Numerical Methods and Programming
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Lecture - 34
Partial Differential Equations

Today, we will continue our discussion on partial differential equations that is differential equations of the form given here that is a del square u by del x square plus b del square u by del x del y plus c del square u by del y square plus f of, a function of u x y del u by del x del u by del y equal to 0.

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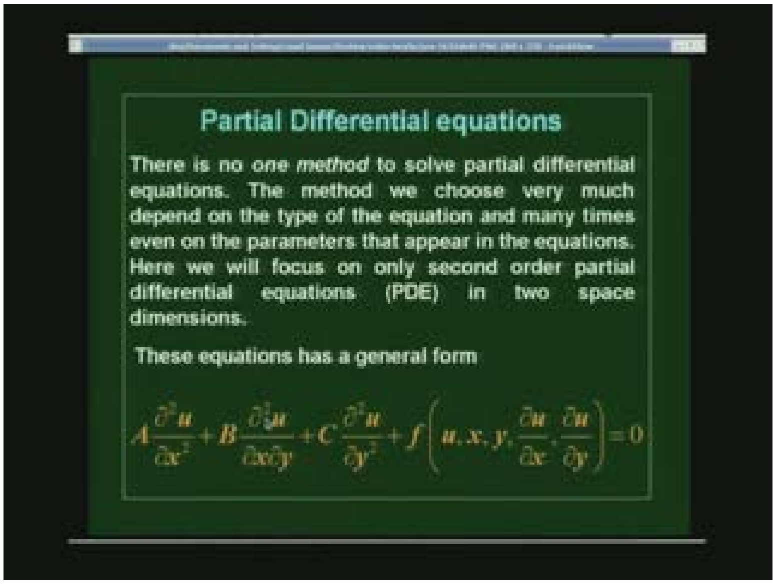

So this is a general form of partial differential equations which we will be looking at where, variables this function is the second derivatives here are linear in u, so that is the kind of functions you would be looking at in fact with specific examples we will be looking at only linear, only equations that are linear in u okay and for different values of a b and c and f.

So with this, we saw that in the last class that with this general form for the partial differential equations where u is a function of both x and y. we can classify this into 3 different categories that is depending upon these coefficients what the value of these coefficients are, that is if b is 0 and when a and c are positive and then we call that as elliptic partial differential equations that is of this form, that is b square minus 4 ac less than 0. So if b is 0 then 4 ac, a and c are positive then you have elliptic partial differential equation an example being the Laplace equation, one equation which you will be looking at and then we could have a parabolic equations where b square minus 4 ac is 0.

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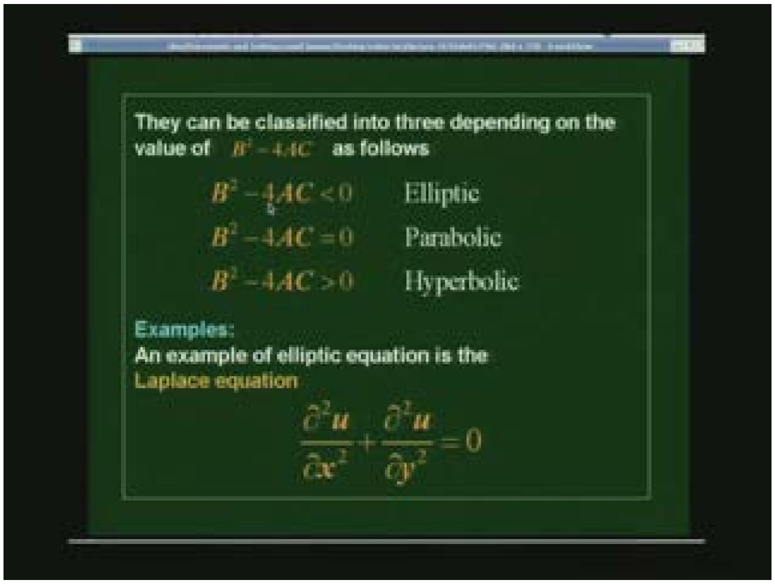

So that could happen if both b and one of these coefficients a or c are 0, okay that is the case which you would be looking at here in the case of a diffusion equation or a heat equation in one dimensions.

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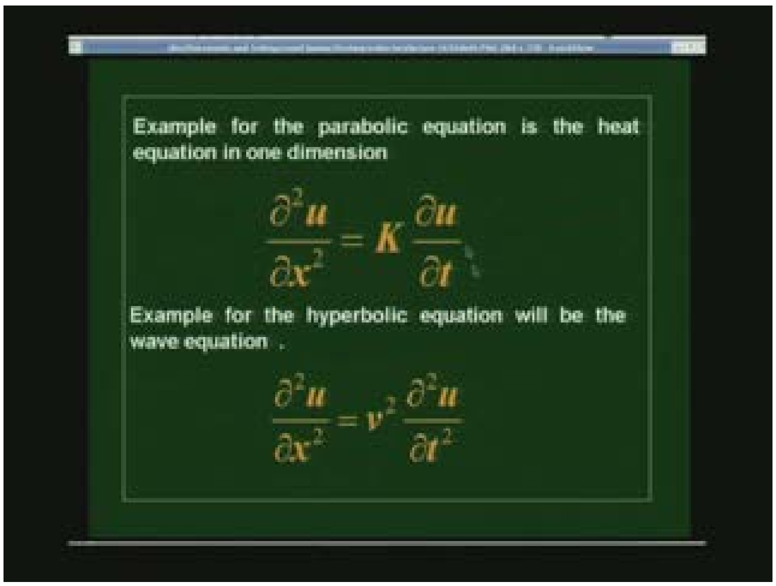

So that is the case where you have a parabolic equation, see that there is only one second derivative is only a function of second derivative is only in x, one spaced dimension spaced variable and then u is a function of time 2 in this particular example. Okay that is a heat equation or you could have cases where b square minus 4 ac is greater than 0 that is, that kind of equations are hyperbolic equations and for example being that of a wave equation. So in this case now here you have again b is 0 but a and c are opposite signs.

Now we are taking one variable as x and other variable as time in this particular example, that is the solution, that is the example of hyperbolic equation, that is the wave equation in one dimension. So we have elliptic equations and we have parabolic equations and then we have hyperbolic equations.

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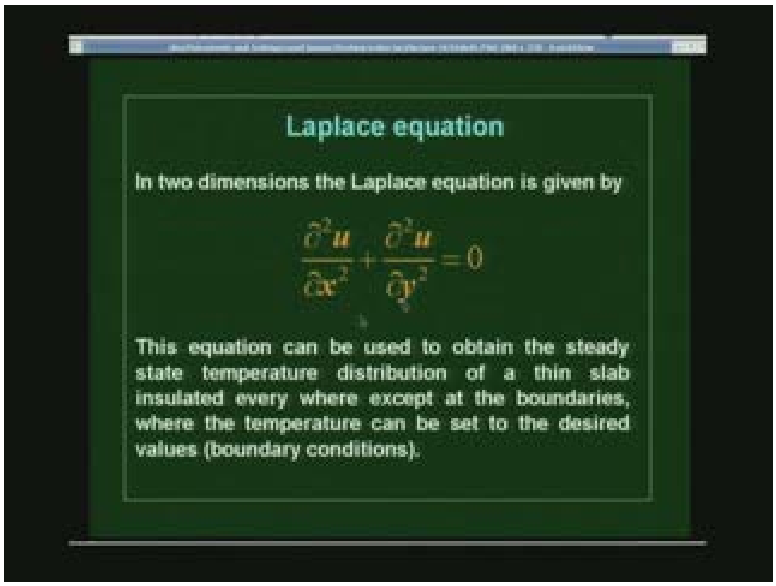

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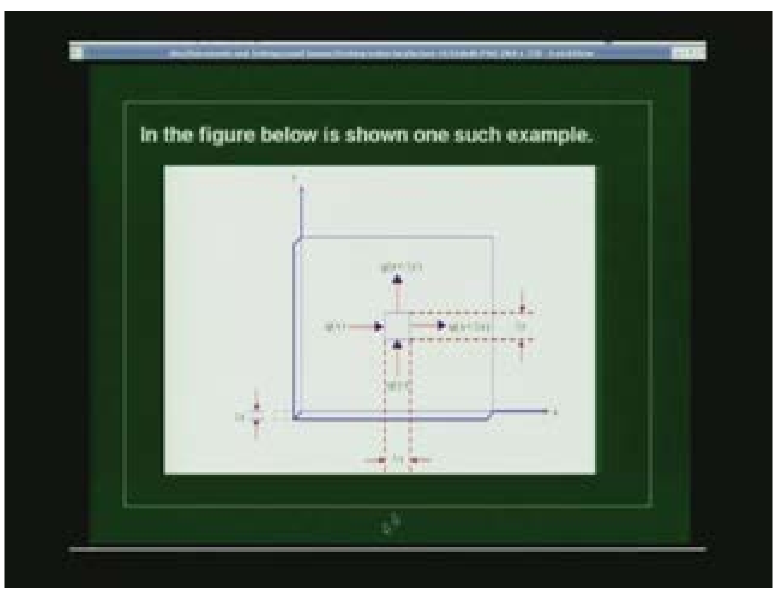

So we look at more specific case of Laplace's equation which arises quite often in science and engineering, for example when you want to deal with the case of the heat flow, the steady state solution of this particular thing, there is a slab here and the slab has a thickness let us say delta z and then now you cover the slab, you insulate the slab on the top and the bottom and then you let the heat flow only through the sides. In this example there is heat coming in from this side and this side and then going out through this and this side.

