Numerical Methods and Programming
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Lecture - 28
Numerical Integration: Comparison of Different Basic Rules

In the last class, we discussed numerical integration and the basic rules for numerical integration or the basic theoretical foundation for numerical integration. So we just summarize before we go into the next step of polynomial approximation to the integration, we will just go through and summarize what we did in the last class. So the idea was to use replace an analytical formula of this type for an integral of a function between the intervals a to between the intervals a to b to by some numerical method that was, that we necessary in cases where this function itself is the functional form is not known or sometimes even when the functional form is not known but the integration is difficult or impossible analytically.

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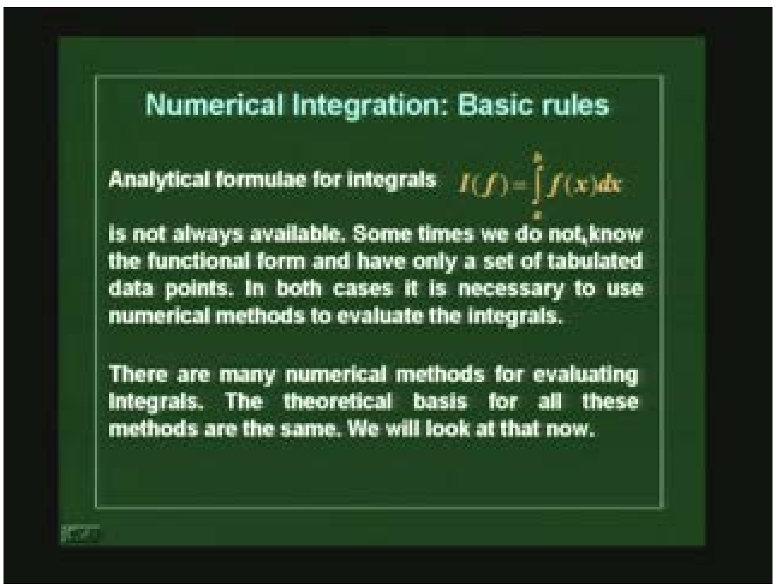

So we said the basic idea here would be to approximate the function by a polynomial of order k and then say the error in the polynomial would be then of this form that is here phi k being the product of x minus x_j between the intervals for all values starting from 0 to k, is order k plus 1 polynomial term here so this is order k polynomial plus an order k plus 1 term and we know what is coefficient is from the divided difference.

So then we said that now, if you approximate it that way then we have the error in the polynomial as the difference between the integrals of these two that is the integral of the function which is the actual function which we do know and minus the integral in the, of the polynomial and that will be this particular term and then we looked at 2 different cases so that is, so in this error 2 different cases were especially analyzed one case where this term psi k of x is of the same sign in the whole interval a to b.

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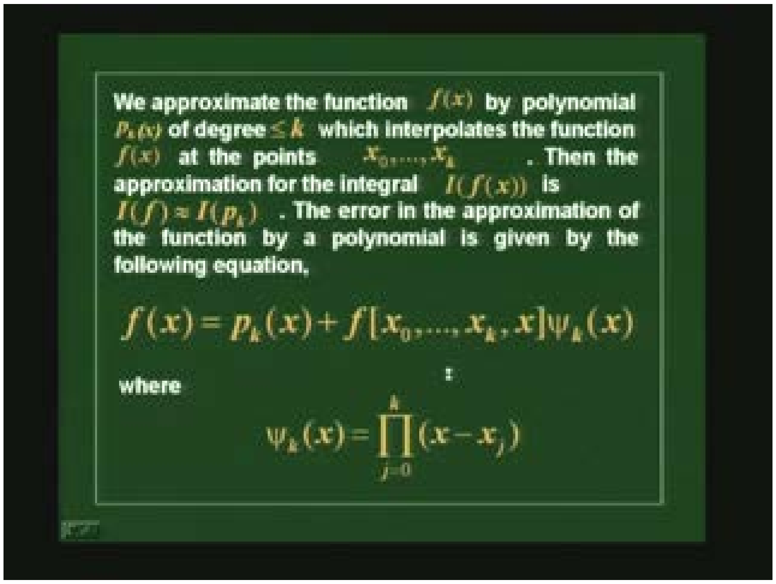

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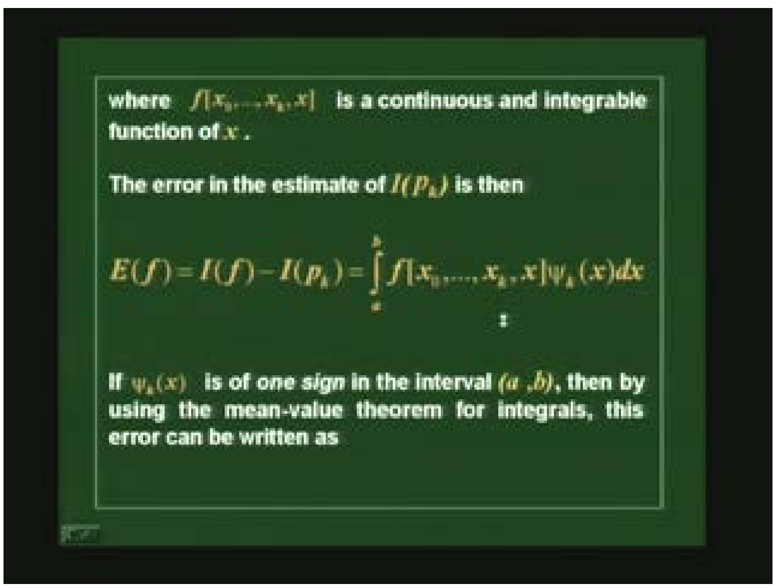

In that case we said that using the mean value theorem I can write the error. So it is of the same sign whole throughout the interval a to b then I can approximate this error by f at evaluated at some values zeta between the intervals a to b multiplied by integral psi k of x dx this is and then using the mean value theorem, we can say that this is given by the k plus one derivative of the function divided by k plus 1 factorial at some point it.

Okay now this true only if this is of the same sign that is psi k of x that is order k plus 1 polynomial is of the same sign in the whole interval between a to b that is in the interval a to b. So that case we can use this okay and then we saw that in another case where when that is not of the same sign but a special case again that psi k of x dx in the interval a to b is 0 in such cases we can again write this that the coefficient the term in the error the first term in the error that is the f this term we can replace by k plus 1or a term is by using the divided difference method which we have learnt in the polynomial interpolation.

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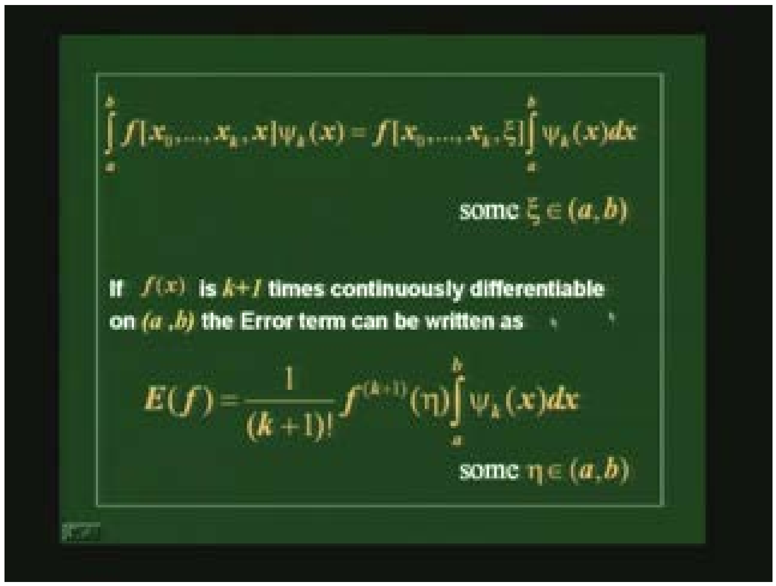

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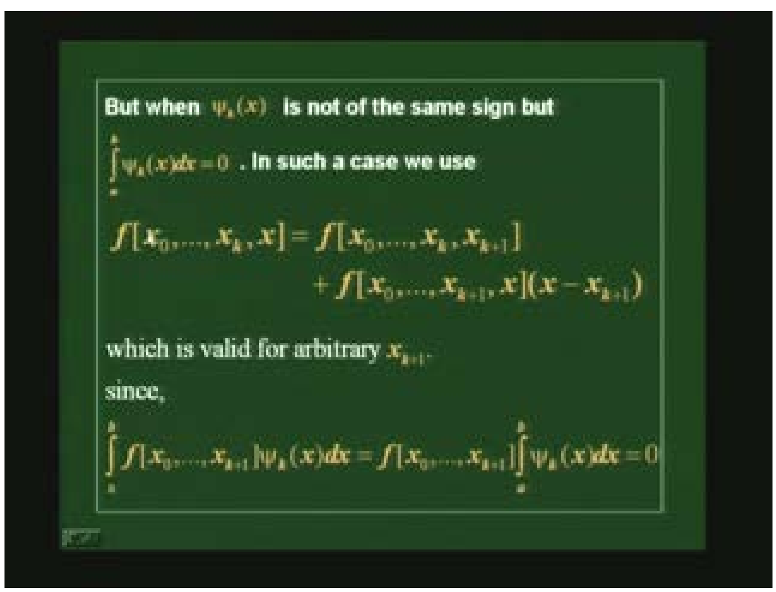

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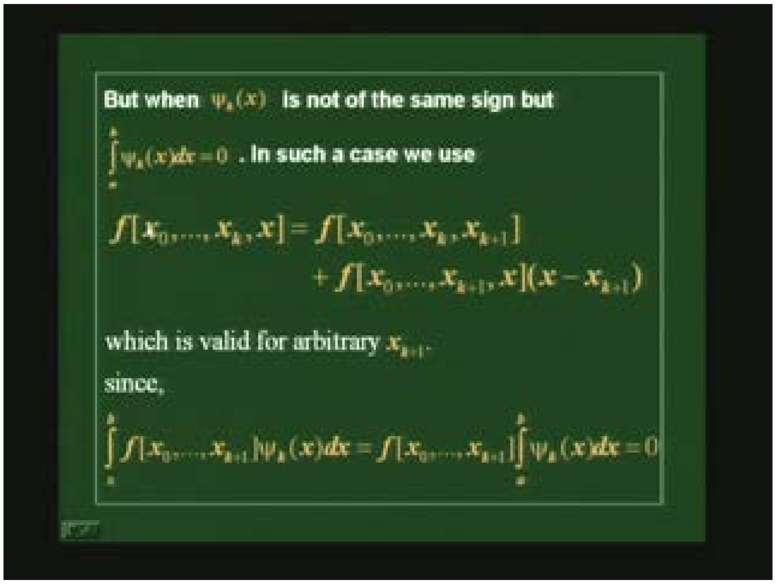

I can use the divided difference method and write that k th order term into k plus 1th order term as a into which is a_k plus 1 th order term, so that is then from that you can say the now the new function that is the psi k into x minus x minus x_k plus 1 that is now this thing would be replaced by psi k into x minus x_k plus 1 and then and then if that is of the same sign then I can write this error as f of f k plus 2 derivative of a function f.

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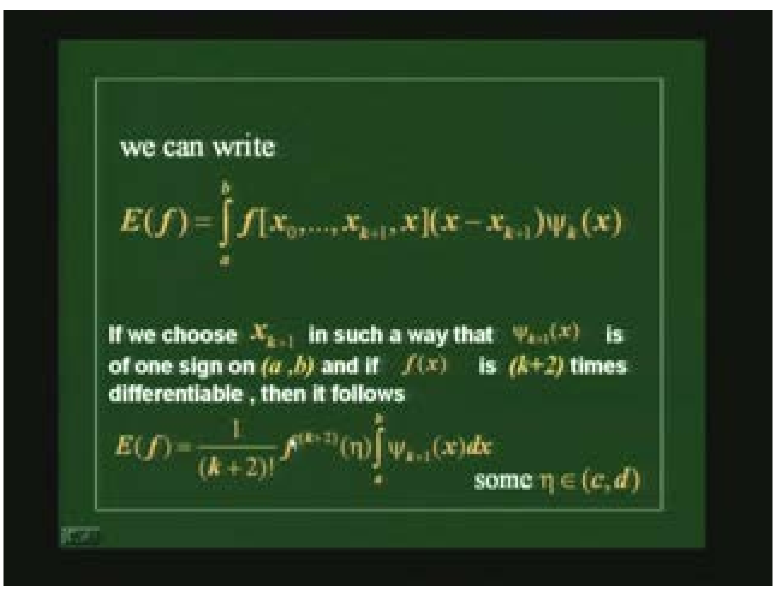

That was the basic foundation and then, we say that now if it is not true that if this term that is x minus x_k plus 1 into psi of k is still 0 in the in the integral of this. Assume that x minus x_k plus 1 into psi of k is still 0 in the integral the integral of this is still 0 and then we write the go to k plus 2 order term and then we write it as the error as now become the error will now become as something of the order the derivative of k plus third derivative of the function. So in that way we could to a large order and accuracy, so that is what said and then we looked at three different cases actually we looked at two different cases yesterday that is one case was what we called a rectangular polynomial, rectangular method for approximating the integral that is replacing the function f of x by p_0 of x. So the idea was that all the integral a function f of x of this form.

