Numerical Methods and Programming
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Lecture - 27
Numerical Integration: Basic Rules

Today we will discuss in the numerical methods adopted for calculating the integral of a function. So we will be looking at some of the rules most of the commonly used methods for computing integrals numerically will be discussed in this lecture. So as we know that we can write the formula needed for the integral in this form. So we represent it by a symbol I and we say I of a function f, so this notation would mean that integral of a function f and that is integral I will say one variable that is f of x dx is that we limit our discussions here to integral of one variable and in the limits some limits a and b. So now, the thing should be discussing is how to compute this integral using numerical methods.

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So we will follow procedures start with procedures similar to what we did for computing the differentials. So that is approximate the functions by a polynomial, as we know most of the time we would need to compute these integrals numerically then we do know what this function, what the functional form is that is, we have again a set of data points. So we have given tabulated data and then we want to find the integral that is most often the case when you have you will be using this numerical scheme for integration.

There will be some cases where the function is so difficult to compute we know the functional form but it is difficult to compute evaluate the integral analytically and then we will again do it numerically there are some cases where that must also be required but so we were for the time being, we assume that what we have is set of tabulated data points and the same method can be applied even when there is known functions.

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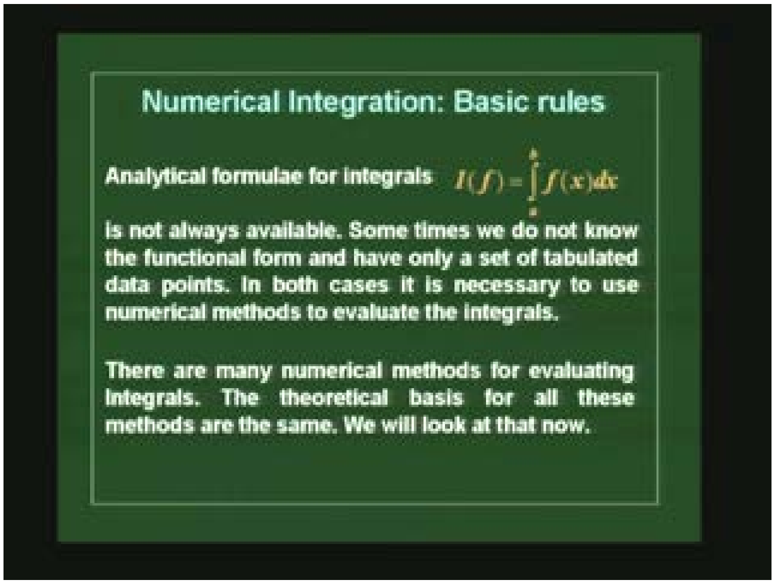

We can always tabulate a set of data points given a function having the function and the value the tabulated points and the tabulated points in this form f of x_i and x_i here. Okay now we want to compute evaluate this integral this x being x_i is greater than a and les than b, that is what we will be evaluating.

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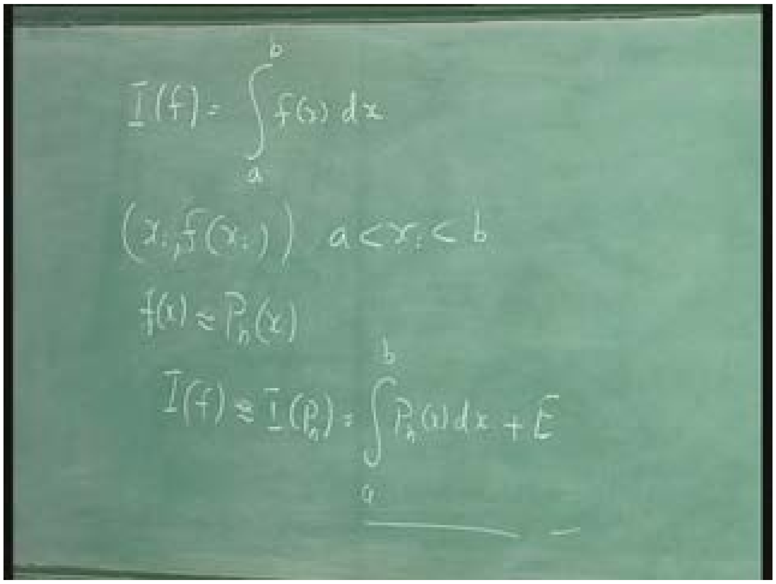

So in that case we would we can approximate this function f of x by a polynomial p of order n right, we will say f of x, f of x can be approximated by a polynomial p_n of x and then we will say that i of f is now can be approximated by i of p_n and then we will write that as integral a to b, p_n of x dx plus some term error some error term in this thing. So our idea in this lectures would be to evaluate find an expression for this quantity and also for that that this quantity which is the error using different schemes. So that is what we would be looking at.

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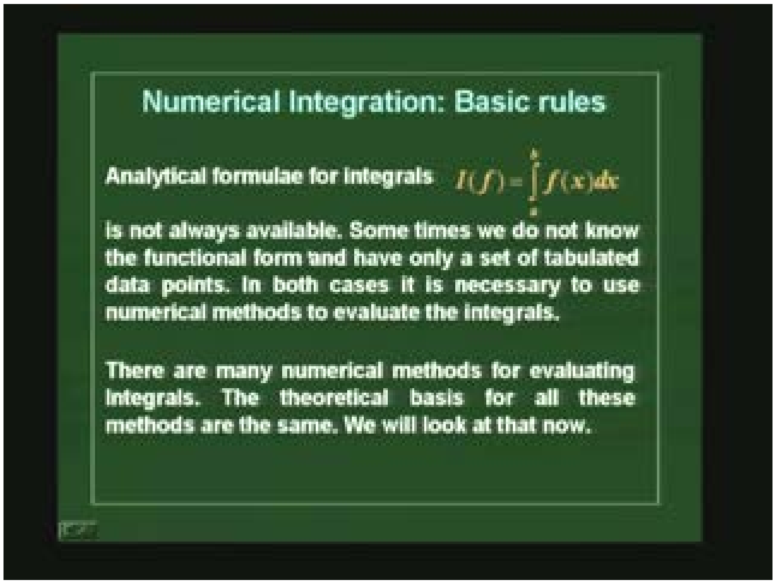

So to summarize the idea is the following or the basic theme for this lecture will the following that that wherever an analytical expression for the integral of a function cannot be found given it could be because we do not know the functional form of this, we only have tabulated set of data points or you have the functional form but it is difficult to do the analyze the integral analytically in both cases it is necessary to use some numerical methods to evaluate integrals.

So we look at now this method to evaluate these integrals before that first write we will formulate a theoretical basis and from and from there on we will look at various schemes which will implement which will allow us to calculate this integral. So that is the basic idea and then, now as I said that we would approximate the first thing to do always approximate this function f of x by a polynomial of degree k here, less than or equal to k plus b which we call p_k of x. So that polynomial interpolates the function through all these points all these points tabulated set of points 0 to x_k between the 2 limits a and b that is a okay and then we can write the approximation for the integral i f of x by i p_k of x.

Okay and then there is an error in this approximation and that error in the approximation is given by this next term in the polynomial. So that is the same procedure which we had used earlier, so remember which we are when we are using that when you say we had p in approximation for by a polynomial approximation, in approximate polynomial that passes through a set of data points.