So that is we keep these two side at the higher temperature with respect to these two sides and then you could ask the question what is the steady state solution of the temperature that is you want to know, what is the temperature profile in this slab at steady state that is when this temperature is not changing with respect to time. So to begin with we will change the time.

So to start with let us say you keep it at some 100 degrees on this side and 0 degrees on this side and then there is a flow of heat and so as you start with there is a temperature in this region and then after some time it reaches a steady state and you want to know what are the steady state temperature distributions in this area is.

So then what would you do, we would write let me say qx is the heat flow in that direction and qy is the heat flow in this direction. So that is the amount of heat per unit area per unit time which is flowing out in this region in this direction and qx is the amount of heat which is flowing out in this direction per unit area per unit time.

So then you take a small section like this and then you say that whatever coming in should go out at steady state because the temperature here is not changing, okay so whatever heat which comes in should go out and we have insulated the top and bottom of this slab, so the flow is only in the xy plane.

So that simplifies our analysis here and then you say what is the heat which comes in, what comes in should be. So we are looking at something like this, so we are looking at a small section and then we are saying that what comes in here is qx and then comes here is qy, qy and what goes out here now things go out here. So now this thickness is delta x and delta y, so what goes out here is qx plus delta x and what goes out here is qy plus delta y.

So what is the total heat coming in here should be qx, q at x that is the heat flow per unit area per unit time. So this is a xy plane this is the x direction and that is the y direction and z is the thickness of the slab. So we have delta y into delta z as the cross section of this plane coming in and in time delta t that will be the heat coming in time delta t in through this and similarly what is the heat coming in here, that will be qy into delta x and delta z, delta t.

So that is the net heat which is coming in and at steady state this should be equal to what is going out so that, what is going out is the following that is qx plus delta x this is y plus delta y this is y plus delta y and this is x plus delta x that is our x direction and this is our y direction.

So qx plus delta x into delta y, delta z, delta t that is the heat going out there and the heat going out here heat going out here and the heat going out there would be qy plus delta y into delta x, delta z, delta t. So that will be the heat which is going out so this equation, so we equate these two right and then we can rearrange the terms and then write it as qx minus qx plus delta x into delta x, delta, so delta z delta t cancels everywhere.

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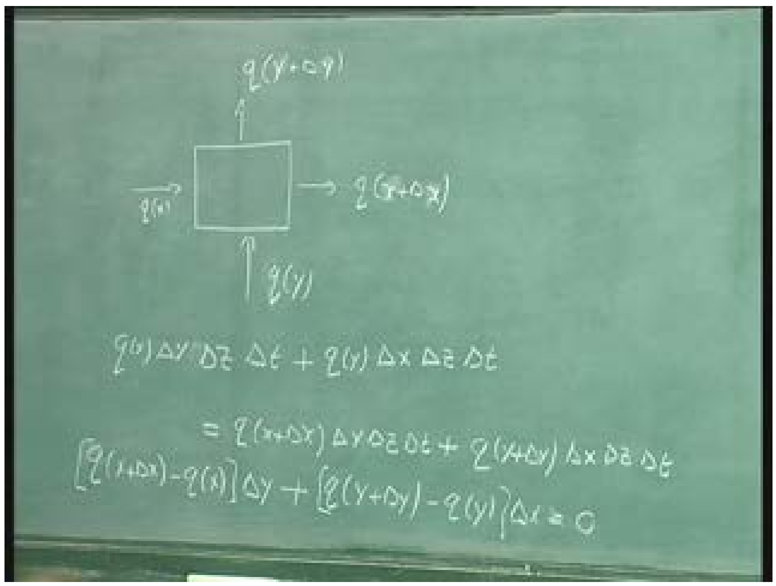

So we can write this as, q into x plus delta x minus qx into delta y plus qy plus delta y minus qy into delta x equal to 0. I just rearrange the terms and I can write that and from this, so I can write this the from here I can write this as qx plus delta x minus qy dividing the whole thing by delta x into delta y I can write this as qx plus delta x minus qx divided by delta x plus qy plus delta y minus qy divided by delta y equal to 0.

So now remember, this is the definition of the forward difference for the first derivative, so when you replace the first derivative by the forward difference mapping this is the definition of the forward difference. So I can write this as del q by del x and then that is del q by del y equal to 0 that is what we get here, were it remember q is the heat flux that is the quantity of heat which flows in per unit area per unit time. So that is what the q is. So now in this case now remember what we are looking here is an example of an elliptic partial differential equation.

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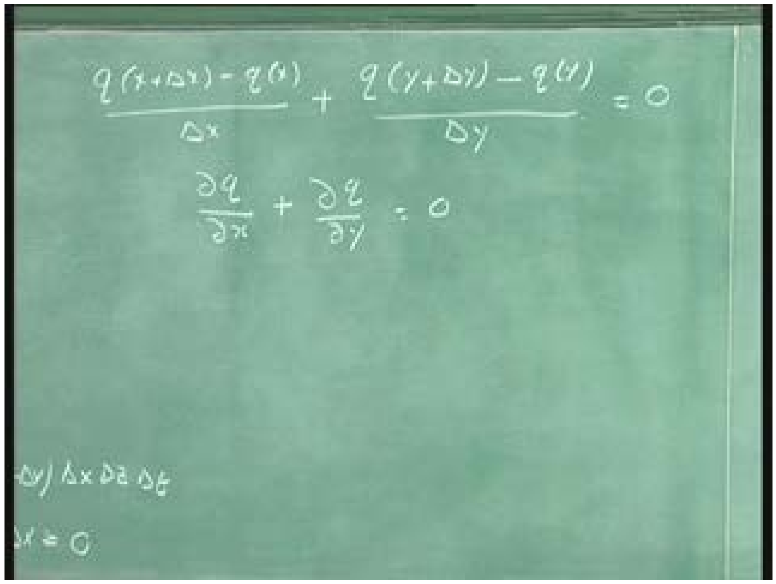

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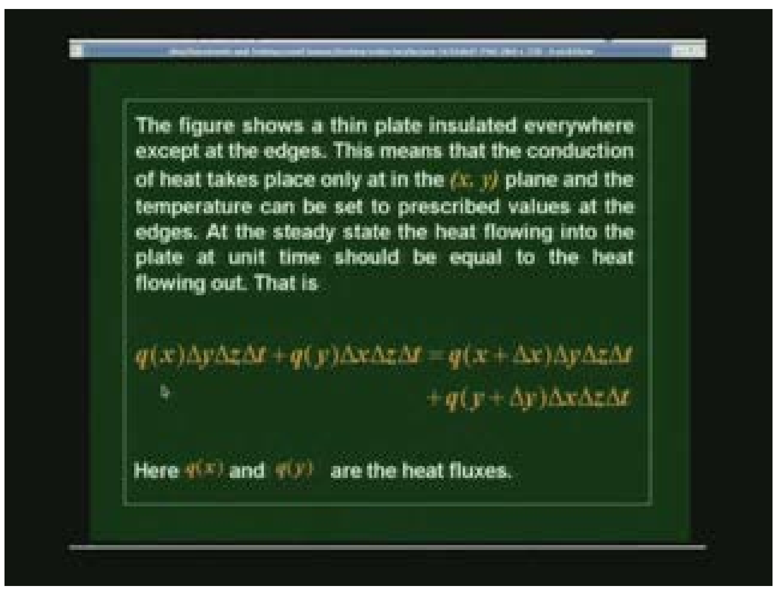

So this is one case where that arises now we have written this equation for the heat flow in terms of the, the problem of heat flow on a slab in the xy plane in terms of the currents the heat currents now, we want to write this in terms of the temperature. So what we have done so far is just say that what is the net heat coming in that is qx into delta y into delta z into delta t that is basically the, this heat current multiplied by this cross sectional area here multiplied by time and similarly, whatever coming in here this q at qy the current multiplied by the cross sectional area multiplied by time that is the flux multiplied by the area multiplied by the time. So that is we are equating this to what is going out, so then we have this equation and from this equation we could write this as del q by del x plus del q by del y equal to 0. That is the equation which we got.