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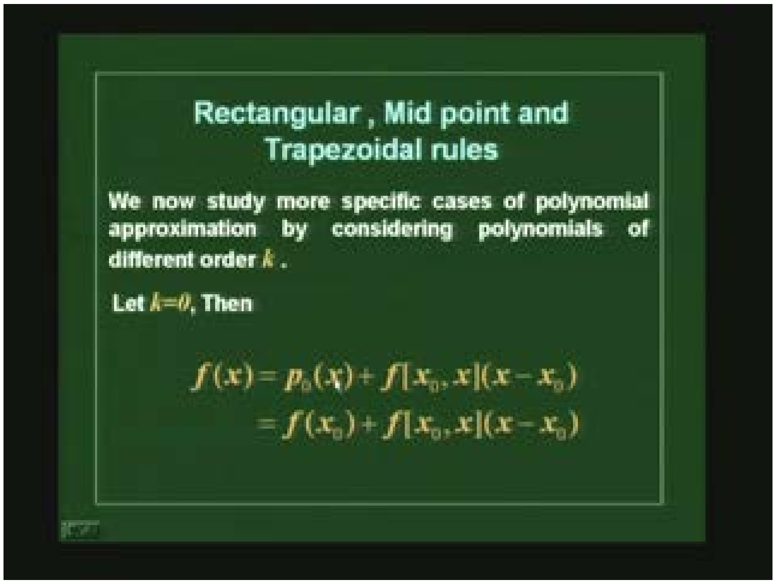

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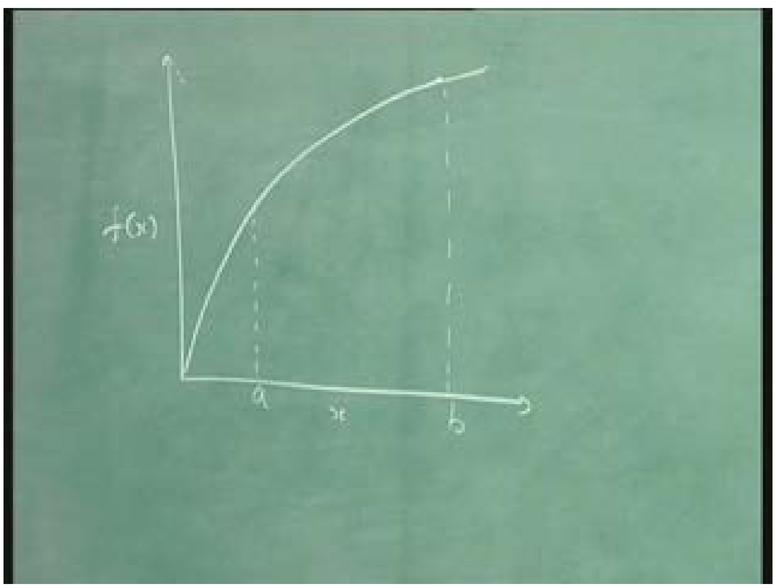

If you have a function of this form and then if you want to integrate that function between the limits a to b, what we mean by that is actually finding the area of under this curve. So that is what we mean by actually finding the integral of this curve right.

So now in the first approximation in the rectangular case where , we approximate the function f of x by p_0 of x we are actually finding the integral will turn out to be i of p_0 of x that turns out to be f of a into b minus a in the rectangular approximation that is we have a polynomial of order 0 and the tabulated value, we said in the beginning in the in the beginning of the interval then it is f of a into b minus a that is the area under this particular box here.

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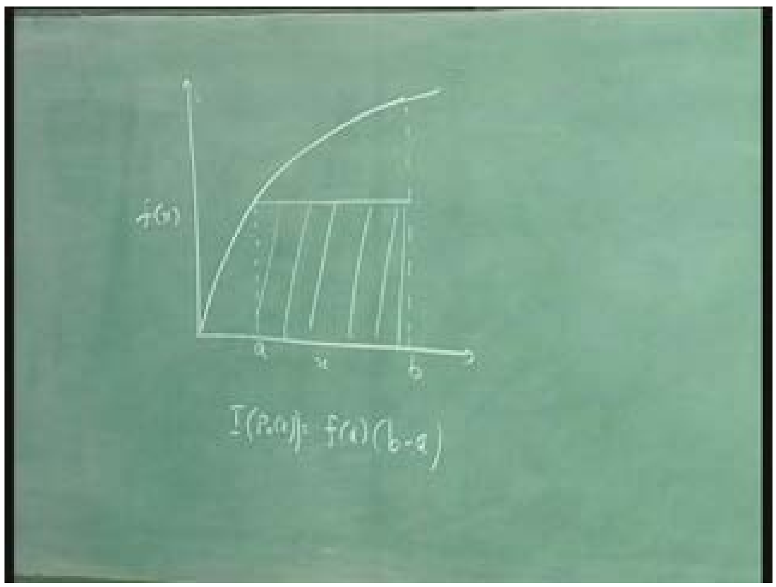

Okay that will be the area under this box that is the one method we looked at another method which we looked at was to say that we will not in this case the error is of the order is the order first derivative. Okay the first derivative of the function that is the error in this order of first derivative and the second case we looked at is to say that we can tabulate we can tabulate the function at the mid point between the interval a to b, in that case, we have a case where x minus x_0 dx that is psi 0 of x minus x_0 dx is 0 instead of being one sign in the earlier case it was one sign that is when we took the tabulated the function at first point of the interval in the beginning of the interval but now we tabulate the function at the mid point of the interval and then psi 0 of x integral a to b dx is 0 in that particular case we can write the integral as you know b minus a f into a plus b by 2.

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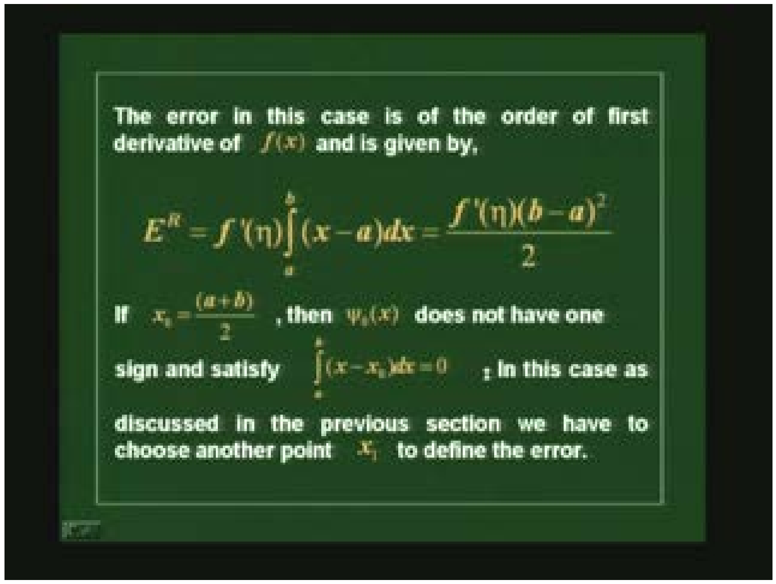

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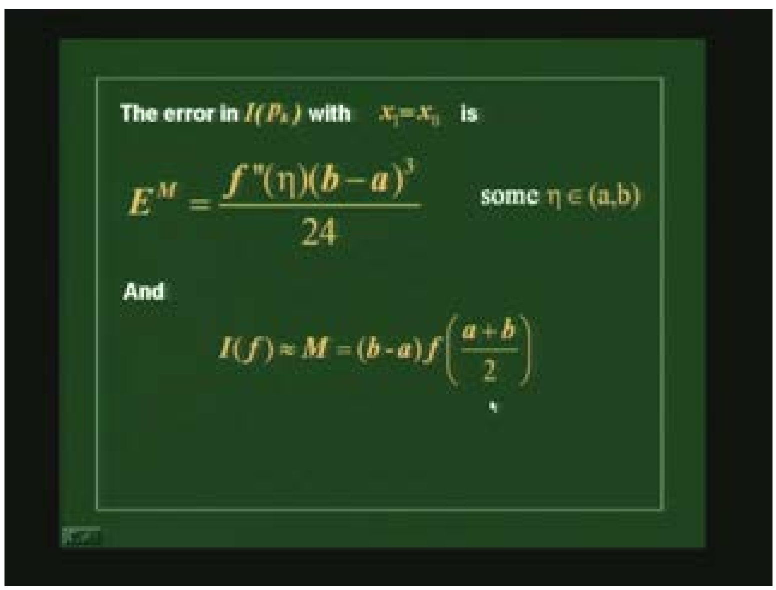

So that is saying that now I will not do this but instead I have the function f of a now I have f of x versus x. So it is a function like this and I am tabulating I have this the interval that I want to do right so that is a and that is b but I will tabulate this function in the middle here f of a plus f of b by 2. Okay that is a plus b by 2 and I have the function value at a plus b by 2 here and then I will compute the f of a plus the i p_0 of x in this case is f of a plus b by 2 into b minus a that is the integral and so, here is f of a into b minus a here its f of a plus b by 2 into b minus a it is a revers different box and the box would be like this see now if the box you are computing the integral as the area of this box is what now we are computing.

You can see both of them I have errors this has the error by this amount and while this has an error by this amount and this amount and you can see that the error here is the error here is of the order of the second derivative error here is the midpoint in the integration

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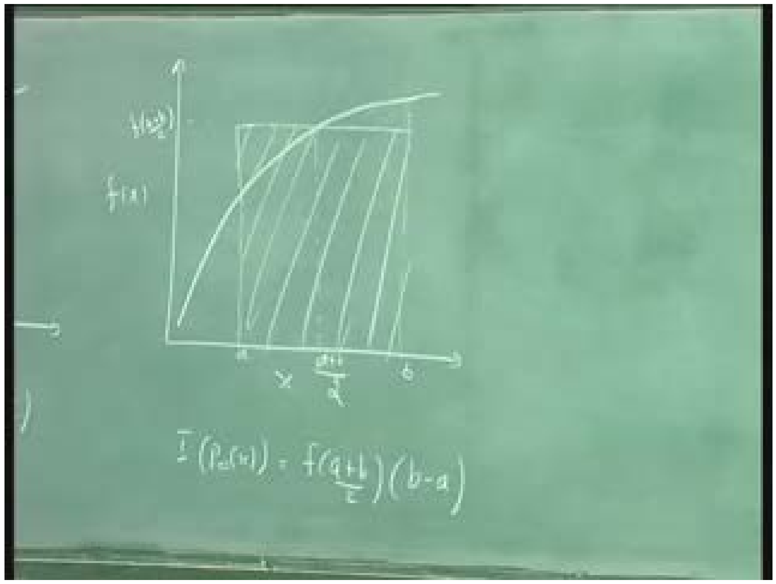

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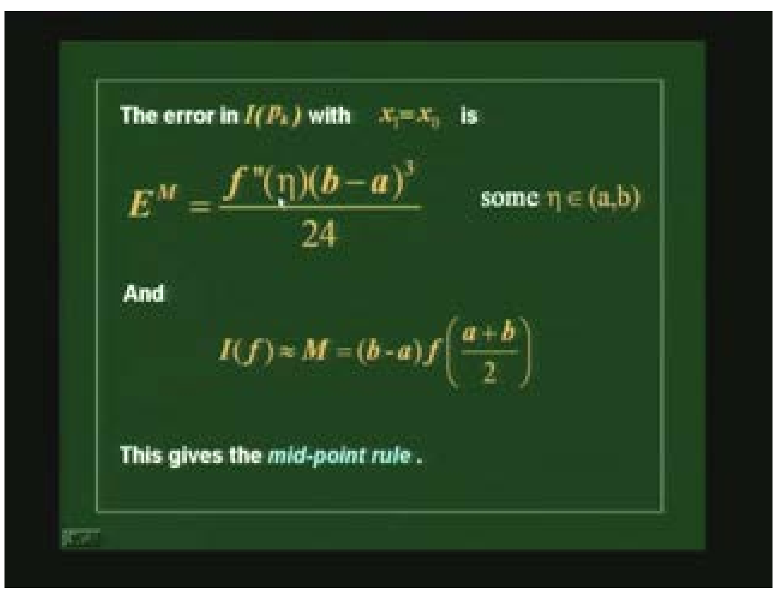

Okay the error is in the second order okay that means if the function is second derivative of the function is 0 then the error is 0 that you can see straight away and here in the function with a straight line here with second derivative 0 and then this positive error will cancel with this negative error here. So then we will have an exact integral. So in both these cases we have error and we are just using simple idea that we just construct a box and then find the integral of area under that enclosed by that particular box and I should say that all these methods these two methods especially can be used as you do not have to cover the whole interval by this, you can split the interval into separate intervals and then evaluate this integral in succession that is I could take this particular case and then say I will do I will split this into many boxes instead of one of them okay.

So I can split this interval a to b now which I have now this interval a to b I have now I can, so I have this interval a to b and I can split that into smaller intervals right a_1 , a_2 etcetera and then compute the area under each of them using the rectangular rule or a midpoint rule okay I could use either the rectangular rule or a midpoint rule to compute the area of the error it is a area under each of this rectangle, if I use the rectangular rule here I will go and do this I do this particular area for first and then I will do this particular area I will still have an error because of this this little boxes here little triangles here will be left out okay but it will be much less than the earlier case.

So you can improve the accuracy of this integration by going into smaller rectangles and same case here too you could go in to smaller boxes and the then improve the accuracy that is to say that.

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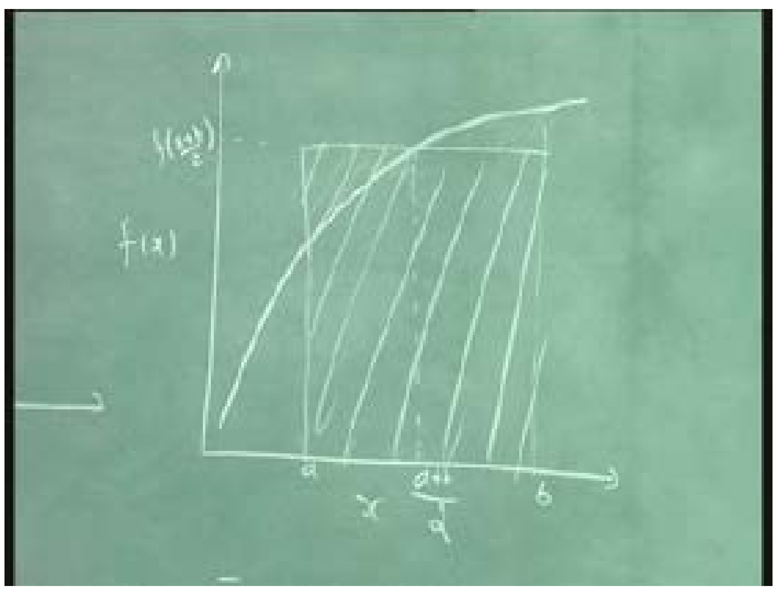

You can improve the accuracy by either reducing the interval b minus a or go into smaller boxes split the b minus a whole b minus a into many intervals and then you can increase the accuracy. So now these are the two methods we have looked at, so what the mid point rule and the rectangular rule that is both order are a_0 polynomial.

