Numerical Methods and Programming
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Lecture - 29
Numerical Integration: Gaussian Rules

We have been talking about numerical integration of a function. So in which we wrote the numerical integral of a function f as, i of f and that is f of x dx between the limits a to b. We wrote this as approximation saying that this can be approximated to integral a to b, p_k of x dx and we said that error in such integration is given by the next term in the polynomial which is f x_0 x_k, x integral a to b of this into psi k of x dx that is where that is what we have done so far and then we said that in the case where psi of k psi k_x is k plus 1th order polynomial which is of this form, which is x minus x_0, x_1 minus x_1 up to x minus x_k and will said that a particular case where integral psi k of x dx is equal to 0 is of interest to us and then in that case I can use the difference, divided difference method to write f of x_0, x_k, x as I can write that this quantity f.

We can write this as f of x_0, x_k plus x_k plus 1, f of $x_{0,}$ x_k plus 1, x into x_k plus 1 minus x that was the important thing which we are going to use again in the coming discussion that is the basic idea, okay let me repeat this again. Let the numerical integration is to represent, is to carry out an integral of this form numerically when we cannot do that analytically. So then we represent, we approximate the function f of x either the function f of x or the data points, tabulated data points we have by a polynomial p_k of x and then do the integral of that polynomial and do if you do such an operation, then you have an error in the integration which I represent by "e" here which would be like the next term in the polynomial that is the p_k plus 1 th term in the polynomial which can be written as that.

The co-efficient of the k plus1polynomial coming from the divided difference and psi k of x, which is x minus x_0 into x minus x_1 up to x minus x_k. Then with this special case in which with the points x_0, x_1, x_k are so chosen that we can write, we have psi k of x dx with the limit a to b is equal to 0. In that case we can write this error term using this divided difference and substituting for this quantity here from this equation and using the fact that, this is a constant.

Okay so that constant multiplied by this using the property integral psi k of x dx is equal to 0. So leads to E as a error as integral a to b, f of x_0, x_k plus 1x, x minus x_k plus 1minus x into psi k of x dx, that is basically integral a to b, f x_0 x_k plus 1, x, that is a coefficient into psi k plus 1 time of x dx. So we see that now the error has gone into the k plus 2 order term in the polynomial. Okay so we have a polynomial approximation the k th order polynomial approximation but we had chosen x_0 that the points x_0 to x_k such that this was 0 and then the error would go in to k plus 1 th order in a polynomial, that was k plus 2 term in the polynomial not k plus 1 as it was here, So the integral of k plus 2 term in the polynomial.

So then the k plus 1 th term which we have chosen here was also such that integral psi k plus 1 dx is equal to 0 and then, so then we can use the same argument and then go to the k plus 2 term here that is the k plus 3 term order polynomial order term in the polynomial. So and then by this method, we can go all the way up to 2k plus 1 order in the polynomial and then we get by choosing x_0, x_1, x_k, x_k plus 1, x_k plus 2 etcetera carefully. We can go all the way up to such a up to a term such that integral psi 2k plus 1 dx of x of x dx is equal to 0 okay in that case we our error would be of the order of the 2k plus 2 term in the polynomial.

So the integral of the 2k plus 2 terms in the polynomial, we still have polynomial approximation by k terms but the error would just go to 2k plus 2 terms in the polynomial. So that is the basic idea of the Gaussian rules which we want to discuss today.

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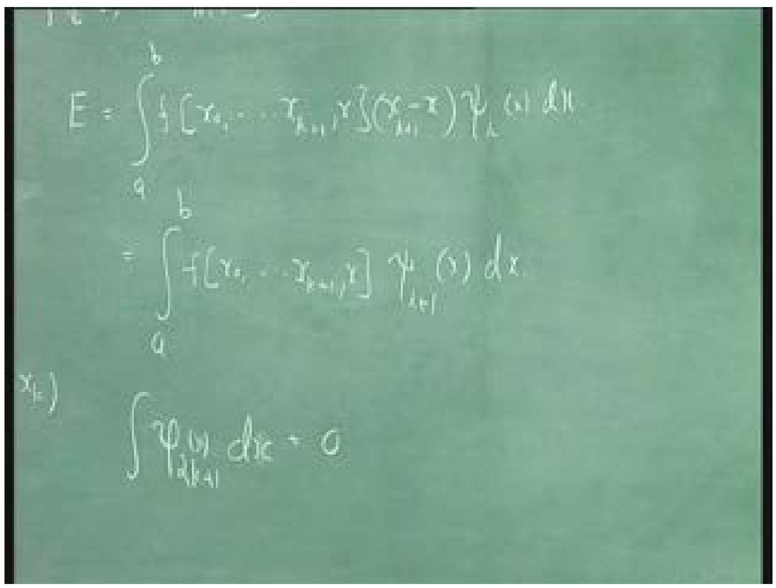

So in the Gaussian rules, when we use Gaussian rules we would write the integral of the function which we have been calling i of f as i of p_k, where p_k being a polynomial of order k and we would write that as before we have seen this in the Simpson's rule and midpoint rule etcetera.

We would write that as a series of, as a sum of a series of terms of this form that is function value evaluated at x_0, function value evaluated at x_1, and function value evaluated at x_k etcetera. Okay so that these ideas would be to actually compute this coefficients a_1, a_2, a_k and then write it as sum of this form. Okay so we want so we need to evaluate this functions we need to know the functions at this at this points x_0, x_k. So now we can choose the points such that, that is what we just said, we have end points we could choose this points such that our error is of the order of 2 n plus 1 th derivative of f of x or we go into that term in the polynomial. So the whole discussion of the Gaussian rules is to find out how do we choose this points such that we have such a low error. So that is the basic discussion which we would have today.

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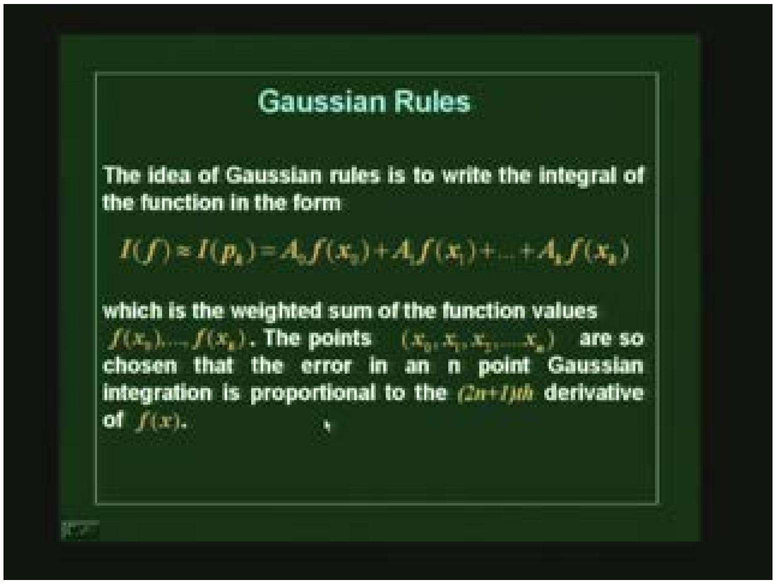

So the idea here is to write f of x dx as we said split this function f into 2 product of 2 functions that is g of x and w of x dx.

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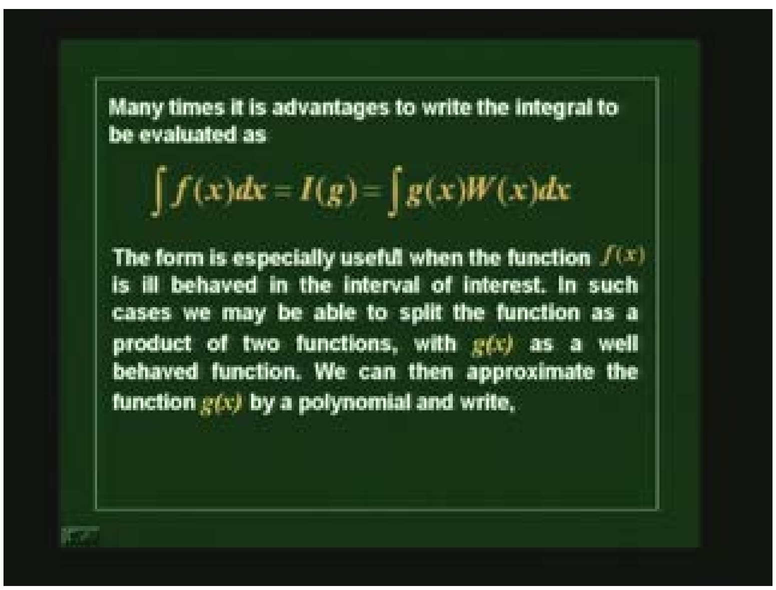

So this is now remember our what we want do is to is choose the points x_0, x_1, x_2, x_3, x_k such that we have psi 2 plus 2 k plus 1 dx is equal to 0 that is what we want to do or in other words we want to write this as psi k of x that is our original function here x_0, x_k

points, the product of x minus x_0, x minus x_1, x minus x_k and then we are going to choose x minus x_k plus 1, x minus x_k plus 2 all these points up to x minus x, x_k plus 1 minus x and x_k plus 1 minus, x_k plus 2 minus x etcetera up to $x_{2\,k}$ plus 1 minus x and x dx equal to 0 that is what we want to write.

So that is the thing which we want to finally arrive at, okay we have to choose we have to have a method of choosing x_k plus 1, k plus 2 up to 2 k plus 1. Okay such that we had this thing equal to 0. So we will have k plus 1 points here to begin with to determine the k th order polynomial and then we need another k terms okay to we have to choose another k point such that this integral is 0. So that our odd error is of order k to k plus 2 th term in the polynomial.

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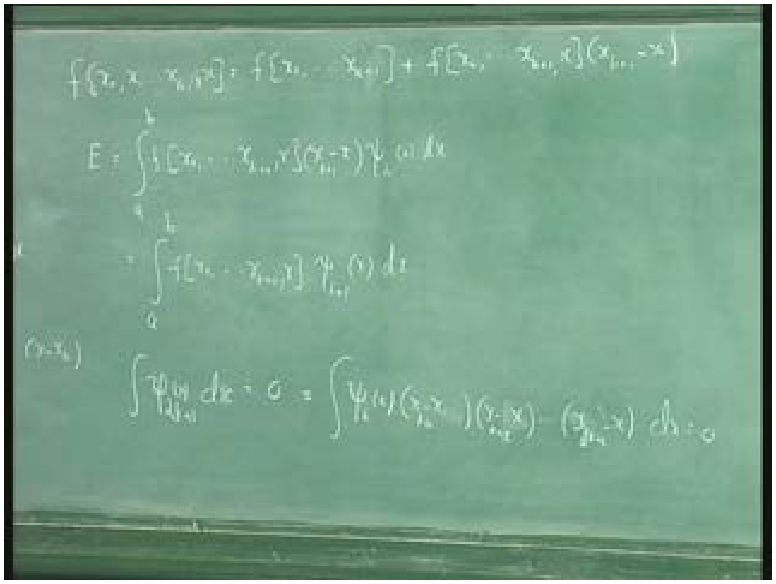

Okay, so for that the first thing which we do is we write this function f of x as a product of 2 function. So we will replace this function f of x as a product of 2 functions, so we going to say f of x or can be integral f of x dx a to b can be written as integral g of x into w of x dx between the limits a to b. So this has a double advantage one is that it will give us this is required for us to find the points that is one thing, another thing is that this g of x to approximate if you want to approximate this function f of x by a polynomial but there may be problems because f of x itself may not be a well behaved function in the interval a to b. So in that case we could represented multiplied it by we can split in to g of x into w of x, where g of x is a well behaved function in the interval a to b.

