Numerical Methods and Programming

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Lecture - 31

Solving Ordinary Differential Equations: Eulers Scheme

In the last few classes we have been talking about finding the integral of a function a numerical, so today we will discuss finding the solution of a differential equation. So what you mean by that is, if you have an equation of the form that is dy the simple equation of this form the dy by dx is equal to f of x and you want to find out the solution to this what I mean by that is to find out what is Y as a function of x. Okay that is what you want to show.

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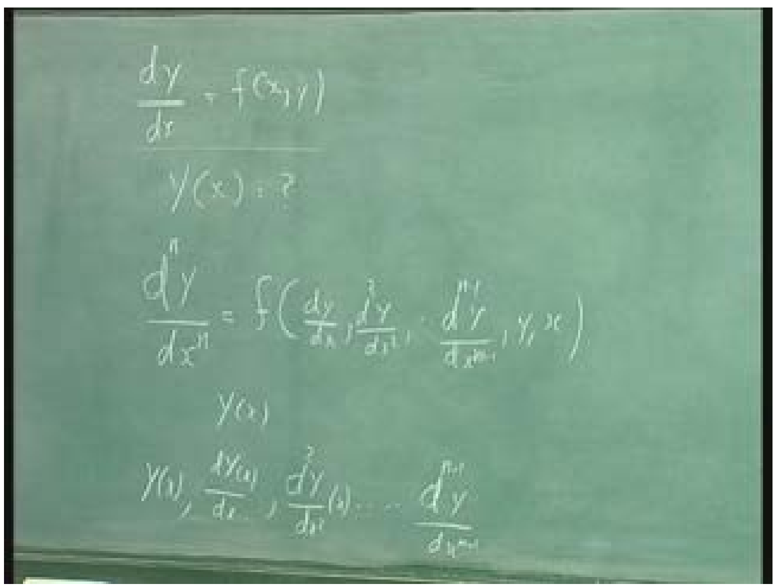

Okay now this is a simple ordinary differential equation of order 1. So in general we could have our differential equation of this form d to the power n that is d x^n there is the nth or the differential, the nth order derivative equal to some function of the derivative of order n minus 1 to, that is dy dx d squared y by dx square is the function of all of these and d^n minus 1 y by dx^n minus 1 y_x. Okay see in general it could be a function of all of these quantities now we want to solve the equation which means that we want to find out what is y of x and what is y dy by dx as a function of x and d squared y by dx squared as a function of x etcetera, all the way up to d^n minus 1 by dx power n minus 1. So that as a function of x. So all find all this quantities and also y of x and all it derivatives. So if you solve this equation what do you meant by solution would be that to find all this quantities.

So that is what we meant by s solving a differential equation and there are many ways doing this now what we are interested here in here is to find the, use some numerical methods of solving such differential equation, such ordinary differential equations. So ordinary in the sense these are the functions of one variable this is only derivative with respect to x ordinary differential equation as a postscript partial differential equation and the right hand side of the function could be linear or non-linear. So methods are generate for this, so it could be linear function or it could be non-linear function it could be a function of x and y are x squared y squared or for any power of x or any non linear function of x and y from the right hand side. So it could be a function of x and y.

So start with, we will look at a simple equations of simple equation of this form that is first order differential equation. So the power on this is tells the order of the differential equation, this is a first order differential equation and we will start with that the first or differential equation then we will see that all first higher order differential equation can be in can be written as a series of first order differential equation. So numerically it is enough to know a method to solve first order ordinary differential equation and then we can generalize that higher order higher order equations. So that is what we you would be looking at.

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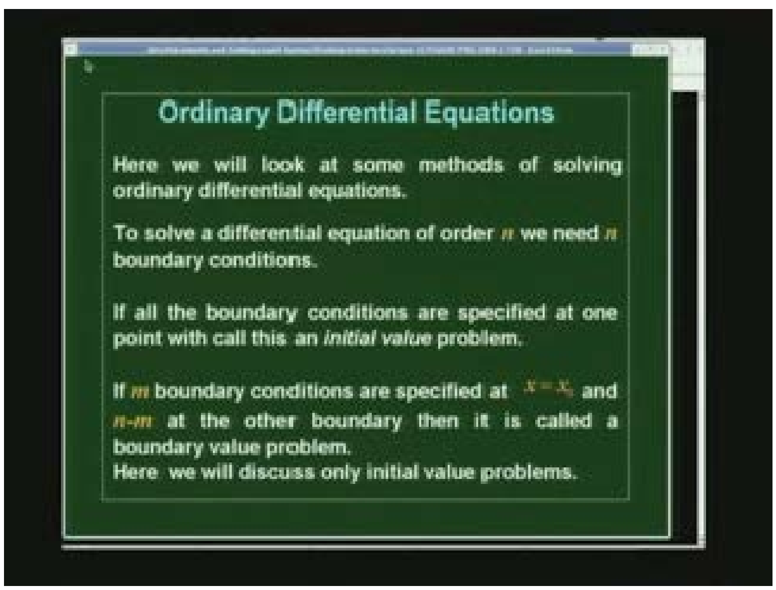

Okay so it summarize this is the, this is what we want to do that we look at some methods solving ordinary differential equations. So todays lecture, we look at a few methods how to solve ordinary differential equations. So now as I said that we will looking at first order equation to start with, so now how do we solve this to solve that we need boundary conditions to be specified is now that is another criteria. So if you have a differential equation of this form then you would say it is a nth order differential equation. So now if we solve the nth order differential equation we need n boundary conditions. So boundary conditions are the value of the function y and derivative specified at the boundary, so we need that to solve this equation. Okay if you want to solve an nth order differential equation we need n boundary condition. So here for example, we need 1 boundary condition, so you could have we want to go in this solution has to be written from starting, some starting x value from some ending x value.

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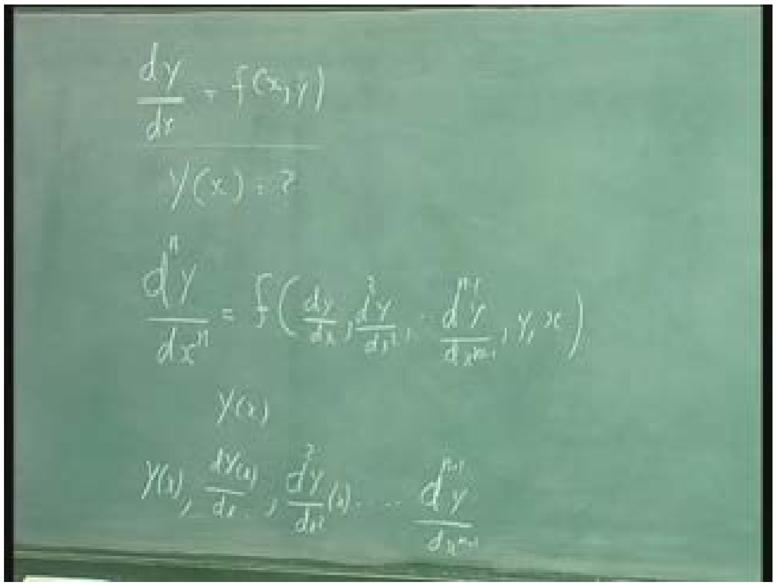

So x_0 to x_1 okay this what would be solving this equations to, okay so you have to solve all this equation first from starting from x value from some end x value and now if some of the conditions specified, boundary conditions of the values of these has been specified, some conditions can be specified at the beginning and some special conditions can be specified at the end that is for example you want to know what is y of x it starting from x is equal to x_0 to x is equal to x_1 let say and then some of the boundary conditions can be specified at x_0 and some can be at x_1.

So in general when you solve an nth order equation of this type and if all the boundary conditions that is n boundary conditions as specified at x is equal to x_0 that is the beginning value starting value then we call that an initial value problem and if some of them are specified here and some of them was specified at the other end then we would call that a boundary value problem. So there are two different kinds of a differential equation, ordinary differential equations then the methods if we adopt for this as slightly different. So you could have a repeat all the boundary conditions specified at x is equal to x_0 and then it is a initial value problem and if some of them specified here and some has specified at the other end then it is called a boundary value problem. So what we are looking at this few lectures would be a boundary value, so initial value problem that is all the boundary conditions as specified here at the at the beginning that is a initial value problem that is what we will be looking at.

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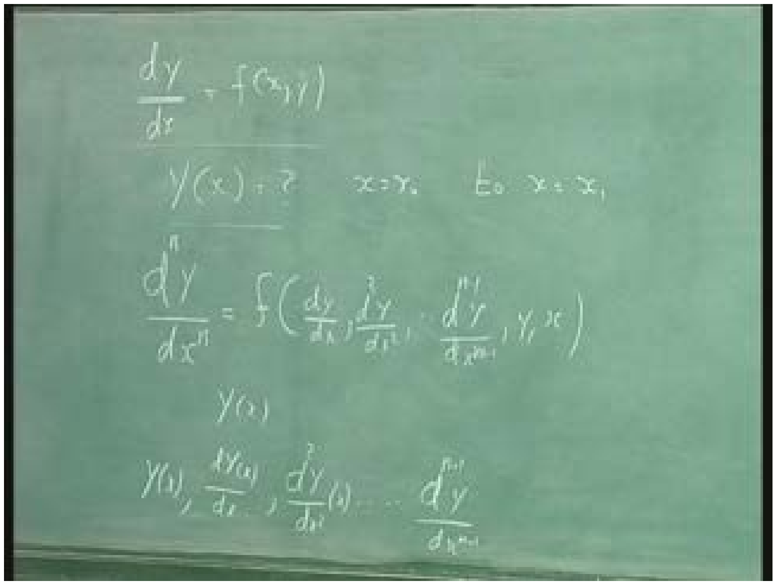

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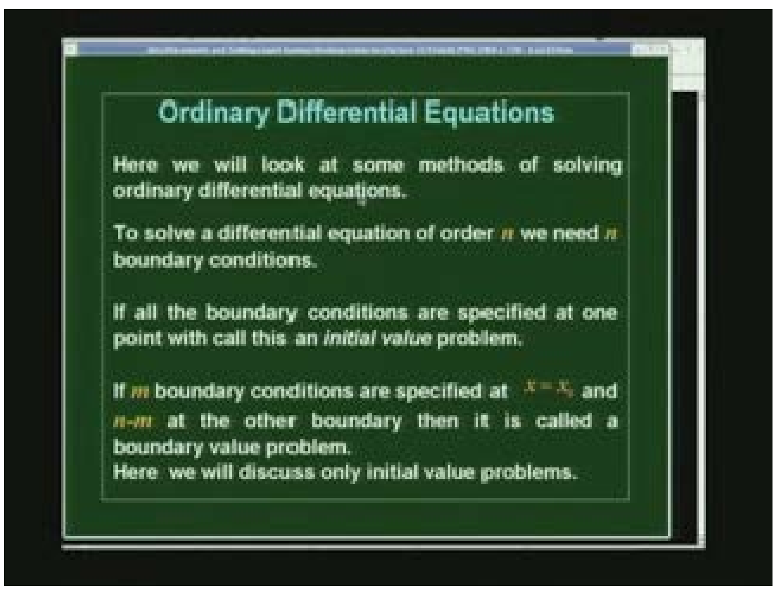

So it is summarize here, so we need to solve a differential equation of order n, we need n boundary conditions. So now if all the boundary conditions are specified at the one point that is a starting value of course not necessary that the starting value has to be the higher value of x then the however it can be anywhere but one for ever we start the integration and that point if all the values are specified then it is called an initial value problem, if the if m of the boundary condition are specified at the starting value x is equal to x_0 and the rest of them that n minus m where n is the order of the equation for the other boundary then it is called a boundary value problem. So I guess this is clear.

So what will be looking at in this few lectures would be solving equations of this form that is general, first general nth order ordinary differential equations, so linear and non-linear with all the boundary condition specified at x equal to x_0. So we will have the boundary conditions in this particular case the boundary case would be when you want to solve these equations the boundary conditions are. So we call them as BC conditions should be y at x_0 right and y prime at x_0. So what do you mean by y prime is the first derivative of y and y double prime at x_0 all the way up to y_n minus 1 prime x_0.

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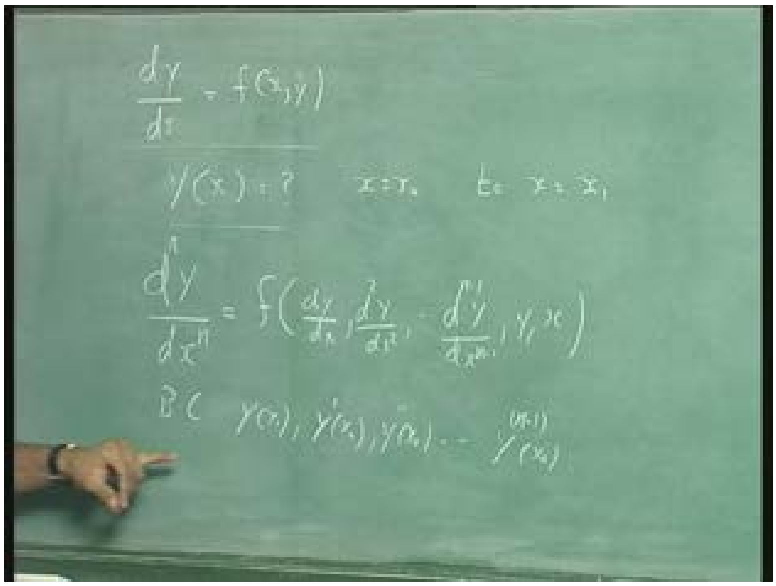

So this has to be specified to solve this equation as an initial value problem. So we starting from x_0 to x_1, so all the values of that function and its derivatives as specified at x equal to x_0. So we need that that is a that boundary condition is required to, so there are n boundary conditions here. Okay now this is a notation which is going to adopt that is prime means derivative with respect to x and for higher order derivative, we put them in bracket if it is if a power is put in bracket that means the derivatives that many times. So this is n minus 1 derivative of the function y evaluated at x equal to x_0. So that is a notation if you going to follow. Okay so in the in the past few lectures we had seen very simple ways, some ways of integrating the function.

So we can use something similar, okay to find the solutions of a differential equations at to so we will start with a very simple scheme to solve an equation of this type, you just check to first order equation. So for the time being we are going to take the first order equation and then we will come to the higher order equations later. So we are going to solve the equations of this form y of x dy by dx is equal to f of x_0 using some simple scheme for solving this equations to start with and we will go to little more sophisticated schemes in the coming lectures sophisticated but more accurate and faster.

So here what do you want to find is y of x if I all values of x starting from x_0, x_1 and we have given y of x_0 as y_0, okay now that is a boundary condition given to us the boundary condition y of x_0 is equal to y that x_0 is y_0 we need only1 boundary condition because it is a first order equations so how do we solve such equation.