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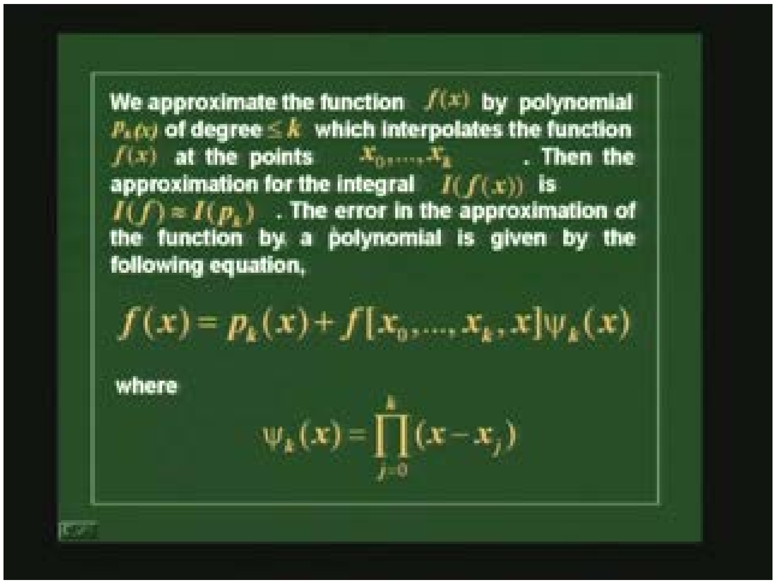

Okay if you have k data points and we use a polynomial of order k minus 1 or if you have k plus 1 point and if you use polynomial of order k and then the error, we can compute as the next term in the polynomial. So we said that we can write p_k, as so remember that is f of x_0 plus f of x_0 x_1 into x minus x_0. So we said that we could write this and write the last term as f of x_0 x_1 up to x_k minus 1, so that is and then we will write x minus x_0 into x minus x_1 up to x minus x_k minus $_1$

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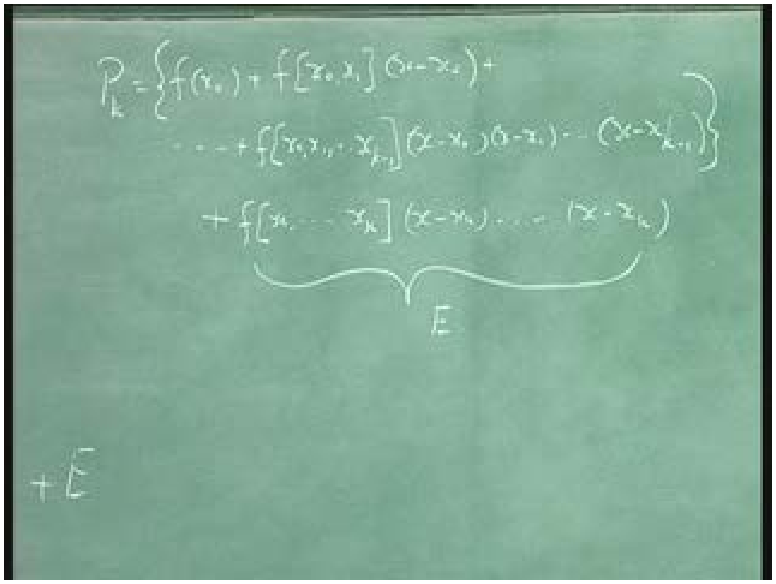

So that is the k th order polynomial and the error in the last term would be the that is the that is the error which I told write as so this is the k th order polynomial here I have and then I have the error can be written as of this order f x_0 x_k whatever be the k point is so I have an additional point x k and now that is the k plus 1 th order polynomial. So that I will have as x minus x_0 going up to x minus x_k. Okay that is that will be my error term right, so that is my error term in the polynomial, so that is what I have been trying to write here so in this case I have written a k plus order term. So that is a k th order polynomial and a k plus 1order term so and the error at x.

So here it is written as now if I want to evaluate this polynomial at x_i just add this point as so that is a k minus 1 term I add one more point here and write this as it is not changed. So x_k x_k plus 1 and then I would write it as x minus x_0 into x minus x_1 x minus x_k and that plus the error term as f of x_0 x_1 x_k plus 1 and then I make the pop polynomial pass through that particular value of x

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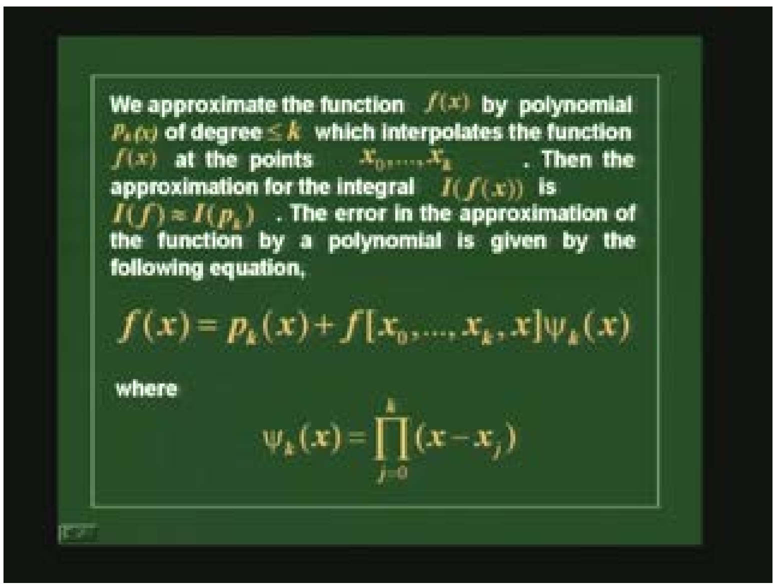

So remember this my polynomial in the Newton's form and I can evaluate this polynomial at any value x right and so there will be some error that error term I can compute by saying that I construct a polynomial that passes through x by putting by that taking that one as the tabulated value and then writing phi, phi going from 1 to n, 1 to k of x minus x_i, so that is my error term in the in the polynomial.

So i going from 0 to k here, it is k minus 1. So that is the that is my error term in the polynomial so what I did was I say that write the error term I said I can construct the coefficient by demanding that this polynomial goes through x the value at which I want to evaluate so that this whole polynomial is exact and then write that as a error term. Okay of course we have seen that this would be of the order of f k plus 1 derivative the k plus derivative 1derivative of the function f divided by k plus 1factorial, we have seen this earlier that this will be of this order this function will be of this order k plus 1 factorial and this is what I will write as psy. So this is written as psy k of x. okay that is the notation we have used here okay.

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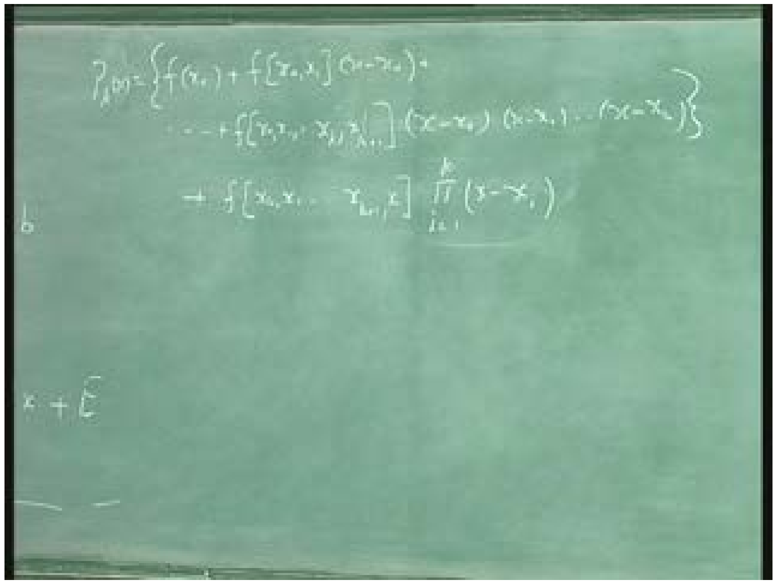