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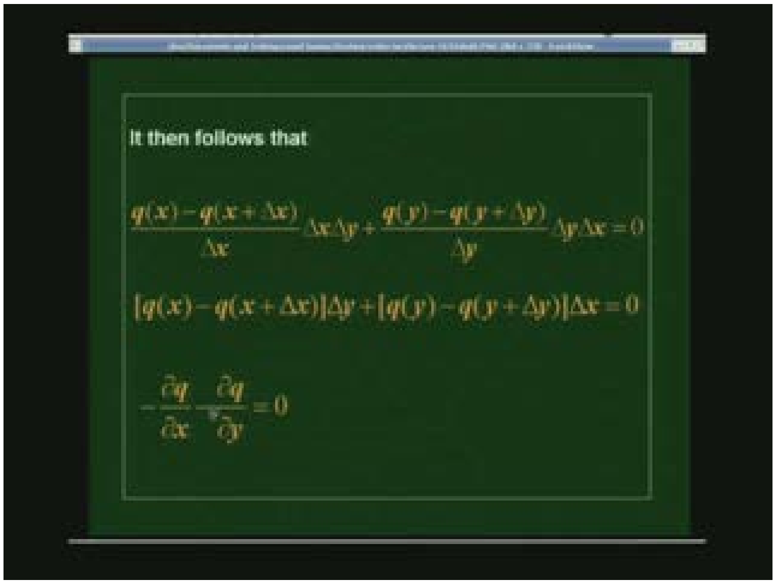

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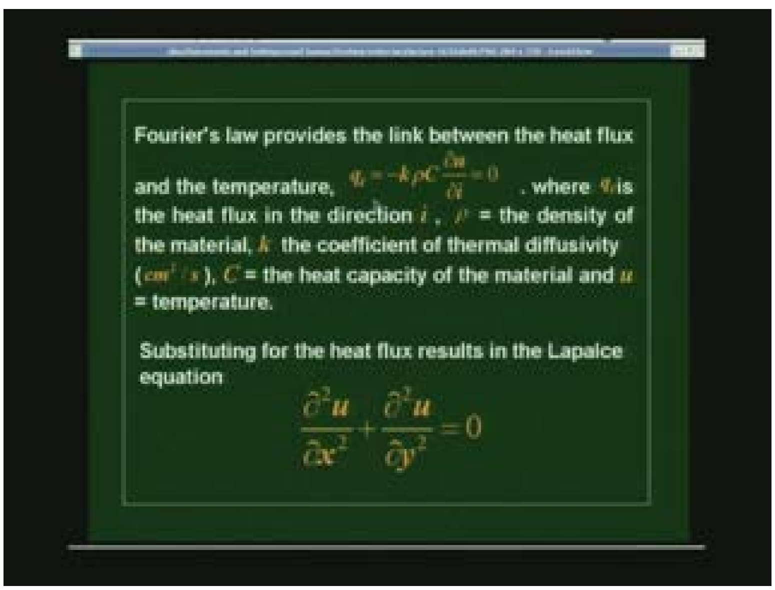

But now the q's are the fluxes and then this fluxes are related to the temperature by the Fourier's law which also we know that saying that q is proportional to this q. So we are saying we connect to these fluxes by the fourier's law to the temperature, so now this is we denote the flux in the x direction by qx here and the flux in the y direction by qy.

So we are saying the fluxes are related to the temperature through the Fourier's law which tells us that qx the flux in the x direction is proportional to the gradient in temperature u, we call u as the temperature here and gradient of the temperature in the x direction. We call we use u for temperature to distinguish it from t which is time so now we substitute this here so that is q is equal to a constant times the gradient in that direction, you substitute here and that will give us del square u by del x square plus del square u by del y square equal to 0. So that is the Laplace's equation which you were looking for. So now this is one case where we obtain the Laplace's equation.

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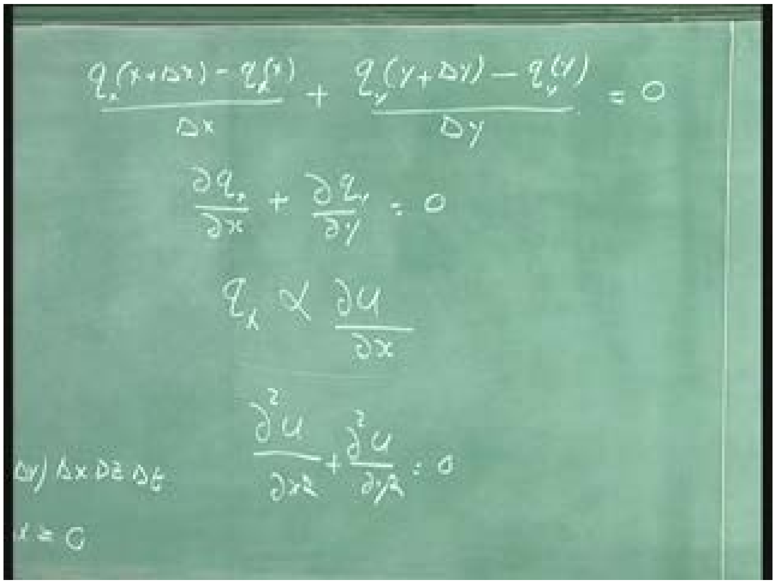

That is the case where we want to know what is the steady state temperature distribution of a slab, when it is when the heat comes in through the sides and goes out through the side that is the top and bottom is insulated and then you have the two dimensional flow of heat and then the steady state will be given by the steady state distribution of the temperature u, the u is the temperature here it is given by the laplace's equation.

So we will look at this as a specific example for elliptic partial differential equation and then see how we can solve this equation by using, replacing the derivatives by its difference equations that is what we will be looking at now. So let me summarize this so we have the fourier's law which says that the flux in the x direction any direction i is related to the gradient of temperature in that direction and multiplied by some constants over here it is k is the coefficient of thermal diffusivity a row is the density and c is the heat capacity and that would lead when you substitute that here leads to the laplace's equation here. So now in the case where there is a source this equation would be Poisson's equation.

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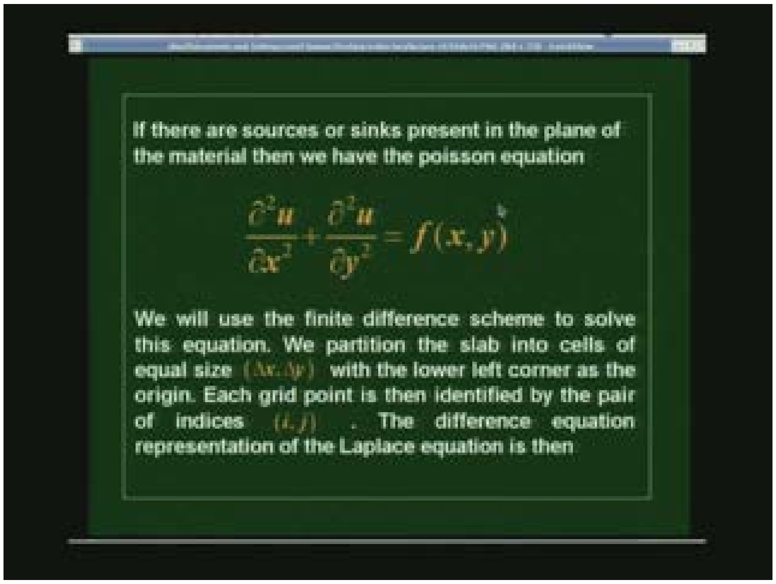

That is also another example so we could have for example a particular source sitting there now that is kept at a constant temperature so if you have a slab where the middle of the slab or some part of the slab is kept at a constant temperature. So that is like having a source in that point and then you have this equation here the right hand side of that would be a source term. So that will give us the laplace's equation so both equations can be solved by similar methods, that is basically replacing these derivatives, these second derivatives by the difference equations.

So now remember what the difference equations were, so when you said that the first derivative as we can see here the first derivative comes from this or when you want to have first derivative you can replace this by a difference equation. Similarly, you can replace the second derivatives also by a difference equation, so we will write that now. So when you want to write equation of this type so del square u by del x square equal to del x square, let us say we will replace this particular term. So that what you do is we can write it as u x plus delta x minus 2 u x plus u x minus delta x divided by delta x square.

So that is something which we can, that is one thing which we can use one scheme in which the, one way of replacing getting the replacing the derivative by this difference equation. Okay so now this is symmetric difference equation for the second derivative, so similarly you can write for y also and replace it in our laplace's equation. So our laplace's equation which was del square u by del x square plus del square u by del y square will now become u x plus delta x minus 2 u x plus u x minus delta x divided by delta x square plus u y plus delta y minus 2 u y. So u y plus u y minus delta y divided by delta y square.

So this term is evaluated at a constant y value while this is evaluated at a constant x value that is the way we do it so we can actually when we write this we should actually write it as x this we should write it as y minus 2 u x, y that is evaluated at a constant y value plus u x minus delta x, y similarly, here we should write this as u x y plus delta y minus 2 u x y plus u times x, y minus delta y.

So that is the equation which we will write. Okay so we have these 2 equations, so we will be placing these we have this equation these terms replacing these two terms in this equation. So now we can write this whole thing as u x plus delta x, y plus u x minus delta x, y plus u divided by, okay so now the special case the special case where delta x equal to delta y, we can simplify this equation and write it in this form. Okay so in this special case we can write this as u x y plus delta y, plus u x y minus delta y.