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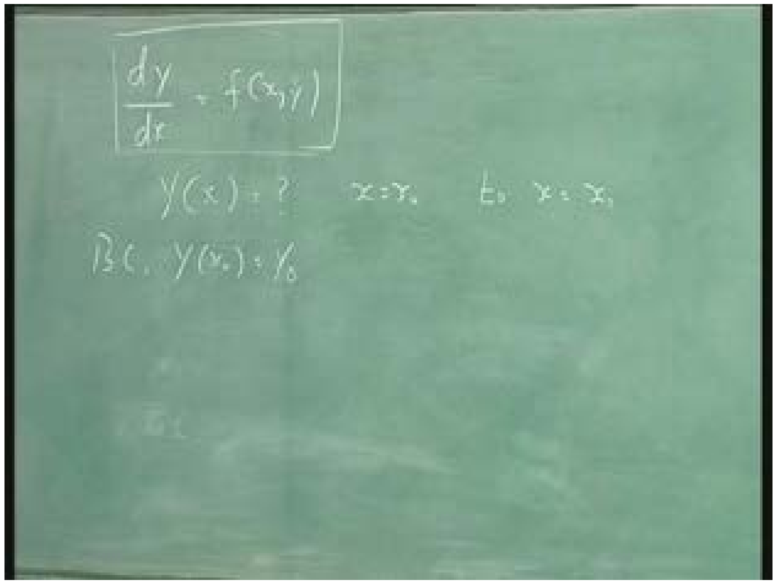

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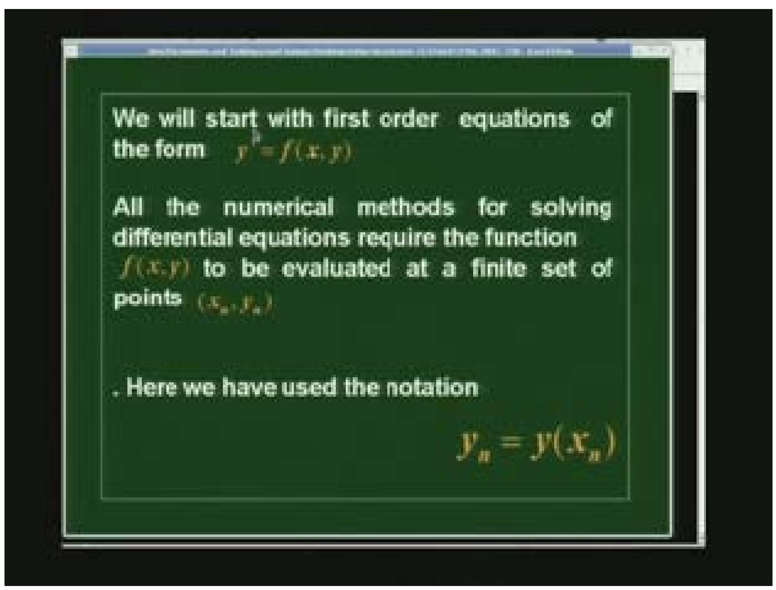

So that is a dy by dx is now written as a y prime and the right hand side is the function of x and y. So we will start an equation of this form, so now we need to desecrates thus function this is at is all numerical methods for solving differential equations require at that that is just like what we did for the case of integration, we need to desecrates these function that is evaluate this function at discrete value of resultant value of x_n and y_n not necessarily equally spaced but some discrete values of x_n and y_n. So what is again the notation please, note the notation as even I say y_n it is y at x of n, x_n.

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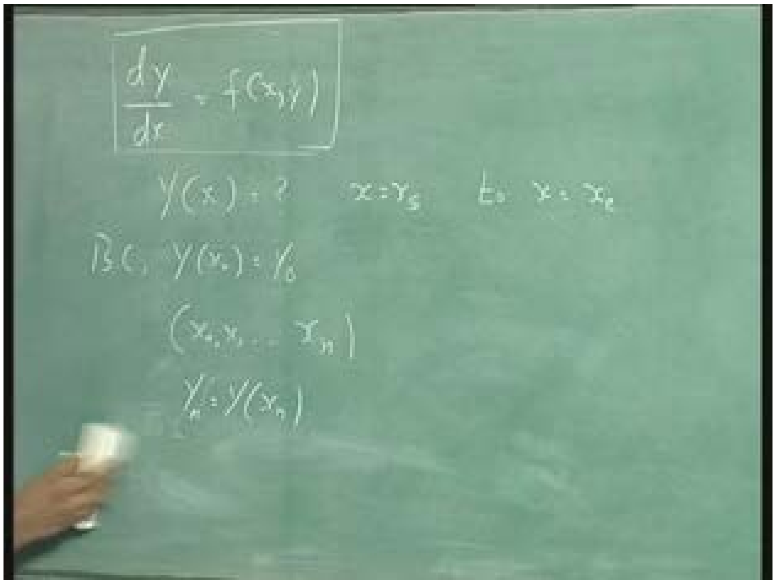

So what you going to do is, you going to split thus interval from x_0 to x_1 as x_i, so we have x values as x_0, x_1 up to x_n this x_n being equal to x what ever the end value is x equal to x_0, x start to x end. So that is being split into n different values when before we call us y_n is y of x_n is evaluated, y evaluate x_n is y of n that so we want to solve and what we know is y_0 is y of x_0 that is what start with and then I can drive write this function and this derivative this equation I can solve as and say that I can solve it like that I can say y as x_0 plus x.

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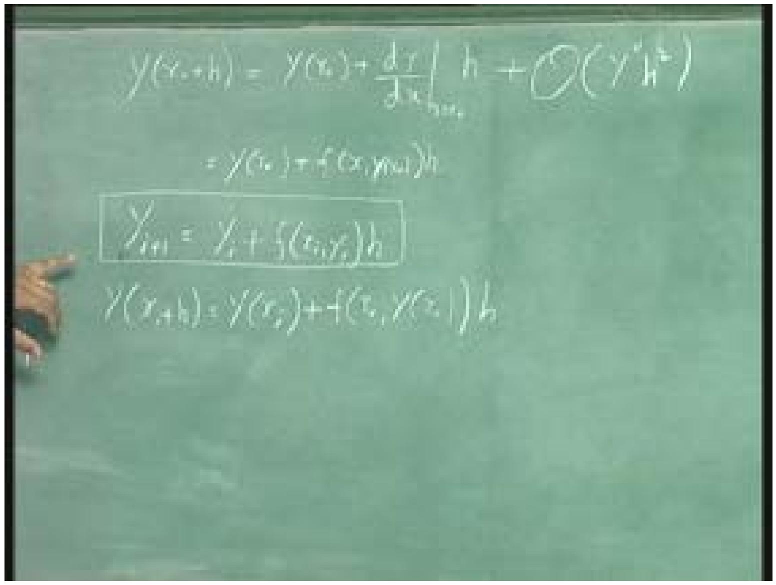

So method is we start with the same y as x_0 which we know and x_0 plus another value h that is equal to y of x_0 plus dy by dx at x_0, the tailors series expansion into h and something of the order d square y by dx square that is y double prime which is left on in to x square. Okay right if something of the order by double prime h square is left off, that is the error in this approximation if you now this looks familiar.

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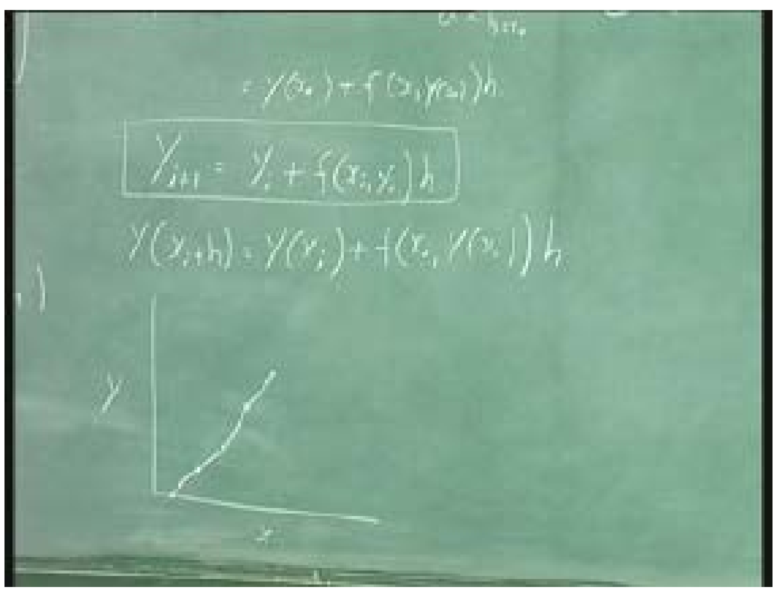

Okay so we now what dy by dx is thus f of xy, so I can write this as y at x_0 plus f of xy f at x_0 that is y_0 into h. Now h is the interval between distance between x_0 and x_1, so in general I can write this as y_i plus 1 is equal to y_i plus f at x_i y_i into h, I hope that is clear that we have, we can take any point and then tailor expander on that point and cutoff the series at the first derivative and then I have the error of the order of h square h being the distance between x_0 and x and then next sound which are taken desecrates here and then I can say that y at x_0 y of x_0 plus h is y of x_0 plus y of x_1 and y evaluated at x_0, y of x_0 into h is the function of that because dy by dx is f.

Okay in general I can do this expansion around any point. So if I know the x_i value why it is x_i and then I can find y at x_i plus 1 which is y x_i plus h which is y_i plus 1. So when you write this again. So let us this is same as writing y at x_i plus h equal to y of x_i the notation y at x_i plus f of x_i y at x_i right. Now this is the same as that this is the general formula which are going to use, okay if I know y at i equal to 0 then I can y_0, if I know y_0 and then I can continue and build up the function value the y value we go along. So the idea service start from some point there is a initial value which we given to us and then we built up using the local tangents. So what we are doing is that you have given x versus y which we want to solve also and we have given the y value at x equal to x_0 some y value given let take it as the and then we can given the local slope that is is the derivative and then we have use that and then use we know the slope so we go there.

Okay using the slope and then we know the slope at that point and so we keep going in a in some using some interval. So that is the general scheme, so we can we can do this this way you can find out the points of all the function for y of x we can so that let me summaries that again here.

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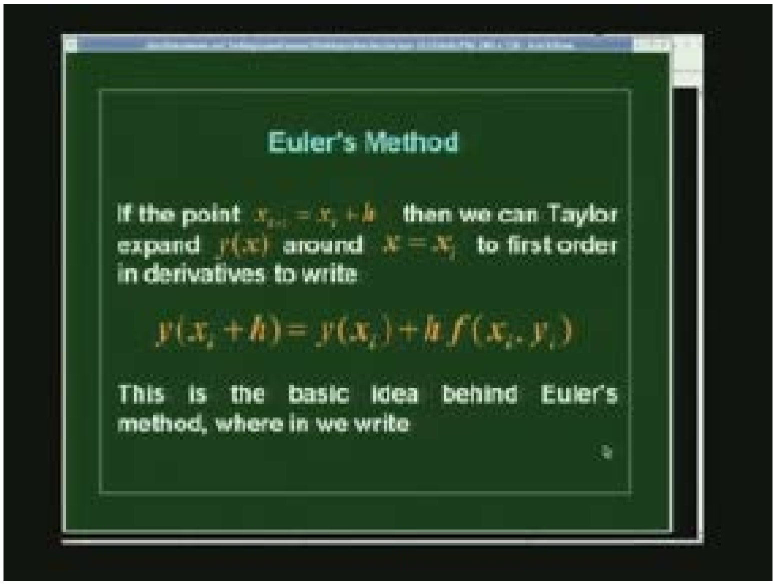

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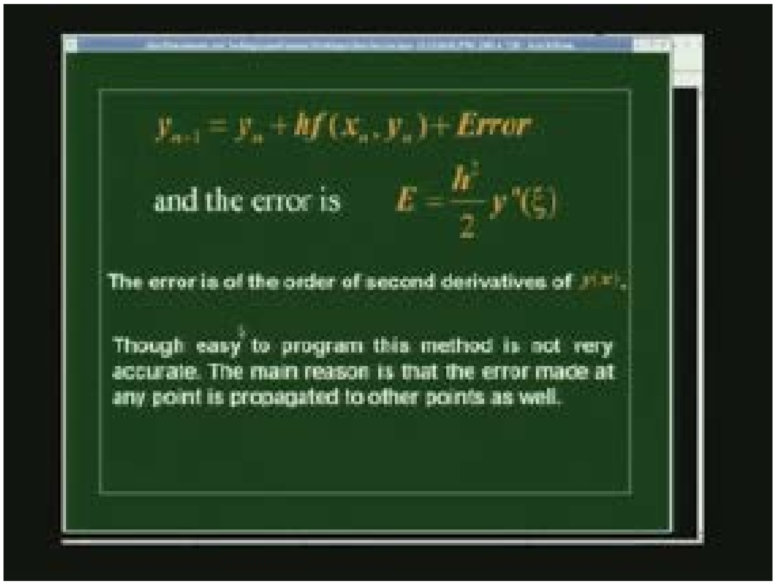

So that is called as the Eulers method is now what we describe this called Eulers method the simplest way to solve a ordinary differential equation of order 1. So that is a following that a we write a point notation x_i plus 1 equal to x_i plus h and we will taylor expand y of x around that point and we write as y of x_i plus h as y of x_i plus h f of x_i y_i now once we have that and then you start with y of x_0 and then we keep building the function one after the other that different points in a serial manner.

So we will write it as y_n plus 1 is y_n plus h, f of x_n y_n plus some error term which is the order h square y double prime evaluated using some mean value theorem. Okay so now this is as you can see is extremely easy to program but not quite accurate and we would introduced extremely small interval of h to actually get this function value you want to use very small interval of h to get the accurate results for using this particular method. So we will just see thus in a in a program it is a very easy to program, so you will as well program this so we call Euler method.

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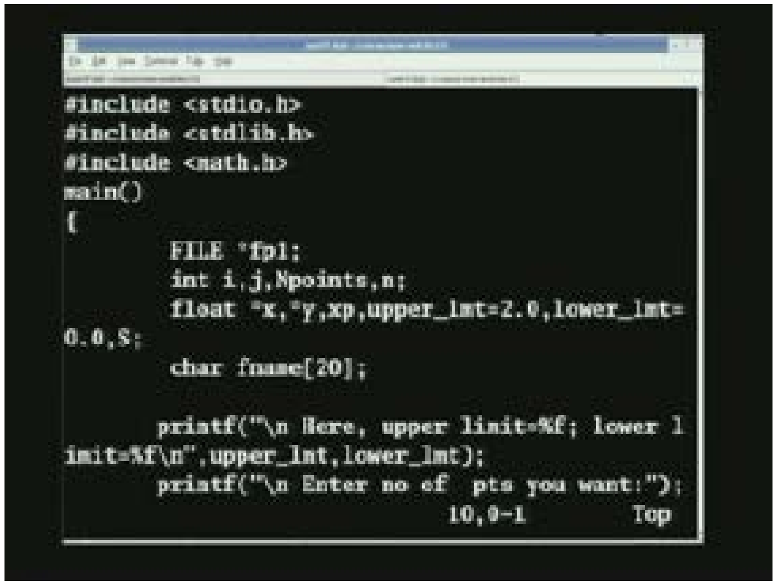

So the program is similar to the integration program which we had we shall seen in earlier lectures that is we need a upper limit and a lower limit and we need to tabulate the function to the x values, okay in our idea is to obtain to the y values. Okay so that is what the summary we need to, we have the x values stipulated between the limits x_0 that is the upper limit and the lower limit and the y values we need to get the y values x and y are pointers here, okay so now we first give some, locate some memory to that points x and y and then we can put in the x value, so we put in the x values here.