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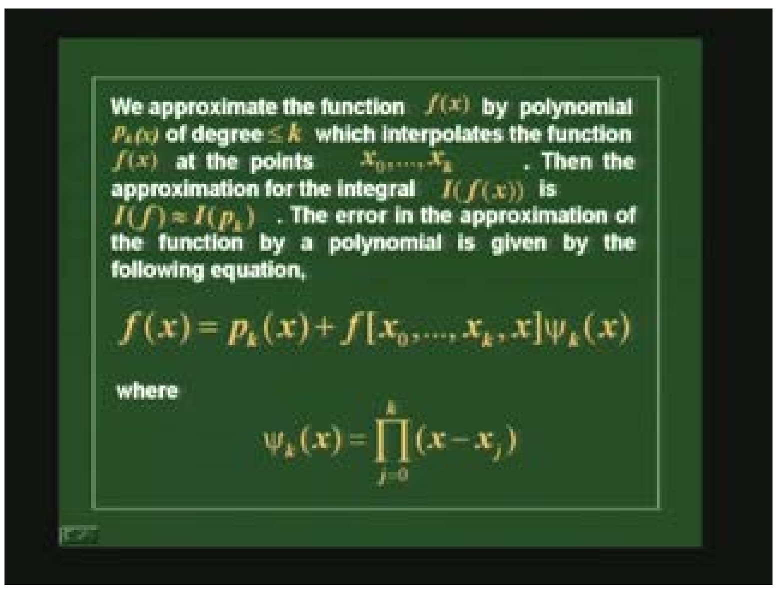

So this quantity is written as psy k of x which is actually going from 0 to k, x minus x_j, j going from 0 to k. So that is the polynomial representation of this function with the error term and this all this we have seen earlier in the polynomial case and again in the, when we evaluating the numerical derivatives and now, we have using this for also computing. or writing of taking this in theoretical foundation for numerical integration and the only new notation, we use is the quantity psy k which is product of x minus x_j for j going from 0 to k that is our notation and that will be the error term.

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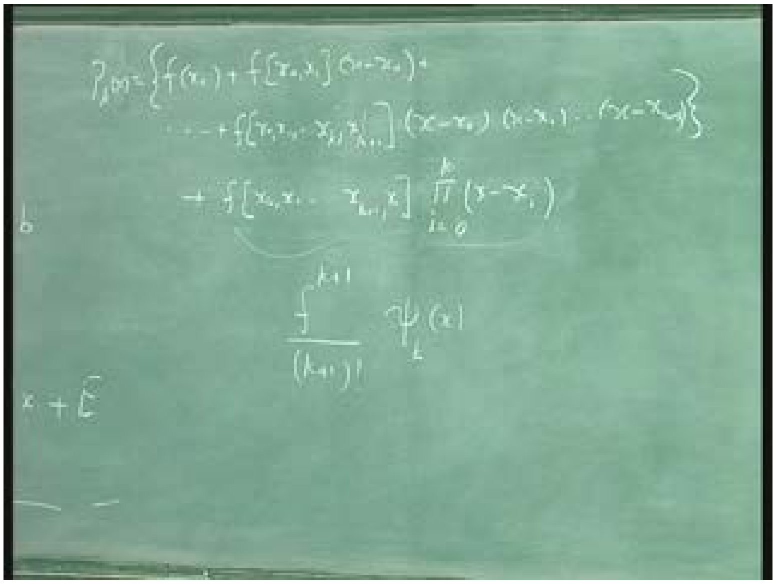

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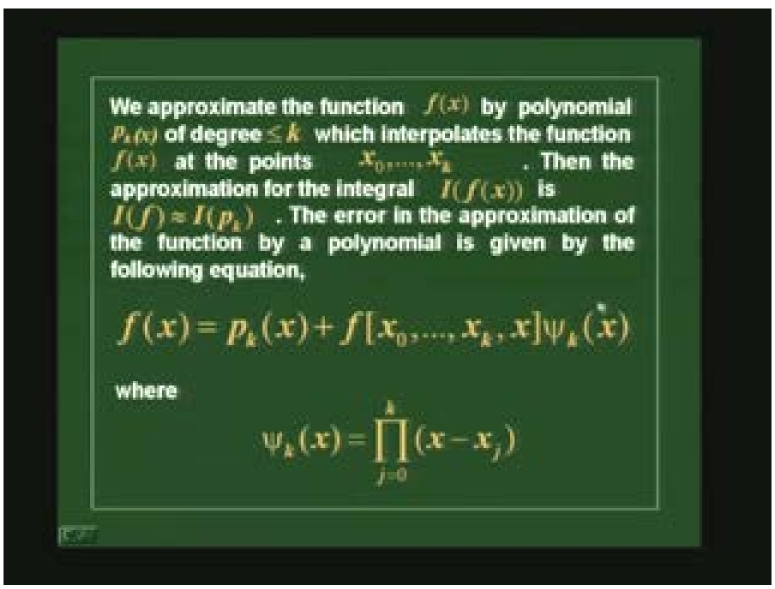

So when we say the integral i of f, we approximate we approximate by integrating i as integral i, p_k of x okay and then the error term and this would be the integral of this quantity. So the error term and this would be integral a to b of f x_0 x_k plus 1 and x and we will say psy k of x. So that will be our error term

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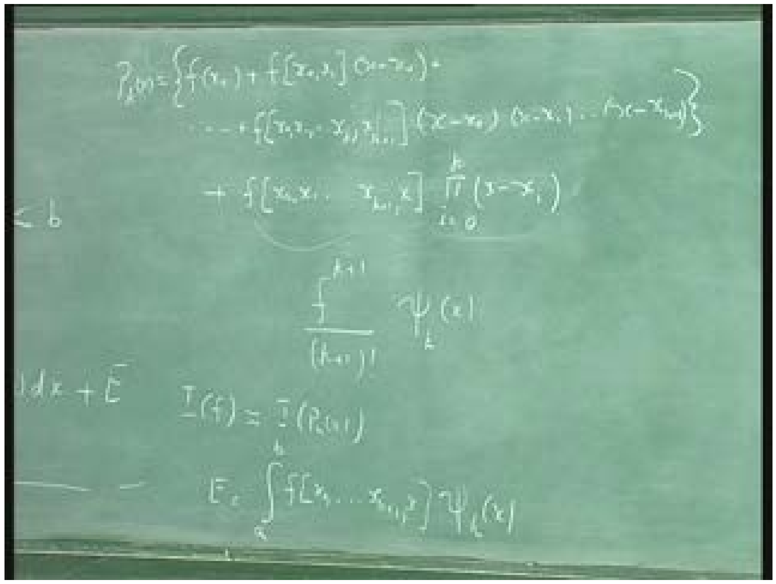

So, given that we can write down the error estimate in this form, now i of f minus i of p of k as this integral, so assuming that this quantity f is continuous and integrable function of x. So that is important, so assuming that this is continuous and integrable in this interval okay we can write the error term as that.

