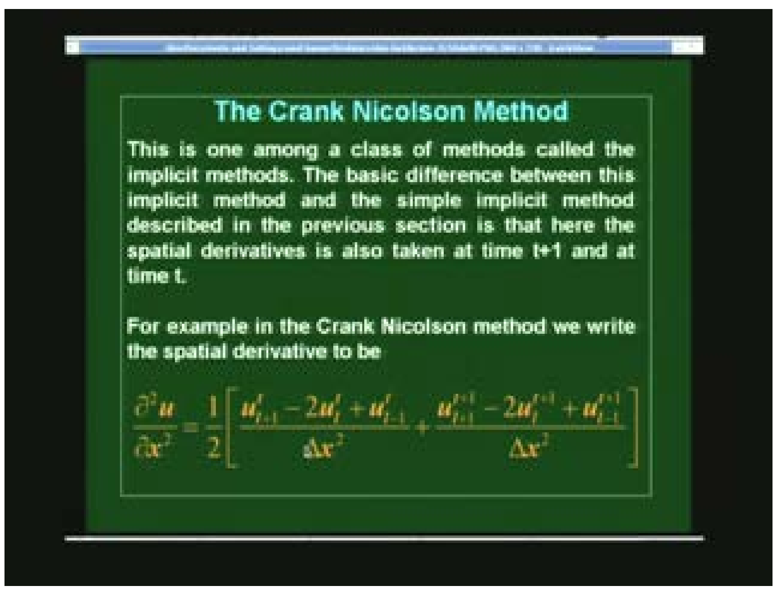


So this is called the Crank Nicholson scheme. So in this method this is again an implicit method in the sense that the right hand side of the equation del u by del t equal to k del square u by del x square is still a function of temperature at t plus 1 at time t plus 1 but now it is a mixture of time t and t plus 1.

So again we could use the same scheme So then we would write the equation now in this form that is $u_l$ at t plus 1 minus $u_l$ at t by delta t, so this is our del u by del t now the del square u by del t square is written as mean of the second derivative evaluated at time t and at time t plus 1 and then we collect all the terms here and write the equation now for the interior points as this one.

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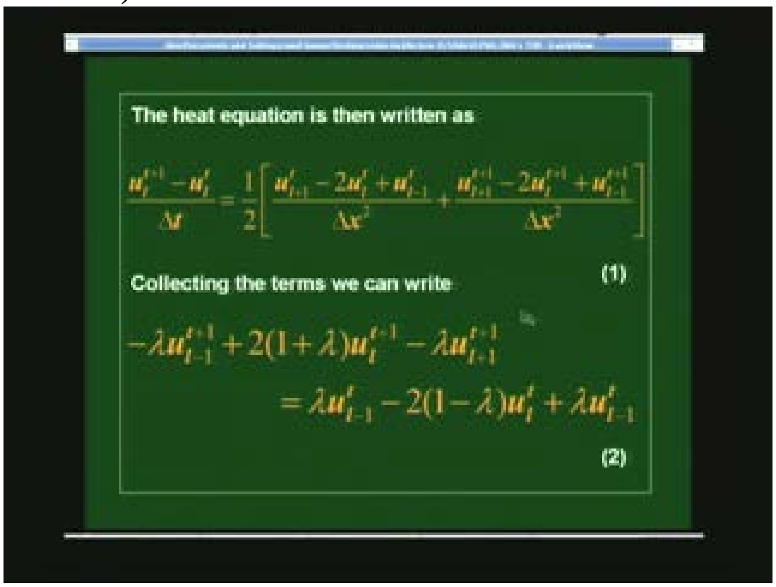


So we have at the interior point equations as t plus 1 minus lambda $u_l$ minus 1, t plus 1, 2 times 1 plus lambda $u_l$ t plus 1 minus lambda $u_l$ plus 1, t plus 1 and the right hand side now is now a function of l minus, a temperature at l minus 0, l and l plus 0. So this is actually l plus 1, so that is the difference so we have instead of simply writing it as so what we wrote in the simple implicit scheme was at $u_l$ at time t plus 1 minus $u_l$ at time t divided by delta t. We wrote that as k times $u_l$ plus 1 t plus 1 minus 2 $u_l$ at t plus 1 plus $u_l$ minus 1 at time t plus 1 divided by delta x square.

So now instead of that now you write $u_l$ at time t plus 1 minus $u_l$ at time t as equal to lambda which is delta k delta t by delta x square into ul plus 1 time t plus 1 minus2 $u_l$ at time t plus 1 plus $u_l$ minus 1 at time t plus 1 plus now ul plus 1 at time t plus $u_l$, $u_l$ plus 1 at time t plus minus 2 $u_l$ at time t plus $u_l$ minus 1 at time t then divided by half the mean.

So we collect terms here and write this as now, so we have all the terms at t plus 1 on to the left hand side and all the terms at t on the right hand side so you have $u_l$ plus 1 at time t, $u_l$ minus 1 at time t plus 1 into, so we have as minus lambda by 2 and then you have $u_l$. so we have1plus lambda into 21 plus lambda $u_l$ at time t plus 1, 1 plus lambda.

So that is 1 term coming from here and one term here and then minus lambda by 2 $u_l$ plus 1 at time t plus 1 on the left hand side and the right hand side then we have equation as lambda by 2 into $u_l$ minus 1 time t plus $u_l$ plus $u_l$ plus 1at time t minus 2$u_l$ at time t then we have that that is plus ul at time t.

So that is our equation now to solve, so again this is in the matrix form. So we have all the time t plus 1 on the left hand side all the t on the right hand side. So the only thing this is exactly the same as simple implicit method except that the right hand side now has to be evaluated using the temperature at l minus 1, l plus 1 and l at time t, that is the only difference and then you could use exactly the same inversion scheme, so once you have set up all the matrix again you could invert that matrix and then for every time you could multiply the right hand side by that matrix and get the temperature at time t plus 1.

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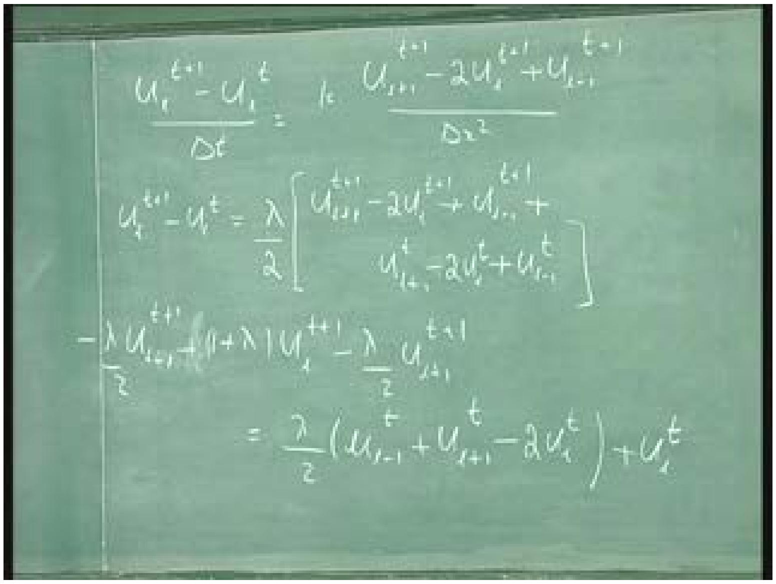

So that is also when it is one spatial variable and we will look at in the next class what happens, if you have a two spatial variables that is x and y and how do we deal with implicit and explicit methods in that particular case that is what you would be looking at in the next class.