So we will have an advantage in this in that context here. So now it is a g of x we are going to represent it by, we are going to approximate by a polynomial p_k of x. So we are going to approximate g of x by p_k of x of a kth order polynomial that is what we are going to do.

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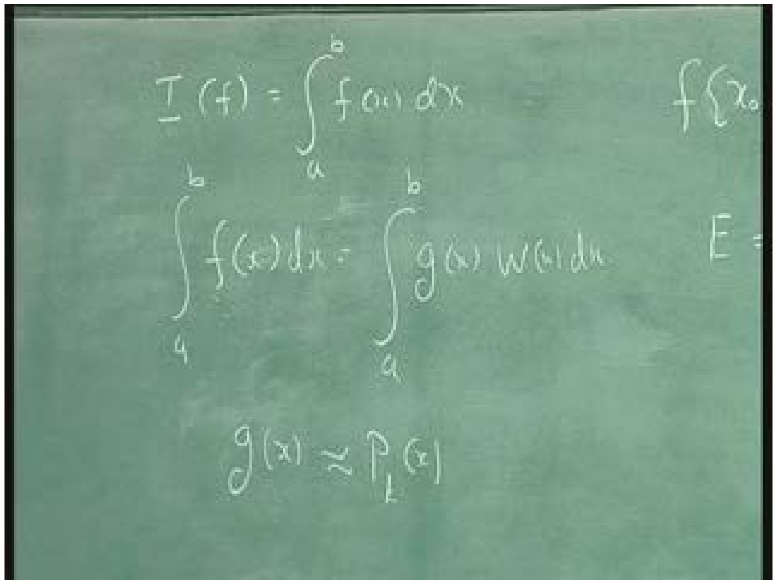

So we would write i of g is as a integral g of x, w of x dx. Okay so when you say i of f, we did not have a "w". So just to distinguish from that we write i of g and we say that it is g of x, w of x dx. Okay so now the form is as I said especially useful when we able to split the function such that it is one is g is well behaved and then we would approximate g of x by a polynomial and then we would write g of x as p_k of x plus the error in the polynomial which is f x_0 x_1 x_k x psi k of x.

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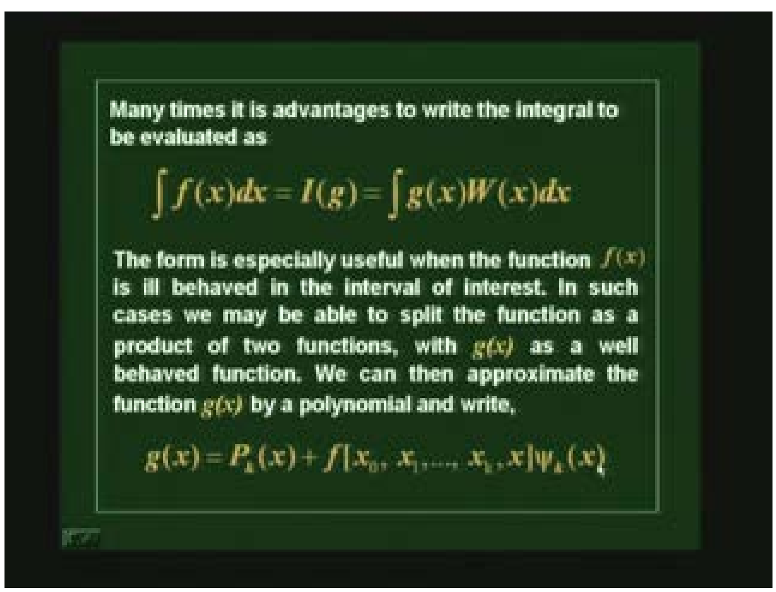

Okay this being the product of x minus x_0 up to x minus x_k. Okay that is what we have and then you can write now the error as this right. So the error in this approximation would be this instead of f now we have g and a “w” here right, psi k of x dx what is psi k is this product is just written down. Okay so now we want to we are choosing the points such that w of x psi k_x dx is equal to 0 right that is what we required. So what we have done earlier in the case of midpoint or trapezoidal rule, we did not have w. So always we are choosing is the point such that psi k of x dx is equal to 0 such that we can go to one order up as I just explained here right.

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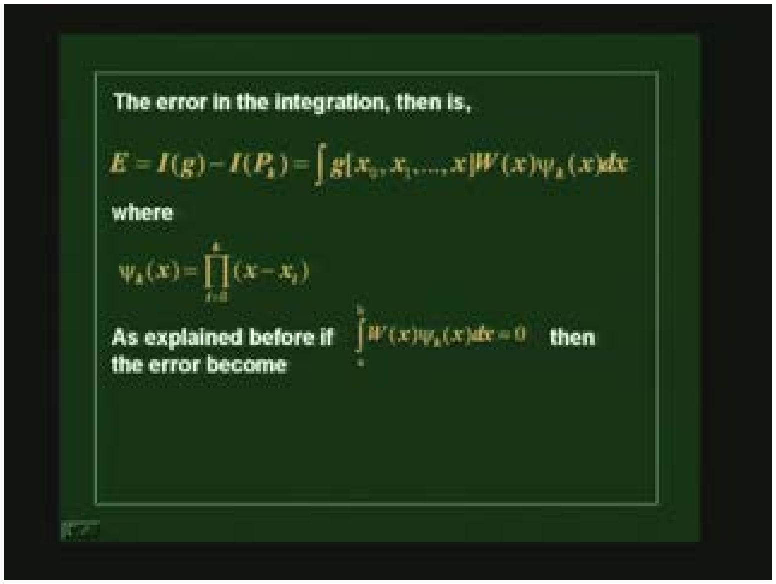

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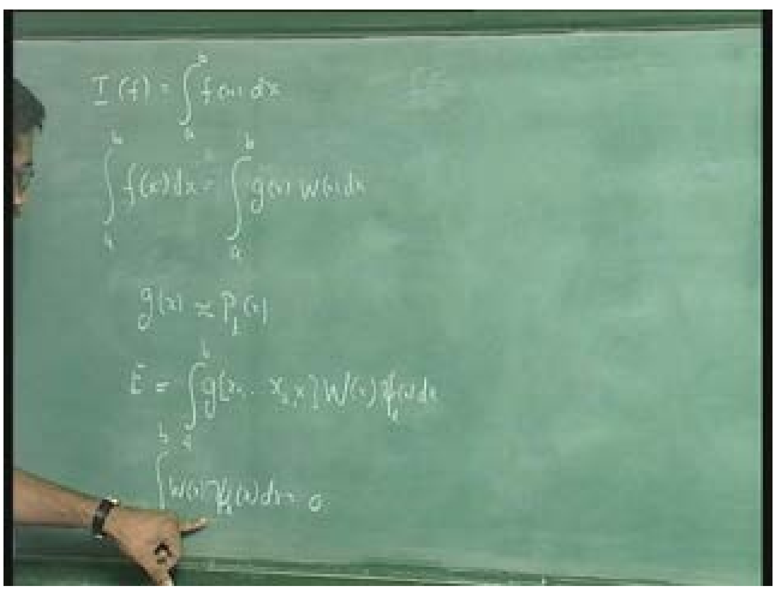

Okay but now our integral itself is g of x, w of x dx. So if you want to go to higher order, okay now error here is, in this case is equal to integral a to b, g x_0 x_k x, now there is a “w” there dx and then psi, that is what we have now, that is the error. Okay so now if you want to go to the k plus 1 th order term, now the error is in the k plus 1 th order term. So that is this is k plus 1 th order term, now if you want to go into a k plus two order term what we need is integral a to b, w of x, psi k of x dx is equal to 0.

Okay that is what we would require now. We want this to be 0.So w of x, psi k of x dx is equal to 0 then we will write this error as g x_0 x_k plus 1, x, w of x, psi k plus 1x dx, if this true, then we can write it like that. So then we will go up and up to 2k plus 1 term. So then the 2k plus 1 term we have to go all the way up to.

So from here we will continue and then write the error as integral a to b g x_0 x_k plus 1, x, w of x, psi k plus 1 of x dx equal to 0 and if this term this the integral w of x psi k plus 1 x dx is also 0 then we would go to higher term etcetera, till we reach the error as this as a to b, g, x_2 k plus 1, x, w of x, psi 2k plus 1 of x dx is equal to 0. So that is what we want to reach. Okay so that is what I have been summarizing here.

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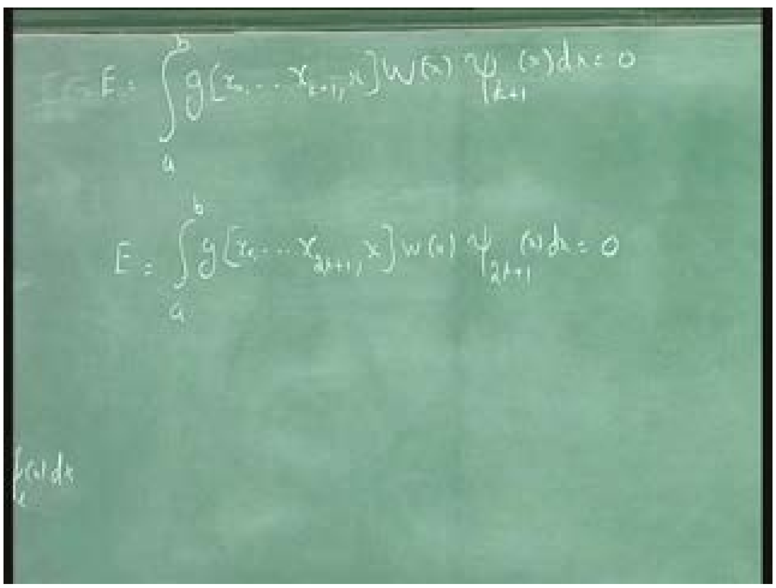

Okay let us we will go to a next level and then by choosing this to, such that this is 0 and then we go to the next level and then so on and then till we reach the error to be 2k plus 1.Okay so now we look at the method which gives us, so now for a k th order polynomial we have a polynomial of order k. Okay, we suppose, we are able to choose these points k plus 1, k plus 2, k plus 3 up to k plus n such that this is equal to 0 that is our aim right okay then the error will become k plus n plus 1 dx. That is what I was trying to tell you here.