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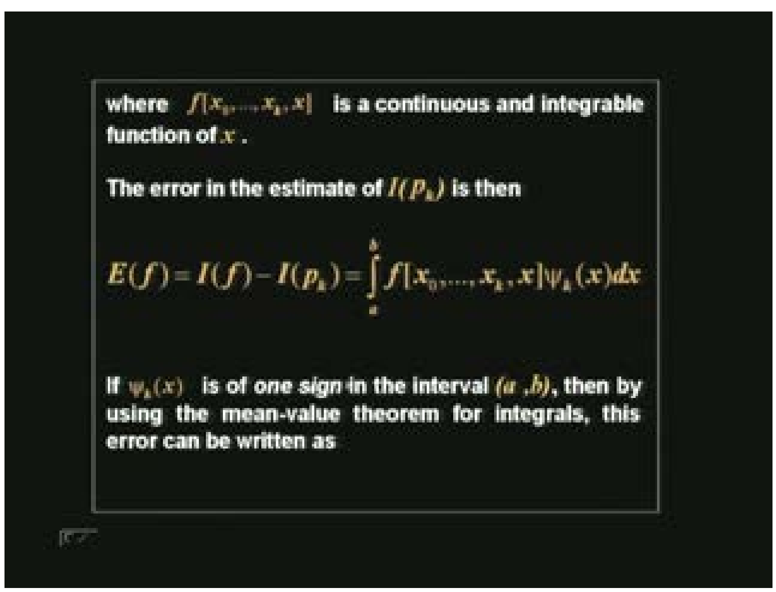

Okay so now as I said that we can approximate this quantity by the derivative of the function at k plus 1 at any point k plus 1 th derivative of the function f at some point eta between a and b.
We have seen, we have said that this is the, this follows from the mean value theorem and we can use this approximation for this coefficient, that is the coefficient can be written as the k plus 1 th derivative divided by k plus 1 factorial for some eta in between a and b to evaluate this error. So that is and then we can write this error term as this quantity, then it will not be a function of x any more. So we just take this out and then write this quantity multiplied by the integral of psy k of x dx.

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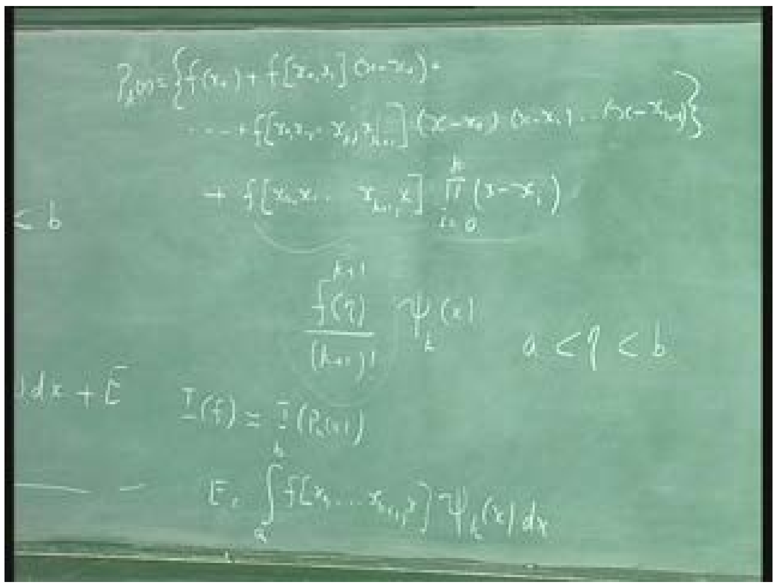

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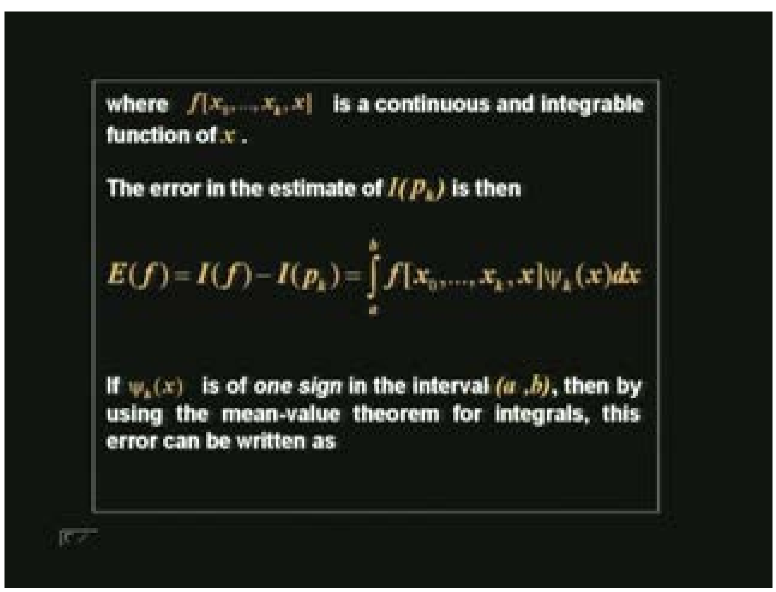

So we can write this error as f_k plus 1 th derivative of f at eta divided by k plus 1factorial into integral a to b psy k of x dx and this is definitely an estimate of the error provided that, so that is what I am trying to write here. So that this this is definitely an estimate of the error provided that this quantity psy k of x has is of one sign it does not change sign.

Okay so we can show that this equal to or can be approximated by f_k plus 1 by that I mean the derivative of f this is the k plus 1th derivative of eta in the k plus 1 th factorial integral of integral a to b psy k of x dx. Okay provided psy k of x is of one sign positive or negative with in the limits a and b and then I can approximate it by using mean value theorem, so that is what we would be using.

So we can see that if I do calculate the integral of the function by approximating it by a polynomial of order k like this and then the error I have is of the order k plus 1 th derivative of the function multiplied by this integral provided that k psy k of x is of 1 sign we ll see specific examples of this that is the first lesson we learn here that is the error can be of this order if psy k, please note that if that psy k of x is of 1 sign then I can write this as the k plus 1 th derivative divided by k plus 1 factorial that is one method of evaluating or estimating the error, okay by approximating the function by a polynomial.

Now this need not be always true that psy k of x need be of one sign in the whole interval okay that depends on what values at what points or the functions tabulated in this interval a to b. So that is, so what happens if can we estimate it in some more general cases it turns out that there is one more special case in which we can evaluate or we can approximate this error or we can get an idea about the error I can say as a special case there is no general case for this.

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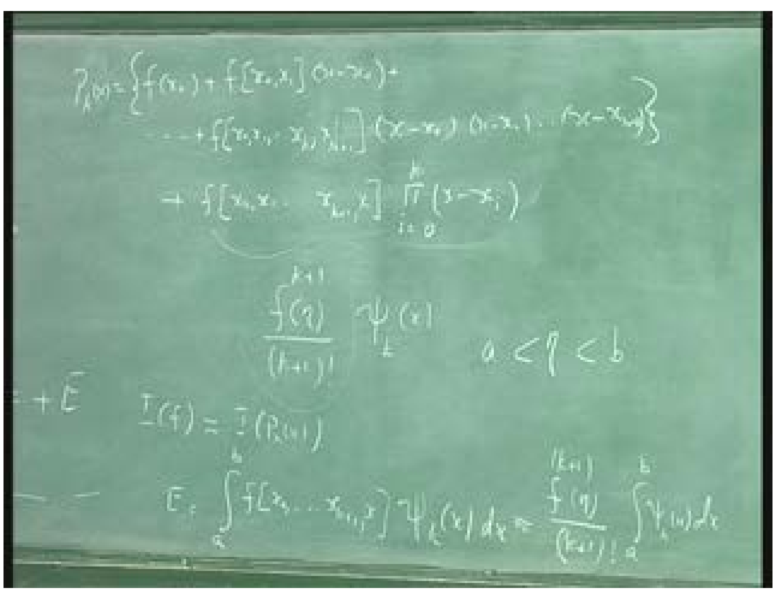

So one case is where psy k of x is of 1 sign now another case is psy k of x is the integral of that is 0 that are the two special cases we would be discussing. Okay if psy k of x is of one sign then as we just now said then I can write this as some function, I take this out outside okay and then replaces by k plus 1 derivative and write that as f k plus1eta by k plus 1 factorial into psy k of x. So, if provided again that f of x is now k plus 1times continuously differentiable.