So plus 4 minus 4 ux, y right. So whole thing divided by delta x square, so this equation in the special case where delta x and delta y are the same that is we are going to replace all these derivatives by the difference equation. so that means we are going to descriptized space and write this in a grid, so that we have descriptized space now that is x goes up by delta x and y goes up by delta y from point to point. So then and let us say we descriptize x and y directions by the same amount that is delta x and delta y.

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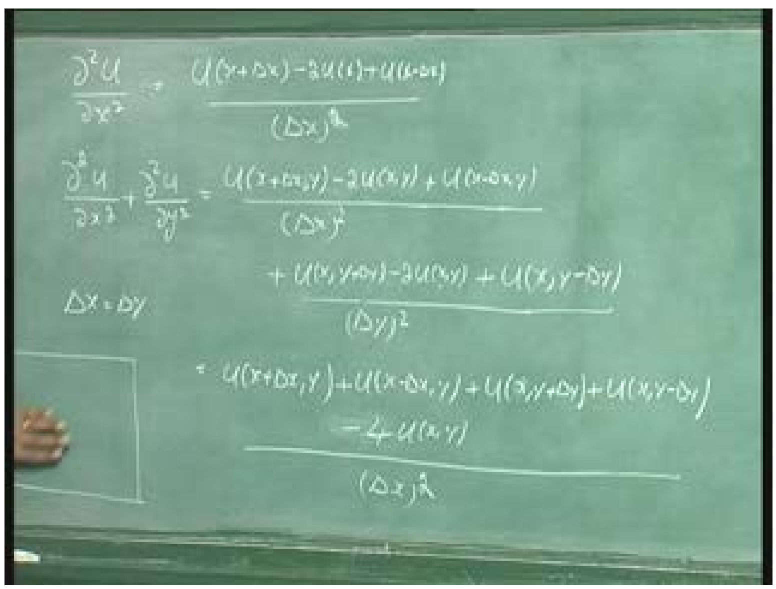

So by this what I mean is you take a piece like this and you want to know now, for example the heat distribution the temperature distribution inside this then what we do is we actually make a grid here and then we try to write the equations for every point on this grid. Okay so we write this grid and we write equations for every point on this grid. So now if this spacing and this spacing are the same we take a square grid and we take this spacing and this spacing to be the same and then we can simplify this equation to, if you use this we can simplify this equation to this form.

So now what is the advantage of doing this is that now we can we know that every point is separated by equal distance and then I can replace all this by an index both x and points by an index and write this in a much more simple form. So this is the basic idea, so that is we replace this equation by its difference equation now this equal to 0 is a equation which we have to solve.

So now let me summarize that once again here. So we have the equation, so the second derivative in that direction now notice that I have replaced them by indices, so indices in this case and again that is i is my x index and j is my y index. So I am going to say that this each point is now represented by an i, j. So each point on this lattice is now i, j, so i goes up in that direction and j goes up in this direction. So that is what this is okay and then you have the equation as x minus delta x is now i minus 1 and x plus delta x is i plus 1 and xy is i, j and then I can write this equation in this form that u_i minus 1, j minus 2 u_{ij} plus u_i plus 1 j and by delta x square and then u_{ij} minus 1 minus2 u_{ij} plus u_{ij} plus 1 by delta y square equal to 0 as my Laplace's equation now.

So then this equation now for every point on this lattice right. So when I write this equation now in the index form and then that is valid at every point on this lattice. So I have to solve that equation, so if I have a n by n lattices a, n by n grid then I have n into n equations to solve in this particular case. So at every point i and j I have an equation, so you will have n into n equation to solve to get this solution so you will write that down specifically.

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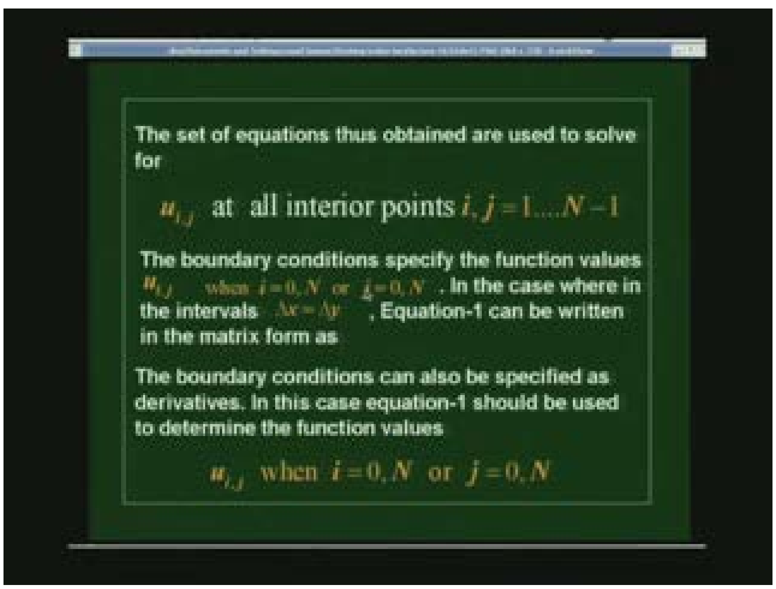

So we choose this particular simplification that is delta x is equal to delta y and to write this in a much more simple form which we just now saw and then you can write equations now as in the index form as u $_{ij}$, u $_i$ plus 1, j plus u_i minus 1, j plus u $_i$, j plus 1 plus u $_i$, j minus . So I repeat this u_i plus 1j, u_i minus 1j, so that is x plus delta x, x minus delta x and y plus delta x delta y and y minus delta y and that is equal to equal minus 4 times u_{ij} equal to 0that is our equation, that is the equation which you have to solve and now this for every i.

So i going from 1 to n and j going from 1 to n we have to solve that. So now this is a differential equation which we have to solve.

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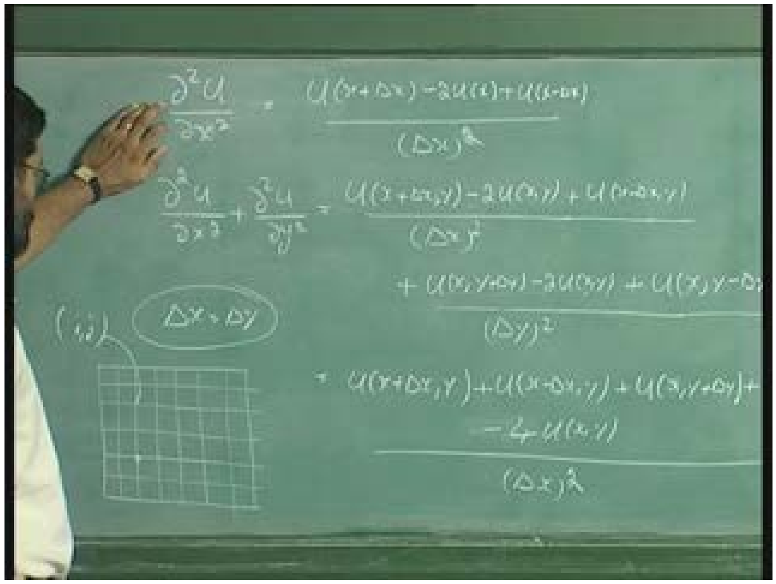

So we obviously need boundary conditions for this and we can see that very clearly when you write to write this as I said for every i and j values, so now if I try to write that this equation for a point on the boundary here and then I will have trouble because on the boundary then I will have to say for example, if I take this boundary and then I have to specify this is j equal to n let us say, I have to specify j plus 1.

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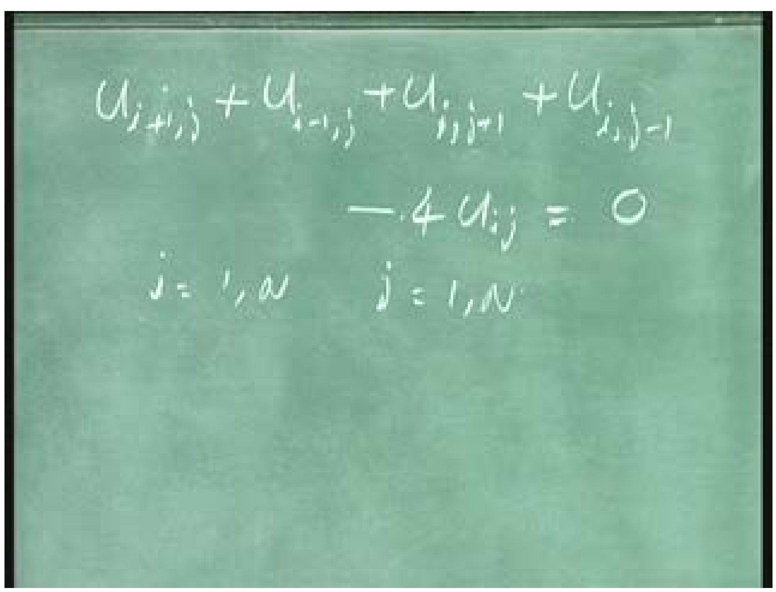

So that is because i need j plus 1 here, so if I specify it on the boundary the j plus 1 is not known so I cannot do that, I cannot write this equation to the boundary. So we can see the need for boundary conditions to solve this equation, so how do we put in the boundary conditions. So that is also we will see now so let us take a small grid and write this down explicitly.