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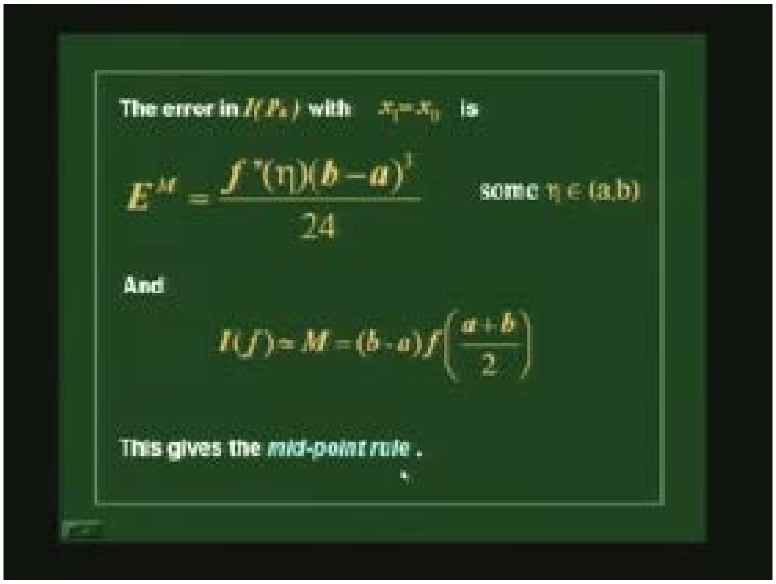

Now we look at the case where we have now, we approximate the function by an order one polynomial. So now we look at polynomials order one that is approximating the function f of x by a polynomial of order 1 as we write f of x as f of x_0 from in the Newton's form f of x_0 plus f of x_0 x_1 x minus x_0. So that is our polynomial of order one and then we will have an error term which would be of this type. So that is the error term

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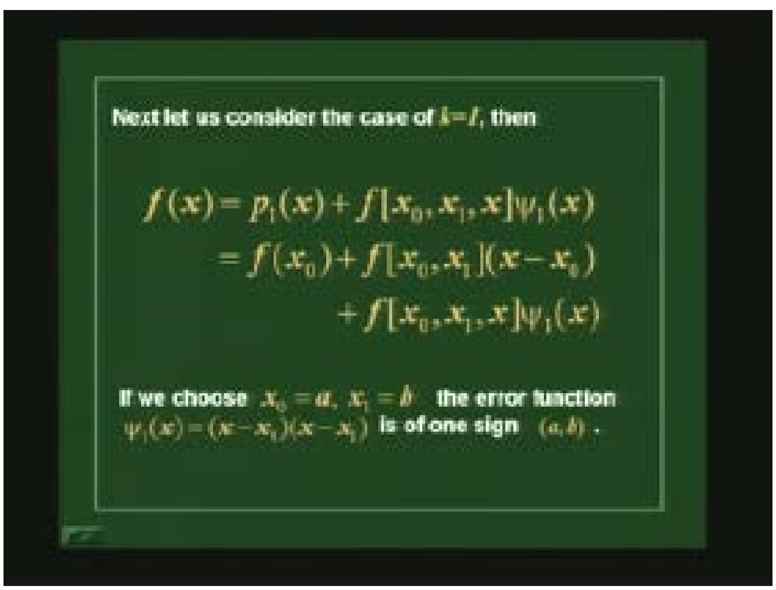

So this is our psi 1 x minus x_1 and x minus x_1. So x_0 x_1 x this our error term so now again we have to make choices for x_0 and x_1 and we look at this particular case where x_0 and x_1

are chosen at 2 different ends okay now that Is the case which we are going to look at so now we are taking the function f of x and then we are choosing the two intervals.

So we ll choose similar type of functions and this my x axis and this is we will have a function like this and then we choose the interval a and b and we are going to take the function value evaluate the function values tabulate the function values at these 2 points f of x f of a and f of b that is what we are going to use in the whole interval a to b. So that means that psi 1 of x is now x minus x_0 into x minus x_1 will become x minus a into x minus b x_0 is a and x_1 is b that is the polynomial which we have and then we can write the integral I of x. So we wrote the polynomial here so the function we have to now approximated by f of a right plus f of b minus f of a divided by b minus a into x minus a. So that is our function and then we have the error term here which is which is f of a, b_x into it is a square bracket it is the coefficient into x minus, x minus a into x minus b. So that is the polynomial.

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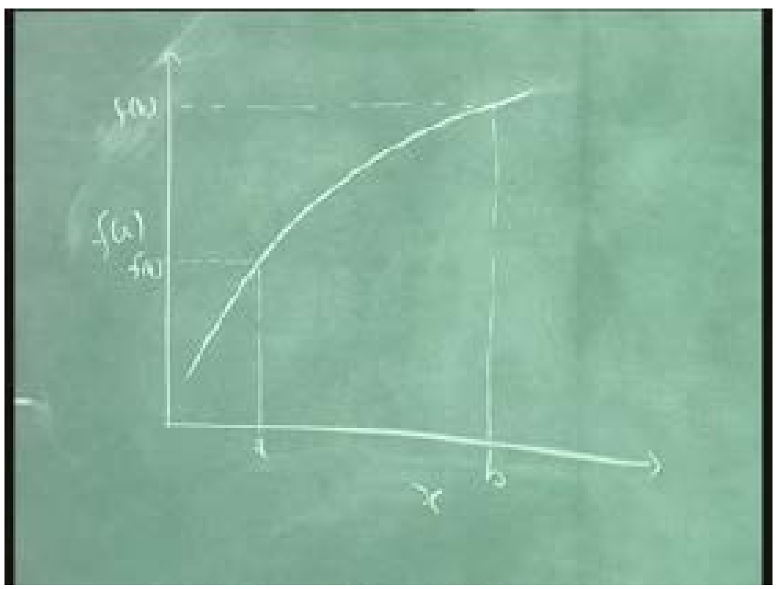

So integral I of f well now this the approximation this the polynomial. So I should write this actually f of x approximated by p_1 of x equal to that. Okay now this i of f approximated by i of p_1 of x in the interval a to b right that is, so that is would be a integral of this right. So that is f of a into b minus a plus x squared minus a evaluated in the interval b to a x squared minus a by 2 evaluated in the interval a to b that is f of a into b minus a plus f of b minus f of a divided by b minus a into x minus a whole squared by 2 in the interval b to a and that is the approximation and that is the error term next to that so you can simplify that and write in this form.

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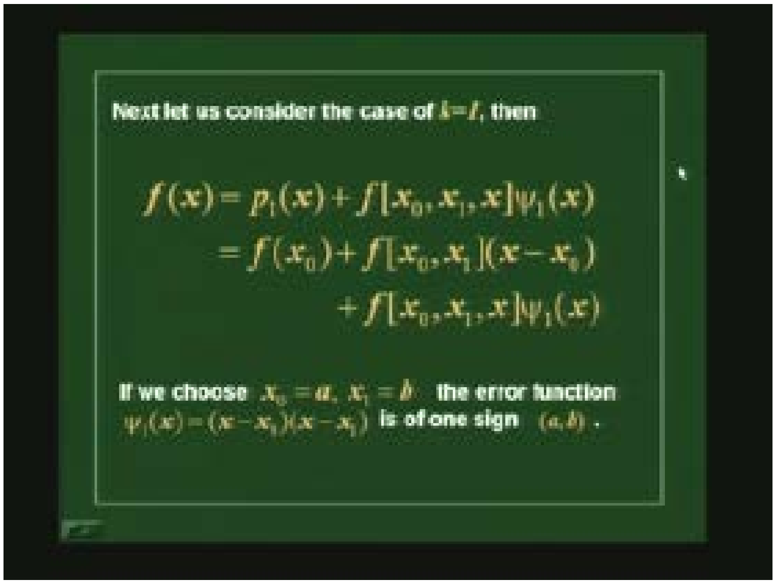

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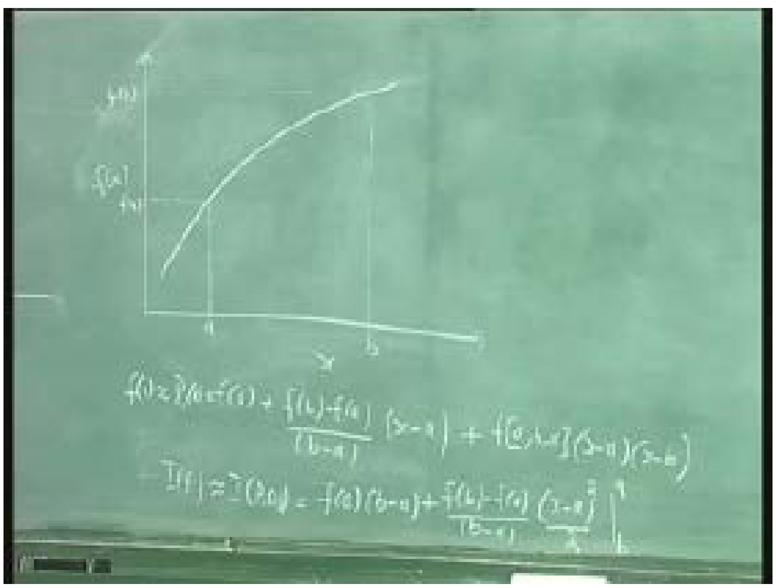

So the integral will be now become b minus a into f of a plus f of b by 2. So, now graphically this is the same as drawing a straight line from here to that okay and taking the area under this curve. So that is what we are getting here this integral which is actually f of a plus f of which which is equal to b minus a by two into f of a plus f of b. Okay that is the same as finding the area under this curve which is under this trapezoid which we get okay hence the name called trapezoidal rule, okay in which we will actually draw a line from here to there at 2 end points and then find the area under that trapezoid, so we can see that actually these terms which we are getting here. So what we are getting this is these two together is actually the area under this rectangle here and then the area under this triangle so that is what we are getting these two terms here that is the sound which is actually the are under the trapezoid. So hence that the name trapezoidal rule.

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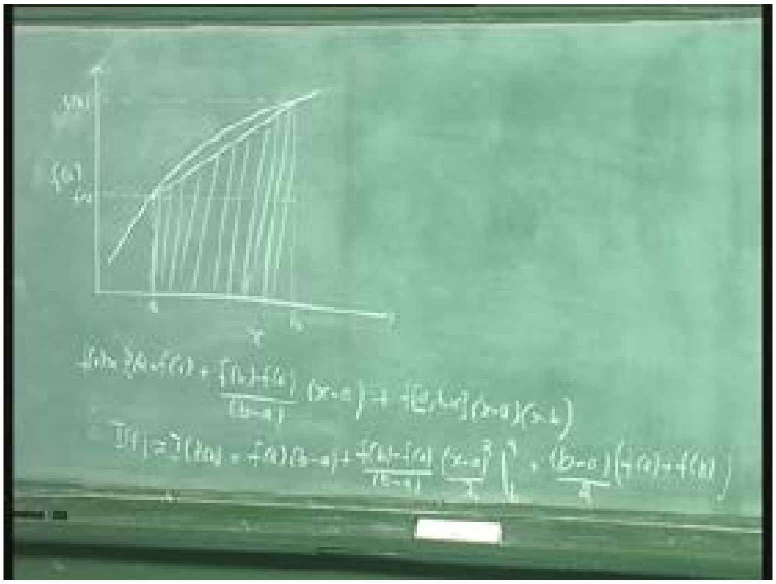

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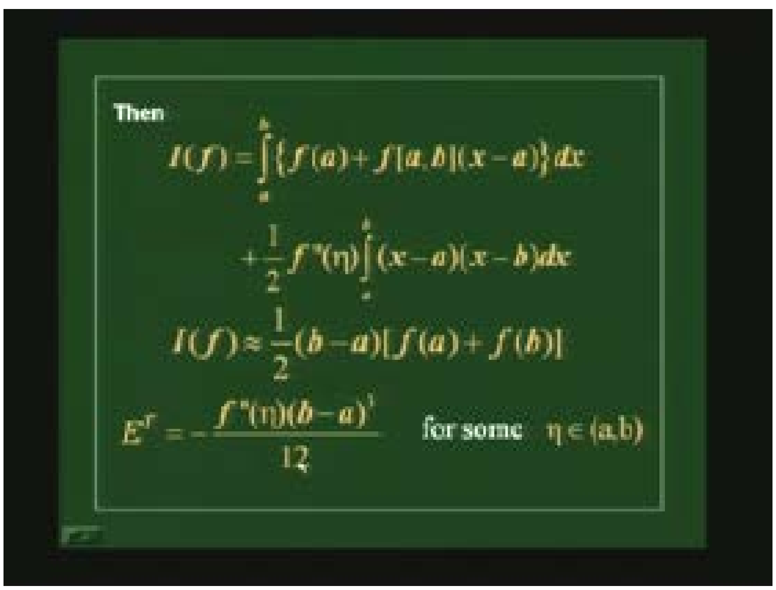

So now the error again here in the trapezoidal rule which is represented by ET. Okay so we have ER for rectangle and for midpoint an ET we have to use this. Okay, that is again of the order second derivative in f into b minus a cube by 12. Okay then the reason being that again as I said reason being that, so the error term this x minus a into x minus b now this is of one sign in the whole interval.

Okay and hence, so x is always less than b okay that is negative and x is always greater than a. Okay this this term is positive the product is always negative which is of one sign and hence we can write this as in this form in the error as in this form and that error is of the order of a second derivative for the interval a to b. Okay so this is another case where we get an order second derivative of the function the error of the second derivative of the function.

So again notice that we could we do not use this method for the whole interval at one short it could again split this into smaller intervals and use them to and find area of each of this trapezoid in between accuracy in that case would be pretty high. So you can see here.