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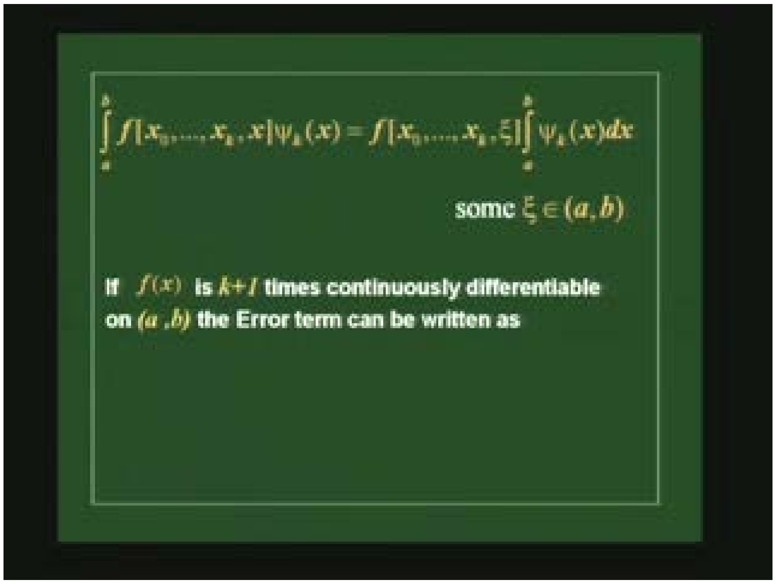

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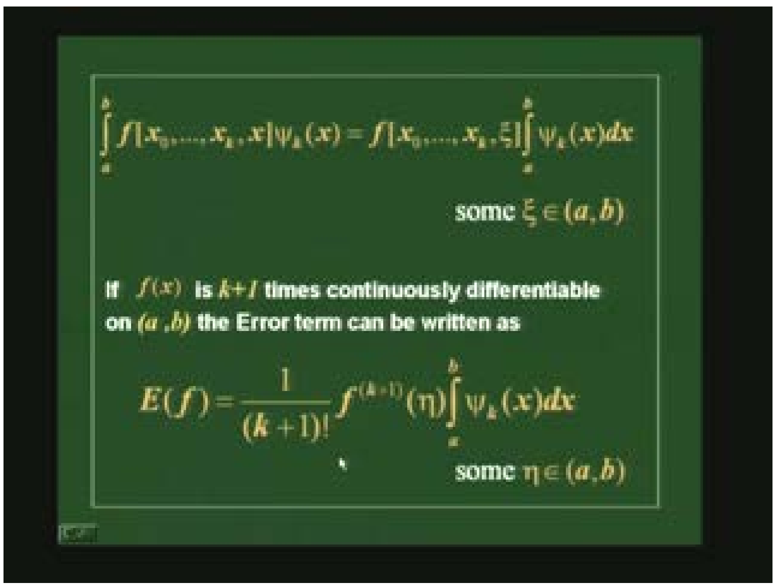

So I would tell you this this is this is can be appear again that f of x is we continuously differentiable and in the interval a and b, then we know the error is of this form of this order for some value of eta in between a and b. So remember the mean value theorem which is used here to say that this quantity f of x_0 x_k x can be approximated by or there is at least one point with in the interval a and b which is called which we call eta at which this derivative or this function can be approximated by the k plus 1 th derivative that is the mean value theorem. So now we look at another case psy k of x is not of the same sign but integral psy k of x dx is equal to 0.

Okay so now here we can use slightly different method that we could say that so now psy k of x dx is 0 we want to look at this special case where integral psy of k psy k of x dx is 0. So that we want to write so, remember that we had the approximation for the integral i of f by i of p_k, we said the error is of the order of integral a to b of this function f the f this this coefficient of the k plus 1 th term of the polynomial which is k_0 x_0 x_k x psy k of x dx where we said the psy k of x is p_i, i going from 0 to k $_x$ minus x_i.

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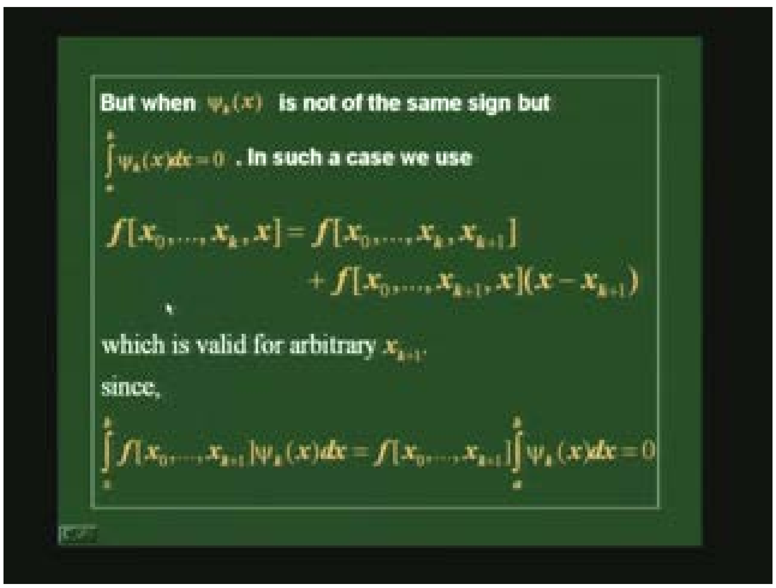

So now what we are saying is that this integral we have the special case where psy k of x dx between the limits a to b is 0. So in that limit can we approximate this this error by in some ways we say that when this psy k of x was of one sign when psy k of x was of one sign I could use the mean value theorem and approximate this error.

So second case is where this integral is 0 so, then how do we approximate. So then we will use the following property of this coefficients of this Newton's form polynomial that f of x_0, x_k, x_k plus 1 x can be written as f of x_0 x_k minus f of x_0 x_k minus 1 then the last term now would be x. Okay so it is x_k, so we want to write x_k plus 1x, so we will write f of x_0 x_k and f x_0 x divided by x minus x_k plus 1 or in short I can write f of x_0 x_k x okay as so if I can approximate f of x_0 x_k into x this function from this divided difference scheme the next order divided difference being the divided difference of the previous order divided difference that is what we are using remember what we have used in the

Newton's form to compute that we said f x_0, x_1, x_2 this this third order is f of x_0 x_1, x_1 x_2 minus f of x_0 x_1 the second order divided difference is divided by the x_2 minus x_0.

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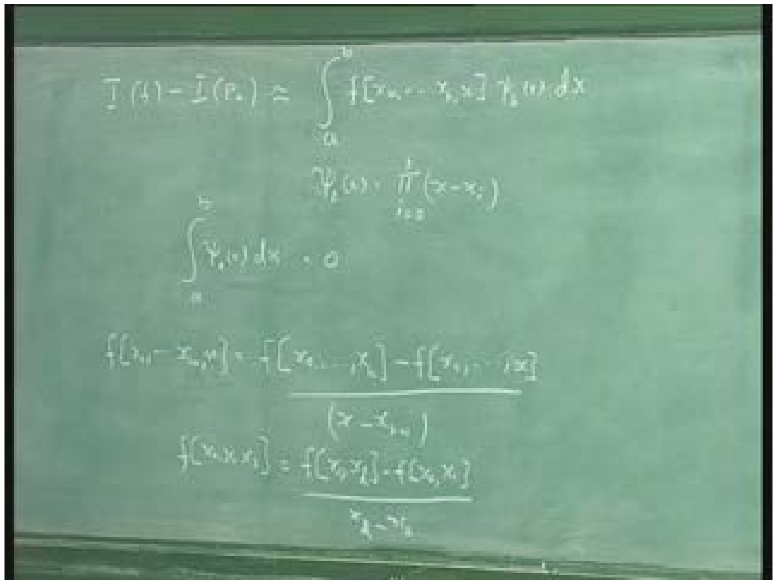