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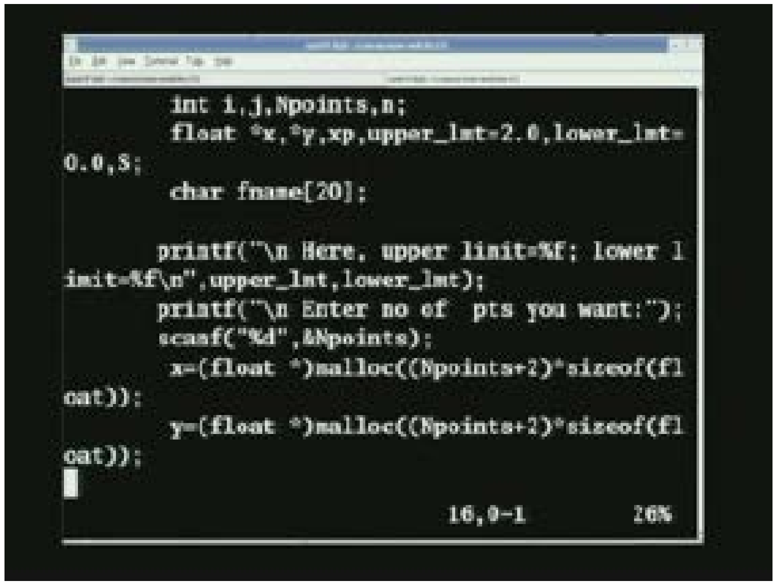

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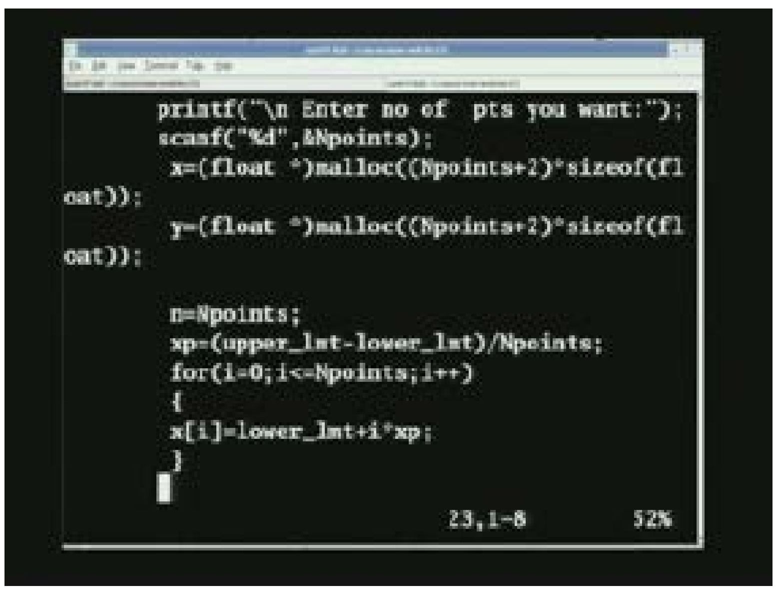

So this is now the lower limit at equal interval. Okay this is exactly same as what we did for integration in the last class. So we have the x of i given by lower limit plus i into xp being the interval are called h and the and the discussion just now this is the tabulated values of x we obtain, so now we have a boundary conditions. So now trying to solve what are we trying to solve, we trying to solve an equation of this form.

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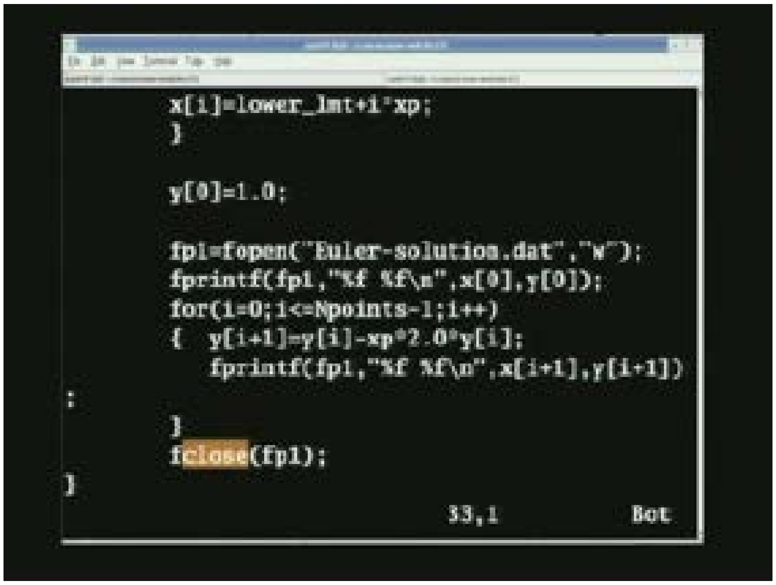

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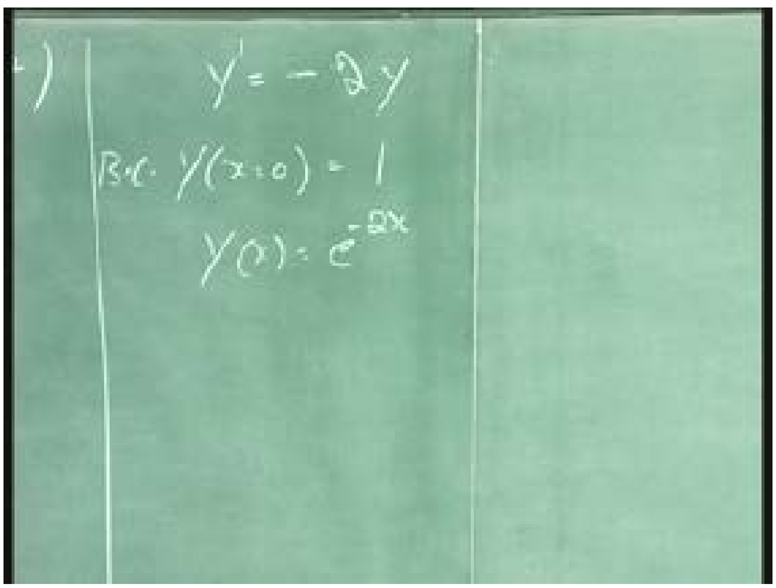

So well and safe, you want to solve, so we trying to solve y prime is equal to minus 2y, okay that is what we are going to solve, dy by dx is equal to minus 2 y that is a equation that we are going to solve. Okay with the boundary condition that y of 0 y at x is equal to 0 that is y at x is equal to 0 is equal to 1. Okay that is all boundary condition, so we know what is the solution to this analytically, so we can compare solution we know that y of x is equal to exponential minus 2x right. So that will be the solution to this equation we know that, so which satisfy the boundary condition that why at x is equal to 0 is 1. So now we can solve this numerically that is what we then compare this results.

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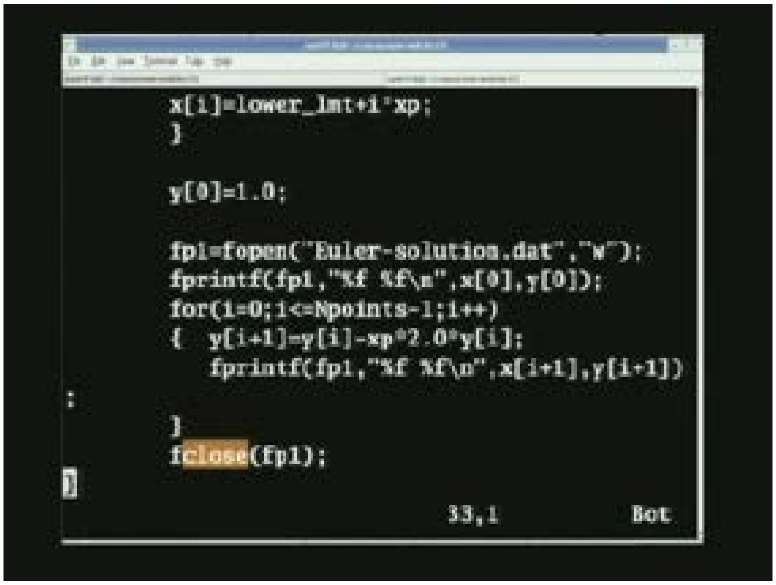

So we are using Eulers method remember, so my boundary condition is y of 0 is 1 and then I go from 0 to n point minus 1 that is a interval, that is a n point is the number if intervals of going to give which is read off here.

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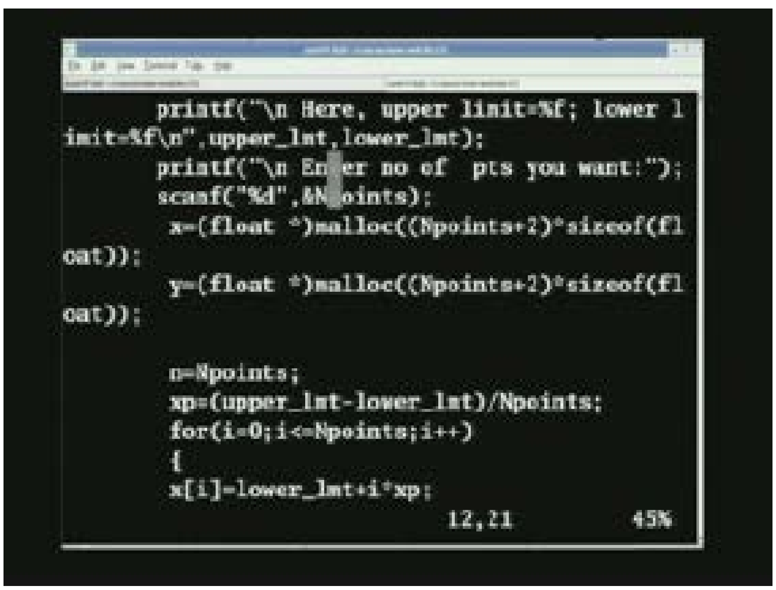

So it would read of here, last few how many points you want to put in between the upper limit and lower limit and then it reads that of in the screen and I will locate at memory according to how many points, I wanted x and y and then I divided by interval 2n points by using this. So xp is usually upper limit by the lower limit divided by n point that is the interval and then I look of tabulated all the x value and then I go here and say that y plus i plus 1 is as the equal interval okay y at i plus 1 is y_i plus the derivative.

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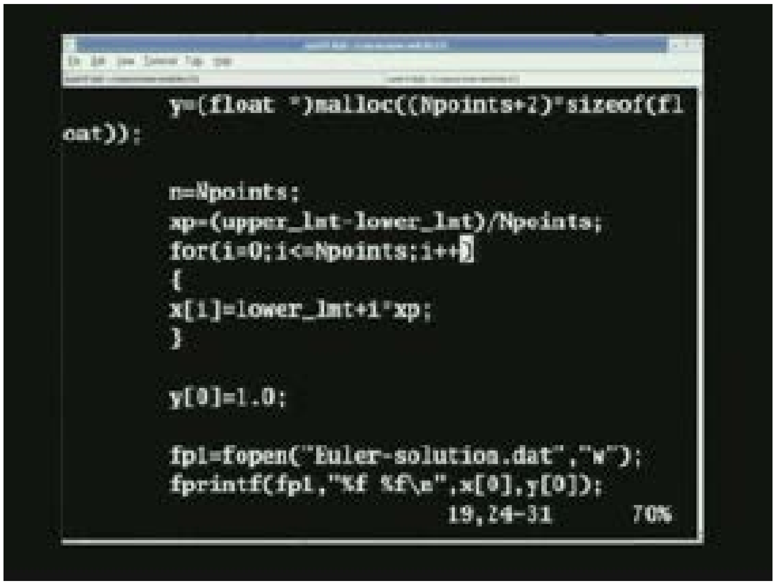

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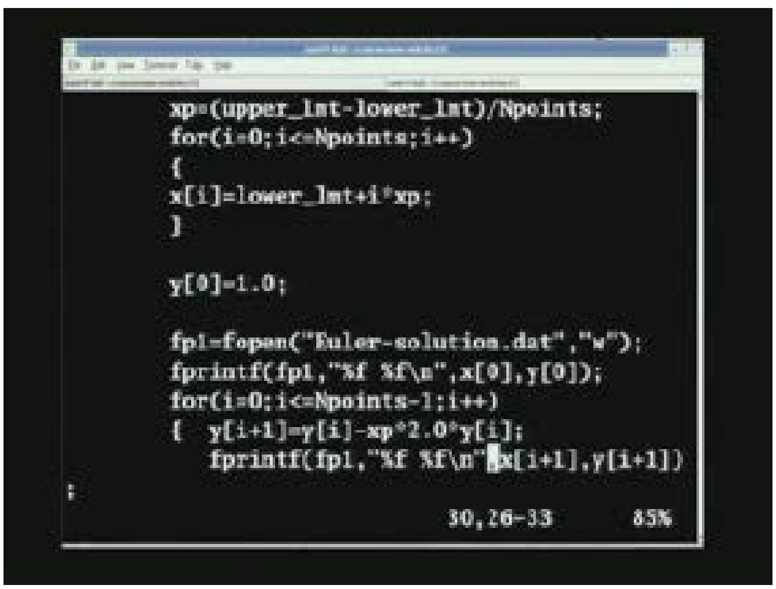

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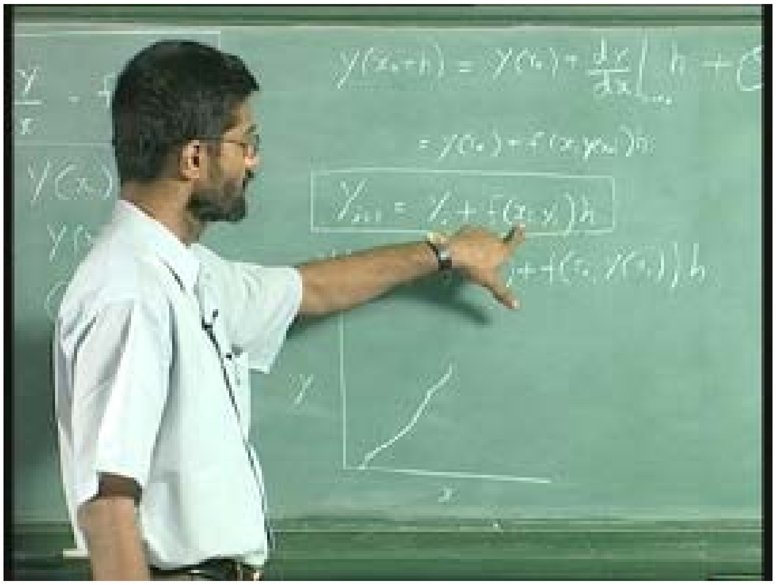

So the derivative the function f is y_i, so what we have we are going to use as the formula y at i plus 1 is y_i plus f into h. So f is minus 2 y is the y_i minus 2 into h is what we are going to use that is what here and just we print out this then we know the actual solution if just run this and then see what will be get.