So we have a let us say we have a grid like this here. So we make a grid of this form, so we have j, the i value going in this direction and j value going in that direction let us say this is 0, 0 and then similarly we have 1. So we have a grid like this and then we put on this grid the points that is this is 0, 0. So this is 0, 1, 2, 3, 4 and this is 0, this is 1, 2, 3, 4.

So we have a grid some grid like that and now you want to solve equations for every point on this grid, so if I want to write the equation here so then that is now 2, 2. So this point is 2, 2 right that is what this point is. So I want to write this equation here I would write it as $u_{3, 2}$ plus $u_{1, 2}$ plus $u_{2, 3}$ plus $u_{2, 1}$ minus $4u_{2, 2}$ equal to 0 that is the equation I would write for this point and now if I want to write for let us say this point that is now 2, 0 if I want to write for this point I can see that I will have so this is for 2, 2.

Now for 2, 0 that is i is 0 i is 2, j is 0 and then I have to write it as 3, 0 plus u, I will have 1, 2 plus u $_{2, 3}$ $_{2, 1}$ j is 0 plus u $_{2}$,minus 1minus 4 $u_{2, 0}$ equal to 0. So now you can see that this point we do not know what it is. So obviously we cannot write equations on the boundary nodes as easily as the interior nodes, so this scheme differentiates the boundary nodes and the interior nodes. So something has to be done to actually specify the points there, now there are two ways of doing this now these points are specified by the boundary conditions. So if you want to write these equations at the boundary and then you need special conditions.

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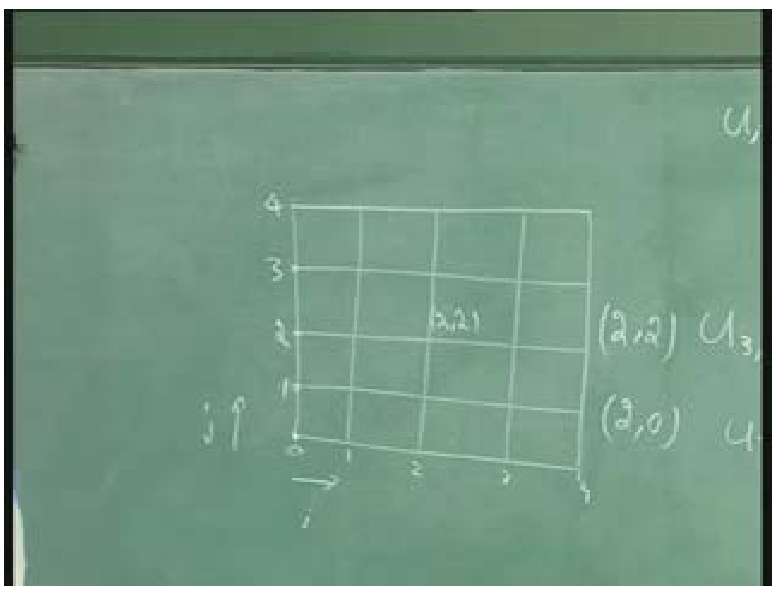

So there are 2 ways of doing this one is that we do not write the equations at the boundary because the boundary conditions specify what is the value of u at the boundary is. So in our example of temperature distribution it is specified what is the temperature at this boundary nodes are that is one case another case is that we do not know what the temperature there is but we know what the derivative of the temperature at that boundary are. So there are two specific examples which we should be looking at.

So let us look at the first example, where we have the boundary conditions specified in terms of the temperature there. So we are given the temperature at these boundary points that is, so now we look at the examples, so let us say we have kept this at 75 degrees here and then this is at 50 degrees and we say this is kept at 100 degrees and this is kept at 0 degrees.

So that is u equal to 50, u equal to 100, u equal to 75 and u equal to 0. So now that is the starting condition that is the condition which we are given. Okay so we maintain this 4 boundaries at, so we maintain these four boundaries at the specific temperatures given and then we want to find out what is the temperature distribution inside is at steady state that is solve this equation with this boundary condition.

So now obviously I do not have to write this because we know I know that the solution for 20 is now that u 2, 0 is 75, I know that I guess do not need to write that equation. So I need to write only for these 9 points 1, 2, 3, 4, 5, 6, 7, 8, 9 the boundary conditions are specified this corner points we either denote with this boundary or with that boundary.

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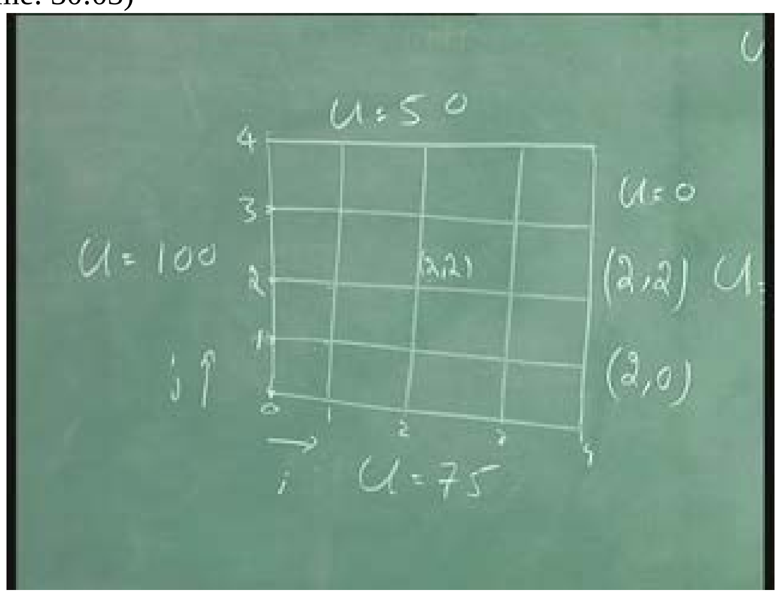

So that is the condition which we can look at. Okay so then we can write the whole equations for all the boundary points. So we can do that here for example if I want to write it for this case that is, so let us write it for 4 points so we will write it for this particular point for 1,1 so we will write it as so for the case 1, 1 I would write that as $u_{2,1}$plus $u_{0,1}$ plus $u_{1,2}$ plus $u_{1,0}$ minus 4 $u_{1,1}$ equal to 0 and now I will write for the next point that is for 1, 2, 1, so when I write for 2, 1 I will have $u_{3,1}$ plus $u_{1,1}$ plus u.

So we are writing this now for 2,1, so then we have $u_{2,\,2}$ plus $u_{2,\,0}$ minus $4u_{2,\,1}$ that is equal to 0. So we get nine equations like this and we can see that they are all couple that is $u_{1,1}$ appears here $u_{1,1}$ appears here. So we get nine simultaneous equations like this and we know how to solve such simultaneous equations. So we know how to solve simultaneous equations, linear equations as I said we will deal only with linear equations, partial differential equations which are linear in the variable u.

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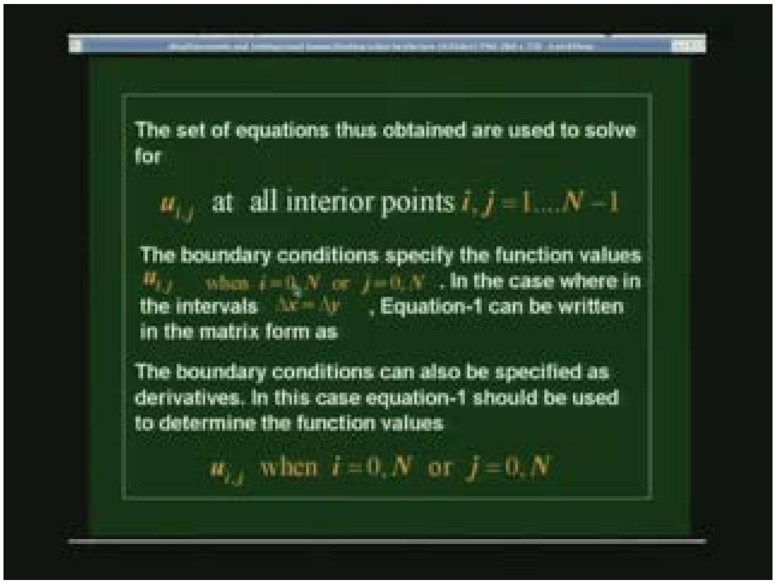