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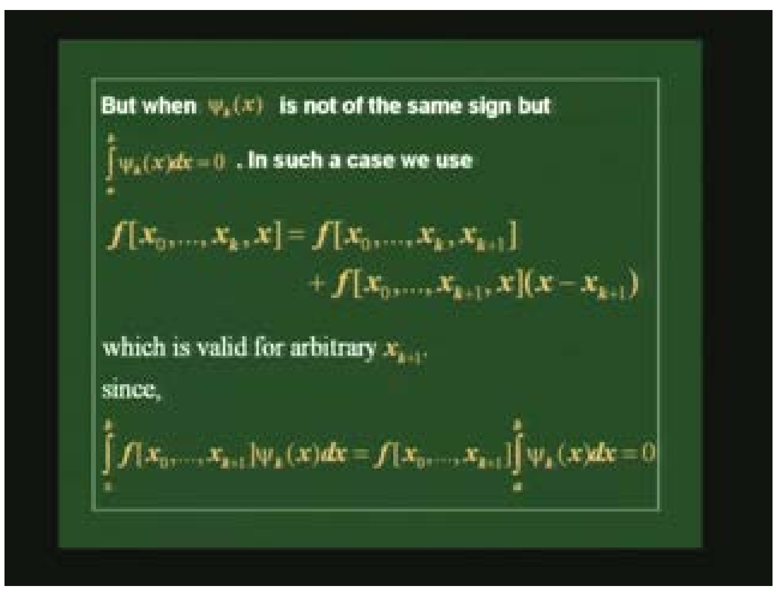

Okay that is what we had used we have used the same scheme here. Okay and then write the divided differences in this form okay so I can then write f f of x_0 x_k as k plus 1 x into x minus x_k and f of, so what I am writing is f of x_0 x_k plus 1 x is equal to f of x_0 x_k x minus f of x_0 x_k plus 1 divided by x minus x_k plus1. So I am using that scheme to write my new value of f of x_0 x_k x. So this quantity here I can replace now by a k plus 1 th order term. Okay so now this I can replace this by f of x_0 x_k plus 1x plus another term. So that is what I want to do here.

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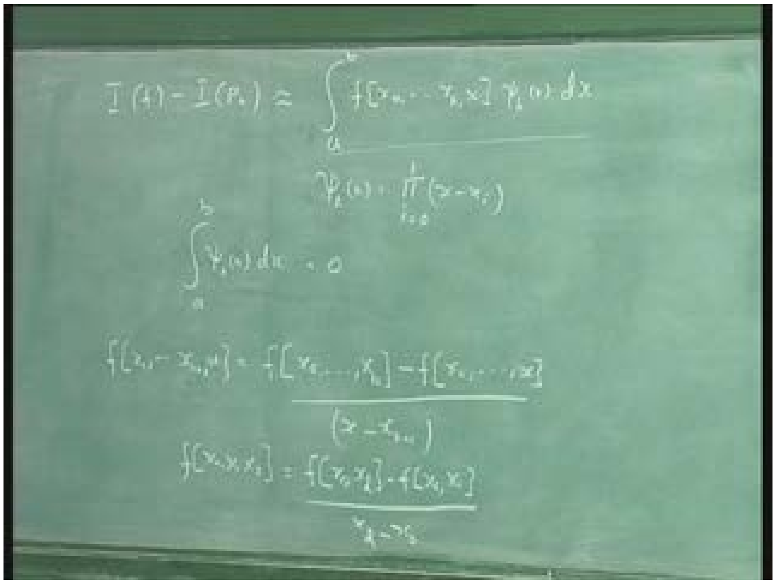

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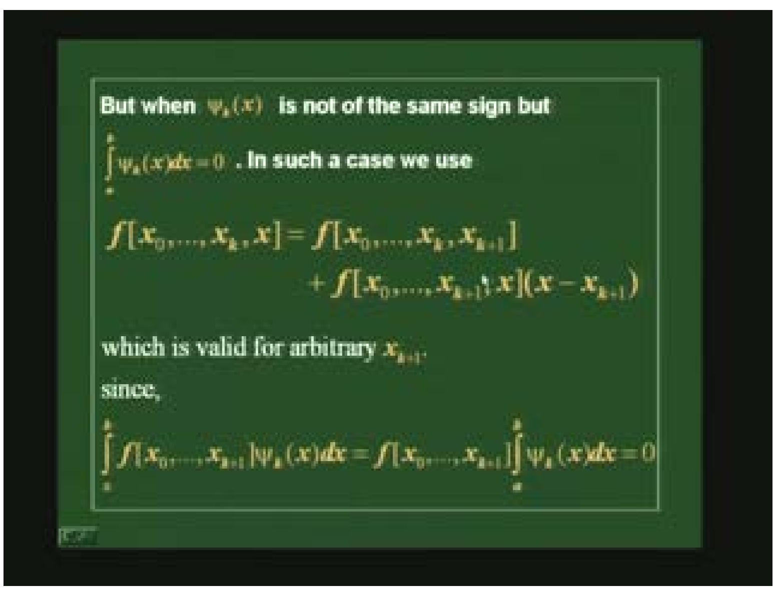

So that is this term, so I am going replace my error in the integration that is by this quantity and then I have so basically I would replace this error here. Okay now using that theorem or using that divided difference scheme I would replace by this integral this integral here by now integral a to b, I will say f of $x_{0,}$ $x_{1,}$ x_k plus 1x into x minus x_k plus

1. I get an additional term there from the divided difference okay then plus I will have an another term which is f of x_0 x_k plus 1 into and then the whole thing multiplied by psy k of x dx. So that is what that is what now I have.

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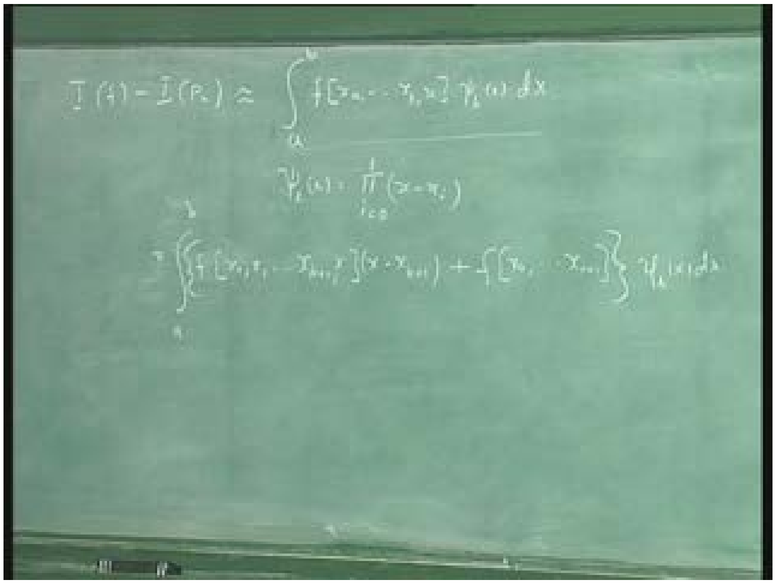

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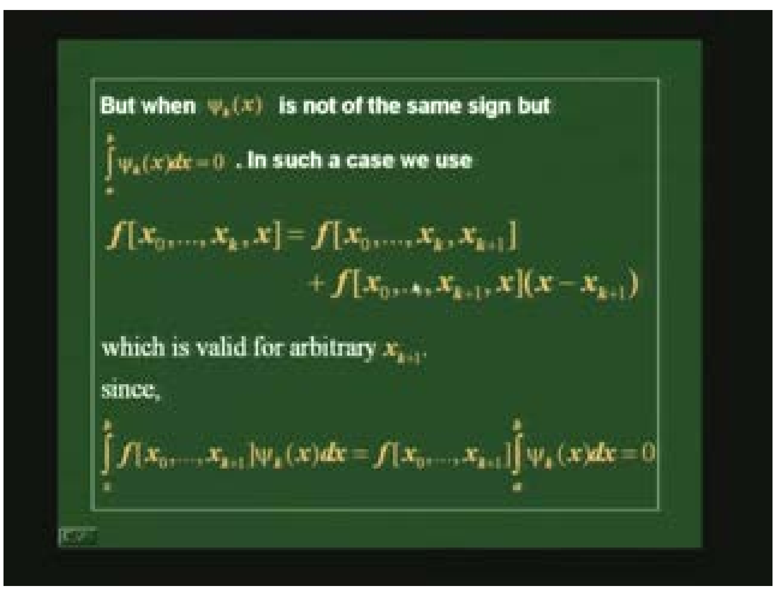