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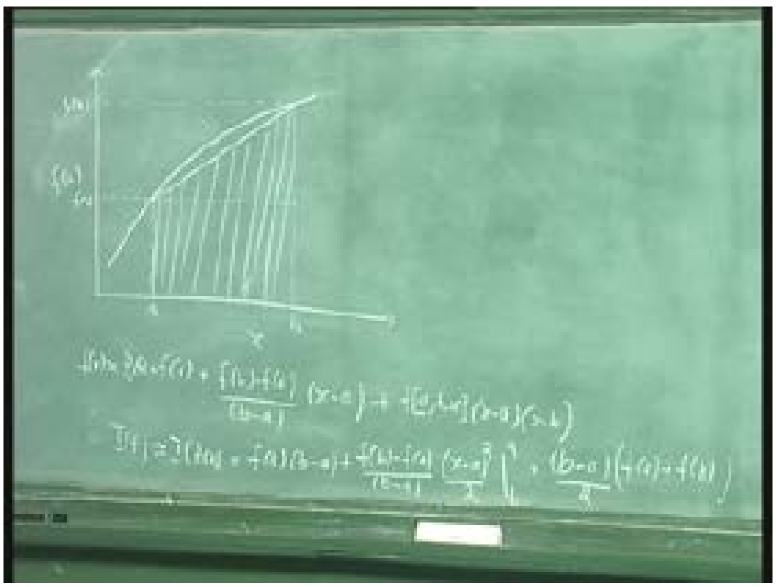

So that is the case of the first order polynomial called the trapezoidal rule. So now so far looked at 3 that is rectangular midpoint and the trapezoidal rule. So you could do this problem that is you could just take cos squared x dx between the limits 0 an phi by 2 and we know what the integral of this analytically and then you could evaluate them using rectangular midpoint and trapezoidal rules and you could calculate the error because you know the exact value of this function and then you could compare them with what you get.

Okay so now so now so far, we have looked order 0 th order polynomial okay that is in the case of rectangular and the midpoint and the midpoint rule and then we looked at the first order polynomial and that is the case of the trapezoidal rule. Okay so now we will look at order two polynomial okay that is the case in the Simpson's rule.

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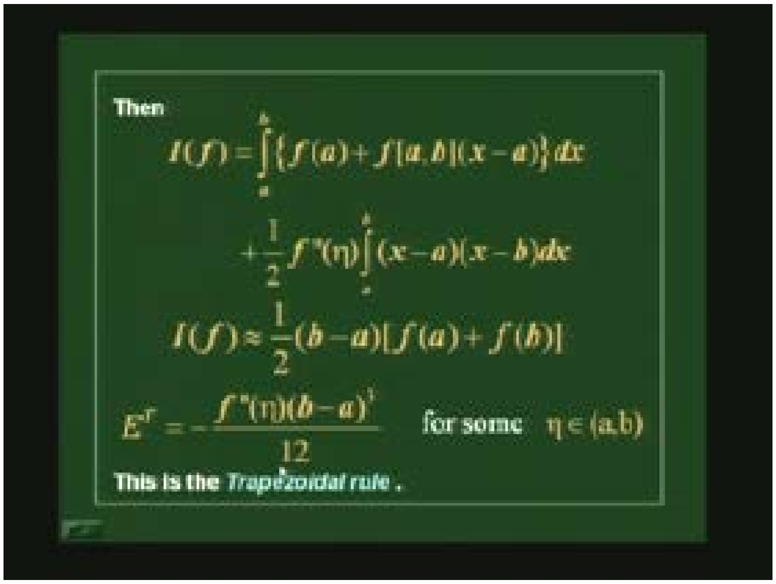

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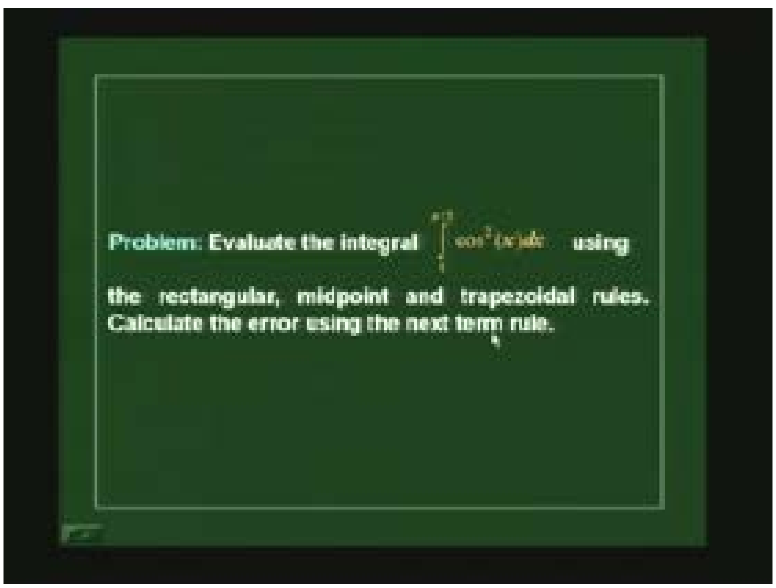

So we will look at order two polynomial are approximation of the function by an order two polynomial and it is integral. So again we will write now the function the method is now we can see that the same method is just that we to go the higher order polynomial and we have to choose, we have to tabulate the function and now more number of values and new number of points.

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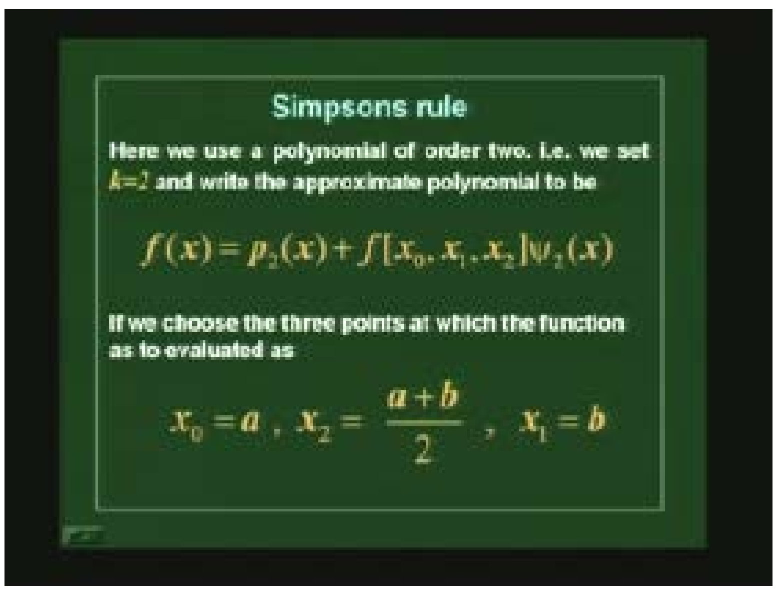

Okay we say f of x is p_2 of x plus f of x_0 x_1 x_2 psi 2 of x. So now this will be now the order three polynomial here is the error term will be the order two polynomial here so we need to now tabulate the function at 3 different values, okay and in the Simpson's rule we choose that points to be a plus b by 2. So we can see that the integral of this would be chi of x would be 0. So that is what we are going to use here x_0 equal to a x_2 equal to a plus b by 2 and x_1 equal to b.

So in this case we are going to use 3 different polynomials we now here substituting it by a polynomial of a 2. So our idea would be here f of x is now p_2 of x plus some error term right. So now the p $_2$ of x is now given by f of x_0 plus f x_0 x_1 x minus x_0 and plus f x_0 x_1 x_2 x minus x_0 into x minus x_1. So that is that this is approximation to the function that is f of a we choosing x_0 to be a, so x_0 is a and x_1 is a plus b by 2 and x_2 to be a plus b by 2 is important x_1 is b, okay that is what we are going to use.

So then we can substitute and right p_2 of x, now as f of a right and x_1 is x x_1 and x_1 is b. So this term would be now be f of b minus f of a divided by b minus a into x minus a right so now this term will now be this term we have three terms that is f of a plus a plus b by two minus f of b minus of that with f of b minus f of a now divided by, so this is b minus a plus b by 2 that is here and this would be b minus a. Now divide the whole thing by b minus a b plus a by 2.

So that thing would be divided by a plus b by two minus a. Okay that is what we are going to get the as the into x minus a into x minus b okay that is our polynomial okay now that is the polynomial we have, okay the approximation and we can now find the integral of that this is this the polynomial we are finding the integral of x minus a x minus a into x minus b between the interval a to b that gives us the okay, so I will summarize shorten this here.

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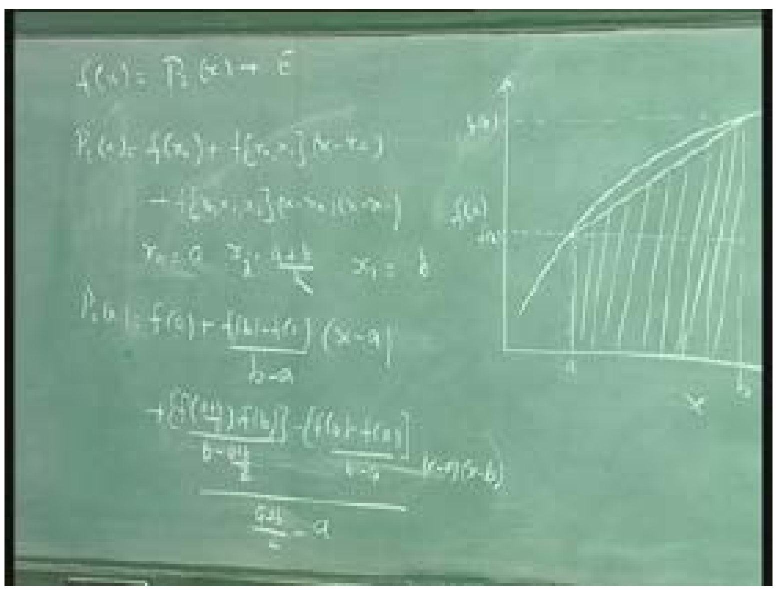

So we do the integral okay we will get this as b minus a, b minus a whole squared by 2 and then b minus a whole cube by 6 okay the this is the this is the last term so we have it coming as x minus a whole squared by 2 and x minus a into x minus b integral. So that will give us b minus a whole cube by 6.

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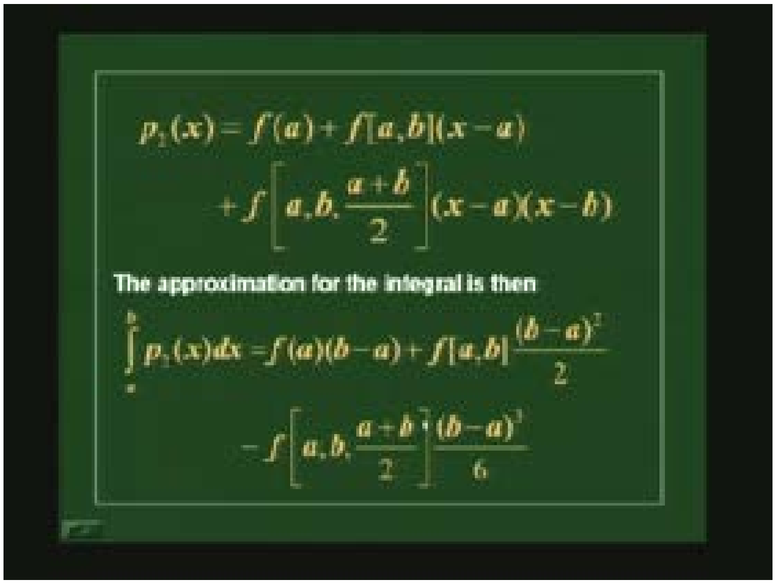

So that is the thing so we are just integrating this function and that is our approximation under the Simpson's rule. Okay so the here we are now approximated the polynomial by a second order polynomial, so its not a straight line any more it is a curved its it is a curved line with it is a curve.

So again this second order polynomial, so we will be now actually using some kind of curve to an to an connect this 2 points right and then calculating the area under that curve. So obviously that would be much more accurate than this in cases where the second derivative of a function is non-zero okay in this particular case Simpson's case, we have taken 3point samples right so we have we have taken a, b and we have taken one more sample here. Okay so that is f of a plus b by 2, we have taken one more sample there, so now our curve now and we approximated this curve by a by another some polynomial of this term.

Okay so that will be the and then find the area under this whole curve that is what Simpson's rule do, okay in the cases where the functions is not so smooth okay, where you have for example the function could be need not be smooth in the interval a to b. So you could have f of x versus x have some curve like that okay not so kinky, but smooth okay in that case we could again split this into different intervals okay smaller intervals and then evaluate the evaluate the integral for each of the intervals okay and then and then compute this total integral as a sum of all those integrals.

So that is the Simpson's rule and here the error would be obviously of the order 3. Okay in the case where now this is this the final answer which you get it as okay the formula, so f of a finally the answer would be get is that p_2 of x would be b minus a by 6 f of a plus 4 f into a plus b by 2, f of b this is just the integral of all of this okay.

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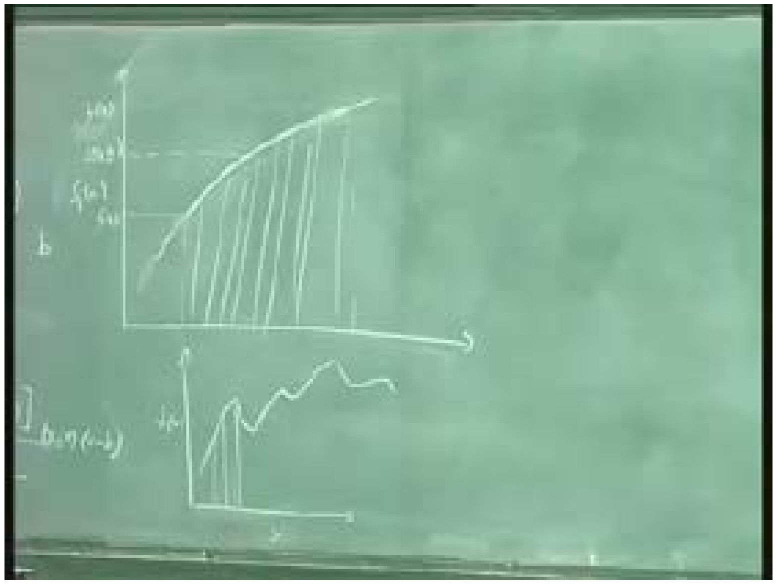

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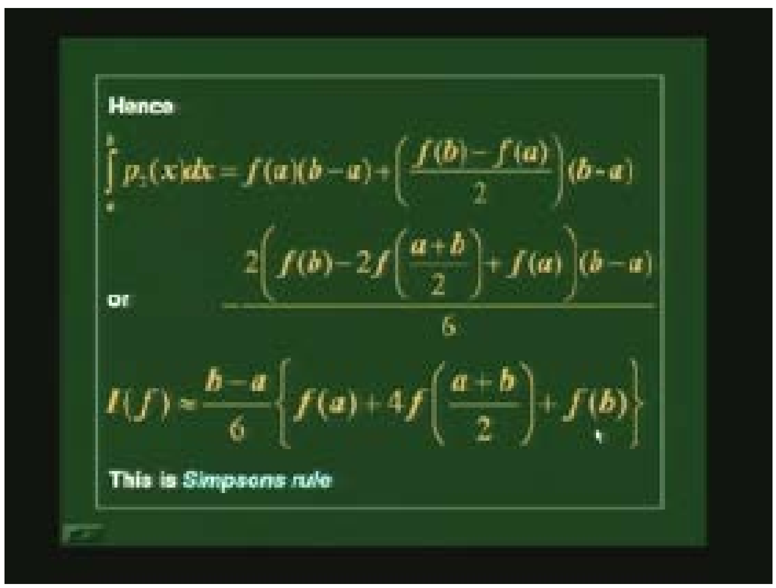

So if I do integral of this and some to integral of this between the integral a to b between the limits a to b then I would obtain some formula of this form a formula of this form.