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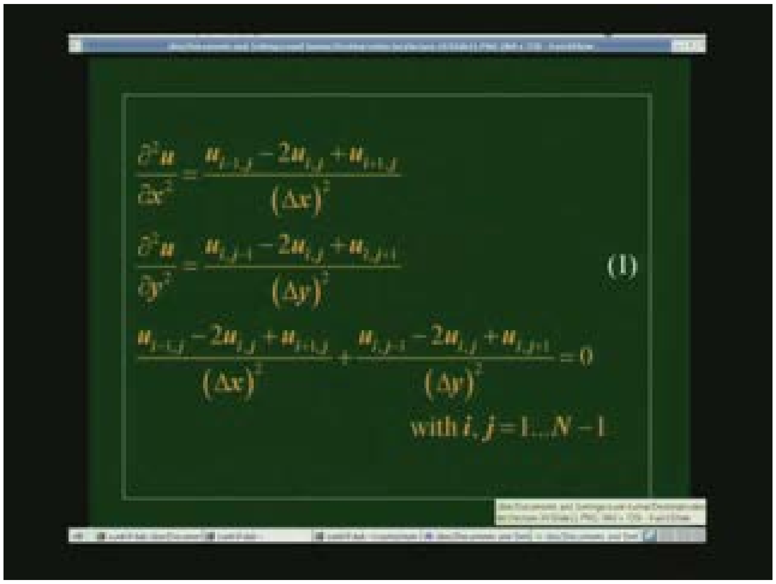

So that we can write them as using the difference scheme as linear simultaneous equations and we know how to solve those equations using matrix methods. We can look at an example of such case in this, so now let me just summarize before that, so what I am trying to do is write into a particular case where delta x and delta y are the same and write these equations these equations in the form that u_i plus 1 u_i minus 1, j minus u_i plus 1 j plus so u_i minus 1 j plus u_i plus 1 j plus u_{ij} minus 1 and plus u_{ij} plus minus 4 u_{ij} equal to 0, when delta x and delta y are the same that is what we are doing in this particular case.

Okay so now let us look at a specific example where we implement this simple boundary condition that is this fixed temperature at the boundaries and then we solve so nine such equations linearly. So it is obvious to us that we can actually solve these equations by the matrix methods we know so we have this nine equations and we can just invert this matrix by using the matrix methods for example, in this case again we know what these values of $u_{0,1}$ and $u_{1,0}$ are, so for example because these boundaries are fixed right. So $u_{0,1}$ is all fixed at $u_{0,1}$ $u_{0,2}$ $u_{0,3}$ $u_{0,4}$ etcetera are fixed at 100.

So this is this is simply this value is replaced by 100 and similarly $u_{1,0}$ so that is when this when the x is1 and this u is 0. So $u_{1,0}$ $u_{2,0}$ $u_{3,0}$ and $u_{4,0}$ are fixed at 75. So I can replace this by that number, so boundary conditions enter through this through these variables at these points.

So only when we write this equation, so for example here you will get it as $u_{2,1}$ plus $u_{1,2}$ minus four $u_{1,1}$ equal to minus $u_{0,1}$ minus u $_{1,0}$ so minus u $_{0,1}$ is 100. So minus 100 minus $u_{1,0}$ so u1,0 is 75. So that is what you would get so this is as minus 1,175, so minus left hand side is minus 175 so all other terms. Okay are 0 in this and similarly here you will have $u_{3,1}$ plus $u_{1,1}$ plus $u_{2,2}$ plus minus $4u_{2,1}$ is equal to minus $u_{2,0}$.

So u $_{2,0}$ is again minus 75 so it is minus 75 the unknowns we can write them in a column matrix form. So we can write that in a column matrix, so then we will write it as $u_{1,1}$ u $_{2,1}$ etcetera and then $u_{1,2}$ $u_{2,2}$ etcetera. So we will have 9 such elements here corresponding to temperatures at 9 points and then we can write this as a matrix form but most of the elements in that matrix would be 0 as you can see most of the elements here would be 0 because the first if I want to write the first column here the first equation, you can see the u $_{1,1}$ is 0, so it is 0 and then I can write it as 0 plus 1plus, so I have u $_{1,1}$ u $_{2,1}$ u $_{2,1}$ is 1 and then you have $u_{3,1}$ and u $_{3,1}$ is 0. So that is 0 and then you have u $_{4,1}$ so I have you see u $_{1,1}$ and u $_{2,1}$ and u $_{3,1}$. So I have to write that so that u $_{2,1}$ u $_{3,1}$ and then I have u 2,1 u $_{1,2}$ u $_{1,2}$ is 1 u $_{2,2}$ is 0 and then I have u $_{2,3}$ which is 0, so I keep going this way. So I have lots of terms which are 0 in this equation u $_{1,1}$ is not 0 that is minus 4.

So I can write this equation when I write this equation using all these variables I get lots of 0s there and then so my matrix equation will have most of the elements as 0 such matrices are known as sparse matrices. So using a inversion scheme to solve such matrices matrix equations where most of the elements are 0 is not very advantageous. So let me summarize this once again by writing it like here.

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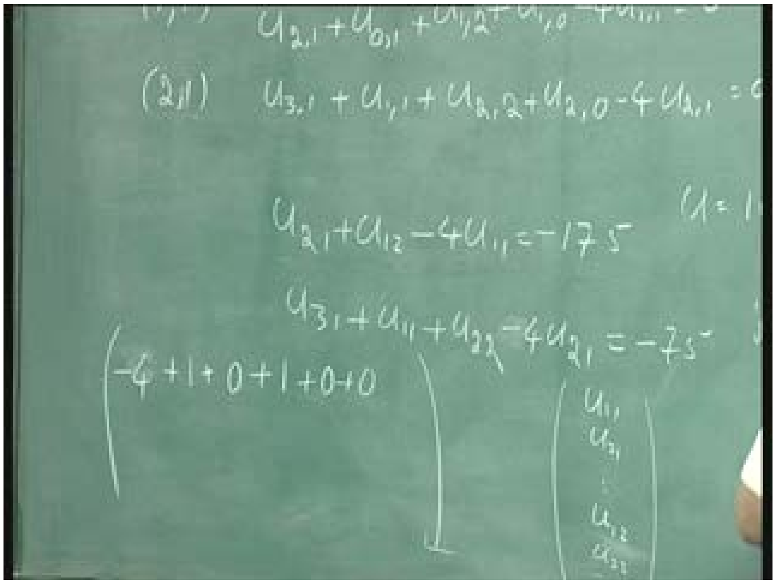

So what I want to do is to write whatever we are doing when we write the equations when we write this differential equation in the difference equation form we would be writing things of this type. So we will write equations of this type one for every point on this lattice we will keep writing equations of this form now these equations I can write in the following form that is I can write it as let us say a_{11} u_{11} plus a_{12} u_{12} plus a, I use 21 here plus a_{13} u_{13} etcetera equal to 0, equal to some function that I call f_1 and similarly a_{21} u_{21} plus a_{22} u_{22}.

I can write it in a matrix form where u_{11} u_{12} u_{13} are the unknowns and a_{11} a_{12} a_{13} are the coefficients. So if I take compare this equation here with the equation which we have which we have here and then I can see that most of those coefficients are simply 0.So that is the point which I was trying to make. Okay, so I have this equation which is now written as a_{11} u_{11} a_{12} u_{12} a_{13} u_{13} etcetera. so that I can finally write this equation as a $_{11}$ a $_{12}$ a_{13} a_{21} a_{22} a_{23} in this form as a matrix multiplied by u_{11} u_{12} u_{13} u_{21} u_{22} etcetera.

I can write it in this form as some right hand side which is given by f_1 f_2 f_3 etcetera. So I can write it in this form that is what we normally do when I have a set of simultaneous linear equations to solve but the point here is that most of these variables most of these coefficients here are 0. Okay so what we get this matrix here will be that most of the elements of this matrix will be so it will be a what is called as sparse matrix.

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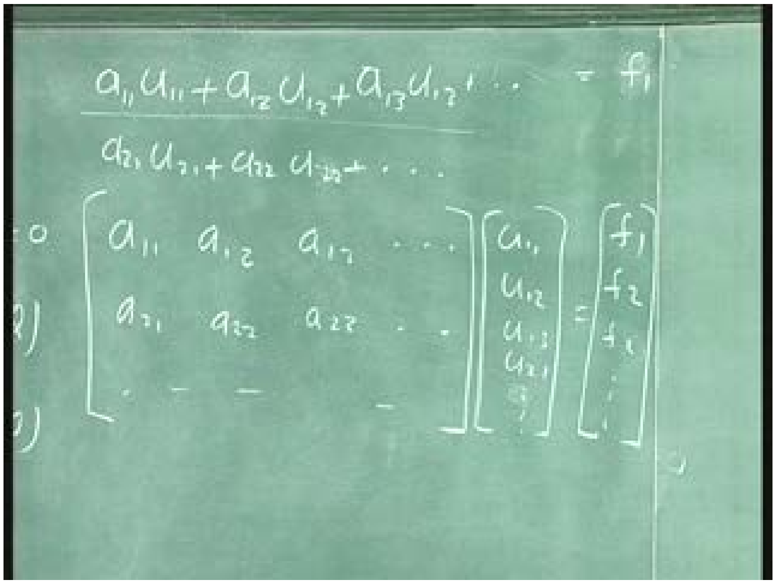

So for that kind of matrices when you have large number of such equations such equations that is we have a grade which is very large such, inverting this matrix to solve this equation is not numerically efficient.