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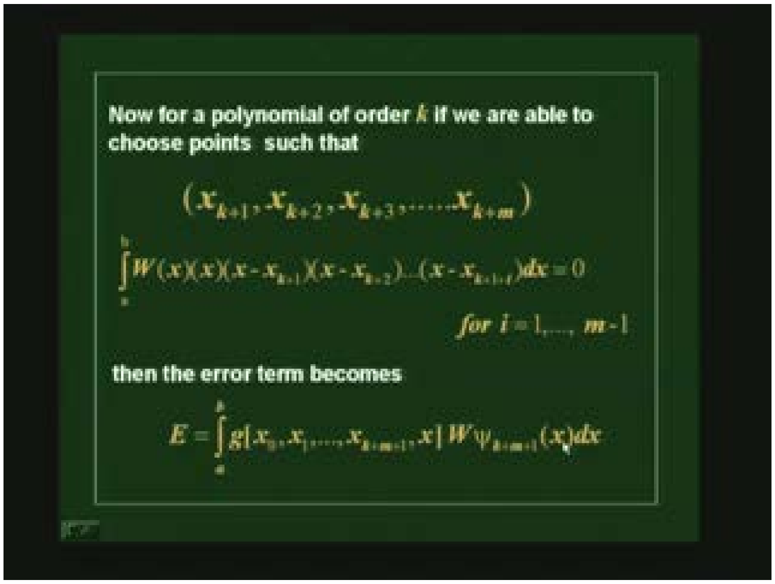

So we choosen k plus 1 term, okay another term here, such that this is 0 and so we went to this and then we choose the next point etcetera. So if you are able to choose m such points apart from apart from x_0 to x_k, we choose another m points, so we have two sets now first we had x_0 to x_k which we use to construct the polynomial of order k and then we have another m points which is x_k plus 1 all the way to x_k plus m. Okay we choose we know, we choose these points, okay such that integral g x_0 x_k plus m. We choose another point k plus m, x, right w of x into phi, i going from 0 to now we have k plus m, k plus m into x minus x_i that product which is psi k plus m okay dx equal to 0.

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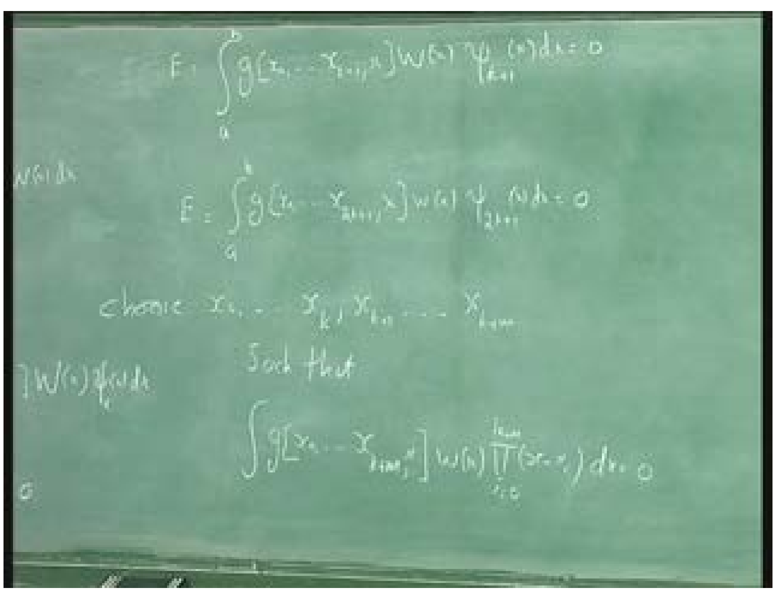

Okay so if you are able to choose that, then error will become of this order. I am repeating this again just to point out that what we want to construct, what we want to construct is the set of points, okay for this we use the orthogonality property of polynomials. So what we want to do is we want to treat 2 sets of polynomials.

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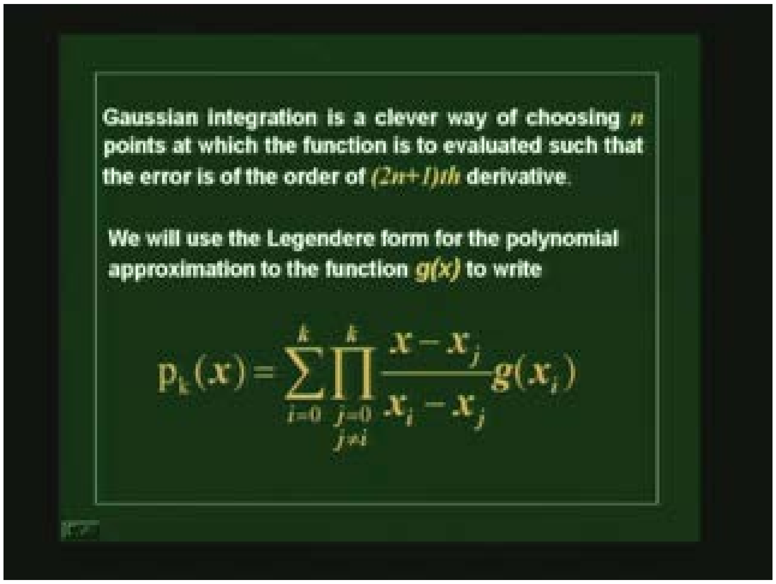

Okay and we want we use orthogonality property of polynomials and construct these points, okay so first let us you choose this polynomial p_k of x and then we will construct the, we will be able to systematically construct these points okay such that this is always true. Okay so we will write p_k of x in the legendary form.

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Okay so we will write p_k of x in this particular form. So we are having approximations g of x by p_k of x and the p_k of x. We are going to write in the form, we will not write it in the Newton form okay instead we would write this p_k of x in the Lagrange form that is sigma i going from 0 to k and phi, j going from 0 to k and j is not equal to i, right. So we will write x minus x_j divided by x_i minus x_j into g of x_i. Okay that is what we are going to write. Okay so this is our polynomial approximations. Okay so remember, so that is what we are going to use.

So in this construction our choice, our idea, our aim is to choose x_0 to x_k plus m okay such that this is 0 okay so we will start with the polynomial approximations of function g of x, g of x is given to us and we will construct p_k of x, the k th order polynomial first okay and then we will construct this polynomial k th in this using this legendary form which we have seen in the polynomial approximations interpolation scheme.

So we will write that as the product j going from 0 to k, x minus x_j by x_i minus x_j obviously even j is not equal to i into g of x_i okay, so sum over all terms of i equal to i going from 0 to k. So that is the so now we have a sum of series of terms of all of them of k th order, there is a polynomial p_k of x.

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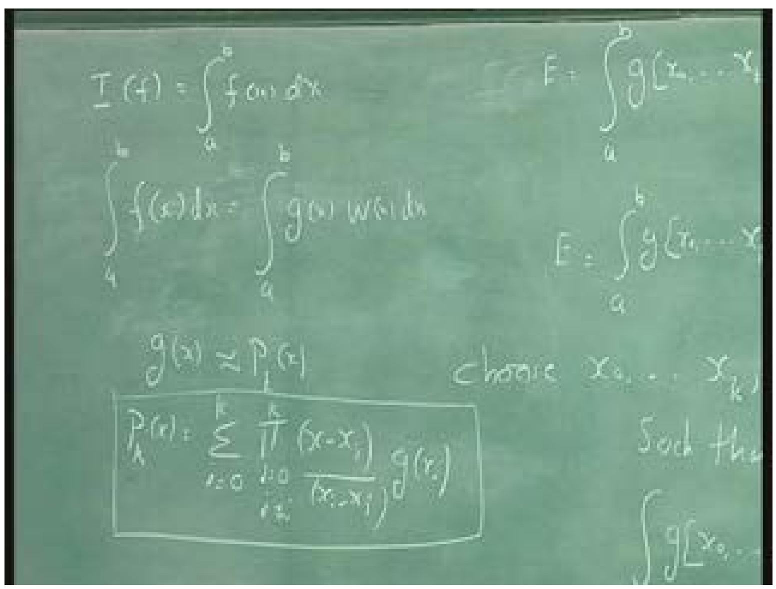

Okay so once we construct that okay the integrals of this are obviously given by these quantities, right so integral i p_k of x is the integral of this. So that is can written down in this form that i p_k of x is given by the sum i going from 0 to k and the product j going from 0 to k, x minus x_j by x minus x_k. So we now have a "w" here and g of x_i dx by definition of i, okay the integral is now defined with this weight w of x dx that is our full function.

Okay so now this is this can be you know so that this can be written as a_i g of x_i. So this integral I can write as a_i g of x_i because g of x_i is not a function of x. So I can pull that out and then write it as sigma i going from 0 to k a_i g of x_i.

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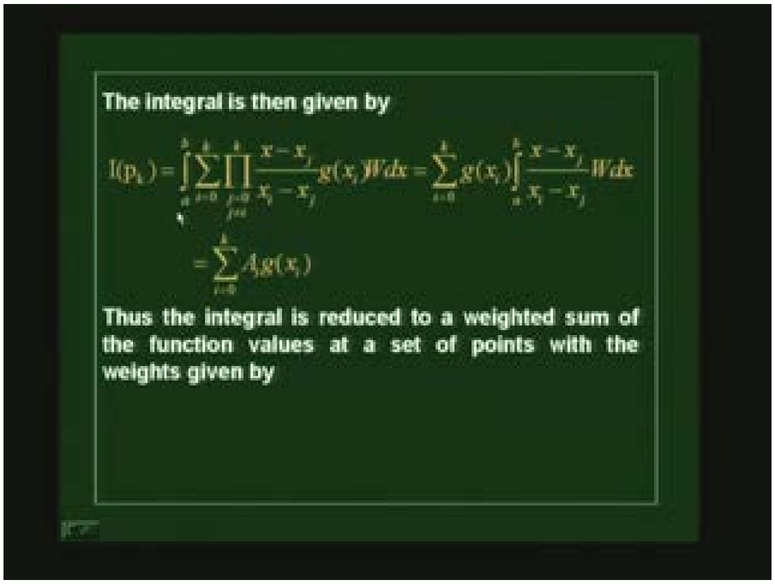

We are write this integral i of p_k as from this. So integral, so before we write the integral over x so I can write g of x_i outside. So I can write g of x_i which is not a function of x and then I would write integral a to b integral a to b sigma i going from 0 to k and phi j going from 0 to k, ij not equal to i of x minus x_i divided by x_i minus x_j that is what we are going to write at the dx okay.

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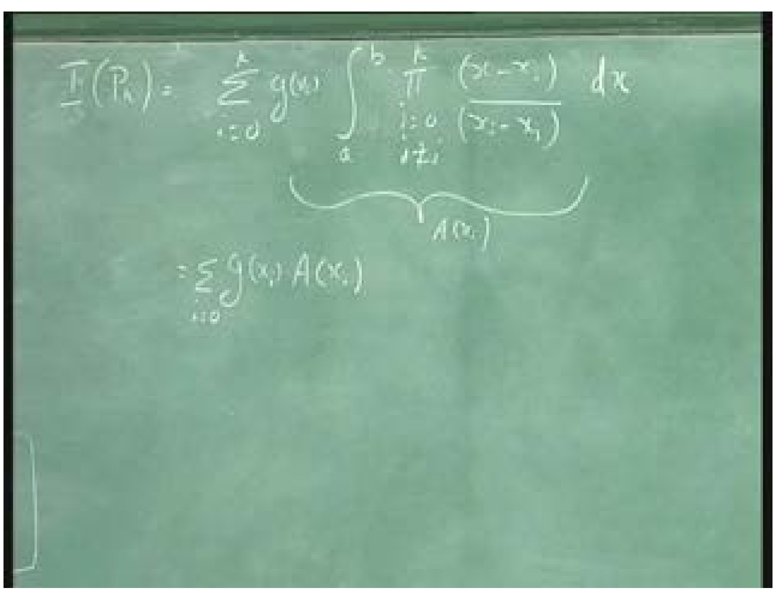

So now this quantity is what I call a of x_i and so it is g of x so it is g of x_i this is a of x_i this integral so it becomes g of x_i. Okay so the sum also can be pulled out so it is the sum going from 0 to this is actually wrong I should not write this here, should sum i going from 0 to k, g of x_i integral a to b. Okay so g of x_i into a of x_i, so a of x_i is this integral. So that is what I am going to write, so that is the form which we want to arrive at it.