So again I can write down the integral of the function in the interval b to a to b as a sum of 3 terms okay each containing the function evaluated at that particular point that is a plus b by 2 and b. So f of a plus f of b plus 4 times f of a plus b by 2 into b minus a by 6 that is the Simpson’s rule.

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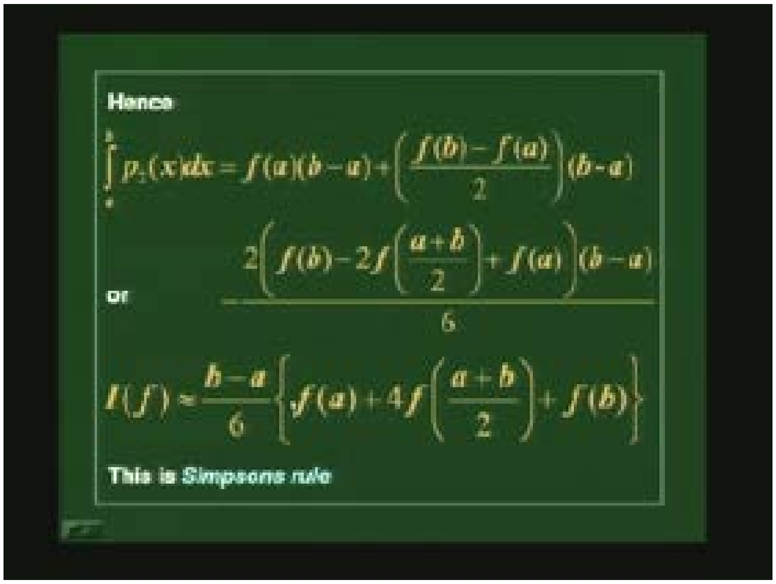

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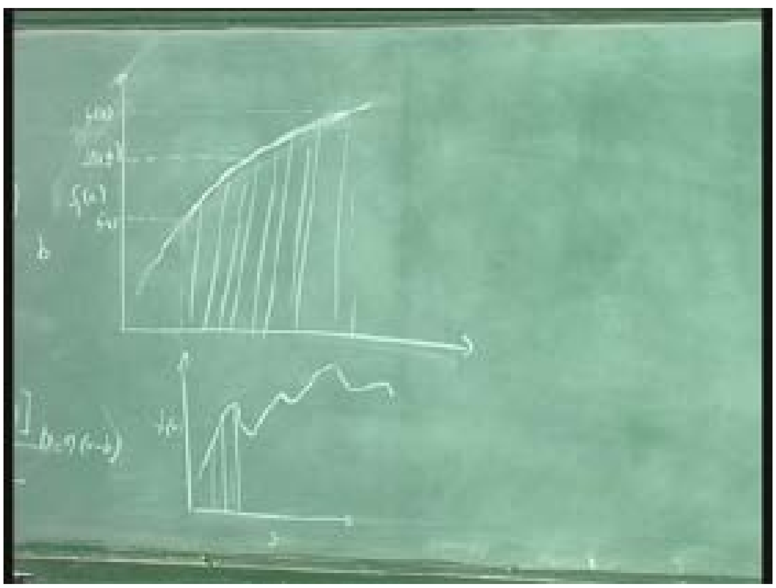

Okay so if you are implementing this this is again a simple formula to implement only thing we need the function value at 3 different points to evaluate this and where the case the function where is much more rapidly than this we will have to split our interval into different smaller intervals and then use the formula this formula in each of the intervals. So we will have to use this formula in each of that intervals, we will at use this formula in each of this intervals that is the idea which the in all of these methods we would more rapidly varying function then we would split them into smaller interval and then evaluate them, of course more computationally intensive and also more probability for round of errors. So that Is why we look at a better method with higher accuracy which we can go for larger intervals because we split them into smaller intervals and then we have to do the many calculations for each of this intervals and then we have more probability for getting round off errors dominating values.

Okay so now we look at the error in the Simpson's rule, so the error in the Simpson's rule is of the order 4 okay now the reason being this that the error term that is x minus a into x minus b into x minus b so the Simpson's rule we had the psi 2 right psi 2 of x that is x minus a into x minus b into x minus a plus b by 2. Okay now this term now vanishes now this integral of this vanishes in the limits a to b. So we have a higher order accuracy here, so what we get here is actually a fourth order term.

We can go one more order up in the in the error so fourth order derivative is the so the error is of the order of the fourth order derivative of the function. So that is this may be by the most accurate method which we seen among the four methods we have looked at so far. So we will just compare now all the three methods which we have looked at okay.

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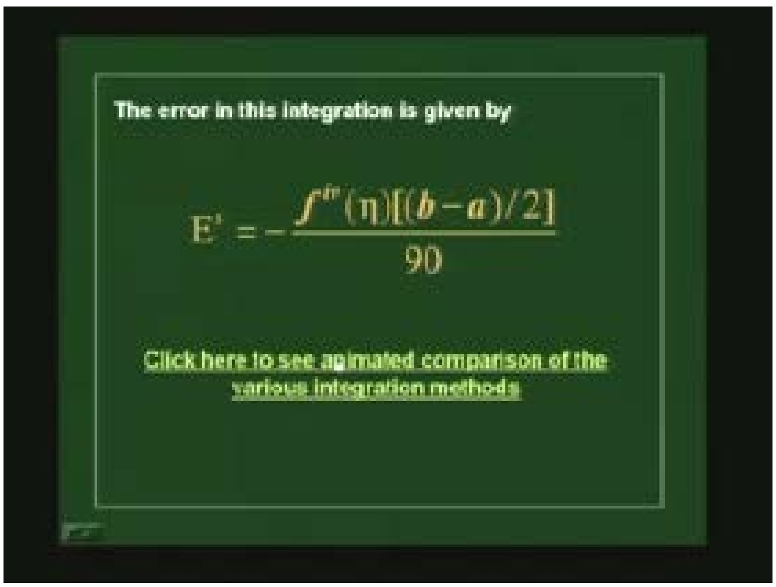

So for a for a integral of a function f of x a numerical integration of a function f of x in the interval a to b okay so we have the first the rectangular rule. Okay which said that it is the this the for the integral is b minus a into f of a that is what the rule said the rectangular formula, so we evaluate the function only at one point and multiply it by the interval that is the rectangular formula. Okay graphically read just this we have a function like this right and then we just evaluate the function at one point, okay and calculate the area under that rectangle that is graphically rectangular rule is and then we have the midpoint formula b minus a into f of a plus b by 2. So again the function is evaluated only at 1 point and then multiplied by the interval b minus a.

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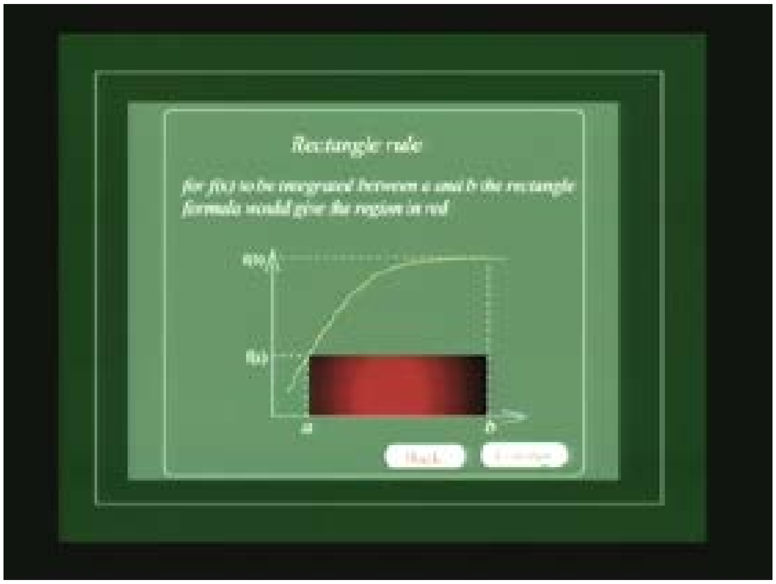

So this calculating the area under again a box but now the function is evaluated at b plus a by 2.Okay that is in this box okay that is the under this is shown in this red color region here and that is the second formula which we looked at the midpoint rule and the we looked at the trapezoidal rule which had the formula as b minus a into f of a plus f of b by 2.

Okay that is again the area under this particular box that is a trapezoid that is a straight line connecting a to b and then the area under that particular line is, so the area under this curve is now approximated by any area under the line connecting f of a to a to b. So in this case again this will be exact second derivative of this function is 0 that is if it is a straight line obviously.

Okay and then we looked at the Simpson’s rule and there we the trapezoidal rule was of the order one polynomial and the Simpson’s rule now we use an order two polynomial that now the function has to be evaluated at 3 different points and we choose them to be a b and a plus b by 2 and then we obtain a formula b minus a by 6 into f of a plus 4, f of a plus b by 2 plus f of b.

So now that is the Simpson’s rule and now that amounts to actually fitting a curve between these points. Okay curved region and then finding the area under that under that curve, so that is the Simpson’s rule. Okay so now to summarize all of them together it will be comparison of all the methods will be this for a function of this form so the approximation to the area would now be given by these 4 different boxes.

So you could have the area under the rectangle, okay area under the trapezoid, area under the another rectangle but now centered at f of a plus b by 2 and then area obtained by fitting a curve of second order polynomial through these 2 points. Okay that is the summary of the Simpson’s of the rules which we have seen.

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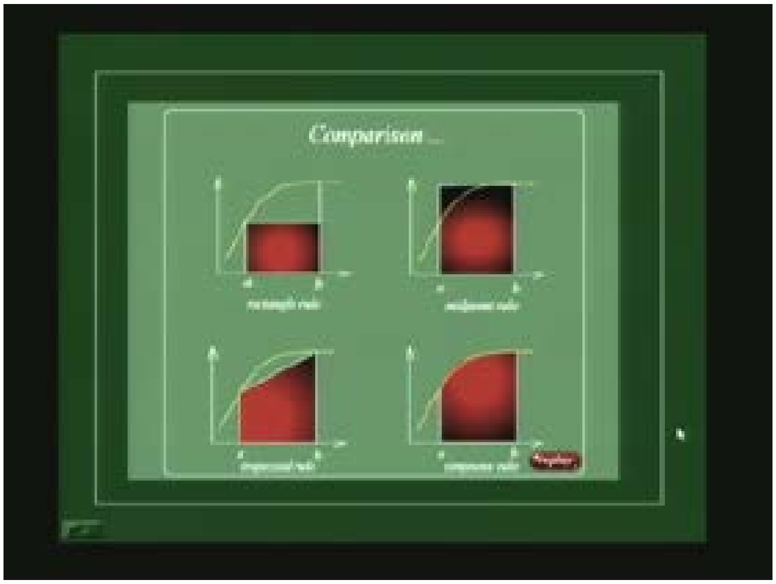

Now of these polynomial approximations, of this simple polynomial approximation now we go to a slightly different method at evaluating integrals numerically but again since using the same basic idea that we can approximate a function by a polynomial. And these rules are called Gaussian rule.

So they are obviously, they are used little more complicated in concept wise but the accuracy is much larger than these simple polynomial this approximations which we saw as a rectangular or midpoint or trapezoidal or Simpson's rule. So now these sets things of rule method are classified as Gaussian rules and we look at them now, that is little more complicated than what we saw now but much more accurate than the simple rules which we saw.

So in the Gaussian rule the idea is to again as I said to write the integral of the function as the integral of a polynomial of order k and we want to write in this method we want to write this as a sum of the function values evaluated at these different k points that is x_0 to x_k, k plus 1 points here and multiplied by some coefficients a 0 a_1 up to a k. So the idea is to write it as a sum of some weight multiplied by that functional value.

Okay that is what I said it is basically the idea of the polynomial of the approximation is the same but now we want to write the formula. Okay in this in this form okay that is the function value evaluated at that point and then multiplied by that weight whole method is to find what these weights are.

So we tabulate the function values at some x_0 x_1 and then we have function values at that point x_0 x_1 up to k and then we use we ll compute the weights of this a_0 a_1 and then we just find the sum of this products that is the weighted sum of the function values now we have to choose the clever idea here is to choose that these points x_0 x_1 x_k etcetera they are not arbitrarily chosen and we have to choose them such that error if we choose k points, okay the error or if we choose the end points the error will be of the order of 2 n plus 1 th derivative of f of x. Note that, in this case of mid rectangular, midpoint rules, we have chosen a function of a polynomial of order 0 or polynomial of order 0 and then we had accuracy of order first derivative of the function or second derivative of a function and in the case of Simpson's rule for example, we had chosen a polynomial of order 2 and we had an accuracy up to fourth order derivative of the function but here we are talking about choosing evaluating a function at n points and getting an accuracy which is of order 2 and n plus one th derivative of the function.