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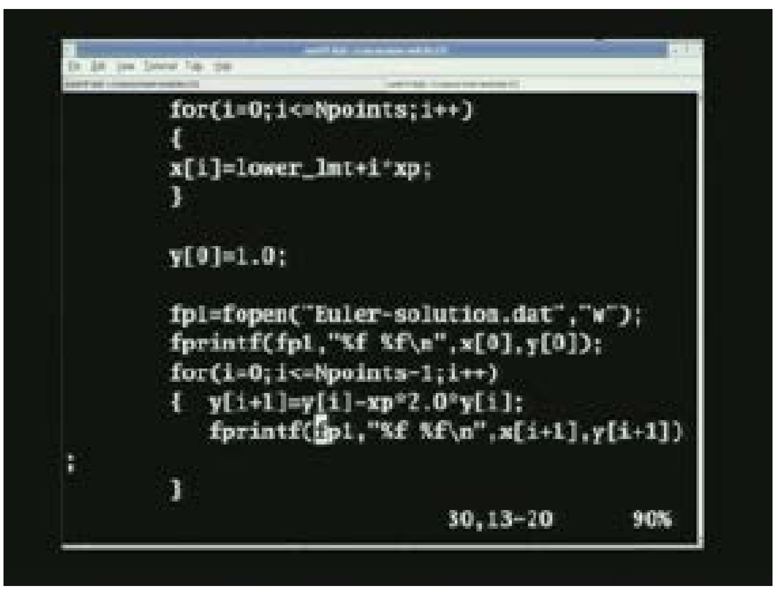

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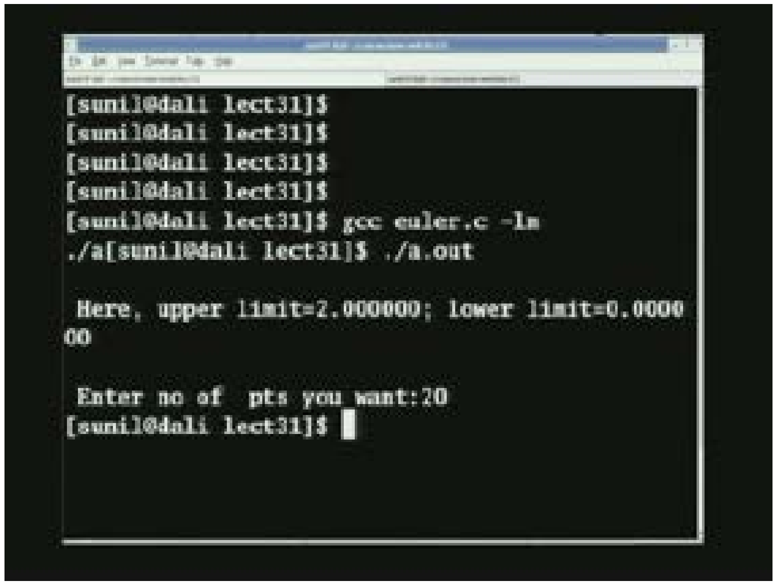

So this is Euler method, so we run this program and so we just run it for let say about 20 points in between 0 and 2 that is a each interval being “.1” h being “.1” so then we will plot that thing if you the program we writing the result into a file called Eulers solution dot dat that what I am writing the x_i and y_i's again writing that into this this file. So we can plot that value.

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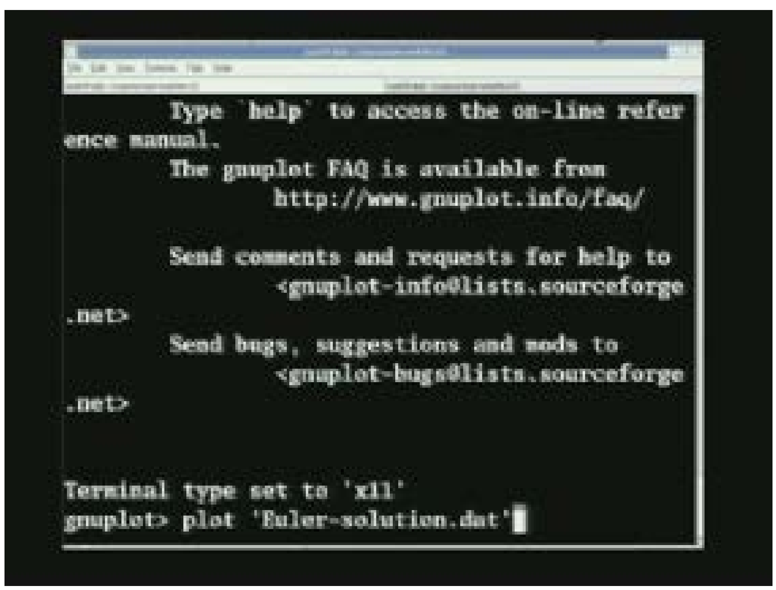

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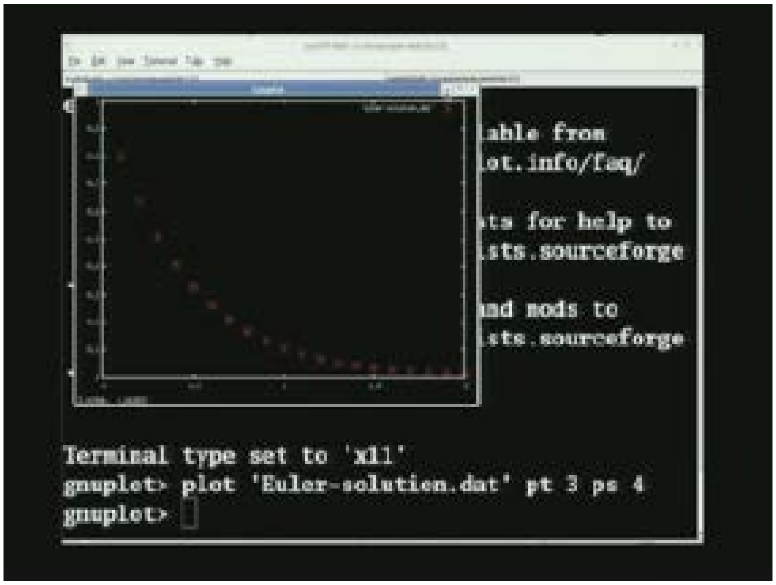

So with, we plot this this Euler solution dot dat. Okay here is my plot of the Eulers solutions, we will compare that with the actual solution that is we know that it is exponential of minus “2.*” x that is a actual solution we know that because we will just compare that with this is what we will get.

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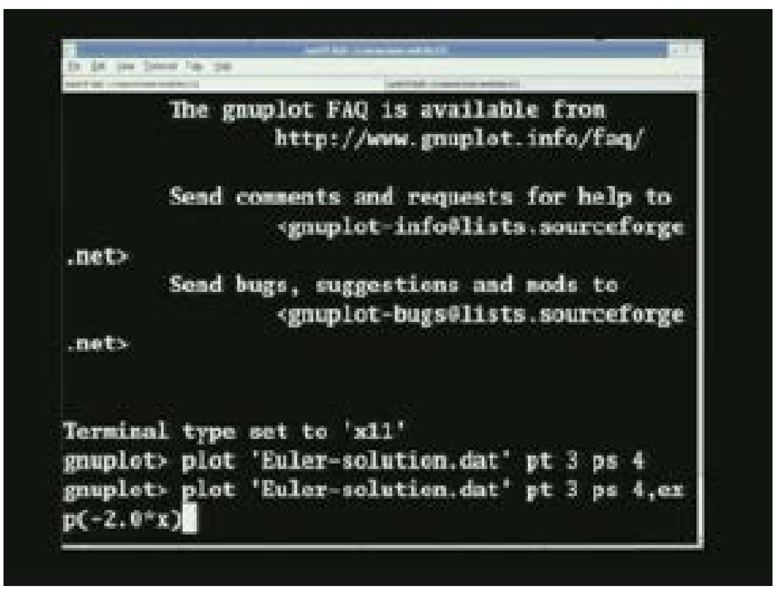

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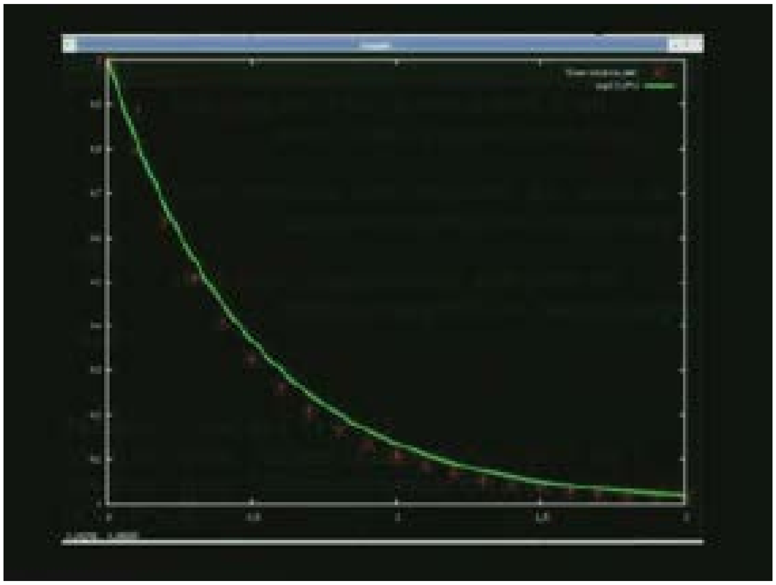

Okay so, we can see the solution this this obstetric which we see here. Okay is the points which we obtain using this Eulers key and the green line which we see here is the one which we obtain the actual solution, the real solution is exponential minus 2x is that we have used 20 point to evaluate this integrant we can see that we have not quite on to the line. We are half here by this line, so let see how many points will require to get to that actual value. So we can run this program again if here now let us run it with some 50 points in between and then go back and plot that think again.

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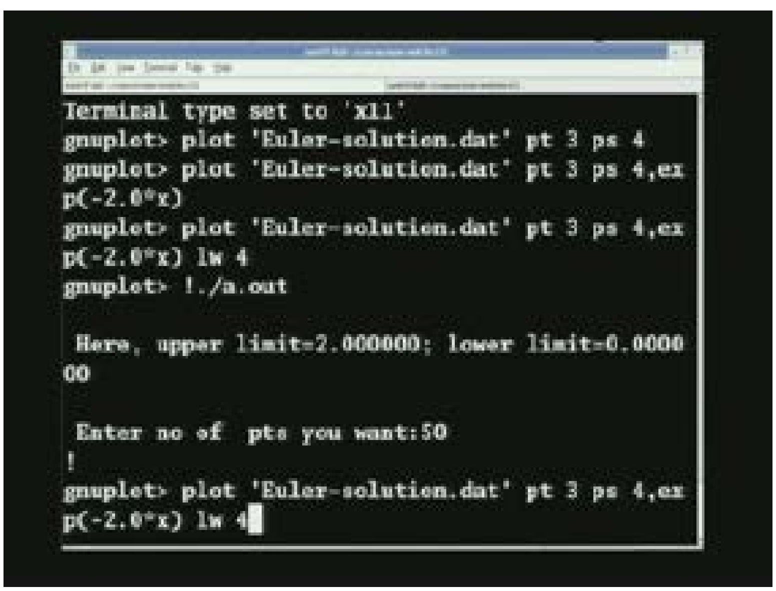

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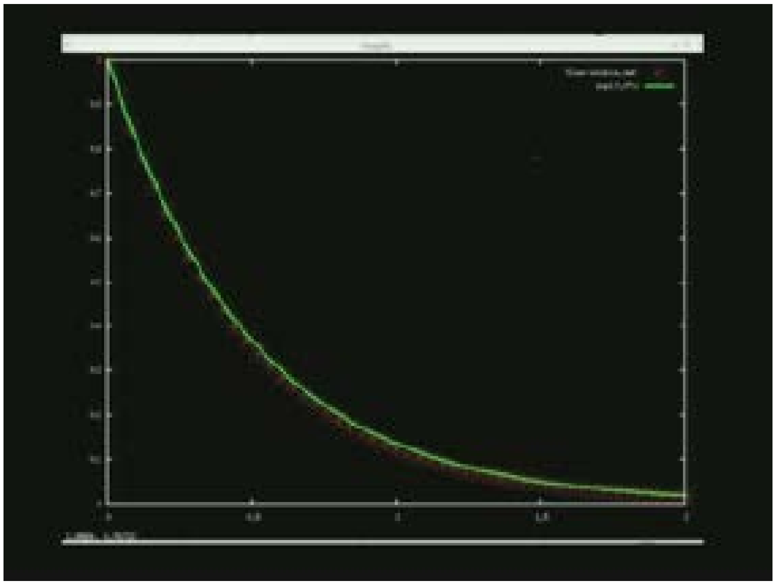

So now we are increasing the number of points. So we are getting better, so we can see thus we increase in number of points, we go more close and close to them actual solution we need to have a large number of point, so we use 50 points within 0 and 2 we get at reasonable accuracy here of course depends on what accuracy you need to get again but I can see that the simple method is not quite accurate we need large number of function evaluation and very small intervals which can lead to other problems like round of errors etcetera in our computation.

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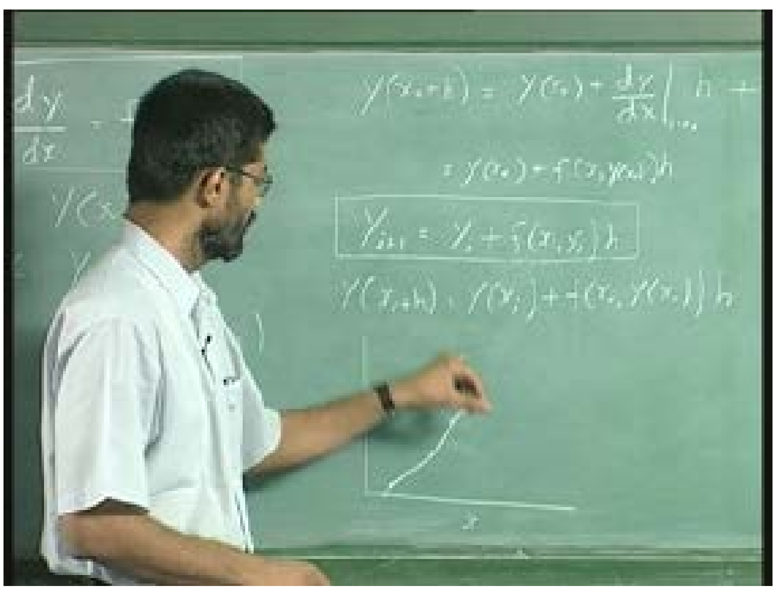

So we are using such part of some other computation, so this large number of function evaluate small intervals could also leads to other numerical errors typing in. So even though extremely simple Euler method is not quite efficient, so we will try to improves this methods slightly, we will try to device the scheme which will improve it around this simple scheme and that is a what we call the predictor, corrector.