So that is a point which I wanted to make it looks like it is clear that I can replace my partial differential equation by a difference equation and write the solution, write the equation in this kind of a matrix form by descriptizing space, I can write it in a matrix form however most of these elements will be 0 and hence this matrix would be what is called a sparse matrix and inverting such a matrix to solve this kind of equation are not numerically efficient.

It turns out a better way of doing this would be simply to use an iterative scheme that is to say that we had this equation that u_i plus 1, j plus u_i minus 1j, plus u_i j plus 1 plus u_i j minus 1 minus 4 u_{ij} equal to 0. So that is what the equation we had that is where, it is from here that we have written all this forms right. So now this equation, I can write as u_{ij} equal to ui plus1j plus u_i minus 0j plus u_{ij} plus 1 plus u_{ij} minus 1 divided by 4right. So I can write this equation in this form, so then what I do is I start with the u_{ij} values.

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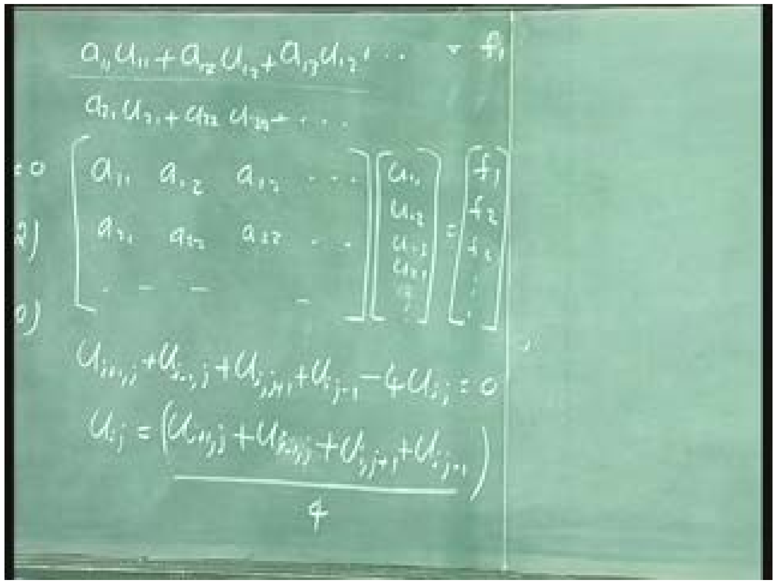

Let us say this particular example what I do is I know the boundary conditions and I put everywhere else as 0 and then I iterate it I obtain what u_{ij}'s are by using this equation I will get the new u_{ij} values for all the interior points and then I put them back again into this equation and iterate it back till I get a steady state solution till this term gives me a value which does not change.

That is so let me repeat this instead of solving a sparse matrix like this the method which is normally used for solving elliptic partial differential equations is the following that you write the equation in this form, that is you write this equation for example, in the case not all elliptic partial differential equations in the case of laplace's equation, you can write the equation in this form as a difference equation and then you say that I use this equation to compute u_{ij} values from by guess values of u_i plus 1 u_i minus1j plus1and j minus1 values, this particular example what we do is we know the boundary conditions.

Okay so we know that the grid on my grid this boundary points are fixed at different u values and then I can write and then I start with interior points are all given some temperature let us say I give it as 0 and then I evolve it in this, using these equations and after some time. So will see that the values inside would keep changing at every iterative loop and finally it will come to a point where the temperatures do not change anymore and then I say that I have reached a steady state.

So that is the kind of equation which we will look at now, so we look at one example of this. So here is a program which does this thing so I have fixed my boundary conditions.
So let us, I have fixed my boundary conditions as 75,50 ,100 and 0. So that is similar to what we have fixed at here, so I had given as you can see the boundary points that is i. So for all values of i going from 0 to n, 1 boundary point 0 is given as 75 and other one that is at n for all i values at n it is given as 50 and all i values at 0, i is 100 and all i values n, i

is 0 and I store the same thing in another matrix called s for reasons which I will tell you just now in a minute.

So this is the boundary conditions specified and so I have this matrix which is n by m matrix n plus 1 by m plus 1 matrix and then what I do is I use these equations very simple program. So this I will just use this equation I have used the equation with my new value of ij is the sum of i plus 1j, i minus 1j, i j minus 1 and ij plus 1 divided by 4. So it is just this equation.

I just use this equation I used a new matrix here "s", so I start with all some values given to you and evaluate the new values of u_{ij} which in my program I call s and then I will see the s and u are the same or not if you have reached the steady state then the s and the u would be the same. That is the values I started with and the new values I got should be the same if it is not the same I go back and then do the iteration again. So I start with all this here it is d matrix elements are called d and from the d matrix I compute the new s values and I compare I am just finding out what is the difference between the d and s are for all points i and j that is my total error.

Okay now if this error is less than ten power of minus 2 here I stop the iteration so I continue the iteration till this becomes less than ten power of minus 2 that is what this program is. So after 1 iteration loop is over i transfer all d values to s values and then all s values to new d values and then I go back and do the iteration again till my d and s are the same. So that is what the program is.

We will just run this program and see, so now that is the what I did was I started with this you can see the top points are all hundred and this side is 75 bottom is 0 and this side is 50 that is what I that is what my boundary condition is and I had started with every point as 0 in the interior point as 0, so now this is after one iteration. So I started with all interior points as 0 as you can see in the program I will show you that again. So all interior points I give as 0, so all points are given 0 except the boundary points. So that is what I started with and then one after one iteration I goes into the interior points have changed.

Now they are not zero anymore they have changed and then I go to the next iteration loop and you can see the interior points are changing further and the boundary points are held fixed and then we keep doing this iteration and you can see that the values are changing.
So the values change and then finally it would reach a steady value, so then after that it does not change anymore now you can see that it does not change very much.

So it has reached kind of a steady value. So when it reaches a steady value this program should come out, so that is what I am trying to do so it runs for many loops and then you can see that the temperature now is decreasing continuously from here to here and similarly, it goes from seventy five to fifty as it goes across here. So that is this method
Now, it has come out once it has reached a steady value it has come out here.

I am printing only one decimal point here but the calculation actually keeps it at many decimal points, now this is a very simple program in which I use this method this method of iterative thing this is called a method of over relaxation. So that is it is a kind of relaxation scheme that is you have some boundary points specified and then you start with some interior value which I put equal to 0 and then you let the whole system relax to it is equilibrium value that is steady state value. Okay so that is what it is done here. So every time you compute from the using the equation the derivative equation that using the derivatives the new values which it can obtain there at that point and continue that till all your second derivatives in this equation has vanished that is the method of over relaxation and there are I should warn you that not all elliptic partial differential equations can be solved using this.

In some cases for example this may not go into, a this may get in to a kind of unstable situation in that cases what you do is to use this equation to get the new value of u_{ij}. So let me call this as s_{ij} here, instead of u_{ij} I call it s_{ij} and I say that my new u_{ij} is actually I would say that my new u_{ij} values are my old uij values let me call this new as my u old u_{ij} values which is u_{ij} itself times some number alpha plus 1 minus alpha times s_{ij}. So, I compute the new s_{ij} values from my old values using this equation but I do not use all of it as the, for the next u_{ij} value.

I mix them with the old and the new with some parameter alpha. So and the limit where alpha goes to 0 it is this relaxation scheme otherwise, it is something slightly different. Okay so now so the limit alpha going to 0 is called method of over relaxation where we are taking the new value as the one which is obtained from this equation straight away or we could in cases where that is unstable it is not going to a steady state, you could achieve better numerical result by mixing the old and the new value with some parameter alpha and the alpha you tune according to your equation, according to stability which you obtain. Okay now that is called method of relaxation and then when alpha equal to 0 we call this a method of over relaxation. So now that is the case where we had the boundary conditions themselves were specified in terms of the u values at the boundary.

So now as I said that this may not be always the case, so you could instead say that I do not know what the boundary values are but I know what the derivatives are. Okay now you can have this equation in 2 different forms, one is the derivative given at the boundary or sometimes the value itself is given at the boundary. So value itself is given at the boundary we can use this and then you have to solve only for the interior points now the value at the boundary is not given to you and then you have to also solve for the u values at the boundary and then we know that if you want to write for example this equation at the boundary here and then you will have problem with terms like u, u_j minus 1, if I want to write here because j is 0 here. So j minus 1 will be u minus 1.

So in that case what we do is the following that we could still use this equation but what we do is we use the boundary condition themselves to specify the, boundary condition themselves to specify the values the u values at these points minus one etcetera. So that is let us say if you are given del u by del x equal to 0 as the boundary condition at points i equal to 0 along the line i equal to 0 and then I have the del u by del x.