So the whole purpose of this calculation would be to compute a of i. So the two now there are two things one is that we should be able to choose our point x_i's such that we have order error of order 2k plus 1, we should be able to choose these points such that error is of order 2k plus 1 when we construct a polynomial of order k. Okay and then we should be also able to compute a of x_i.

So once we have that okay so now if we have this pair that is x_i, a of x_i pair right. So given that then the polynomial the integral of the function of this function this this integral. Okay just simply reduces to g of x_i, a of x_i. So if you know g of x and if we are given a set of points x_i and its weights a of x_i. Okay then I can write the integral as sigma i going from 0 to k, g of x_i, a of x_i. So now note that a of x_i is the is this quantity this integral right and that does not have the function value anywhere.

So this method as its real advantage that if you are given a set of points x_0 to x_k and then and the corresponding weights a of x_0, a of x_k which does not depend upon the function value. So given that you can integrate any function within the interval a to b. Okay so this quantity weights a of x_i depends only on the limits and this is the general polynomial here. Okay which does not depend on the function value g or f for that matter, so I can just tabulate if you give me this x_i and a of x_i. I can do any integral which is smooth and continuous function integral of any function which is smooth and continuous in the in that interval a to b of course this is will depend between the of limit a to b which a to b we use

So that is the advantage, so once you tabulated this a of x_i and a of x_i. So that will depend on what order we choose of course there is a product here right. So if we choose if we choose k or k plus 1 or k plus 2 order k or m whatever order you choose the weights will differ okay because there is product here. So if we use a second order polynomial to do this and there will be 2 values of x and 2 values of x and 3values of that is x_0 x_1, k equal to 2, I mean so x_0 x_1 and x_2 and a of x_0 x_1 and x_2 and if you choose a fourth order polynomial okay that if you choose x_0 x_1 and x_2 x_3 and x_4 and a of x_0 x_1 and x_2 x_3 and x_4 the first 3 values may not be the same it will not be the same because there is a product here inside the integral.

So it will depend on 2 quantities that it will depend on the limit and it will depend on the order of the polynomial which we are interested on, so given that we get the set of x_i's and a of x_i's and we still now learn how to do the compute the x_i's how to get the x_i's but we give the x_i's we can compute a of x_i and from that we can determine the integral i right. So that way it is already we see some the advantage of using this Gaussian method.

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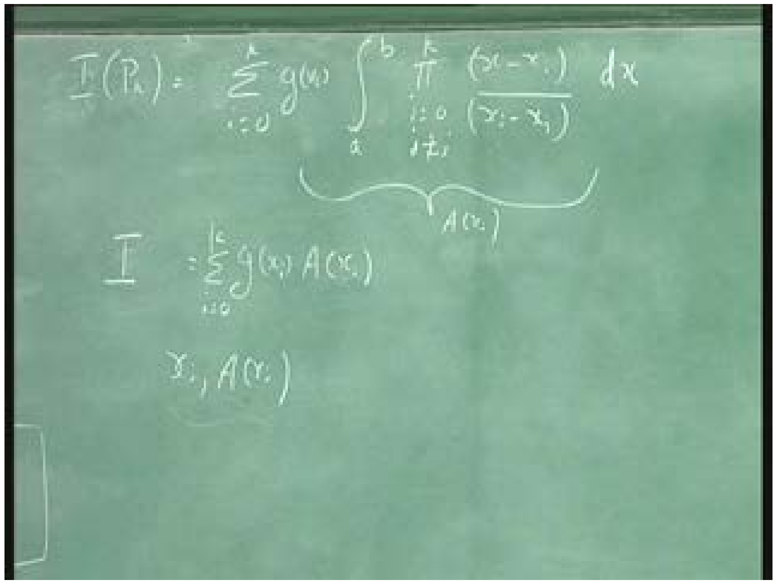

So the integral then reduces to a weighted sum of the functional values and the weights a_i and points x_i has to be given to you or given x_i's we know now how to determine a_i's from this integral, okay so that is where the point is now again now we reduce to only finding the values of x_j's of this x_i's once we know that, we know how to compute a_i's and once and given the order of the polynomial and then we can construct the integral as a weighted sum of the functional values at that points.

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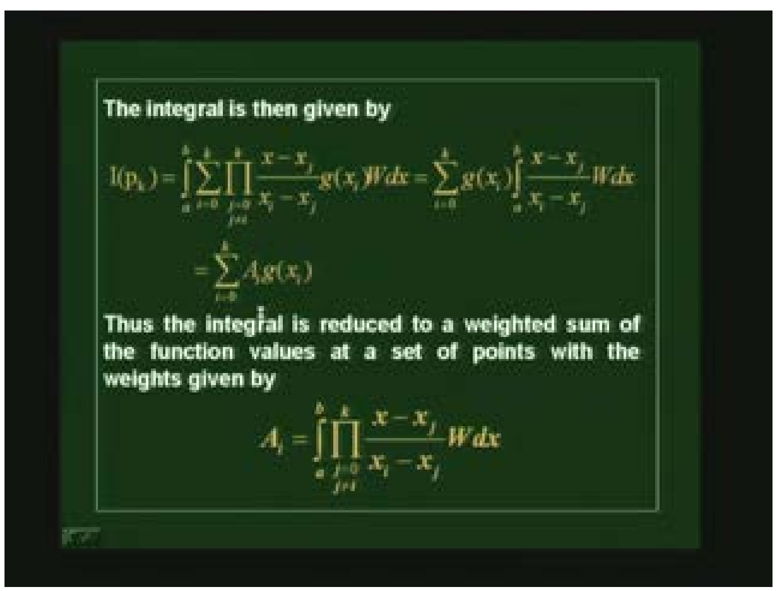

Okay so now we go down now we will see, how we construct that point. Okay once we have that we this quantity written in that form that our error will now keep going down rite k plus 1 k plus 2 k plus n up to 2 k plus 1th derivative. Okay that is we have already seen.

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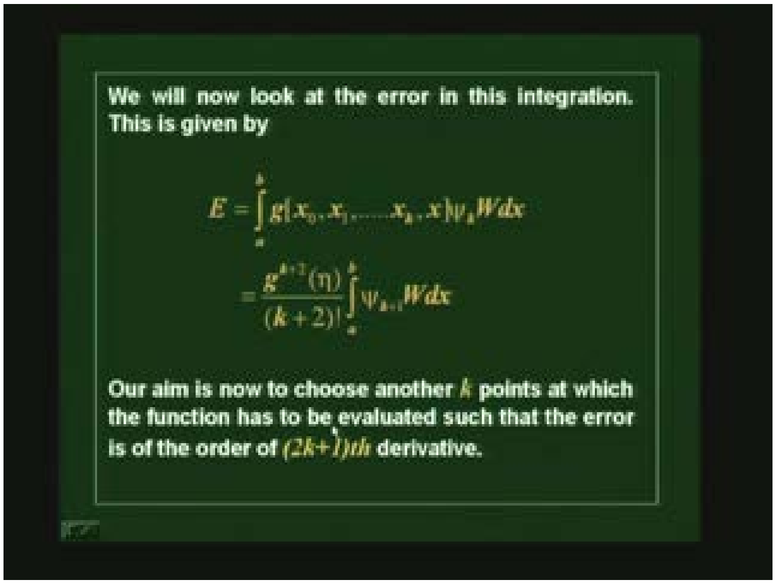

So we have written the now the error would be in this form that is psi k assuming that we can choose all these points that the error is psi k into x minus x_k plus 1, x minus x_k plus 2 x minus 2k plus 1, w of x dx equal to 0 that is what the error would be that is our aim.

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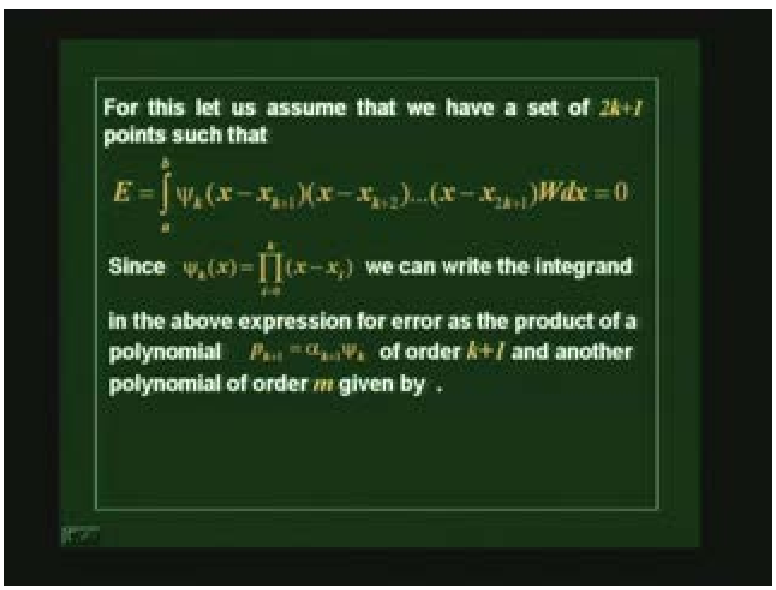

Okay so now since psi of x is this quantity okay we will write this as now we have one polynomial here psi k of x which is the polynomial of order k plus 1, this is of order k plus1and then we have another k plus 1 k plus 1 term another k terms here. So we have another polynomial of order p_k of order k, okay so let the error which we are going to write will be in this form now.

So the final error assuming that we choose all the points starting from x_0 to x_k and then x k plus 1 to x_{2k} plus 1 assuming that we can choose all this our error would be or we choose this points such that a to b right psi k, which we know is the product of x minus x_0 to x minus x_k and then you have x minus x_k plus 1, x minus x_k plus 2 up to x minus x_{2k} plus 1into w of x dx. That is the error which we have okay that is what we going to make it to 0, so that is what we want to make that is equal to 0 that is the aim right.

So now this is a k plus 1th order polynomial okay so we call that as p_k plus 1 as polynomial of order k plus 1. Okay and that we call as psi k of x, okay that is divided by some alpha, some quantity alpha k_1 that some number there just to get our error to be done into alpha k 1 we can choose it in either way so we choose it as alpha k_1 we just construct a polynomial okay alpha k plus 1 psi k. Okay now this is of the order k plus 1because this is the product of x minus x_0, x minus x_1 up to x minus x $_k$ that is k plus 1 terms so this is 1polynomial p_k plus 1x and then we have another polynomial, okay which would be this one. Okay now I call this polynomial, okay I will call it q so I will of the order k.