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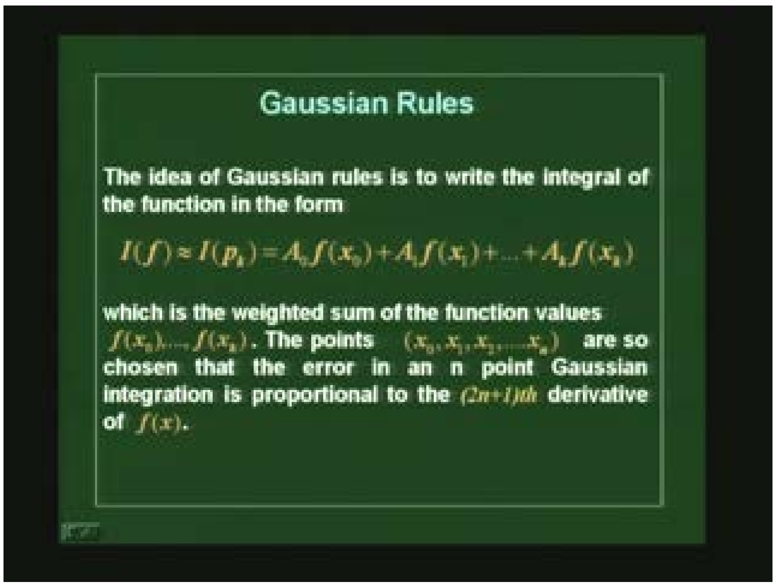

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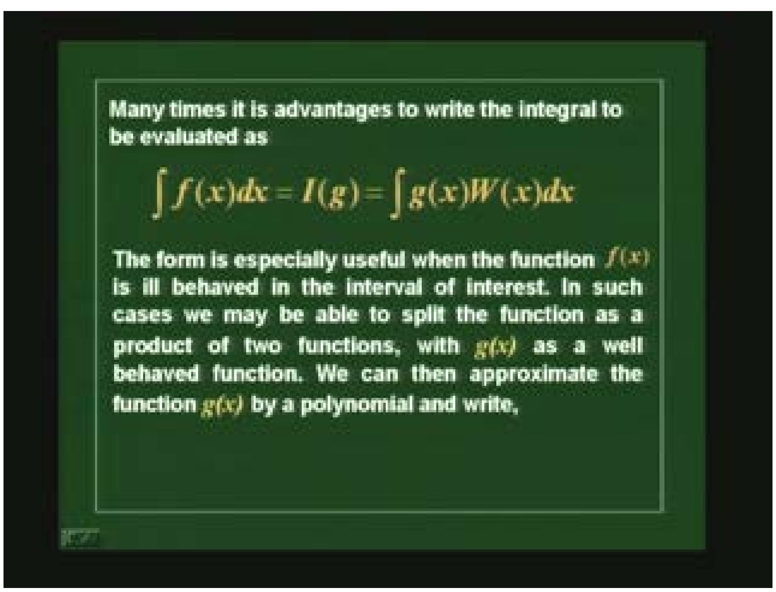

So you can see why these method are the best to use for numerical integration where we have the freedom to choose the point at whish the function we have to evaluated. So it may not be possible in all the cases in cases where it is possible that is where we can evaluate the function at the points of our choice okay and then these are the best methods to use. Okay so now we will look at what the basic idea of this method is, so the method is the following that we write the integral f of x dx that is I of f now is approximated by I of g, where g is another function. So I will write g f of x as g into w. Okay so let me write back here again.

So we want to use we want to we want to find integral of f, so I of f okay so which is actually integral some a to b f of x dx. Okay so now I want to approximate this function by integral a to b here, g of x some other function g into w of x dx.

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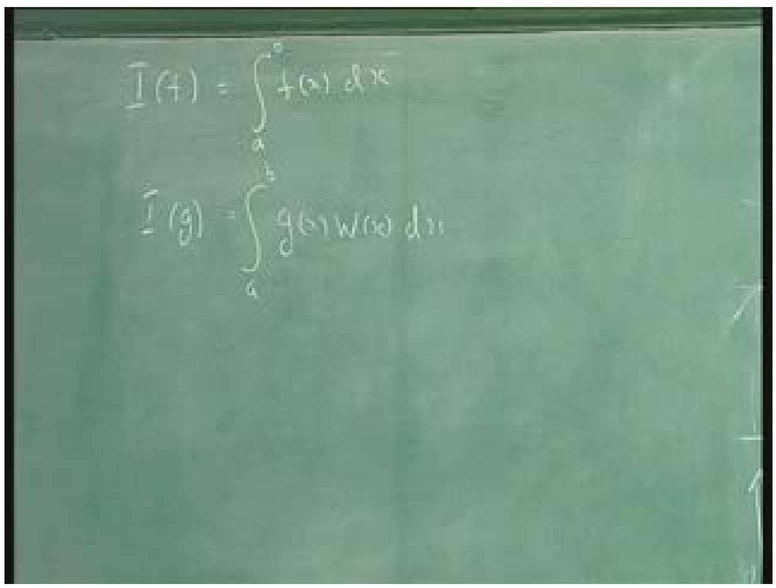

So I want to write this as a product of two such x functions okay and evaluate then then I want to now write the integral of this g that is what I called, I of g so this is equal to now I would use I of g to denote that I have actually split this function f into 2 values g and w. Okay now this is particularly of interest where you know this f of the function f of x itself may not be very well behaved but the integral is finite okay for example, if we have root x sitting in the denominator and you want to integrate it to the limit 0 to something 0 to 1 or minus 1 to plus 1. Okay then you will have trouble evaluating this function at 0 if it is 1 over root x but the integral itself is finite.

So in that particular cases this s actually useful to write this as a function as a well behaved function in the interval a to b multiplied by weight at some a weight w of x dx.
So as I said that there will be cases where f of x is not well behaved but integral is finite, okay so in that particular cases where in that kind of cases this is particularly useful.
So then what we do is now we do not approximate the whole function by the polynomial we will just approximate the function g by the polynomial. Okay that is the idea.

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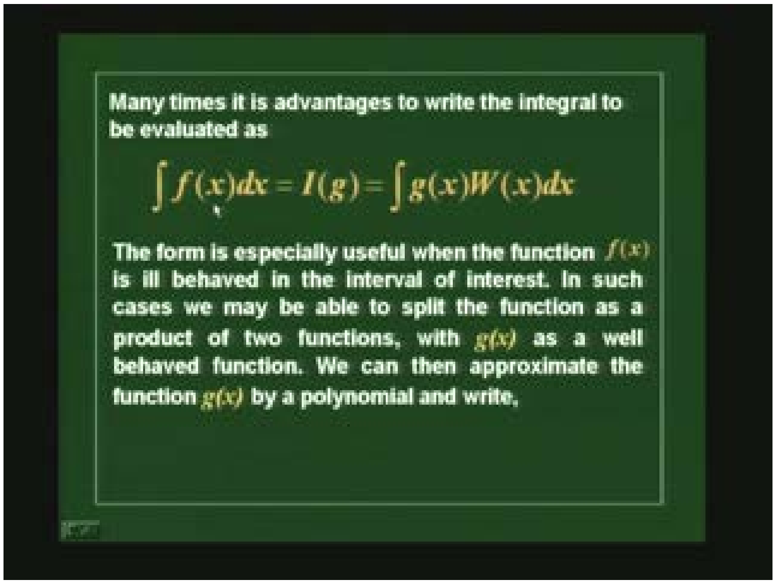

So earlier we were approximating f as a polynomial, okay but now we will not do that, we will approximate the function g by a polynomial. Okay this is Gaussian rule this is what the first step okay we have split the function f into another g and w okay now we use the polynomial approximation but not for the whole function of f but only for g okay that is and that is as I said this particularly useful when the function f of x is not well behaved in the interval but the integral is a finite integral.

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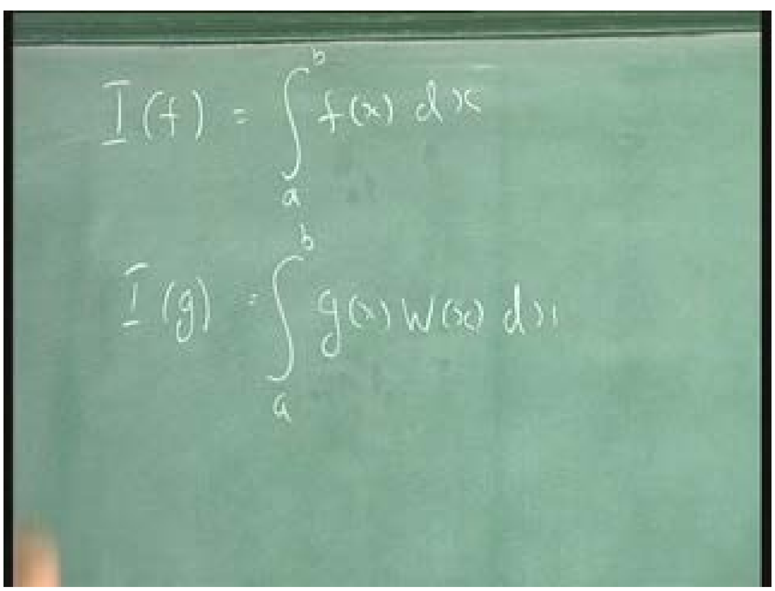

Okay so then we can use that and write the polynomial now in this fashion as before, okay that is the first type, okay now we are familiar with this then now we write instead of f of x we write g of x as p_k of x that is some polynomial ordered k and in some error term into psi k of x where psi k of x is now order k plus 1 polynomial.

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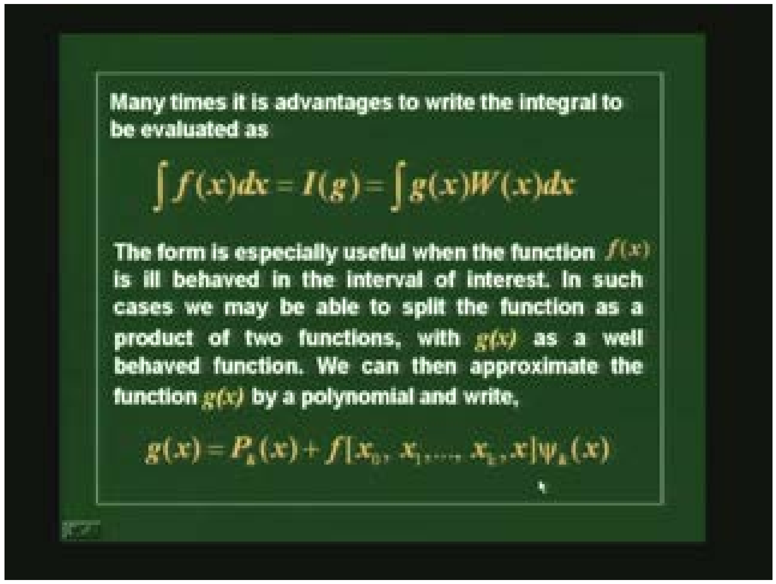

Okay so now given that we can write the error e as I of g minus I of p of k with error in the integration as I said i of g is, i of f but now the error is slightly different from what we have written before because now g was f into w. So the error is actually g x_0 x_1 x_k x into w of x psi k of x dx.

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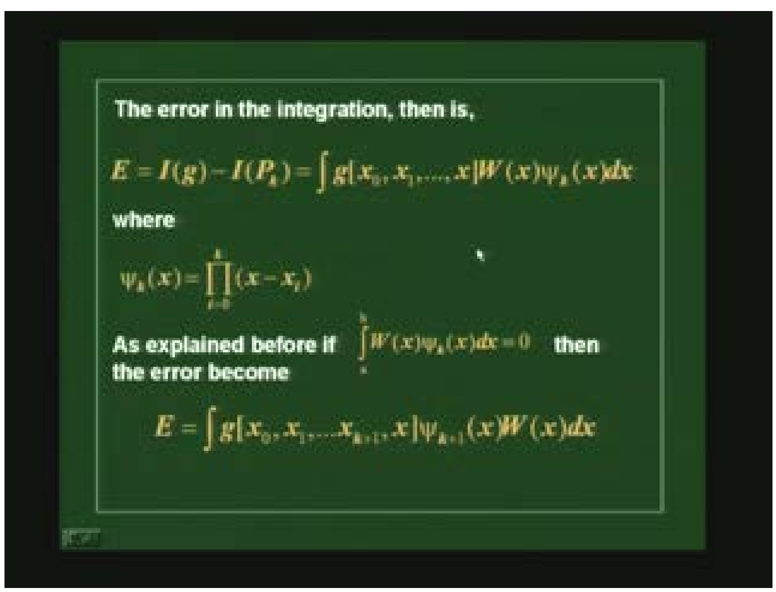

Okay so now we have this extra function sitting here psi k of x is same as what we have used before. So then if we have now that is we done before now we can again argue that if this whole quantity the integral of this quantity that is integral w of x psi k of x dx is 0 then I can as before write g x_0 x_1 x_k x as two terms that is g x_0 x_1 x_k plus g x_0 x_k x_1 x_k plus 1 x into x_k minus x_k plus 1 minus x_k, that is if integral g integral psi k of x into w of x dx is 0 between the limits a and b okay if this is true then we can then we can use the fact that g of x_0 x_k x can be written as g x_0 x_k then plus g times, plus g of x_0 x_k plus 1 x into x_k plus 1 minus x using this limit using the divided difference method right.

So I can replace this quantity by this quantity and we know the integral of this is 0 when this is 0 right because now my error term is now which is g x_0 x_k x into psi k of x_w of x dx, now we are going to have two terms one is integral g x_0 x_k w of psi k of x_w of x right that is the first term dx that is coming from this and we have another term which comes from integral g x_0 x_k plus 1 x, x_k plus 1 minus x, psi k of x_w of x dx.

Okay so we had we had these 2 terms, okay so we have a term which is this integral now this integral is now split into two terms, okay one contains one from using this, this divided difference thing and we know that this is 0 because this is constant and this integral is 0 then we have that this term psi k plus 1 of x. So we know that the order of the accuracy has gone to the k plus 2 th derivative of the function f.