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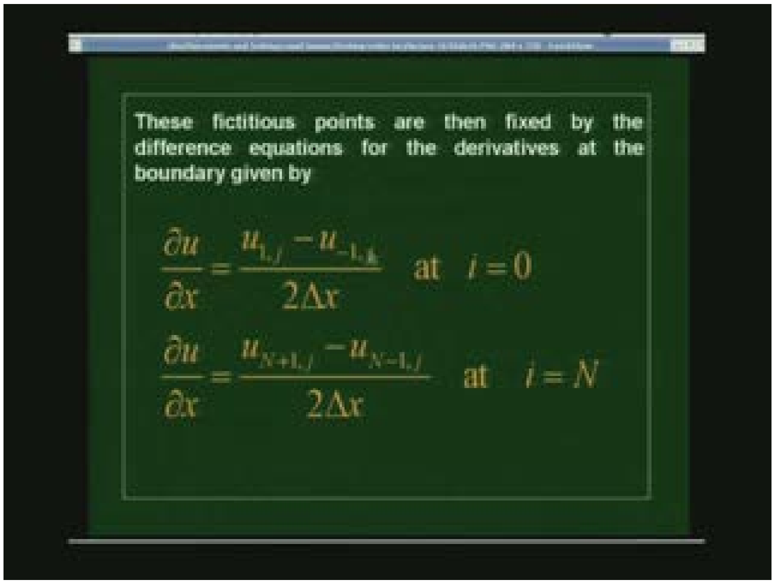

I will replace by a difference equation and a symmetric difference equation and then I would say that that is u_i, u_{1j} minus u minus 1j by 2 delta x at i equal to 0 and from this since, I know what derivative is I can get the value of u_i minus 1. So let me show you that here in this particular example as I said now let us say we have a boundary condition on this line given as del u by del x at x equal to 0 is equal to 0.

Let us say that is the boundary condition given to us. Okay so this is this derivative of u at the boundary is specified or some value c it does not matter some value c is given to us now that means we do not know what the u values here are we have to solve for this now, if I write the difference equation for this point that is 0,2. So if I write it for 0, 2 in this equation and then I have remember my equation had the first point was u_i plus 1j, so that is u_{12} and the second point was u_i minus 1j, so you will have u minus 12 and then I had u_{ij} plus 1 that is 03 and the next point was u_{ij} minus 1,so that is 01 and minus 4u $_{02}$ equal to 0 that is my equation.

So I could be using method of over relaxation and I could write this as equal to 4 0 2 but then I do not know what this point is right. So that is what so if you solve for the u value at this point I need to know what u minus 12 is so what do, how do I get that I know that del u by del x at this point is 0. So what is del u by del x at this point, so I will write now del u by del x as u x that is del u_i by del x_i will write it as u_i plus1 j minus u_i minus 1j divided by 2 delta x whatever may delta x was I can write it like that.

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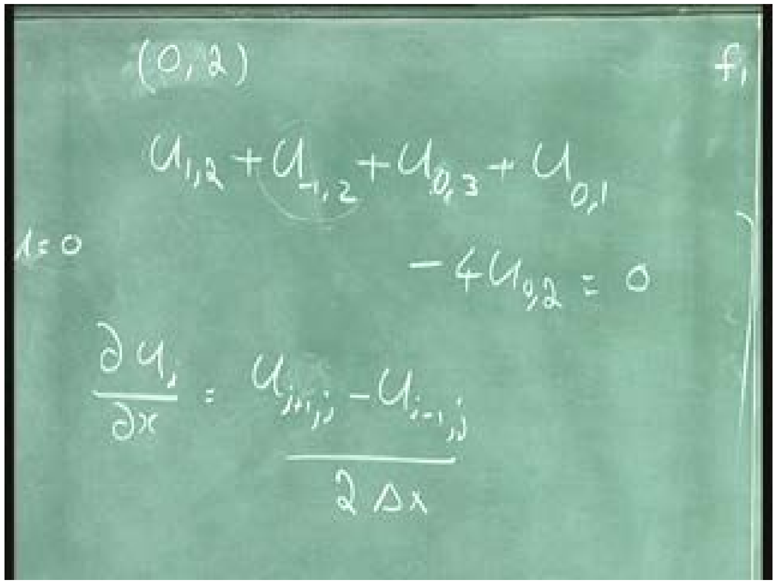

Okay now this is called the symmetric equation for symmetric difference equation for the first derivative where it is symmetric about i that i plus 1 and i minus 1divided by 2 delta x. So I can use this equation here and then from this for example, for i equal to 0 I could, and this equal to c right, so for i equal to 0, I will get this as 21, j minus 2 minus 1, j as c times 2 delta x.

So from this I can obtain u_i minus 1j for all j values and I can substitute that here. So that is one way of doing it and I can do this for all boundary points. Okay so now that is the derivative boundary condition you see that we can either use boundary points themselves as being specified. So it use it or you could use boundary conditions given by, so even if you want to solve this equation you got to either get the boundary points themselves boundary points specified as boundary conditions then we solve this equation only for a set of interior points or else you have the derivatives specified at that case.

Okay so if the boundary values are specified then you solve only for the interior points and you could use relaxation methods or you could use matrix methods to solve this equation to solve for the temperature inside, now if the boundary is not specified and the boundary values of u are not specified but its derivatives are specified and then you could use this equation the difference equation for the derivative to obtain these fictitious points which are outside this grid and then substitute them back into this equation.

You could have even mixed boundaries that is you have derivatives specified at one point but the value is specified at other points and then you would have to use a mixture of these two and then you may have to solve for the fictitious points, the boundary values here using fictitious points beyond this and but only interior points on this side etcetera.
So that is the simple method of solving an elliptic equation this particular example of the Laplace’s equation for using these relaxations methods.

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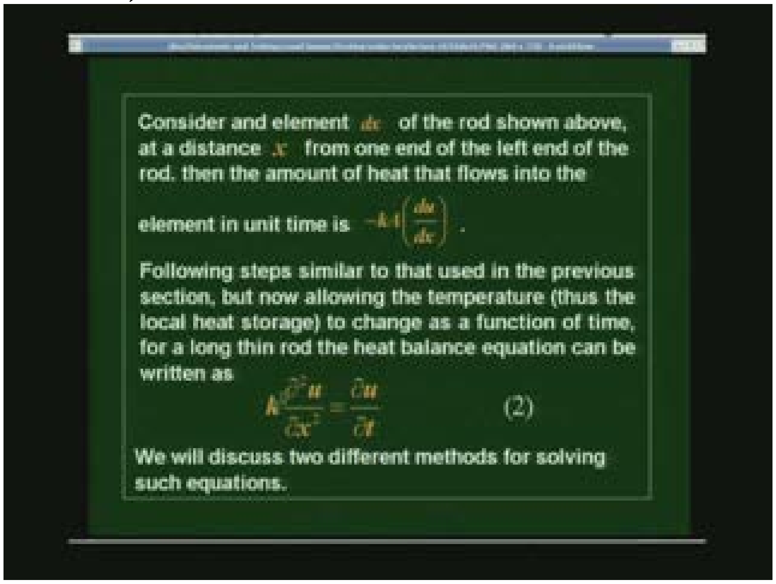

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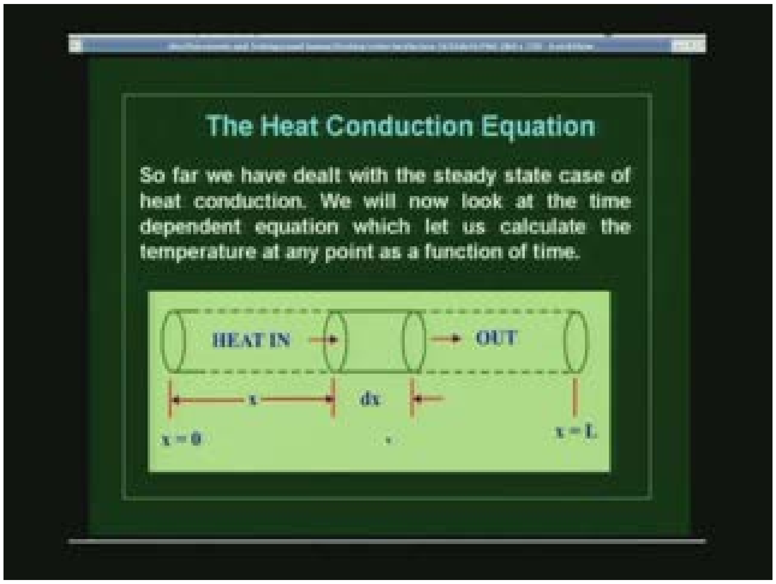

Okay so now next what you will be looking at is the heat equation that as an example for parabolic differential equation. So you will be the next class you would be looking at equations of this form that is when you want to look at what is the heat flow across a one dimensional wire. So you would have to solve equations of this form and then how do we solve that equations is what we would be discussing in the next class.