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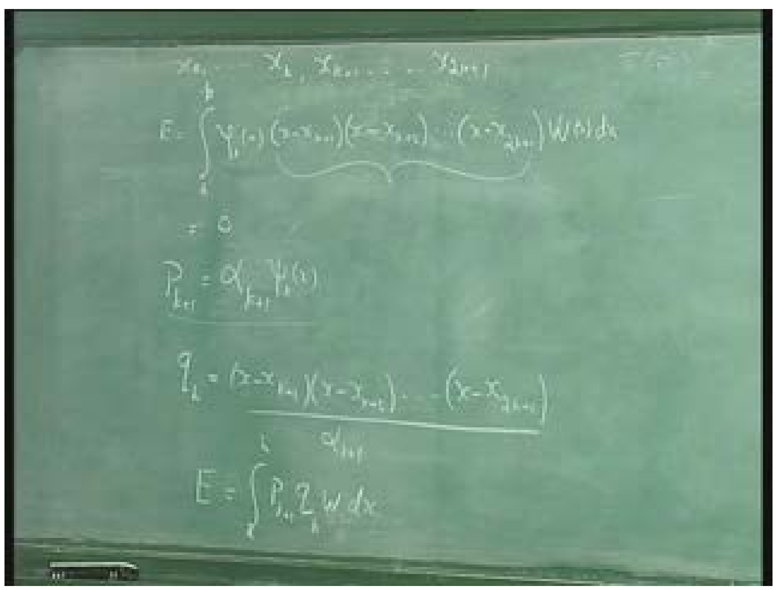

So there are k terms in this, so that will of the order k so we call it a q_k polynomial. Okay I construct a polynomial from that which is x minus x_2, x minus x_k plus 1 into x minus x_k plus 2 all the way to x minus 2k plus 1 divided by alpha k plus 1. I just construct this polynomial this a_k th order polynomial this a_k plus 1th order polynomial and then I can write my error E okay such that it is integral a to b right it will be p_k plus 1, q_k w dx okay, that is the construction. So that is the polynomial p_k plus 1 this the polynomial q_k and I write the error as integral a to b p_k plus 1 q_k w dx. So that will be my polynomial that will be my error now. Okay so I constructed two polynomials 1 is p_k plus 1 from this and the other is a q in this case I written as q_m. I can go all the way up to k_m can go all the way up to k. So that is 2k plus 1 terms, so there are 2 terms, so this is this is one polynomial and that is a another polynomial coming from psi k.

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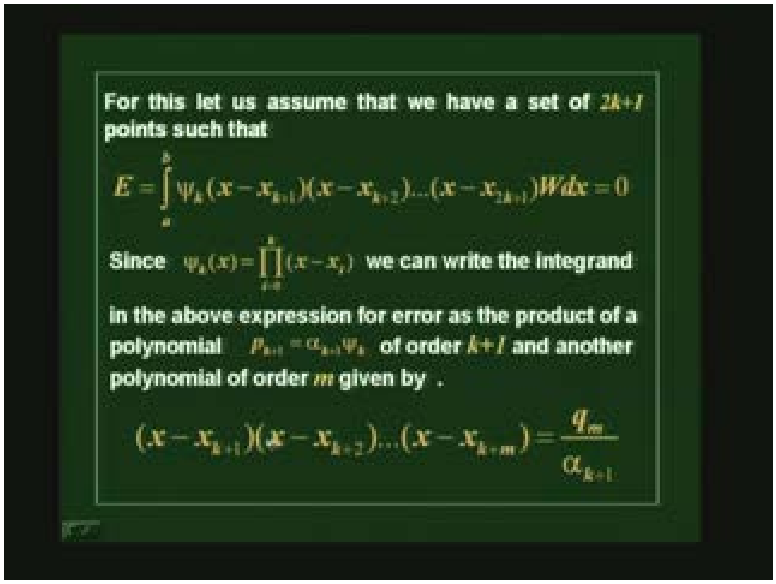

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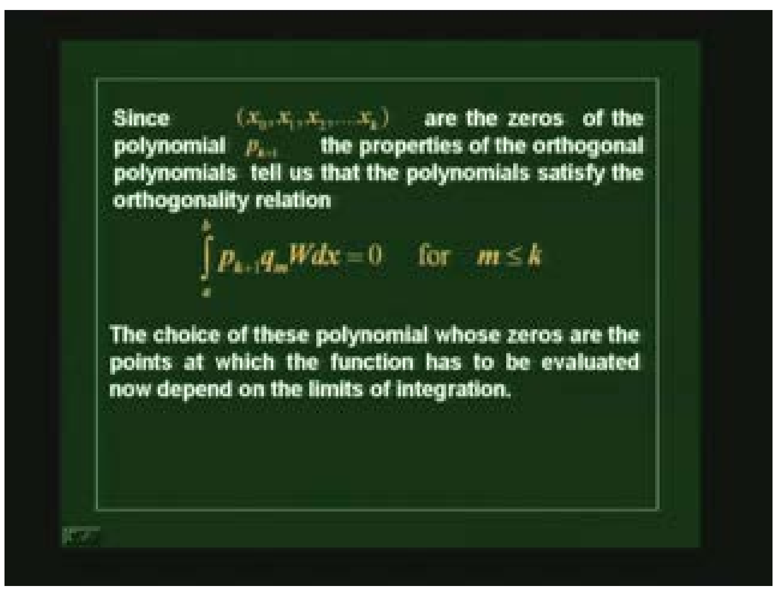

So I can write the error as p_k plus 1, q_m in this case so its general in general I do not have to choose k points here, I can just choose m points then it will be m th order polynomial and then this will be p_k plus 1 q_m w dx, I choosen here k points. In general I can choose m points and then write this quantity as the error then as k q_m look and then we have the orthogonality property of the polynomial okay.

So we have the orthogonality property of the polynomial which tells us that integral of integral we can we have the what is called the inner product, so which says that we can write a polynomial p_k and q_m and then we can always write this depending upon the limits then weights. Okay we have to choose the polynomial such it satisfies the limits and the weights dx is equal to 0o for all m less than k. That is the property which we are going to use.

Okay that is if you have a polynomial p_k and another polynomial q_m and then we can find w, okay for a for a given limit a to b we can find a w such that this quantity is 0 for all polynomial m which is of the order less than k. So okay that also comes from the fact given a polynomial k, order k. I can write that as a sum of polynomials which of order less than k or I can write it in a. I can expand this polynomial as in terms of polynomials which of order less than k and this form an orthogonal set that is what basically this means. So I can write this in this in this form. So if I can choose my p_k, okay that is the idea if I can choose my p_k then I can also determine q_m such that this is 0 from that from that quantity. So that what we are going to use.

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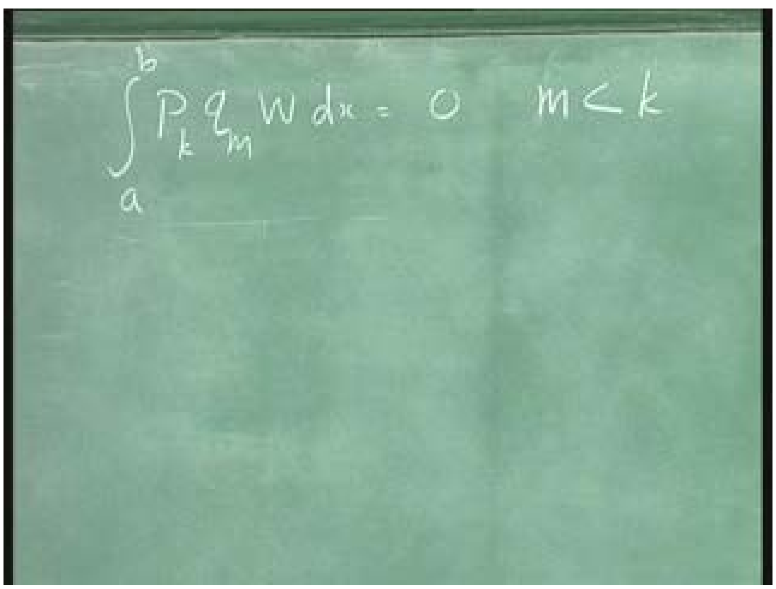

So this orthogonality property is crucial in this analysis. So what we have done so far is to write the error in this term in this case k. So that we k_i said m should be less than k here. So for this to be satisfied this has m should be less than k. So I can go all this is if this is of the order k plus 1 polynomial then I can go up to k th order polynomial here. Okay that is what I was trying to write here.

So what I am trying to say basically is that we can always find this function q_k's given this polynomial p_k plus 1. Okay I can find the polynomial q_k such that p_k plus 1 into q_k w dx is equal to 0. Okay since k is less than k plus 1. So I can find this one using the orthogonality property, so we choose error is this and then we write this as an this is the k plus 1 th order polynomial and this is the k th order polynomial in general this is a some m th order polynomial with m less than k that is all okay. So k is the maximum order of this polynomial which we can write and then we can write this as in this in this form.

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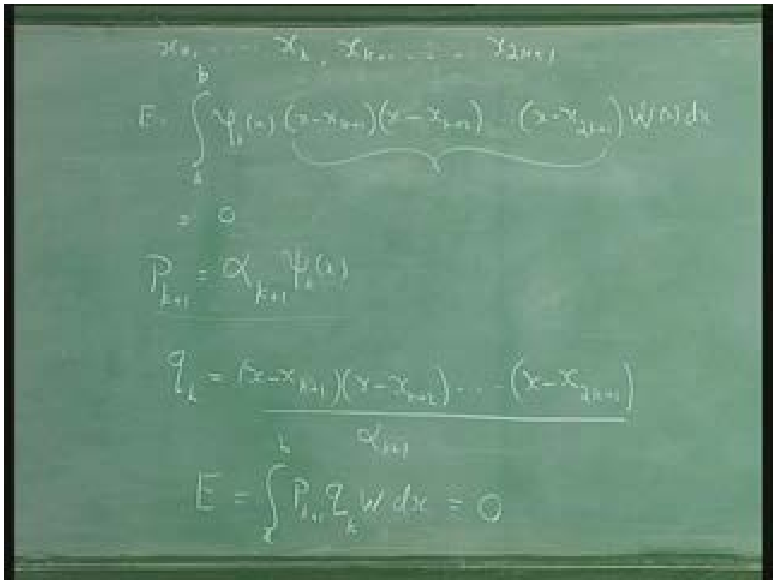

So the choice of this okay now that is what we are using it in the orthogonality property summarized here. Now, remember there is one more case where why this all can always work is in this particular case as I said that we can find a_p a_w and for an a and b such that this 0 okay and in this particular case it will always work because the construction of p of k where we construct p of k is important to make this satisfy or to make it satisfy this condition, we construct the p of k as by using tabulating the points. Okay so no we to use this p of k we construct it in such a way that the points which you choose that is x_0, x_1, x_0 x_1 up to x_k okay or 0s okay 0s of p_k of x.

Okay that is what we are using remember what we have in construction of p the polynomial p_k plus 1, x which we constructed as p_k plus 1 of x, we constructed that as sigma i going from 0 to k. So if its k it is k minus 1 here and then p_i j going from 0 to k_j not equal to I, x minus x_i divided by x minus x_j into g of x_i. Okay these points which you choose x_i's are the 0s of this polynomial.