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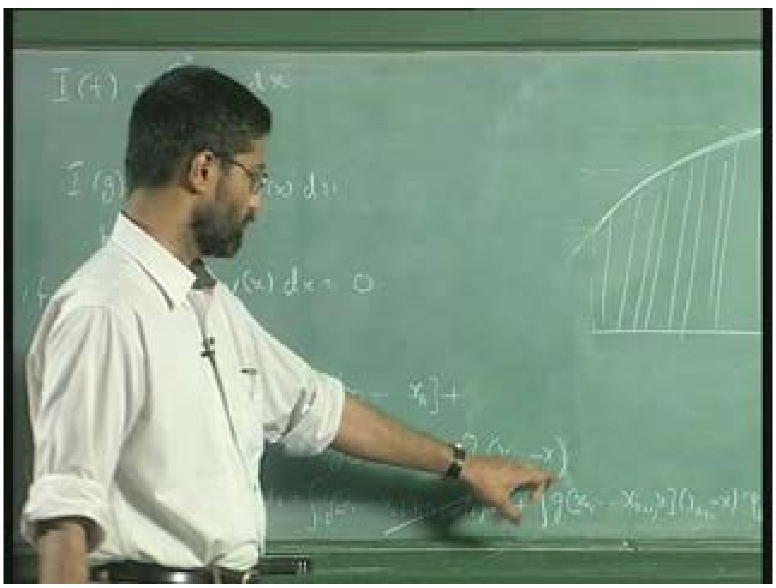

That is what been summarized here. Okay so we use the side here that if this 0 we can go one up in the accuracy in the derivative the error becomes of the order of the next derivative that is k plus 2 derivatives. Okay if this is 0 so now the idea is to choose the functions of the values x_0 x_1 x_2 etcetera such that the order can be pushed up to the 2 k plus 1, okay that is what it is mentioned earlier.

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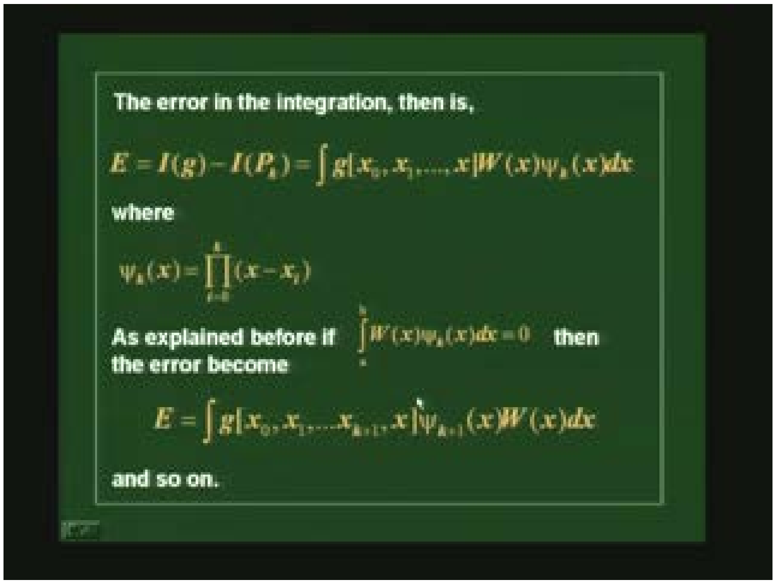

So now I want to choose a now I choose new point k plus 1 right now I choose a_k plus 1such that w of x there is a mistake w of x, x minus x_k plus 1 x up to all these things is 0 okay so w of x. So now the idea is as I said now we choose this function such that, so we had this w of x psi k of x dx equal to 0.

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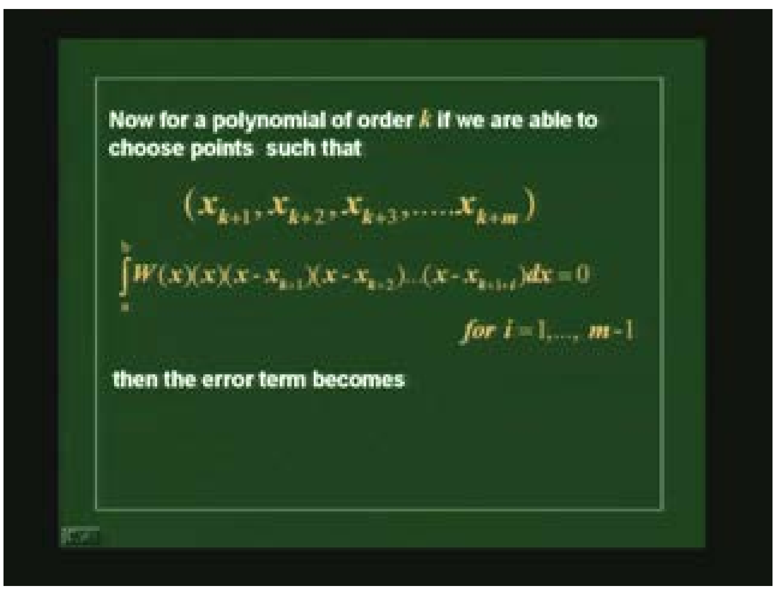

So that led to choosing a new point k plus 1and then evaluating the error w of x psi k plus 1 of x dx. Okay so now let us say which was the k plus 1 point, okay and so this had g which goes from x_0 to x_k plus 1 x_w of x psi k plus 1 of x dx. Now we choose a new point k plus 1, now we choose it such that this integral is again 0 that is integral x_w of x psi k of x dx is 0 and then this will lead to integral g of x integral g of x_0 x_k plus 2 now w of x psi k plus 2 of x dx.

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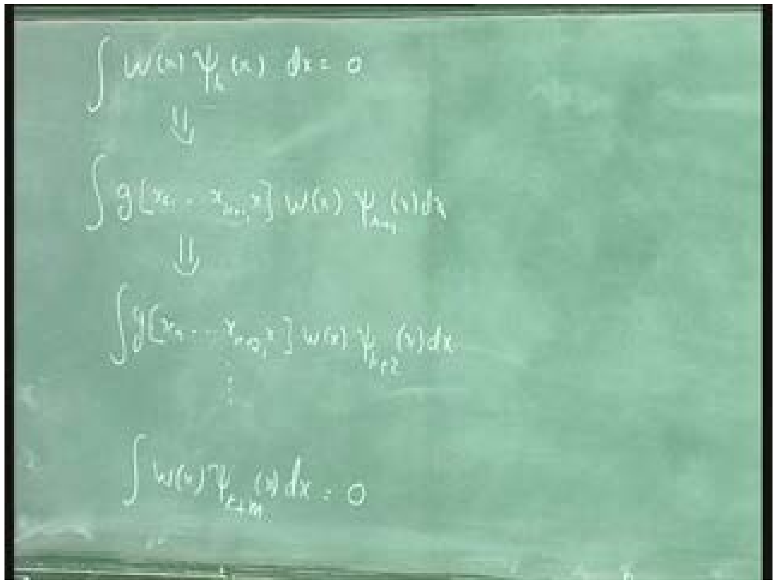

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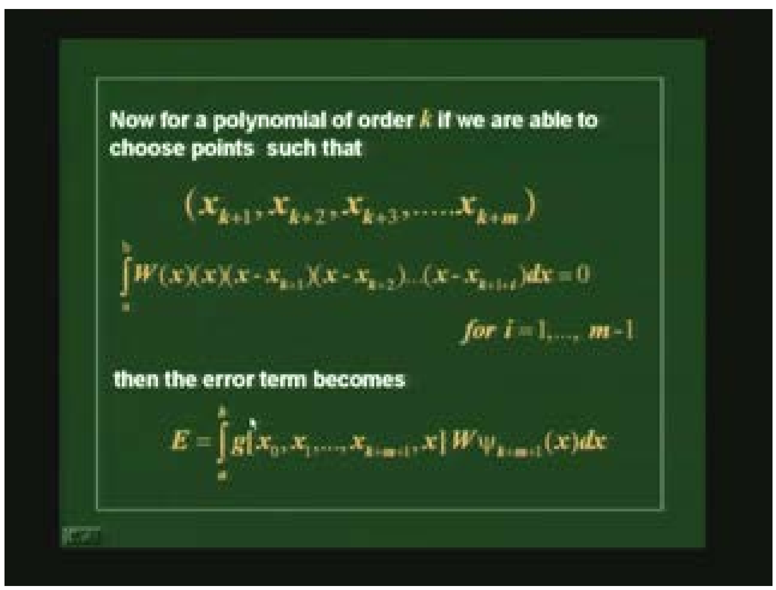

So we can we can continue with that, okay and then go up to 2 k plus 1 order which we want to do that is the idea right we have to choose this points k plus 1 k plus 2 k plus 3 up to k plus m that say okay, such that this integral that is w of x that is integral w of x. Okay so we want to continue with this. So we will, so we continue with that we choose point up to such that w of x psi k plus m of x dx is equal to 0. So we will choose points up to that okay that k plus m would mean that w of x into x minus x_k plus 1 x_w of x into x minus x $_k$

plus 1, x minus x $_k$ plus 2 x minus k plus m dx equal to 0, right 0. Okay then we have an error which is of this form. Okay and then we have the accuracy would now g this k plus m plus 2 derivative of this function g now if m is k that is 2 plus 2 derivative of g that the idea.

So now the question is how we choose these points so the idea of Gaussian integration or Gaussian methods is to choose the points is to choose provide a method of choosing these points that is this points x_k plus 1 k plus 2 k plus 3 k plus m. Okay such that we the error an accuracy of that order. So that is what we call it as the Gaussian method otherwise it is not very different from what we have done earlier.

So the 2 differences one is we split the function f into g and w that is g into w and we choose the points such that we have this this method provides this in the Gaussian methods provides a way of choosing these points k plus 1 to k plus m such that our accuracy is of the order that is 2 k plus 1 th derivative of the function.

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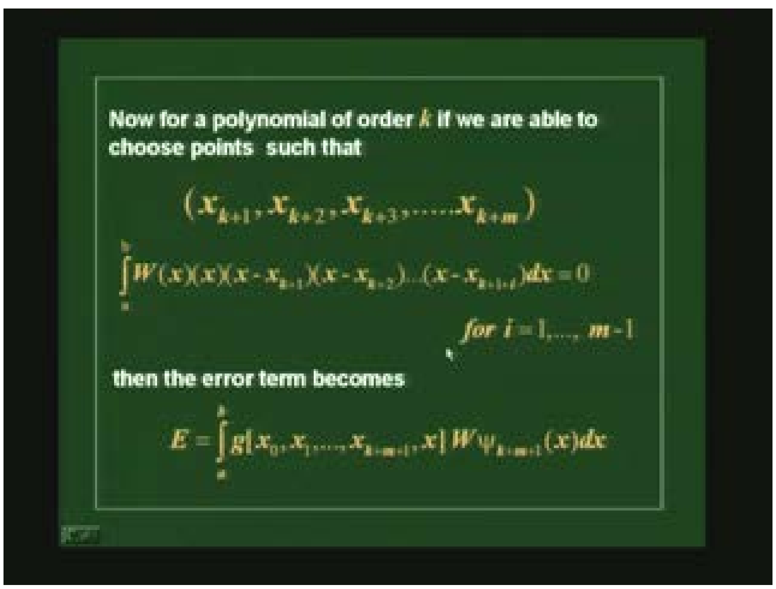

Now what is the polynomial is to do that, we use the write the polynomial in the legendary form we have seen this in polynomial interpolation. Okay so we write the polynomial in the legendary form. Okay that is now we now want to write g of x as p_k of x of x right the two steps.

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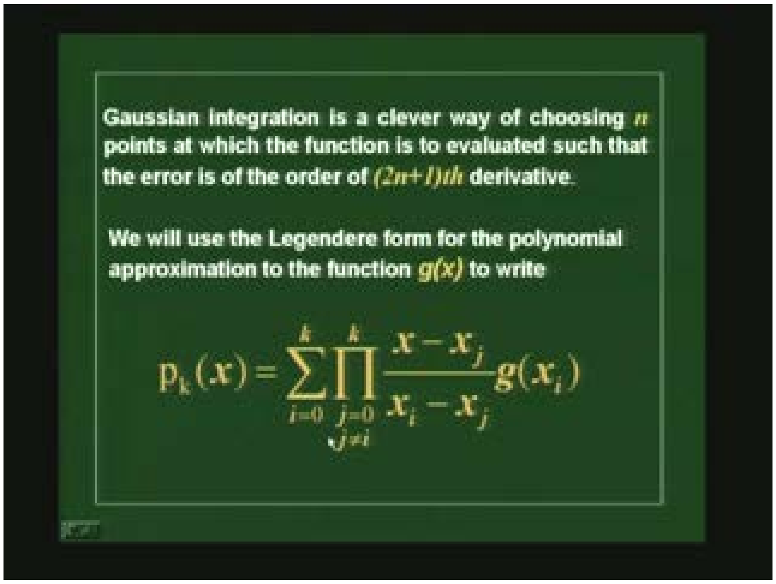

Let us write the 2 steps that is the function f of x which we want to integrate right, we write that as g of x into w of x, okay and then we approximate g of x by p_k of x. Okay so now that we write in the in the legendary form that is integral i going from 0, sum i going from 0 to k and then phi into x minus x_j divided by x_j minus x_i for i for j not equal to i. So for j going from 0 to k for j not equal to i. Okay that is what that is what we have to write okay j_x of i minus x_j into g of x_i right.

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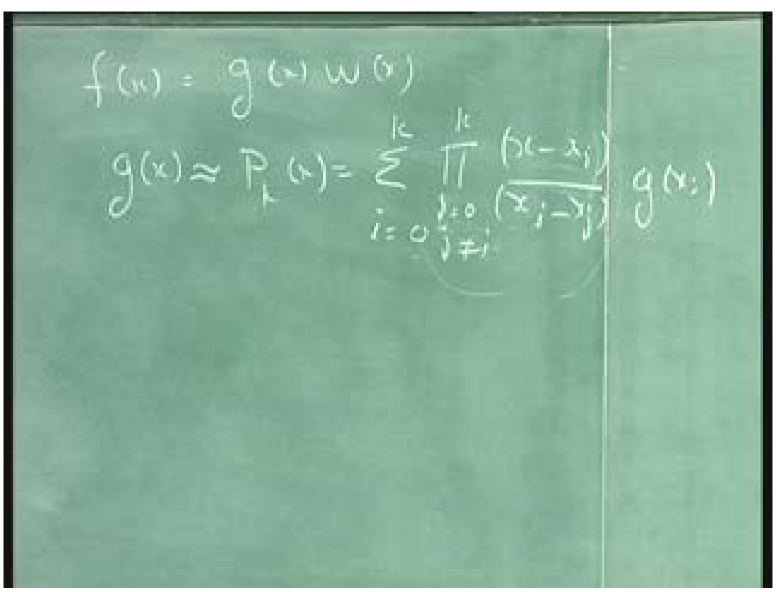

So that is the that is the idea of a form we have written this in polynomial interpolation case we have a sum of the products so the products are x minus x_i by x_i minus x_j into g of

x_i that is the we have to evaluate this product and then multiply it to the function and then take the sum that is the polynomial that is the legendary form of a polynomial approximation. So now we know the, we can choose the, this legendary polynomial approximation.