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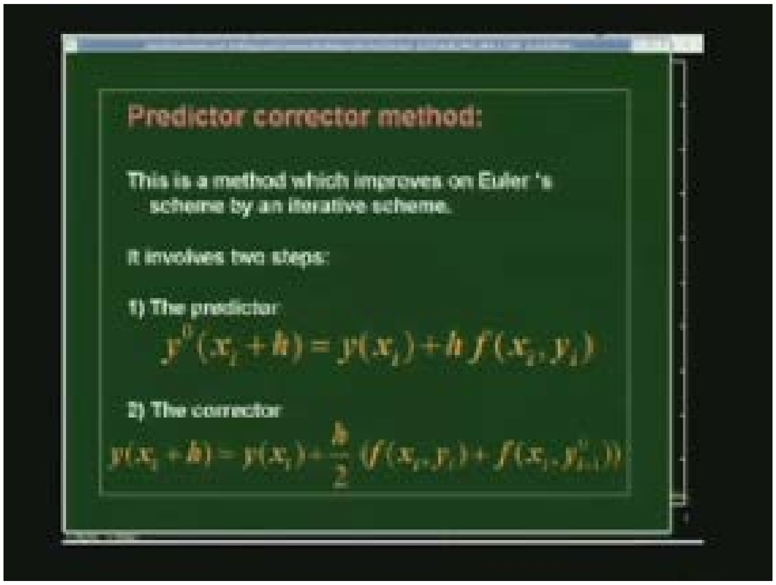

The next method which is called the predictor, corrector method. So this is in different from the Eulers scheme it just using the Eulers scheme but now we use an iterative scheme which involves two steps, the first step you would evaluate just like in the Eulers scheme y at x_i plus h that is y_i plus 1 as y as x_i plus h times f of x_i y_i is that the Eulers scheme right. We call that as the predictor equation, okay that predicts what is the y value at x_i plus h_i so to distinguish it from the true solution you put a simple y_0. Okay so it does not mean that it is a 0 derivative it just y_0 just show that it is the predictor it is now the actual solution. Okay so then the corrector okay the corrector is then what we do is now instead of using just one derivative at the function at x_i y_i, now we will use an average derivative of the function at the 2 at the 1 is the initial value and the other is the predicted value, at the predicted value and then use that the average value as the derivative and get the corrected y of x_i plus h.

So graphically this means the following, so let me just write that now here we are doing the same problem with boundary condition initial condition etcetera given. Okay and then what we are going to do is to write y_i plus 1as I set the predictor which is y_i plus function value f of x_i y_i into h. So function being the this derivative, so now that is like saying that and then I use this quantity is the y_i plus 1 and evaluate the derivative at that point y_i plus 1, okay graphically it something like this. So this is my y and this is my x_i start from somewhere here okay I have my slope there now the slope is let say like this and then that is my distance h, if I go to this point using the slope, along the slope let I start from here and I go to along using the slope along the slope I goat to this point by using the predictor key and then I evaluate thus slope there. So I would now the next step would be after this may be to evaluate y f at x_i plus h comma y_i plus 1c. So I will evaluate at that point.

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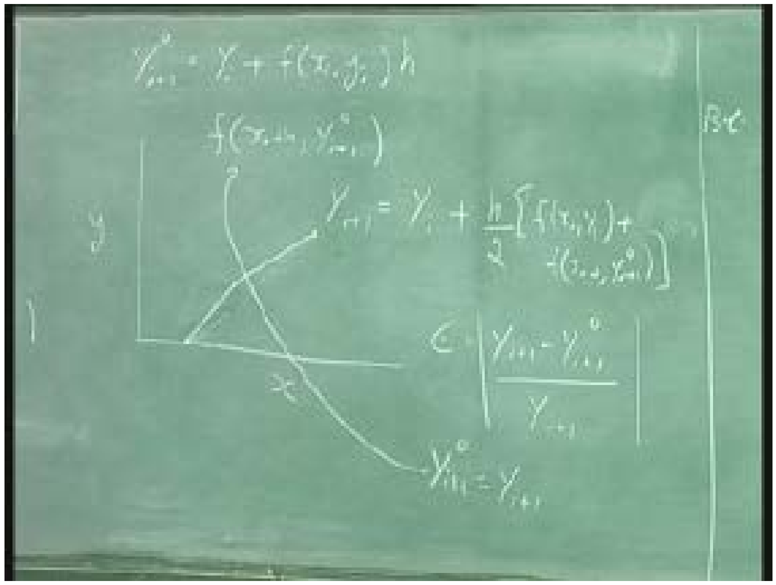

Okay so that is what I should I will do okay and then that is evaluating the slope at this point and then I would takes as let the slope as somewhat like this point slope is like that okay I take this function value the slope here and the slope here, okay and take the average of that slope and the average of that slope okay and then use that to go from here to here. So I will replace this segment here by average slope which I get which will be now like that. So let me draw this little bigger, so I hope this point and then I choose this is slope at that point. So then I would have an average slope which is like that okay then I use that average slope go to the next point.

Okay that is what the scheme that is the corrector scheme is to say that y_i plus 1 actual y_i plus 1 is then equal to y_i plus let me call the two function that is h by 2 now at f value evaluated at x_i y_i and plus f value evaluated at x_i plus 1, f value evaluated at x_i plus 1, y_i plus1 0. So now use the average value to go to that, so now this may not give me the correct answer at so there will be difference between this predicted value and a corrected value, okay then what I do is I will again go back here okay I will now substitute my y_i plus1, 0 as y_i plus 1 obtain from here.

Okay and then evaluate this function f right that is the derivative at x_i plus h, y_i plus 1 and then I use that average derivative to compute y_i plus 1 and now if there is a difference between y_i plus 1 and y_i plus 1, 0 then I replace y_i plus 1, 0 by y_i plus 1 and I go back here again and again compute this again come back here and check is there any difference between y_i plus 1 and y_i plus 1, 0. So I do this as do this iteration again and again till the difference vanishes difference between y_i plus 1 and y_x power I can define an error epsilon as y_i power 1 minus, so eliminate to the next step to be achieved look at the difference that is epsilon as y_i plus 1 minus y_i plus 1, 0 divided by y_i plus 1 module. So now if epsilon is not equal to 0 and then I will go back just saying y_i plus 1 is 0 is y_i plus 1 and then I go back to here to here. So that is what I will starts from here and evaluate y_i plus 1 using this mean derivative and look at the difference and if it does not if the difference is not in this epsilon is not satisfactorily 0 something less than the predetermine value and then I will replace y_i plus 1 to y_i plus 1and go back here and then again do the step till left the epsilon is sufficiently small.

So now this method is called the predictor character method. So we can see this how this would improve the simple Eulers scheme of just doing this one, you can have a simple predictor character method also just do this one and do this again ones again it has another scheme but the iterative scheme would be do this repeatedly till we get y_i plus 1 till the epsilon below some predetermine accuracy. So it is smaller than predetermined for satisfies our accuracy conditions, so we look at that implementation of this number.

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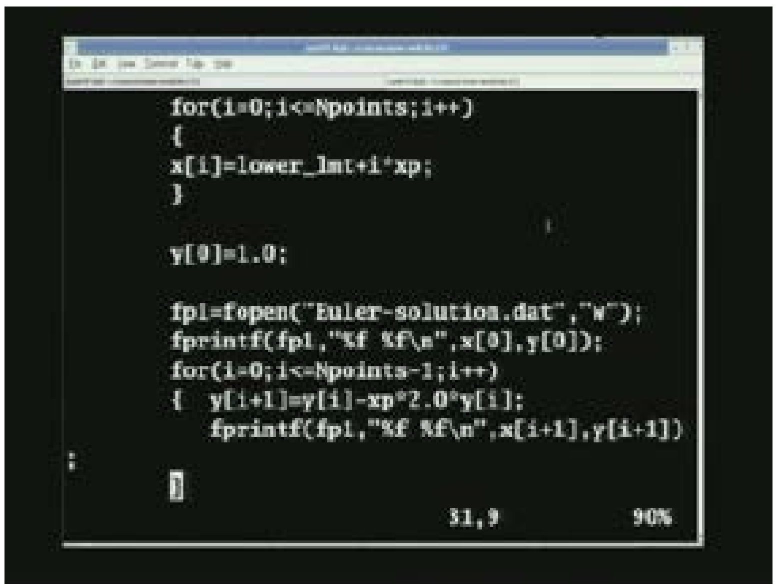

So here is which would which implements it is that is a predictor collector code and then again earlier the part of thing is the same you have the x values y values in the upper limits etcetera specified. Okay now this x notifies this program little more general so I have some subroutine here or function here I call this function. So this subroutine will actually returns to me every time I call this one return with arguments returns to me the function value and derivative value at that point. So I have design some more arrays here which is xx yy dy dx and dy dx1. So that is this is a derivatives this is now I need a define 2 derivatives here because I want to distinguish between the actual derivative at x_i y_i and this is the derivative at x_i plus 1 and y_i plus 1, predicted value. So that is dy dx and dy dx1 the 2 derivatives, okay and this is the value which we are going to evaluate the function xx and yy that is x_i y_i.

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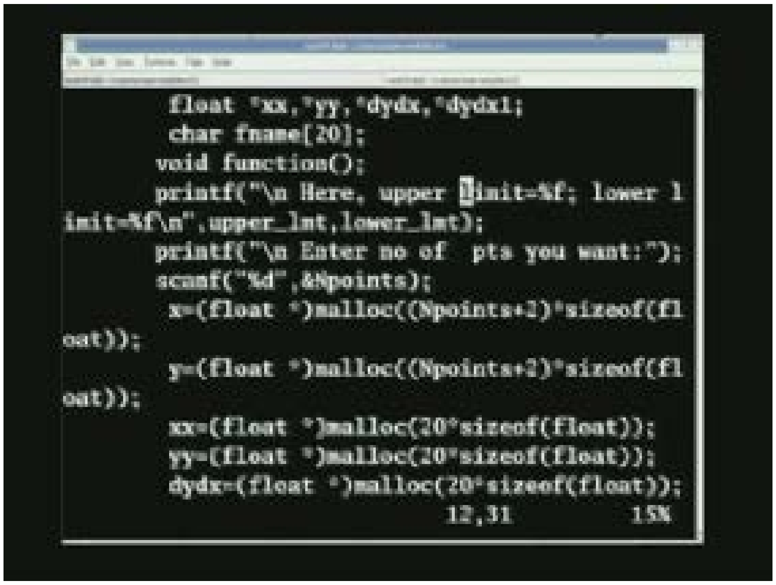

So then we reallocate some memory here using the malloc function for the x and y as before we also allocate memory now to a xx, yy and dy dx dy dx_1 etcetera. Okay so we need that and then we will compute this what we will do is again we have a initial condition we have divided this interval between the upper limit and the lower limits divided by n points. So that is xp is the intervals and equal intervals.

Okay I have tabulated the function though the function values into this value the x values into this array call x_n in equal intervals xp. Okay so now start with the boundary conditions which is y of 0 is 1, so right now we will do that so we will first write this x of 0 y of 0 that is our initial values which is given to us that is written to this file called predictor collector solution of that. Okay now you will write all this solution into solution into this particular file. So now I start with error is equal to 1 to this solution at every point starting from 1, i is equal to 1.

So at every i plus 1 point I need to solve, so if I start the first point put the error is equal to one and I doing the xx and yy at x of i and y of i that is x of 0, y of 0 its the starting value now I evaluate the function here I evaluate the function here. So I will get the derivative at that point, so if I pass to this function the x value and the y value I return the derivatives this is general. So if I want to do this program for a different differential equation right now I am doing it for dy by dx is equal to minus 2 y but I do this for a different differential equation in that case of I would replace this function.

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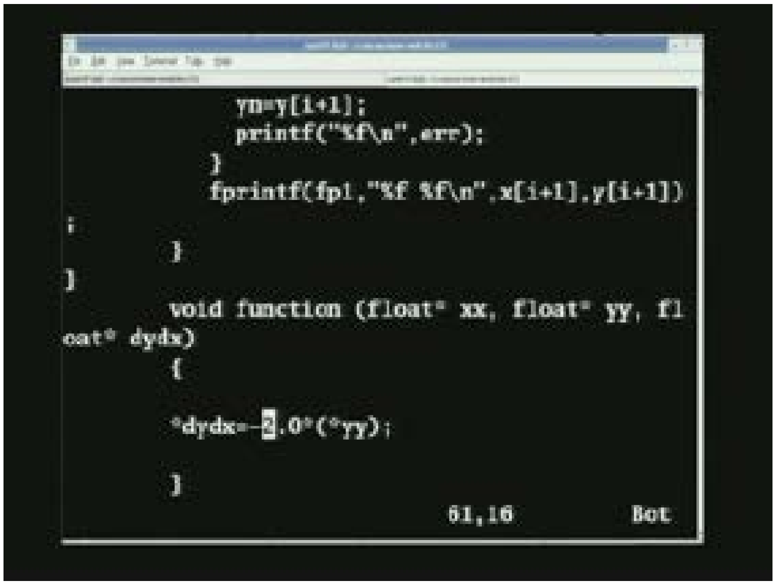

So this function here is simply here is simply returning minus 2 yy minus 2y. So remember this are all of them are pointers. So I return them as that is why as an xy is coming okay this dy by dx is minus 2y that is what it returns. Okay so I can, I will start with that. So that I have a predicator equation here.