Okay so when these points x_0, x_0, x_k are chosen as a 0s of this polynomial or 0s of this polynomial p_k plus 1 because we are choosing k plus 1 points that is 0 to k plus 1points. So when these are choosen as the 0s of this polynomial x_k points are choosen as the 0s of this k plus 1 polynomial and then this q_k this function will always satisfy this. Okay for any arbitrary values of x_k plus 1 x_k plus 2 k plus1. So it does not matter okay so orthogonality theorem tells us that we can always find a polynomial of order m less than k for this equal to 0 now, this left to us to construct this p_k such that this is this true.

So now we construct the p_k by choosing points which are the 0's of this polynomial tabulating this function as the 0s of this polynomial. We have to choose the points okay x is x_0 to k plus 1 and then those points we have chosen by construction has the 0s of this polynomial because this is the way we construct this polynomial here. So once we have that polynomial constructed that way that is we have tabulated x_0 up to x_k the 0s of this polynomial then we have this quantity that is p_k plus 1 q_k w of x dx equal to 0 okay for arbitrary values of k plus1, x_k plus 2o up to 2 k plus 1.

So I hope this is clear that the two points one is that we have the orthogonality property of the polynomials okay and from which this is what we are making use of here and we are saying that we are tabulating x_0 to x_k, we choose this x_0 and x_k as the 0s of this polynomial that is by construction and then we have this relation is always satisfied for arbitrary values of x_k, x_k plus 1, x_k plus 2 and 2k plus 1.

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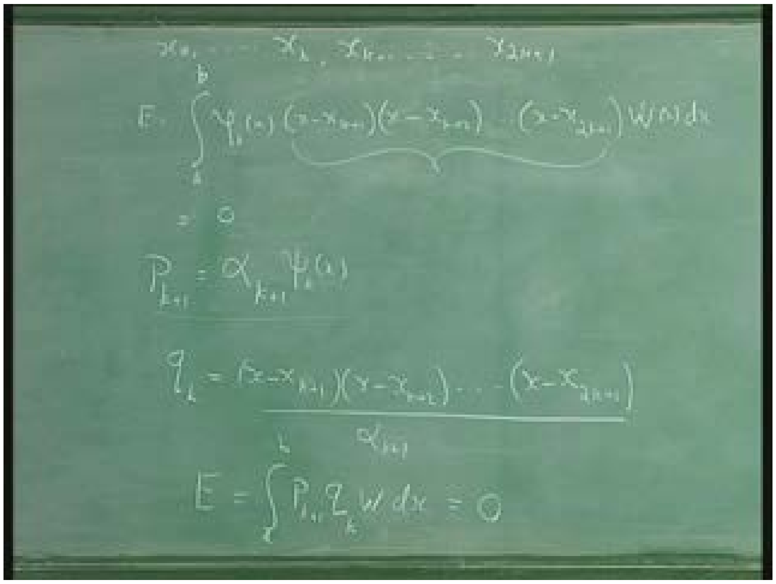

So let me summarize this once again, so we have choosen these points as the 0's of this polynomial p_k plus 1 right then the ortho properties of the orthogonality polynomial tells us that the polynomial satisfy this relation for all values of m less than k for arbitrary values of x_k plus 1 to x m x_k plus m, it does not matter what is the points at which this q_m is constructed what is the points what is the values of x at which this used to construct this q_m. Okay this is always true provided it is continuous then choice of this polynomial is, okay so of course these choice of this thing depends on the limits and the weights etcetera, but we already stated. So the assumption here is that I can always find this w at for a given limit a to b, okay such that this orthogonality property is valid and that is always true.

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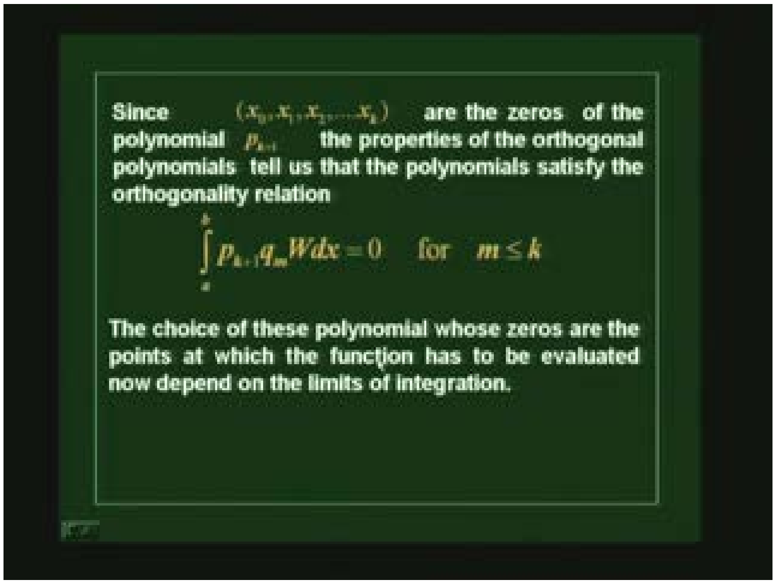

Okay so now given that we can construct now k plus 1 to x_{2k} plus 1 such that this is 0. So now we also want one more thing so we wanted two things, one is that we want this to be 0 such that this is 0 right this is same as that and this is to be 0 and we also want that the next term that is when I add 2k plus 2 term, okay and then this is of this this is of one sign right, this quantity is of one sign.

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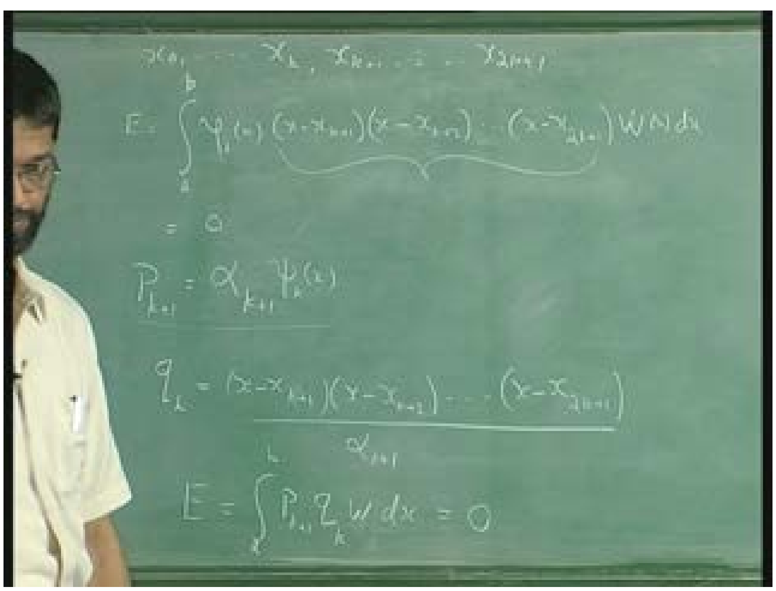

So that I can add error, so that I can use I can say that my error is actually is of the order of the order k plus. 2k plus 2 derivative of the function g right. So for that we will choose this x_k plus 1 to be x_0, x_k plus 2 to be x_1 etcetera. So that this will become this this p_k plus 1, q_k can be written as a square.

Okay that is what we are going to do here we just choose x_k plus 1to be x_0 x_k plus 2 to be $x_1$etcetera such that the polynomial the two polynomials not only satisfied orthogonality condition also that we can write this quantity that psi k in into q_m that is this this quantity p_k plus 1 into q_m as a as a square of the as a function, the square the product of the squares and hence we will one sign within this interval and then we get the order the error in the polynomial the integral to be the order of the 2k plus 2 th derivative of the function. Okay that is the summary of the quantity of the of that integral and then we had a some examples.

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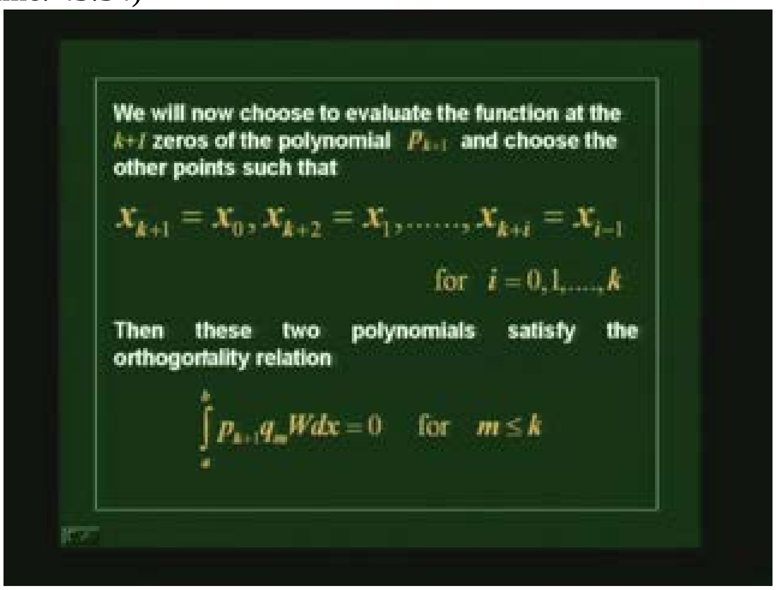

Okay so where we can use the ortho the this quantity the orthogonality relations are given here okay so for example if its in the case of legendry polynomials, okay and this is the orthogonality relations and so note that the limits are from minus 1 to plus 1 right, the legendry polynomials we are limited by the saying that this limits has to be minus 1 to plus 1.

Okay and if you use instead of and then the weight is 0, weight is 1w is 1 while in the chebyshev it is again minis 1 to plus 1 but the weight is 1 minus 1 by 1 minus x square to the power half. Okay so if you have a function this actually goes to this diverges at x this this quantity 1 by 14 minus x square to the power half diverges at x equal to 1, these 2 limits we may be having if you have a function to integrate which diverges like that we will have difficulty integrating but you can use and this will be a well behaved function.

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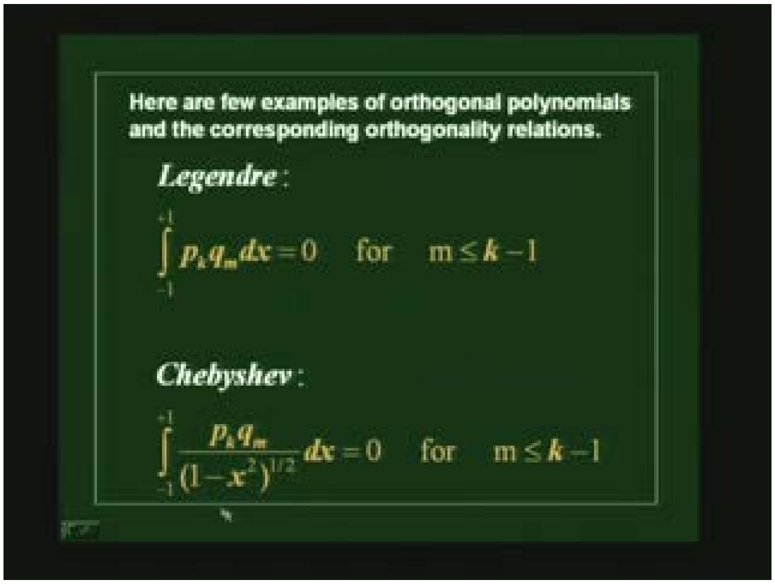

Okay and then there are other examples for example we could use Laguarre and then we have the weight as x to the power of alpha e to the power of minus x and the limits are from 0 to infinity depending upon what the limit is, okay if the limit is from minus 1 to plus 1 you are going to use one of these but if the limit is from 0 to infinity, then you could use this form to construct the polynomial the weights right.