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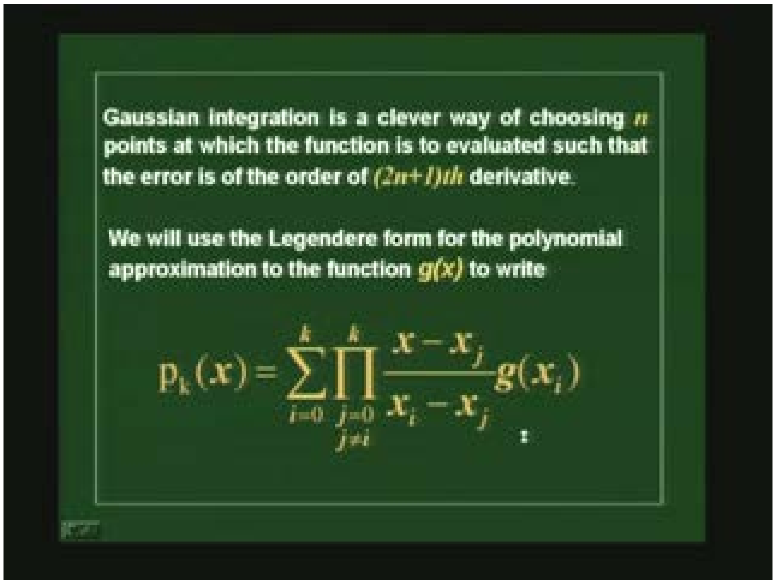

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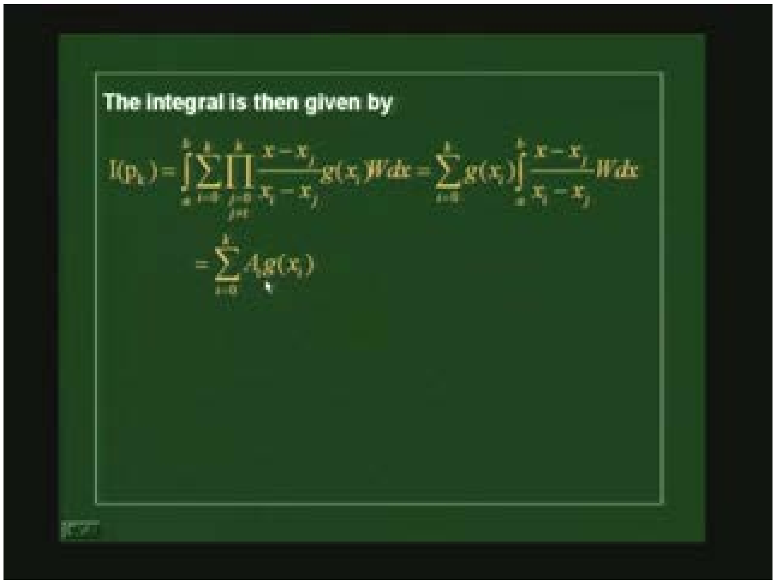

So we can choose the points x values to which the function has to be evaluated as the 0s of this polynomial. Okay that is the, that is the basic idea of Gaussian integration. We will see that now, so first we write polynomial this approximation into g of x in this form that is i going from 0 to k, going from 0to k where j is not equal to I, x minus x_j divided by x_i minus x_j into g of x_i that is the polynomial approximation to g the function g of x. Okay and then the integral can now be written as this right. So I of p_k is of this integral of that particular function which is which is we just wrote down. Okay multiplied by w in the limit a to b dx, okay that is the function that is the integral, okay so we want to evaluate this integral in the in the in this level.

Okay so now so now this function is evaluated at x_i. So that comes out of the integral, okay so I can interchange that I can take out this integral, so what I have to do is evaluate this integral this product for integral of this product, so that is what we are going to do okay and then I will write that as that as the function okay.

So I am going to write the integral I of f as integral of this sigma from i going from 0 to k p_i, j going from 0 to phi this the product going from 0 to k_j not equal to i, x minus x_j divided by x minus x_i minus x_j into g of x_j x_i into w of x dx so I can take out this. I can write that as sigma i going from 0 to k ok integral of integral a to b now a to b right phi j_0 to k_j not equal to I of x minus x_j divided by x_i minus x_j w of x dx.

Okay now I will have here sorry one term I missed that is sigma i going from 0 to k g of x_i, okay so this is the integral and then there is a function we can write this finally as sigma i going from 0 to k g of x_i into some w_i, g of x_i. So w of x_i this w being different from that okay this is okay this w is different from that this is this is the way this is the integral the whole integral I am representing as some w of x_i. So I can write it as a sum of the function value evaluated at these points x_i points multiplied by some integral, so each of these x_i, I will get one values from this integral and I substitute in that okay. So that is okay, so I use the a there to distinguish I write it as a of x_i some weight okay to distinguish it from w_i use it as a of x_i okay that is what is shown here okay.

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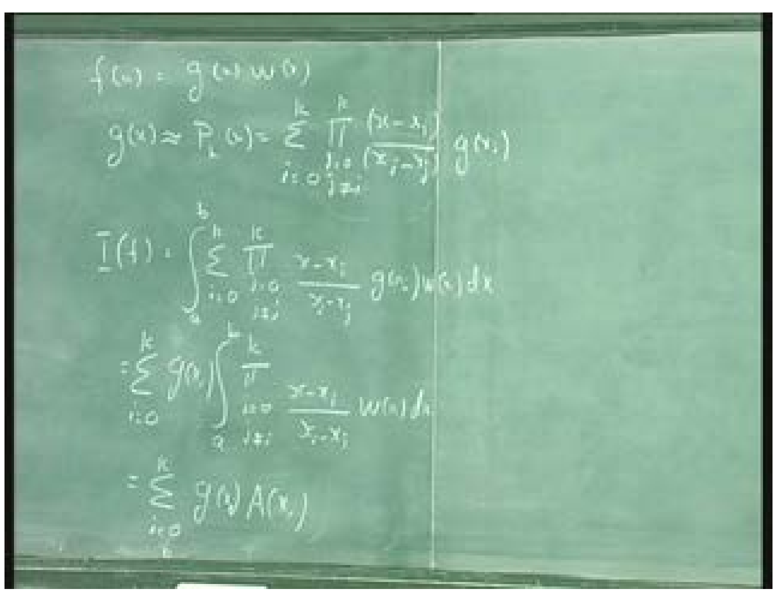

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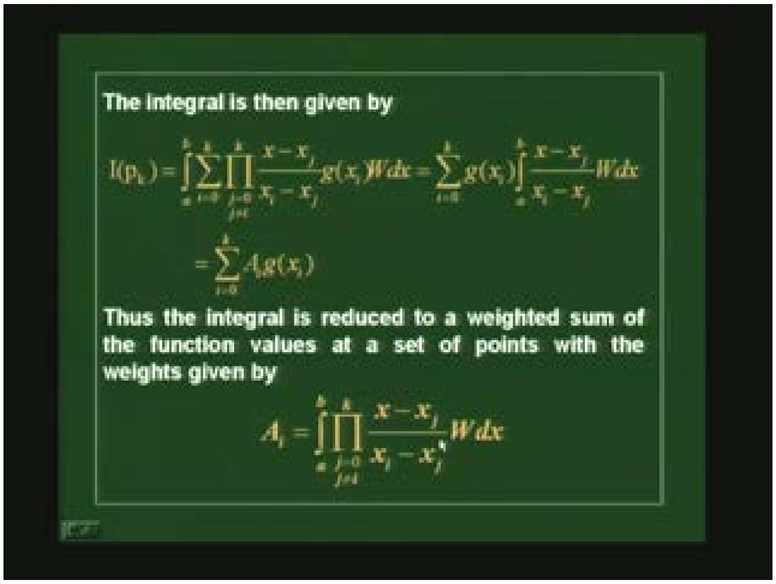

So now there is a integral is now reduced to a weighted sum of the function value a set of points and the weights are given by this this integral. Okay, so x minus x_j divided by x_i minus x_j, w of x dx. So that is the weight okay so now we will now just look at the error term okay with the last. Okay the error term will now be of the order of k plus 2, okay we have chosen such that the error the integration will k plus 2, okay and then but we have chosen the k points okay now we have to choose the k points such that the function value this integral is 0 okay.

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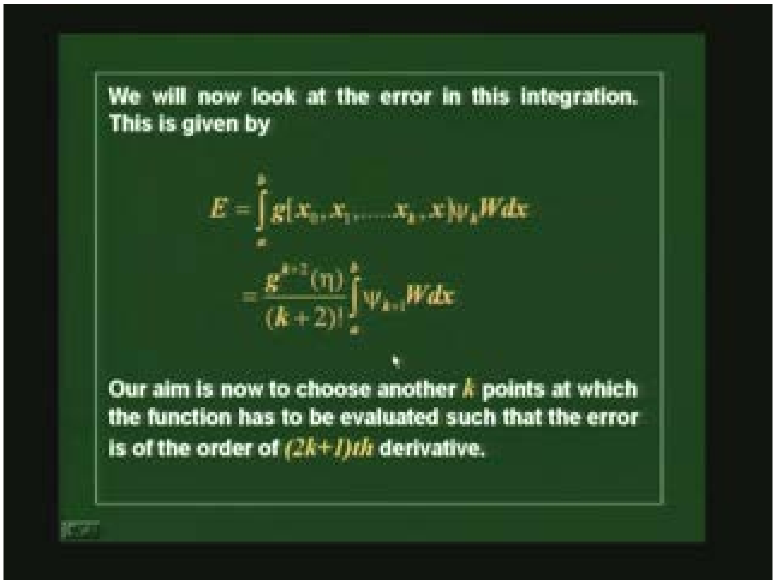

So that is done by choosing the this okay we will choose the k plus points such that they are 0 to i itself that is we will choose k plus okay, we will see the idea is that we will choose this k plus x_k plus 1 x_k plus 2, x up to x_2, k plus 1 as x_0 x_1 and x_k.

So if you do that then we can get the order of 2 k plus 1 accuracy. So first let us assume to how do we arrive at that okay, we just to see that lets assume that this 2 k plus 1points to be of this form such that this integral is 0. Let us assume that we can there is some way of choosing this k plus 1 points that is that is x $_k$ plus 1, x_k points, so that is x_k plus 1 into x_2 k plus 1 such that this integral is 0.

Okay and then since this chi k of x is this product then this that means that this is of the order of 2 k plus 1 this polynomial this whole polynomial is of the order 2 k plus 1 because this is already of the order k plus 1. Okay and so we will write this as two different polynomials, okay we will write this as psi k which is of the order k plus 1, we will write that as some alpha k plus 1 into psi k.

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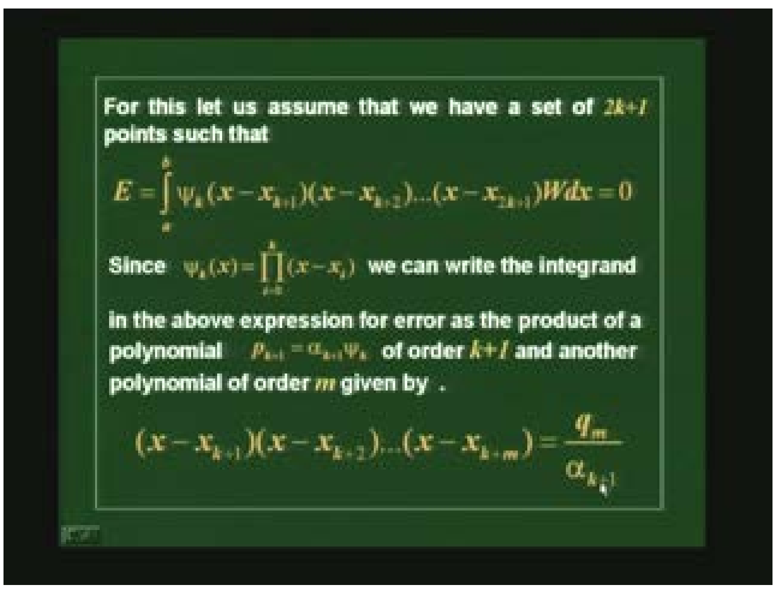

Okay and then we write this whole product here as another polynomial. Okay and then we will we will demand that this this sum should be 0 that is we have two polynomials 1 is k plus 1 order that is m which is also of order k and we demand that this integral should be 0 and then choose the points x_0 x_1 x_k and we see that this product is 0, if we choose x_0 to be x_k plus 0 to be x_0 x_k plus 2 to be x_1 etcetera and then we have an accuracy of the order 2 k plus 1.

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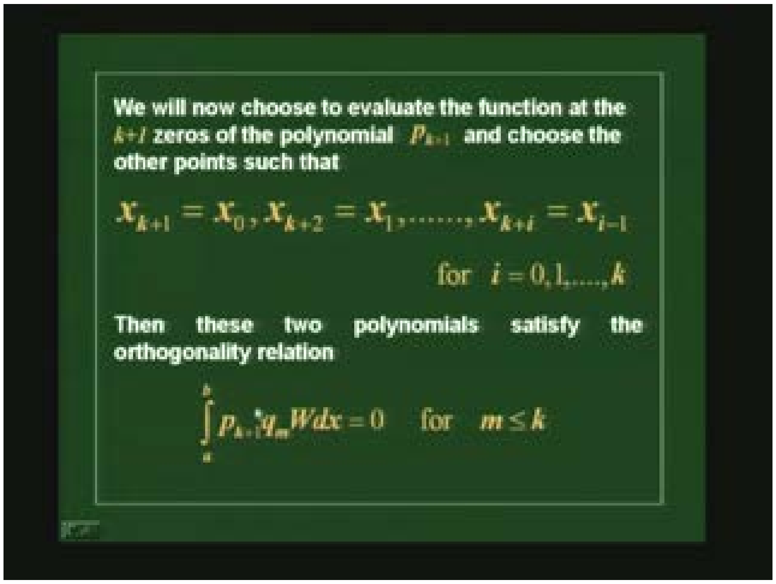

We will see more of this in the in the next class. We will go through this, this last part once again and look at how we arrived it this particular form for this and the implications of that in the in the actual calculation of this integral we will do that in the in the next class.