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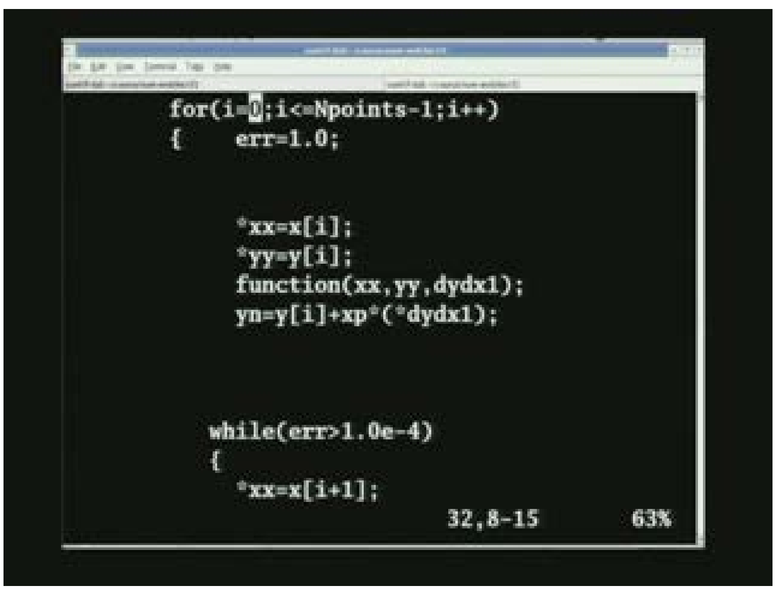

So before here it is starting, so in this loop so this particular i value given that i value, I can impute the next y value at the next point y_n is the y next that y at i plus 1 predicted value that is is y_i plus the interval multiplied by the derivative, if the derivative is written by this function dy dx1. So I have the function value written there correct, okay so then I now go into the corrector loop.

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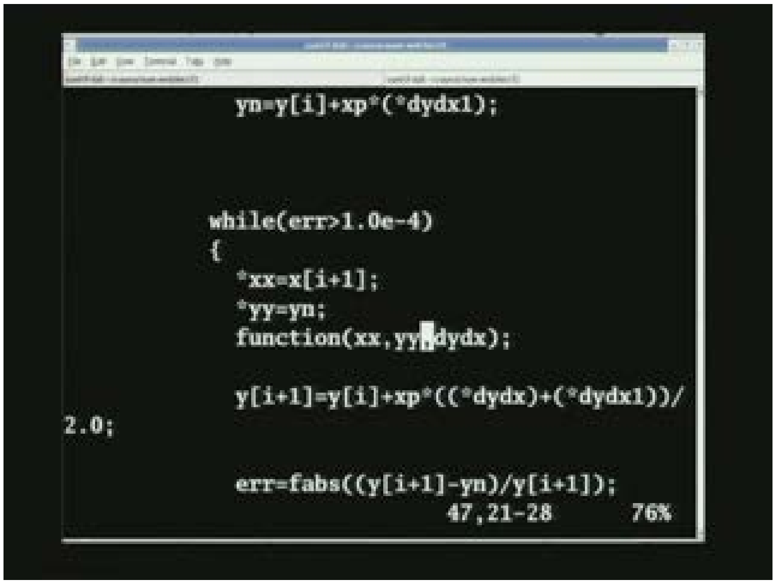

Okay now this is my corrector loop, so this is being predicted okay I then go into the corrector loop under the corrector loop is operates still my error is less than 10 to the power of minus 4. So that is a scheme here, so this is my predictor okay I just predicted the value to the next value as y_n is equal to y_i plus xp time dy dx and then I go into do the corrector here that is xx now is x_i plus 1.

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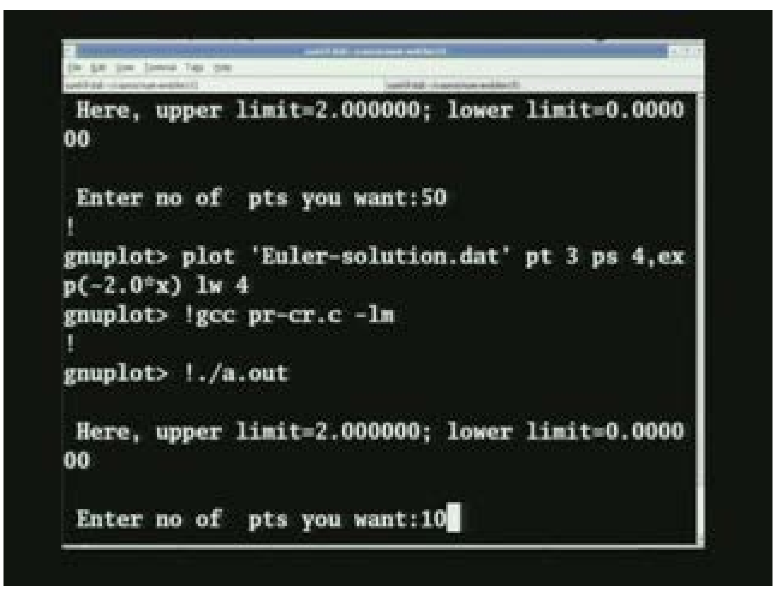

So next value and yy is y_n the next value and then I call the function again I get the derivative here. Right now I got the derivative and then I go to the next function value. So now this is the corrected function value it is y of i plus the average of this 2 derivatives that is why dy by dx plus dy dx1 by 2 remember dy by dx is evaluated at i plus 1 and the predicted y value and this is evaluated at i and y of I, then I compute the error here as y_i plus 1 minus y_i divided by y_i absolute value of that and then put the y next now the next predicted value as this value which you obtain from here is the next predicted value and will continue this loop till thus error is less than 10 power of minus 4. So let us write this program here again so I would run this again as okay now we will run this program we will run it again with ten points let us run it with 10 points. Okay and now that is the now plotting that what you get from the predictor collector alone and we will compare that with value which we get from compare that with actual value.

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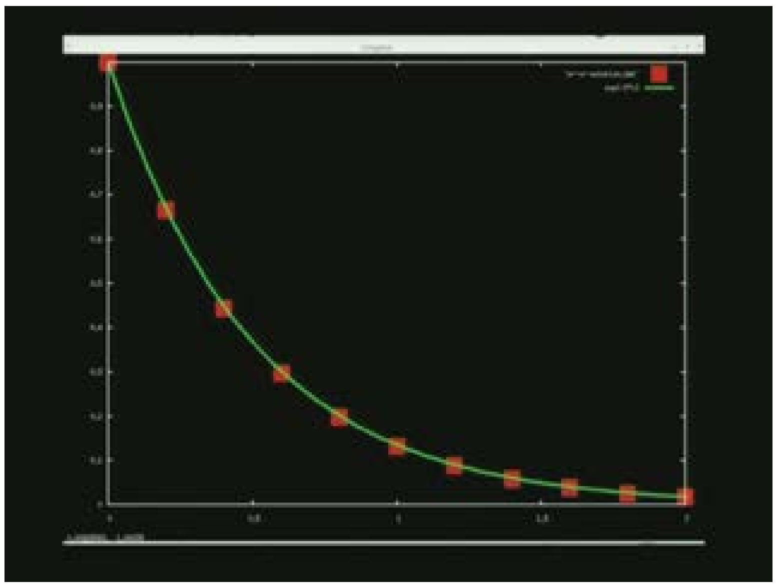

So this the green line here is the value which is obtain from exponential minus 2i that is the real solution and the red squares here are the 1 which obtain from the predictor corrector equations. So we can see that the accuracies but we high let you the 10 points and we can get this accuracy remember in the Eulers scheme a simple Eulers scheme, we need to go to very large values of point that is where to use 50 point to get this kind of accuracy. So the predictor corrector scheme definitely is much better compare to the Eulers scheme but here again we are doing this similar technique that is we are actually evaluating the function value at many times to get the to get the correct solutions even though we have used only 10 points but you evaluate the function point at function value at different many time each of the character loop, this character loop evaluated many times to get the correct results, so it could also have the same type of numeric accuracy problem you too round of errors as that the Eulers scheme but this is much better scheme to that simple Eulers scheme. So we will go now to a slightly more sophisticated way of doing this that is using what is called the Runge Kutta scheme.

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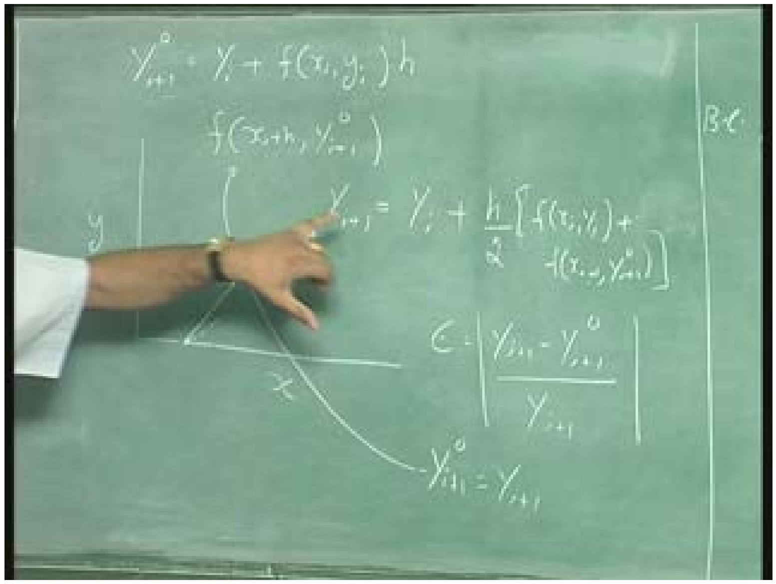

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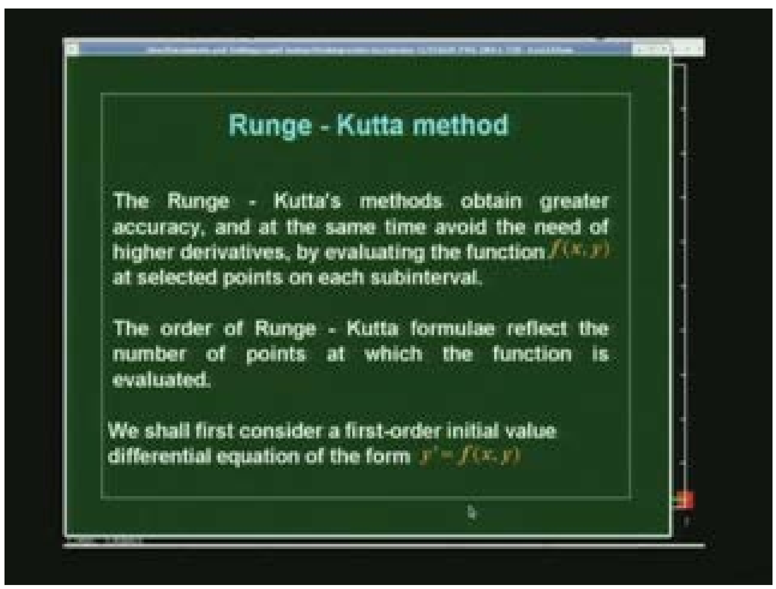

So now this is again a general method for solving first order differential equations, initial value problems and this method is specifically for first order initial value problem and higher order method derivatives can of course be split into the higher order differential equations can be split into first order equation and same methods can be used to solve. So I will just concentrate on the first order equation here so that it definitely provides the much greater accuracy and now the same time we will not do any we will not do any higher derivatives so something.

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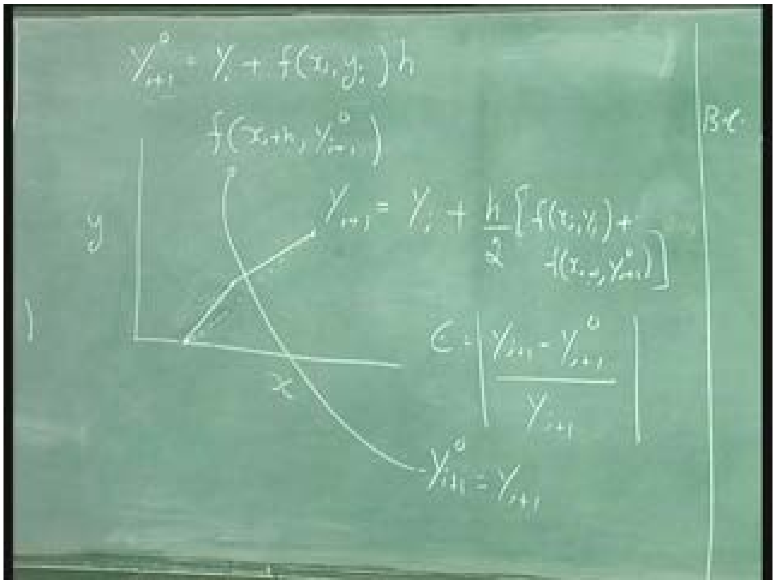

So we just again going to use something similar to this actually work with the first order derivative that is to evaluate the function evaluate i plus 1, I will be using only the first order derivatives of course we can got the point here that is, that we can improve the accuracy of this method by going to higher order derivative for function which is the this may not be available to us. So we may have to compute them numerically etcetera but we will not do that we will just stick to this first order derivative and then try to improve the accuracy of this scheme. So that is 1 of those methods is what this is Runge Kutta method.

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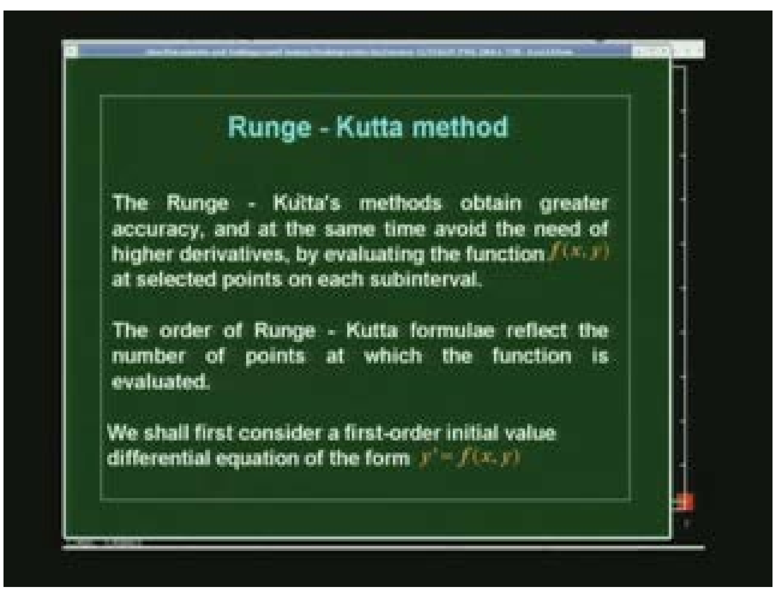

Now there are very there are different orders of Runge Kutta. So we will be first looking at a second order Runge kutta to actually explain the basics idea behind this self method and then but normally all the standard routines which we would see they will be using a fourth order runga kutta, I will just explain what that it means actually in a few minutes.