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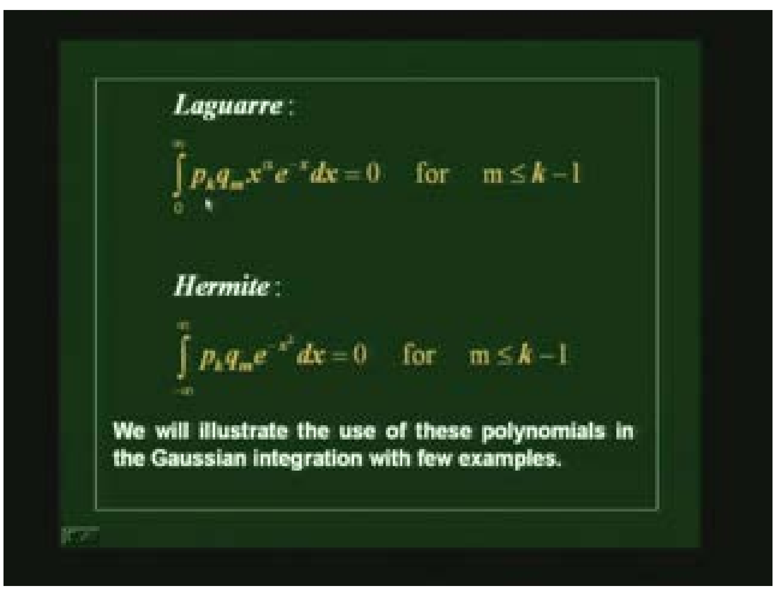

So all this will go in the construction of the weights and the and choosing the values the ta the values x_0 to x_k because we want to choose those to be the 0s of the polynomial p_k, our approximation p $_k$ correct that what we want to choose. So what the way we construct so the when you do the integral we have to first choose the poly the points at which the function has to be evaluated and then calculate the weights. So to choose to calculate the weight as we just saw the weight will depend on the limit right. So the weight here, the weight is the weight depends on the limit and also on the, so these are the weights right the weight depends on the limits and also on the order of the polynomial.

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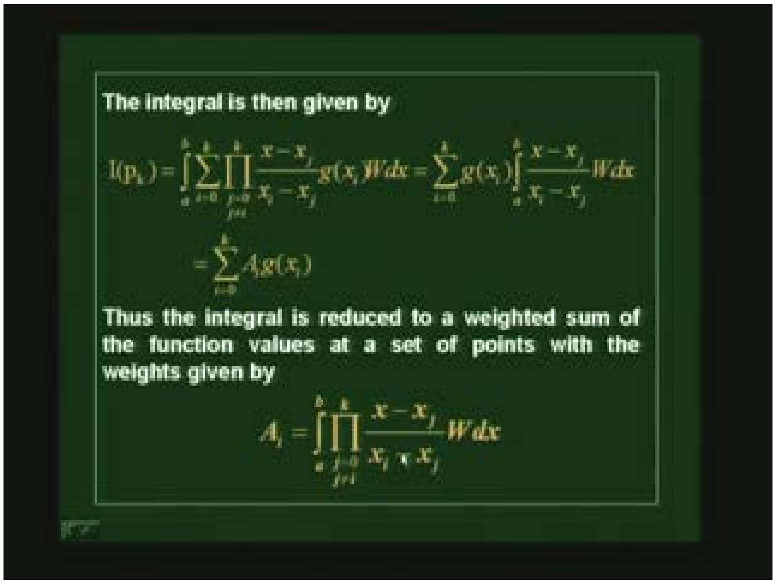

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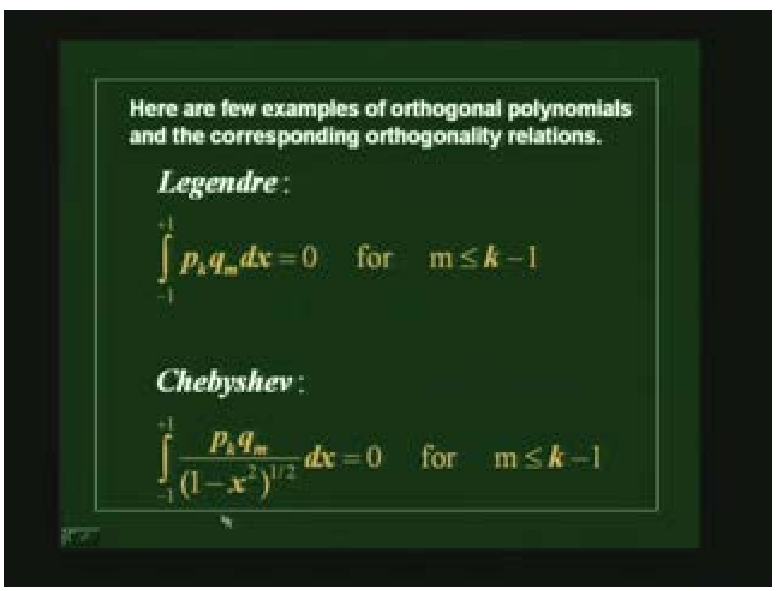

Okay and the values at which we have choosen the function to be but it does not depend on the function value itself. In case remember, that if it does not depend on the function value but then but we have to know the limits and so what we see is the orthogonality relations is coupled to the way w the function w and the limits right. So depending upon what limit we want to into what do the limits of our integration is we have to choose the corresponding method either Legendre chebyshev or Lagaurre or Hermite. Okay so this is also depends on what your functional form has, okay the function has 1minus x square to the power of half kind of singularity then we use this because the function will be the integral will be well behaved we know that and it will be easy to tabulate the function because the what we have to tabulate is g not f that will be well behaved. So that is tells us the choice we will look at some examples of such quantities in the in the next class.

Okay so and we want if we have the limits from 0 to infinity, if you want to do a integral 0 to infinity, then we would use laguarre okay or minus infinity to plus infinity then we would use Hermite. Okay so these are different examples of orthogonality relations which we would be using to determined. Okay so the method is very simple finally.

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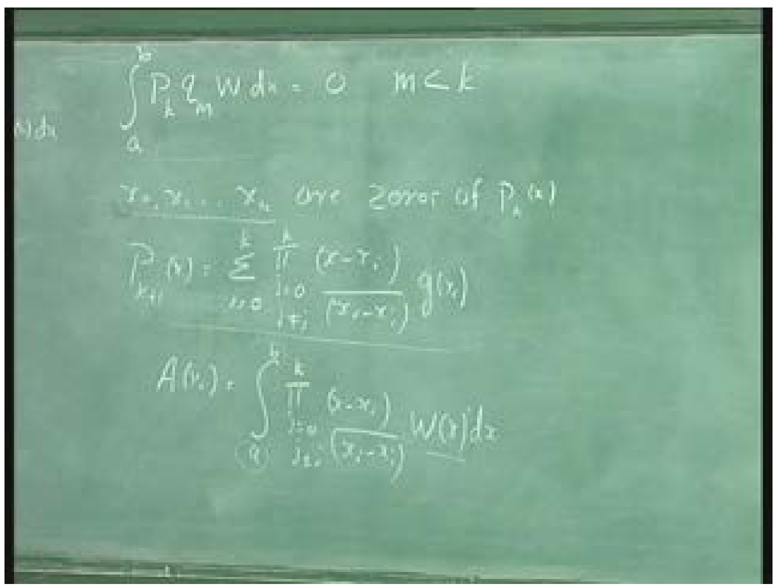

So once we have decided what form of what method we are going to use whether we are going to use Legendre or we are going to use Hermite or whatever so one of this methods then then we will use the x_0, x_1 up to x_k and we have decided what order we want to integrate or order we want to what order of approximation we want to do and then we would choose that many points as the as the first that many 0s of the polynomial p. So we are using the Legendre, gauss Legendre and then we would say that and then we want to do a k th order gauss Legendre then we will choose x_0 to x_k as the first k_0 is of the of the Legendre polynomial p_k. Now if you are if you are choosing it to be a Hermite and then we will choose this to be the first k_0 is of the of the Hermite polynomial and then we can construct the weights. So we will construct the weights by just integrating this quantity, so we will construct the weights a of x_i as integral a to b now the limits affix by what we are going to use there and that will gives us by j going from 0 to k_j not equal to i of x minus x $_i$ divided by x_i minus x_j and whatever the w is w of x dx that gives us the weight.

Okay so this quantity and the limits and this i's will be determined by whether we use gauss whether we use gauss Legendre or gauss Hermite or gauss laguarre so what whatever the method we want to use and that also will be decided by the kind of function which you want to integrate and the limits. So the choice would be depending upon both the limits and kind if functions for example, here in this case the limits are the same but the you can see that this has a weight has a 1 minus x squared to the power half okay, that would be helpful in cases where the function itself has a that kind of a singularity, if a function given to you was of this form let us say f of x has some e to the power of x divided by 1 minus x squared to the power half and then you want you are asked to compute i of f numerically rite between the limits.

Let us say minus 1 to plus 1 and then you want to compute that quantity. Okay so that normal method would be difficult because this limits would diverge. So but then you can write this as integral minus 1to plus 1 g of x which is now our e to the power of x which is well behaved and w of x dx, w of x is 1 over, 1minus x squared to the power half and then we can use straight away this chebyshev form.

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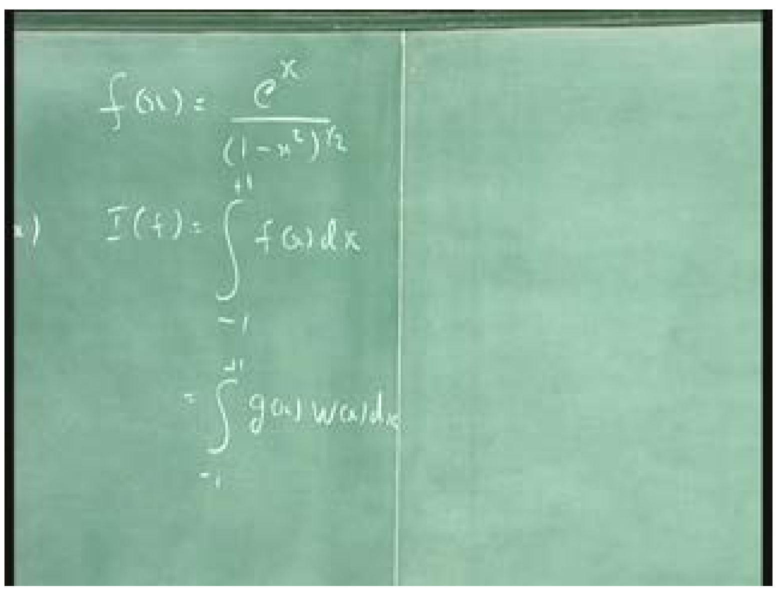

Okay right, so that is the advantage of this method. So let us look at one example here, okay that is to evaluate an integral of this form. Okay now you see it has a limit 0 to 1 x squared e to the power of minus x dx okay, so now we are we want to look at the four point this also called the quadrature scheme. So this is a this is a four point Gaussian method.