Okay so I have two terms, okay now one coming from so this quantity is going to be replaced by two terms one is the k plus 1 into x term this is the new term then I have an extra term here but I know that if I write this as two integrals okay I know that this quantity will vanish because this is not a function of x so this will be simply become integral a to b f of x_0 x_k plus 1 times x into x minus x_k plus 1into psy k of x that is actually psy k plus 1 of x now right

So psy k of x is already a product up to x minus x_k starting from x minus x_0 but now I have another term x minus x_k plus 1. So this becomes psy k plus 1 of x and then I have another term into dx d x plus integral a to b f of x_0 x_k plus 1 into psy k of x dx, okay now this is a constant right this psy k of x dx integral is 0 that is the special case which we are looking at. So this term goes to 0 and then we have what is left with is integral a to b, f of x_0 x_k plus 1 now comma x into psy k plus 1 of x dx.

So we started with error being a k th order term okay and we ended up with a k plus 1th order term. So that is the idea we have actually k th order up to up to this this k and here now it becomes k plus 1. Okay so went up so if you now if I say that okay psy k of x integral was 0 but psy, I now chosen my k plus 1 th point in this. Okay I have added a new point here k plus 1, I have chosen this point such that psy k plus 1 of x which is now I going from 0 to k plus 1x minus x_i is of 1 sign.

Okay two cases here, so I had started with the case that integral a to b psy k of x dx is 0 so now I am saying that I had um because of this property, I could write my error in this form now all I have chosen was to I had a k plus one th point into my calculation okay and write f of x_0 x_k x as f of x_0 x_k plus 1 into x minus x_k plus 1plus f of x_0 x_w x_k plus 1 using the divided difference table.

I added a new point and use the divided difference table to write from this one. So this function here this coefficient of the polynomial is replaced now by this and since psy integral psy k of x dx is 0 this term goes to 0 and then I have x minus x_k plus 1 into psy k_x that will become psy k plus 1 of x. I am saying that I choose this x_k plus 1k plus x_k plus 1 this k plus 1 th point such that this product psy k plus 1 is of one sign. Okay it was this was 0 now I say that psy k plus 1of x is of one sign that is either plus or minus throughout the interval a to b.

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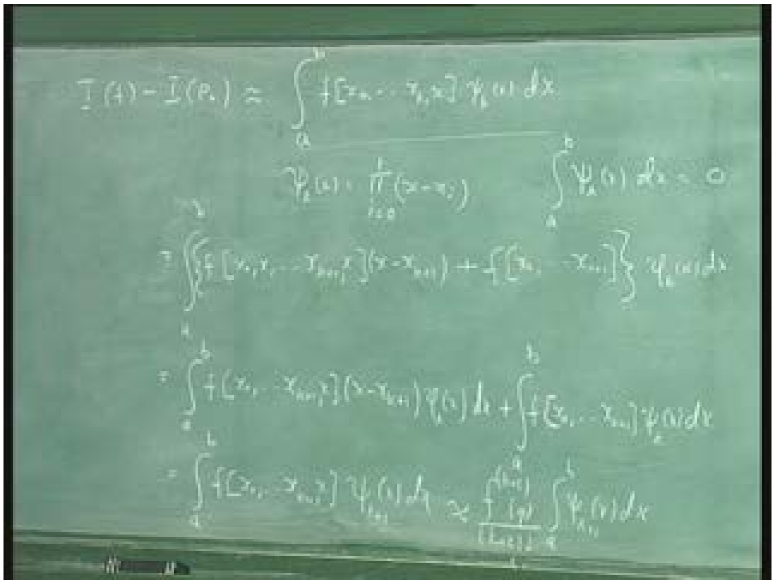

So in that case I can again use the mean value theorem and write approximate this as f of order k plus two derivative divided by k plus 2 factorial right at some eta between a and b i integral a to b psy k plus 1of x dx. Okay now that is the point. so by choosing my k plus point plus 1 th point cleverly that is by saying that I choose this k plus 1 th point such that psy k of x psy k plus 1 of x is of one sign I can write the error of the integral. Okay as of the order of k plus 2 derivative k plus 2 order derivative I have not changed the polynomial assumption here I still have a k h order polynomial but my error is of k plus 2 order, okay that is the which I am going to summarize here.

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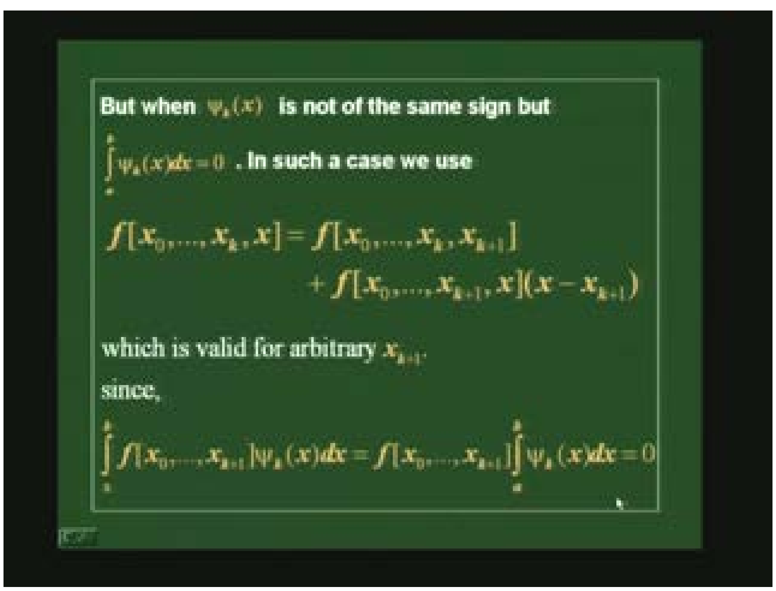

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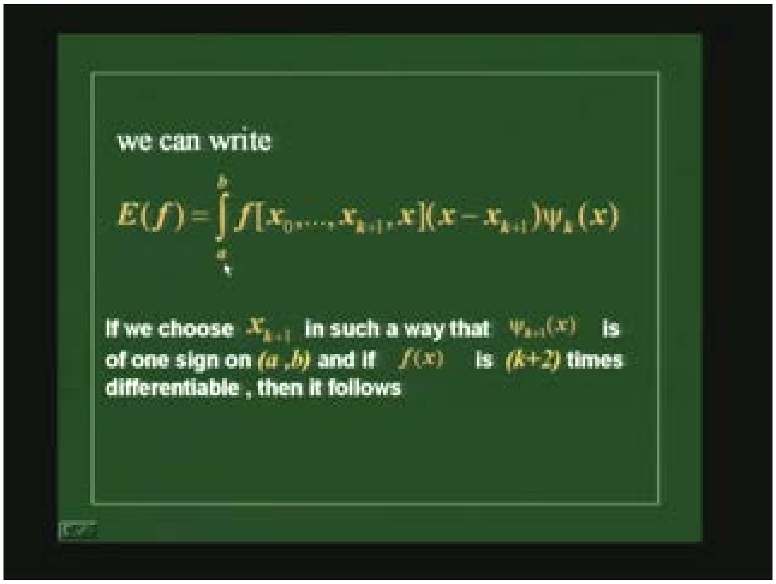

So I can write that thing and then since this integral can be now, I can pull this out right that is what I have done here. Okay now this is 0 which we know that and then I can write my error as of k plus 1 th order by writing in this fashion if this is of one sign again this has to be of one sign this psy k plus 1 here. Okay if this of one sign and then this will become a function which is by mean value theorem something of the order k plus 2derivative provided again that this multiplied by x minus x_k plus 1 is k plus 2times differentiable. Okay then I can write it in this form.

okay so we have found that there is case where I have the k th order polynomial approximation to the function where the error is of the order of the k plus 2 th derivative of the function Provided the function is differentiable k plus 2 times in in the interval a to b and I can choose my points in such a way that psy k plus 1 of x that is x minus x_0 the product x minus x_i where i going from 0 to k plus 1 is of one sign in this interval.