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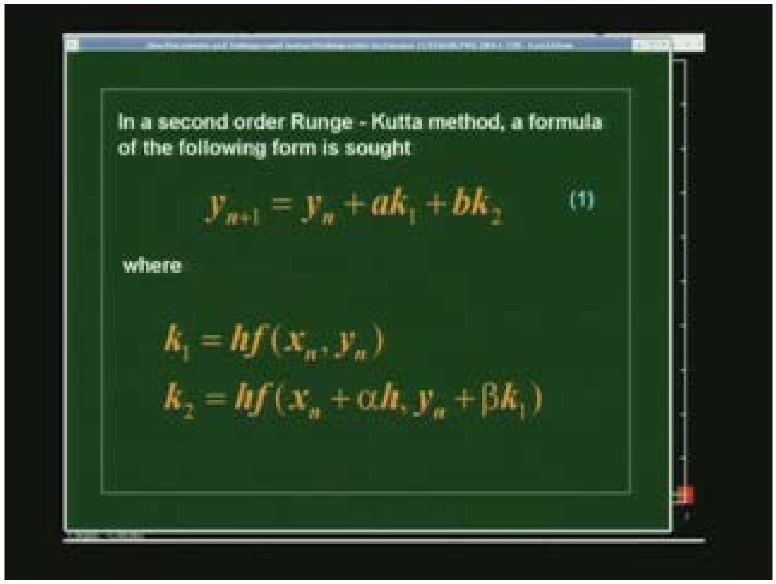

So let us look at a differential equation of this form as we have doing this and then write the solution in this form that is we will write the solution that is y and plus 1 y and plus ak_1 plus bk_2 that is the basic idea of the Runga Kutta.

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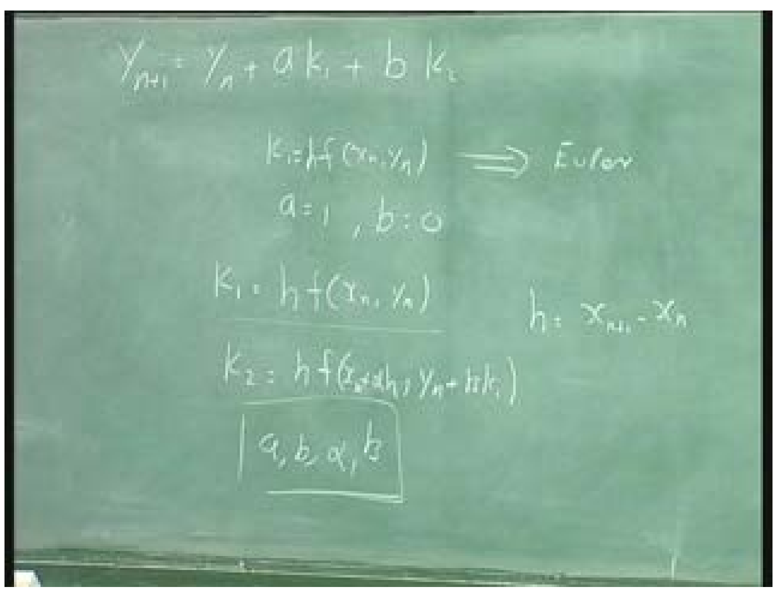

So we want to write the solution that is y x_i y_n plus 1 that is y_n plus 1 is y x_n plus h as y_n plus some coefficient time a function k_1 plus another coefficient times the function k_2, so the case k_2 is equal to 0, k_1 is equal to f of x_n y_n we are back to the Eulers scheme which is right. So that is the Eulers scheme, so that is so this method is a_k you can see is generalization about that k_1 is equal to this a is equal to h, a equal to 1 okay or a equal to 1 and k_1 equal to h times f of x and y introduces the Eulers scheme.

So you want to write this as slightly differently you want to say that this is k_1 is h times f of x and y and so we will we are going to write this in this fashion $k_1$is h times f of x_n y_n and we say k_2 is h times f of x_n, we are going to write that x_n plus h, y_n plus beta times k_1. So that is want to we want to write is a alpha times h and so, x_n times x_n into x_n plus alpha time h and then y_n plus beta time $k_1$that is the scheme we are going to discuss and then our idea would be to determine thus alpha beta and a and b, if you remember in the limit a goes to 1 b goes to 0, a goes to 1 and b goes to 0 that is enough then we have enough to determine k_2. So and then k_1 is equal to h times f all x_n y_n it is Eulers scheme for that is equivalent to say in that alpha is 1, a is 1 and b is 0. So then we have the Eulers scheme and in the case, the general case to the second orders Runge Kutta in the second order Runge kutta.

So we are going to write y_n plus 1 the next value of y as y_n plus ak_1 plus bk_2 with k_1 as a function of this form h is the interval between x_n plus 1 and x_n. So you remember h is x_n plus 1 minus x_n and then the function value at x_n y_n and k_2 is h times function value now evaluated at x_n plus alpha h plus y_n plus beta k_1.

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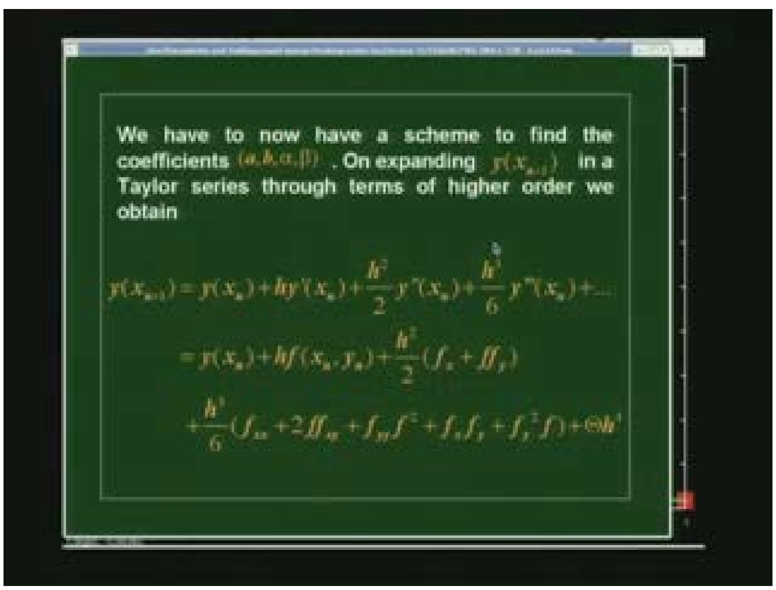

So again you can see then you have choose the appropriate value of alpha, beta and you can recover the predictor character from this. So we have only evaluating the first derivative we are not going to any higher order derivatives so now the question is what is what is the ab alpha beta. So we have to now determine this context such that gives you the correct solution of good aquatics. Okay that is a scheme we have to find out okay how do we determine ab alpha and beta. So what we have to do is the tailors expand the function the function y at x_n plus 1 that is what we called y_n plus 1 you will tailor expand that function that is just a while we do that here.

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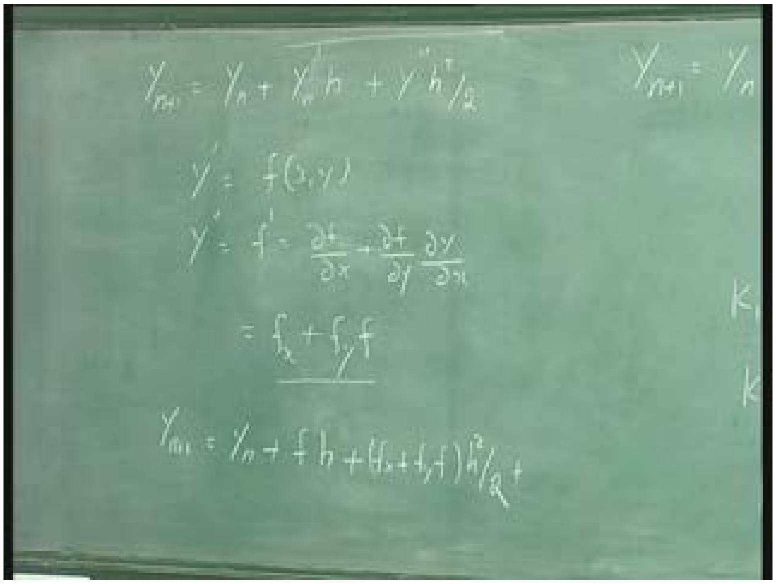

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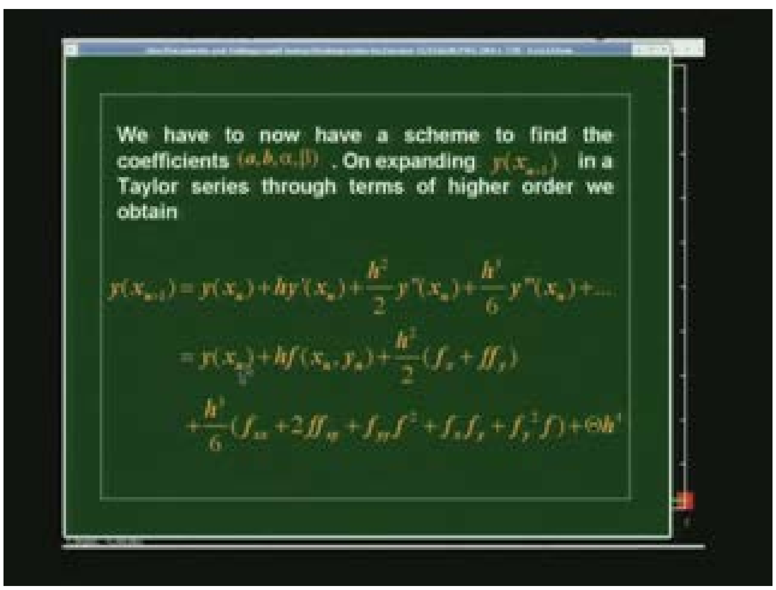

Okay so what we are going to do here is, we are going to write y at n plus 1 as y_n plus y_n prime that is derivative evaluated at that function times h right and then we can write at the y_n double prime at h square by 2 etcetera right. So now y_n prime is f and y_n double prime, so y prime is f the function f, so y double prime is you know this is function of x and y this 1 is okay, we have two terms one is fx that is derivative of this function with respect to x and other term would be f times fy. So I am going to write this as y double prime as a f prime as del f by del x as got 1 and del f by del y into del y by del x, f is a function of both x and y remember right. So that is now fx is a notation for that and this notation is fy that is f. So these two terms I can substitute that here, okay and then I have some order h cube term in this equation. So we can write this as at the up to order h square would be y_n plus 1 as y_n plus f times y_n prime that is h plus fx plus fy into f and h square by 2 similarly, I can write the next term which is of the order h cube term, okay that is today way summarized here.

So you can write this in a derive expansion and can have the order h square term and order h cube term. So the order h square term and order h cube term, so okay order h square term is this one that is y and hf, h square by 2 and then I can again find the derivative of this with respect to x. So I get a second derivative of f with respect of x and then I have the derivative of the this terms, a second derivative of this with respect to x that is fx and then I have fx, f xy into f and similarly, this expansion. So I have to find out derivative before the third derivative, okay so that gives as a more complicated equation you have f of x to 2 f, so 1, 2, 3, 4, 5 terms we can give this we can gives as and then what we can do is so we have this equation here is a let keep this equations here.

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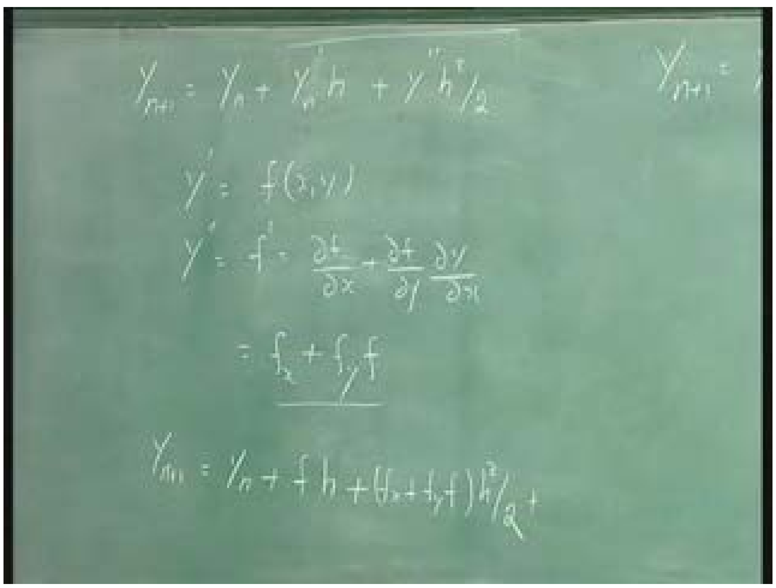

Okay some terms of the order h cube and then what we can do is we can take this k_2 terms, okay now we have this y and plus 1 here and then what we can do is we will expand this k_2 term around x and y_n and substitute that here and then compare this 2 equations now this is a simple tailors expansion of this function this y_n plus 1 then I will now similarly expand this equation by expanding k_2 around that point let us do that here.

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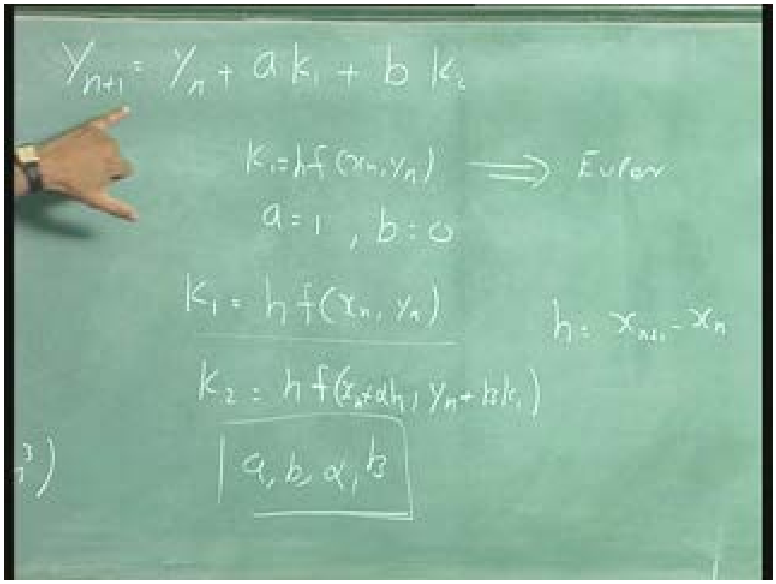

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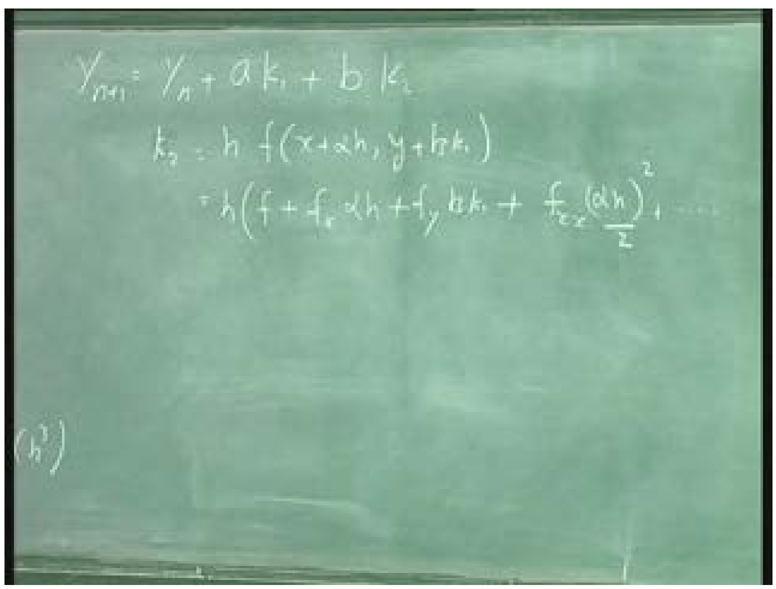

So k_2 remember is awe have this equation with k_2 given as h into f of at x plus alpha h, y plus beta k_1 so now I can expand this and then write this as h times f of x, y that is f. So f of x, y called as f therefore f is here fx del x by del y that is fx times alpha h plus fy times beta k_1 right that I am expanding around x and y and then I have the second derivatives that is I would have the second derivative function. So that will have f of x at alpha h squared by 2 and then the higher terms. So I can write it is in full series so I would get the series like that.