So we want to use four tabulated points that is the polynomial will be of the order 3, okay so we are going to four tabulated points is what we are going to use x_0 to x_3 okay so but the limits are 0 to 1 and we have looked at this all the 4 methods which we know right now they all have limits from minus 1 to plus 1 or 0 to infinity minus infinity to plus infinity etcetera.

Okay so we have what we have to do is to first convert the limits we have to change the variables such that the limits goes from minus 1 to plus 1, okay that is the first step to do right so we will we will do that here. Okay so we will first take you make a slight changes of variables we say we are constructing a new variable t which is minus plus two x so we see that x goes to 0 when x equal to 0 t is minus one when x is 1, t is plus 1,so limits are changed from minus 1 to plus 1 in stead of 0 to 1.

So then we have the integral i which is 0 to 1 x squared e to the power of minus 2 x dx will now become in this changed variable form right. I will substitute here because 1 plus t whole squared e to the power of minus 1plus t dt. Okay so 2 x becomes 1plus t right. So t to 2 x is 1plus t minus 2x becomes 1 plus t, so x squared become 1 plus t whole squared and dt is 2 dx. So here is 1 by 2 etcetera coming here. Okay that is what we would get this here okay and minus 1 to plus 1 is the limits.

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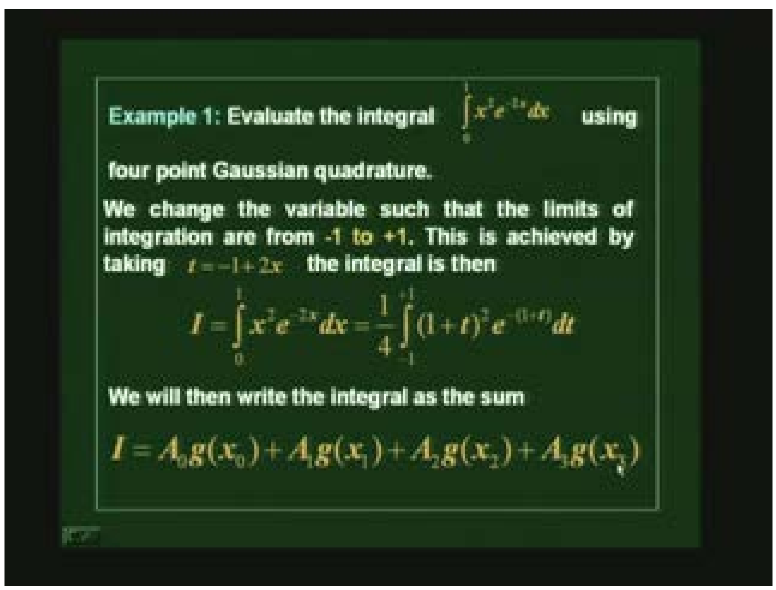

Now we can use the standard form for gauss Legendary, okay so because so now our function is this okay and limits minus 1 to plus 1 so I can use the gauss Legendary form and write it in this form. So now x_0 in this form and these are the weights a_0, a_1, a_2, a_3 right four points and x_0, x_1, x_2, x_3 are the points at which i am going to evaluate this function. So what I mean by x_0 is actually okay this changed variables at t_0, t_1, t_2, t_3 etcetera. So now I want to evaluate this function this is the function which I am going to evaluate not this because I have to make the limits minus 1 to plus 1, okay and then I have to conclude calculate the weights. So now these points 0, 1, 2, 3 will be chosen as the first 4 0s of the Legendary polynomial. Okay that is what I would choose.

Okay so now if we choose so the legendary polynomial the fourth order Legendary polynomial which we have to use now right. So that will that will be this look, so we will choose the first 4 zeros of that Legendary polynomial we need 4 points. We choose the first four zeros of that legendary polynomial has the 4 points of that okay and then the weights will be then given by this integral correct that is what we would be using, so now it is in the t its in the changed form changed variable form.

So please pay attention to that that is we are in the changed limit form so we are going to use 1 plus t squared e to the power of minus 1 by 1 plus t by 4 that is what our function is. Okay that is what I am going to use as a function here right so and then finding the we are going to evaluate this function at the first four 0s of the Legendary polynomial. Okay written the Legendary polynomial here, normally these points are these are always tabulated in many standard tables which lists the 0s of this polynomial.

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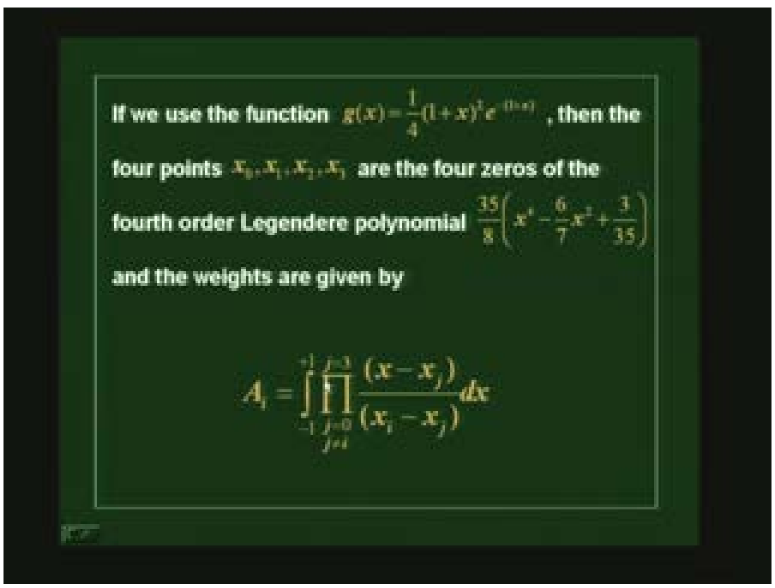

So we do not have to spend time constructing that and so we have this we just read off the 4 zeros from some tables. Okay and then we are going to construct the weights from this, okay and in fact many of the weights are also listed in in standard tables of integrals and series. So only thing which we have to make sure here is that the limit is always minus 1 to plus 1. So we have to do the appropriate change of variable, so our x squared e to the power of minus x squared has now gone to 1 plus t squared e to the power of minus 1 plus x because we want to make the limit from 0 to 1 to minus 1to plus 1. So that is the point that is the point where we have to pay the attention to.

Okay so but many times we do not want to do that so we could write the program such that it is you know transparent in that sense. So we would just construct we could use this function itself but then we have to evaluate this function at one plus x_0 by 2, 1 plus x_1 by 2, 1 plus x_2 by 2 etcetera.

We could not evaluate earlier because we have to evaluating this function could do it to x_0, x_1, x_2, x_3 but otherwise we have to find, we have to do change a variable and then evaluate the function. So that is also possible but then then the limit calculation would be from 0 to 1okay so that is weight calculation will be from 0 to 1.

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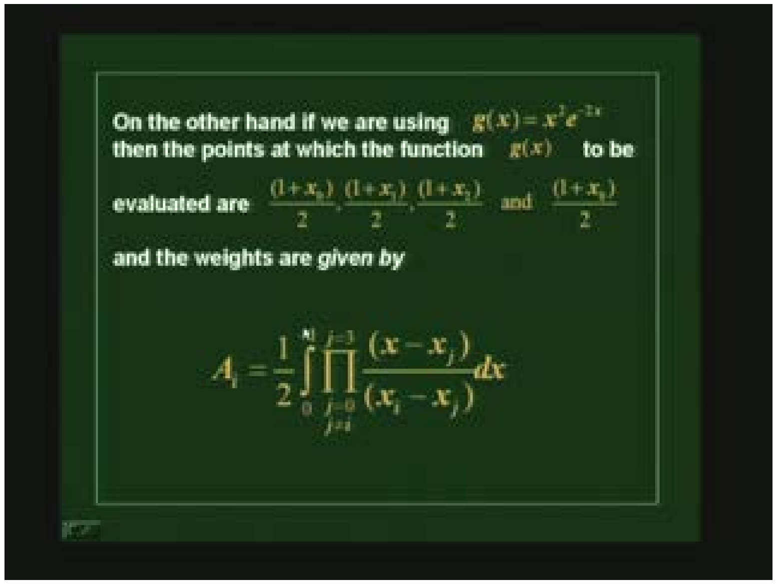

Okay so there are 2 different ways of doing it, but basic idea is that we are going to change the limits, okay so you could either change the limits and evaluate the integral okay change the change the variable and evaluate the integral from minus 1 to plus 1 to calculate the weights or you could change the tabulated points itself. Okay the value of the tabulated points okay now x_0, x_1, x_2 are the 0s of the legendary polynomial but you evaluate the function at 1 plus x_0 by 2, 1 plus x_1 by 2,1 plus x_2 by 2 etcetera, so that is the 2 different ways of doing that.

Okay so now in general if you have an integral, okay just to summarize in general if you have an integral of this form okay f of x dx and if you decide to use Legendary form and then you could construct the weights by just doing this that b minus a by 2, a to b choose the same a to b, we do not change the limits here. Okay but then evaluate the function at a plus b by 2 when you are evaluating the function you change the variable there.

Okay so now this x_j's are the 0s of the Legendary polynomial but you are going to evaluate this function at x_j prime which is given by a plus b by 2 minus a minus b by 2. So pay attention to that again, so you compute the weights we do not change the limits, okay but when you evaluate the function you evaluate them not at the 0s of the Legendary polynomial but the 0s of the Legendary polynomial multiplied by this a minus b by 25 and subtracted from a plus b by 2 our a plus a and b are the limits your integral and as long as a and b are finite, we can do this change of variable and then calculate the calculate the integral.

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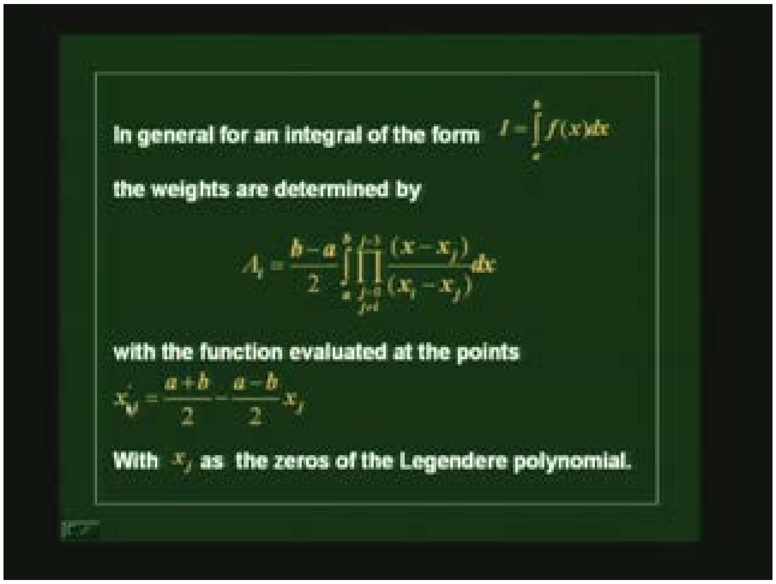

Okay so we will see actual computer code which implements such a thing in the in the next class.