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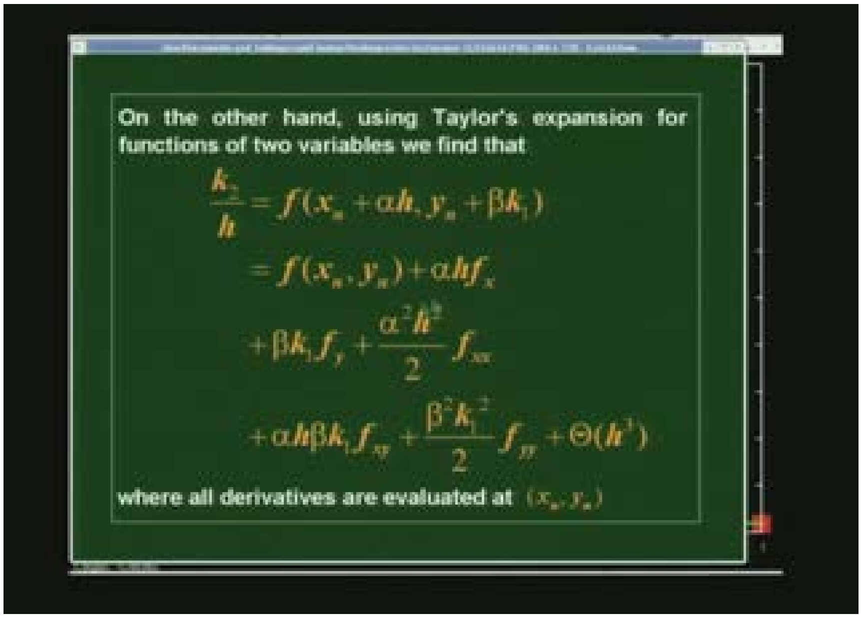

Okay I can expand this k_2 around the whole function value in this fashion. So I have beta k f of x_n y_n that is called f right and then I have the next term is the derivative of the function with respect to x into alpha h and then the derivative of this function with respect to y into beta k_1 and then the second derivative of this function with respect to x is alpha square h square 2 and then the derivative of second derivative of this function with respect to y that is f_{yy} into beta square k_1 square by 2 and the cross derivative which is alpha h beta k_1 f_{xy}.

So that is all I have written here, complete that, that is f_{yy} into beta k_1 whole square by 2 and then I have 2 cross derivative that is f_{xy} that is derivative of this function with respect to x and y and multiplied by alpha h, beta k_1 and then higher values. So that is what I have so now what I will do is I have this equation and I can substitute that here right so then I will get y_n plus 1 now from this as y_n and plus ak_1, k_1 is this f remember k_1 was just x times f, so I have k hf because k_1 is h times f and plus b times k_2 now k_2 in this. So b times hf plus alpha h into fx plus beta k_1 into fy plus alpha square h square by 2 times f_{xx} plus beta square k_1 square by 2 times f_{yy} plus alpha beta hf. So thus k_1 is f k_1 is just f, we can write f there. So this is f times fy k_1 is f

So f times as alpha beta h now this term that is the alpha, beta h k_1 is f into fxy higher terms. Okay so now I have 1 equation which comes straight from the tailors expansion of y and plus 1 and then I have another equation which comes from the tailor expansion of k_2 which is all f and x plus alpha h plus y plus beta k_1. So now I get this two equation 1 and 2 compare this 2 equations okay I compare this equation with respect to this equation so I will compare this equation which I obtain equation number one.

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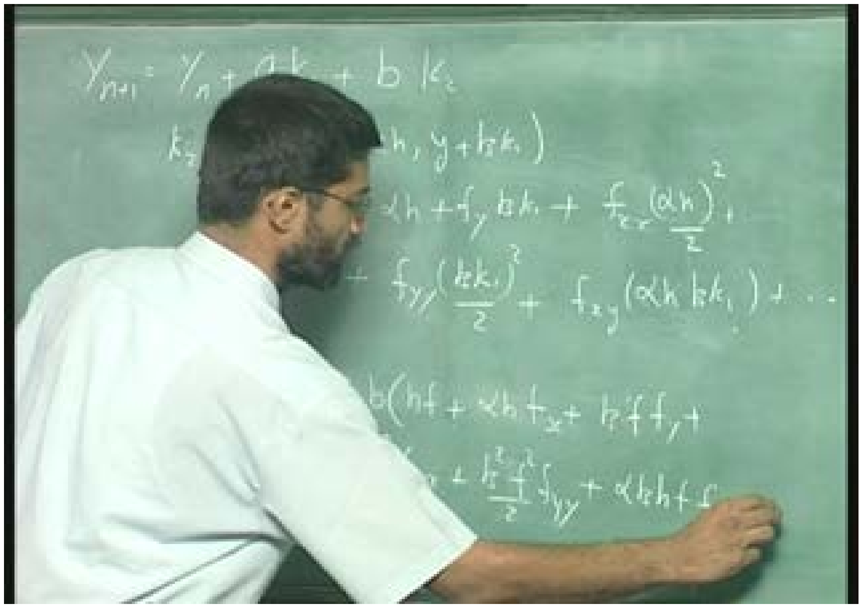

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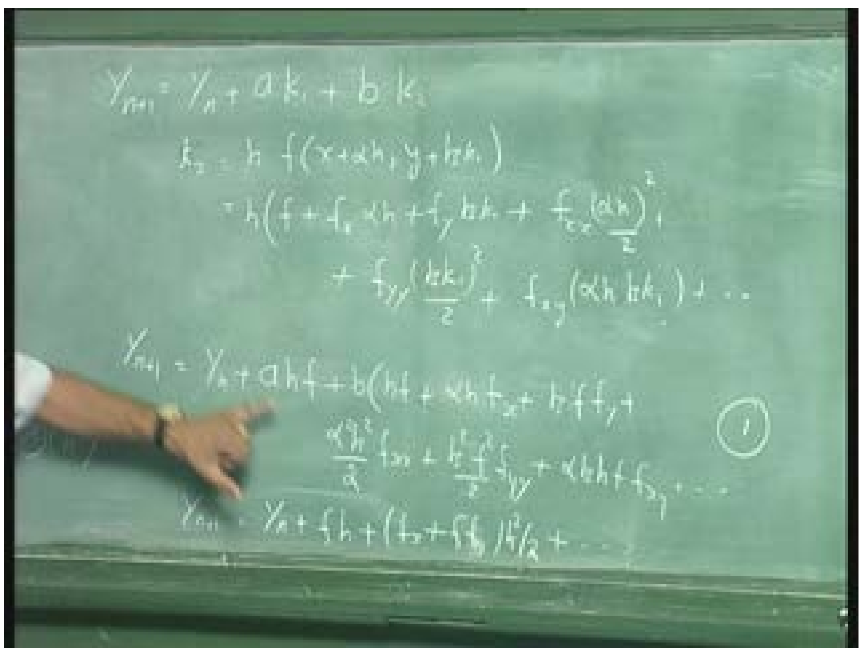

Okay which I can write here again so and then I have the equation which obtain from n plus 1 this is okay now I have compare this one with this. So let me write this equation here again this is 1 equation no one and another equation would be obtain from that y_n plus 1 is equal to y_n plus f times h plus fx plus f times fy f times fy times h squared by 2 plus higher value.

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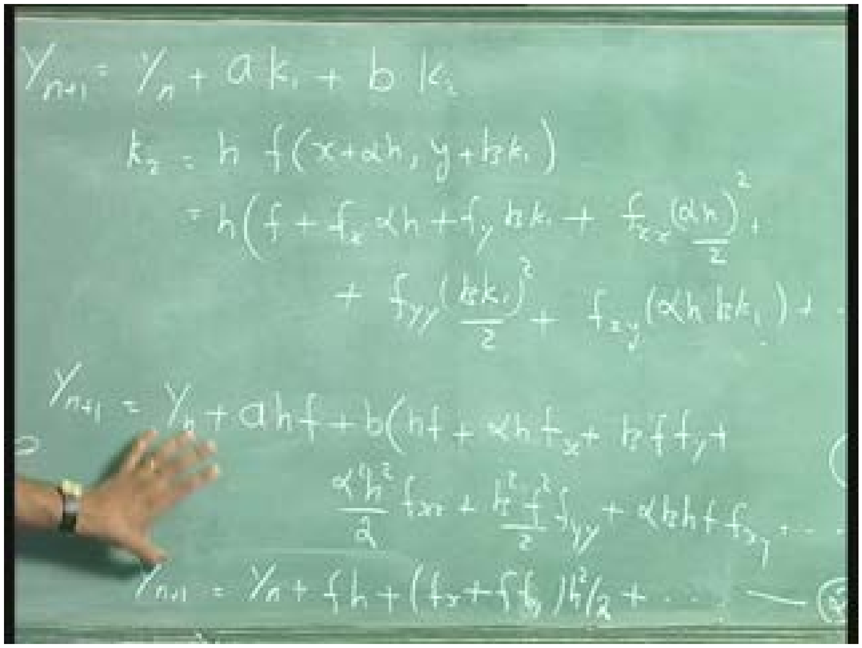

So I can compare order h term with in this and order h square terms within this 2 equations 1 and 2 so remember this comes from the expansion of the k_2 term and this equations comes straight from the tailors expansion of y and plus 1 and I can compare this two terms and then fix the value of the a and alpha this two equations and the same. So that is what the method is basically.

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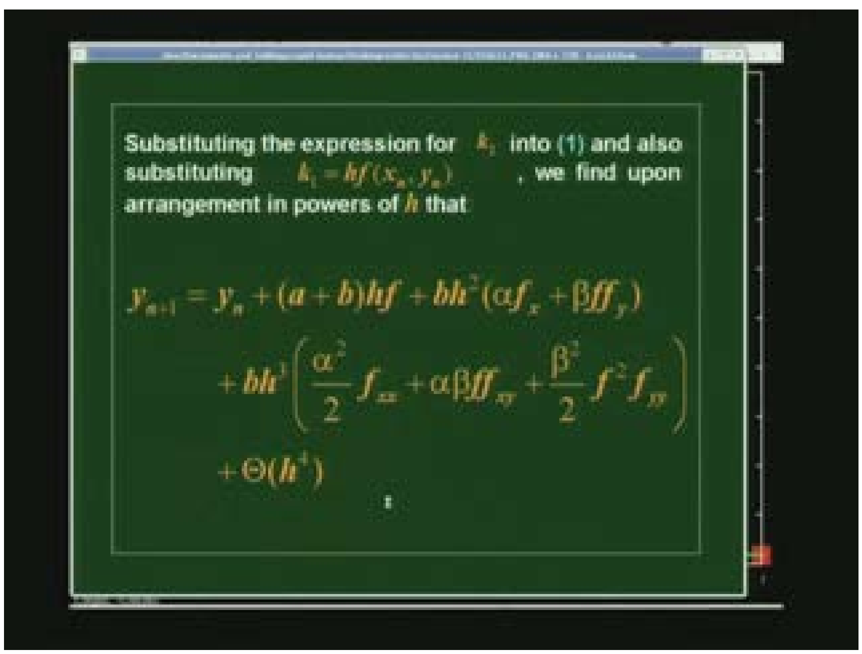

So I can compare I can write this equation in this form of x square and order h and then I can compare this with the simple tailors expansion of y_n plus 1 this equation actually comes from the expansion of k_2.

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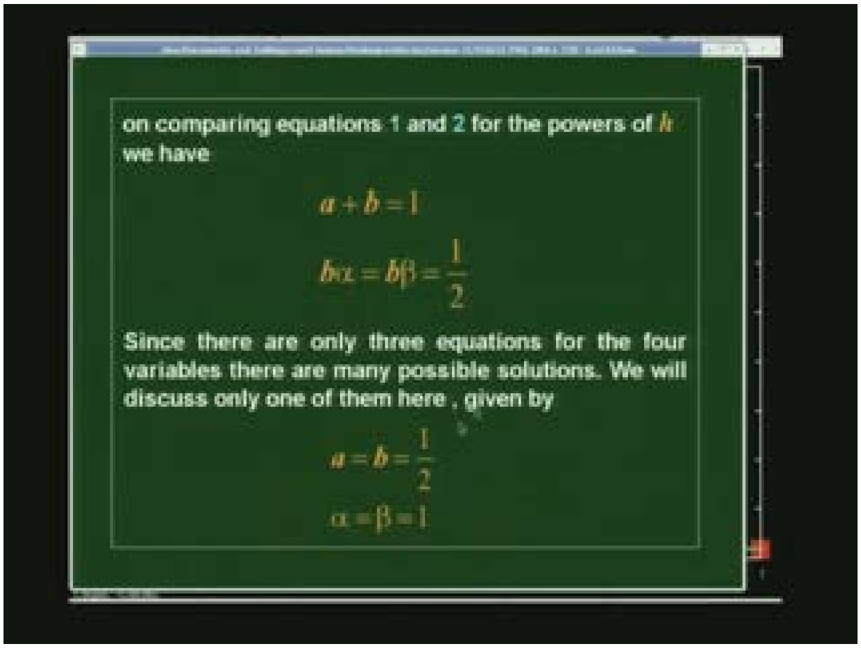

So if I do that I would get the following equations a plus b is equal to 1 beta alpha b alpha b beta equal to half, so now this is the two equation I will get now I have a 4 nodes, 2 equations so I have some freedom here to choose. So what we will do is we choose a equal to b equal to half and alpha beta is equal to 1, okay we will make a choice like that which satisfies this equations okay and that gives as what is called a second order Runge Kutta equations. Okay we will see the application of this into a simple to solve a differential equations which we just now solve that is dy by dx is equal to minus y and compare that with others schemes which we have just done and then we will also go into the next order, write down the equation for the next order Runge Kutta scheme that is what we will do in the next class.