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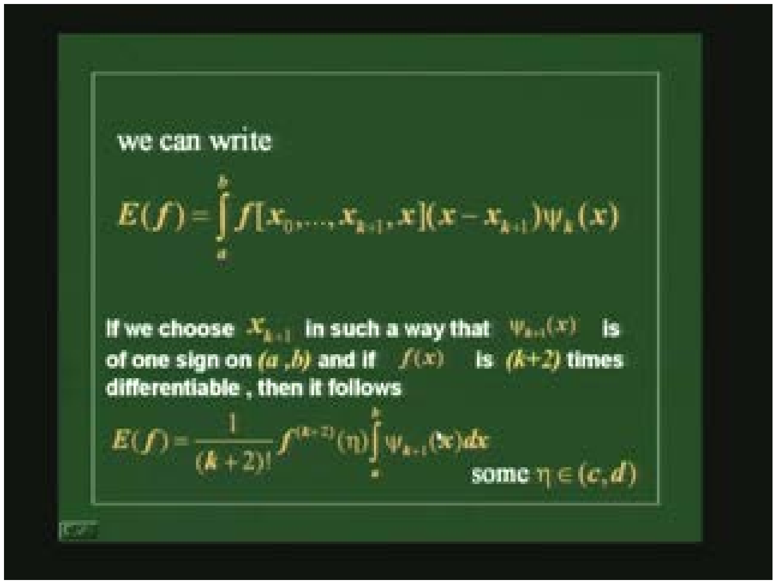

So that is a special case which we have seen, okay so now I can generalize this right I can say that okay I choose my k plus 1 th point. Okay such that again integral psy k this is the case where as we said psy k plus 1 of x is of one sign but I could choose my k plus 1 th point such that the product of that is x minus x_k plus 1 into psy k of x is also 0 that is the integral dx of x minus x_k plus 1 psy k of x is also 0 this was 0 but we choose the k plus 1 th point such that it is again 0 this this integral of this is also 0 and in that case I can go through the same argument and add a new term there and go to a k plus 2 term right.

So I can continue this process I can as long as I can come out with points x_k plus 1 at which I can tabulate the function such that this integral of this quantity is 0 okay I can go to the next term. So this this is particularly useful when we actually know the functional form of f f of x we actually know the functional form of x then we could tabulated it at some special points such that this this integral is always 0.

So if you if you do that okay then the derivative also the error in this thing can go down to a higher order to go to a higher and higher order derivative of the function so that is the useful thing to do and that is what we would use in the in the contragial schemes.

Okay something this idea is been used in the contragial schemes, okay so for the time being let us understand this that I can approximate this integral by a polynomial and the error in the polynomial approximation the error in the integration because of the polynomial approximation is of this order which if psy k of x is of 1 sign between the integer limits a to b can be written as the k plus 1 th order derivative of the function multiplied by integral psy k of x dx but in the case where psy k of x dx integral is 0, I can use the divided difference idea and go to and add one more term in to this error.

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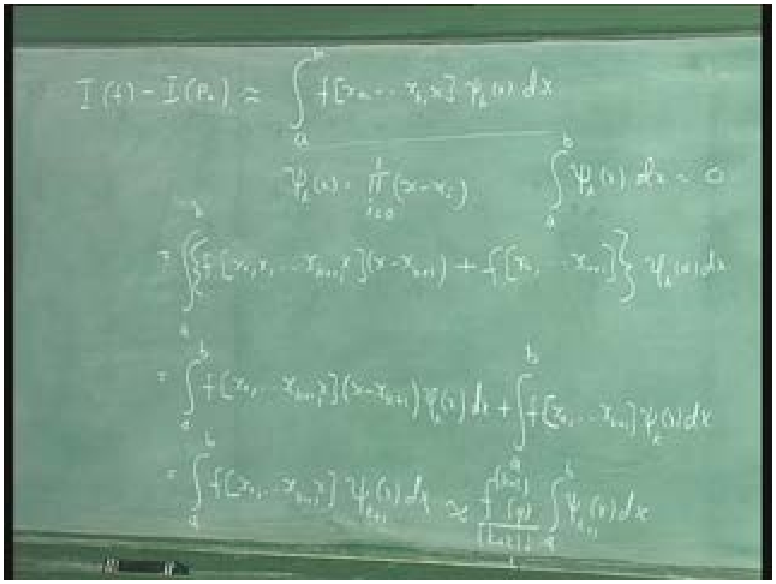

Okay I can split this term into two such terms and then I can see that this integral this part is 0.So I can write this as a k plus 1 th order term a_k plus 2 order term. So k plus 2 order derivative term and then if I can choose more points then I added here a point x_k plus 1 but if the point added was also such that integral a to b psy k plus 1 of x is dx then I can go to k plus 2 from k plus 2 to k plus three order derivative etcetera. So I can continue this.

So I can go from k plus 2 order derivative to k plus 3 and so on right I can I can go up to very high accuracy provided I can actually choose my points in and the function is differentiable up to that order. So we will now look at some special cases of if of integration and we will evaluate the error using this basic theoretical frame. So we will look at to start with 3 different cases called rectangular midpoint and trapezoidal rules to integrate a function Okay, so let us look at a function of this form let us say we have function of this form. I plot that function f of x versus x and then, I have something some function of this form okay and now I want to find the let us say this is my this is this a and that is b, this is x equal to a, this x equal to b now I want to comp compute the integral of this function.

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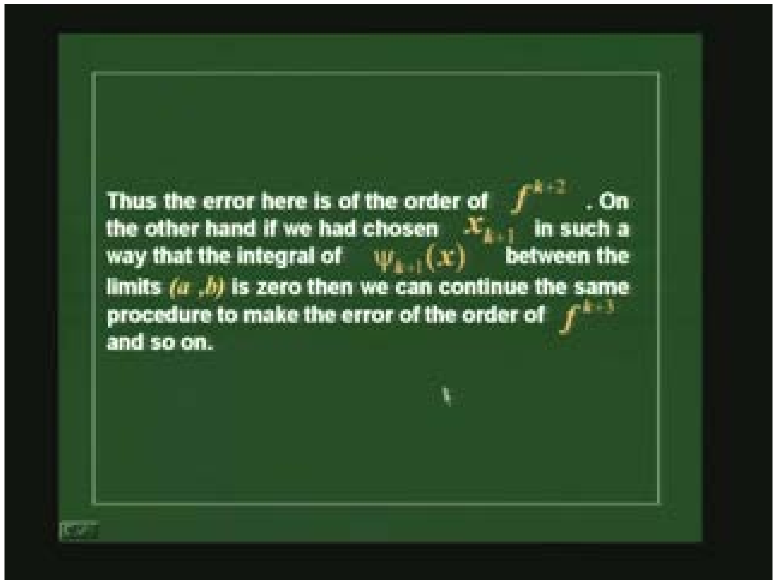

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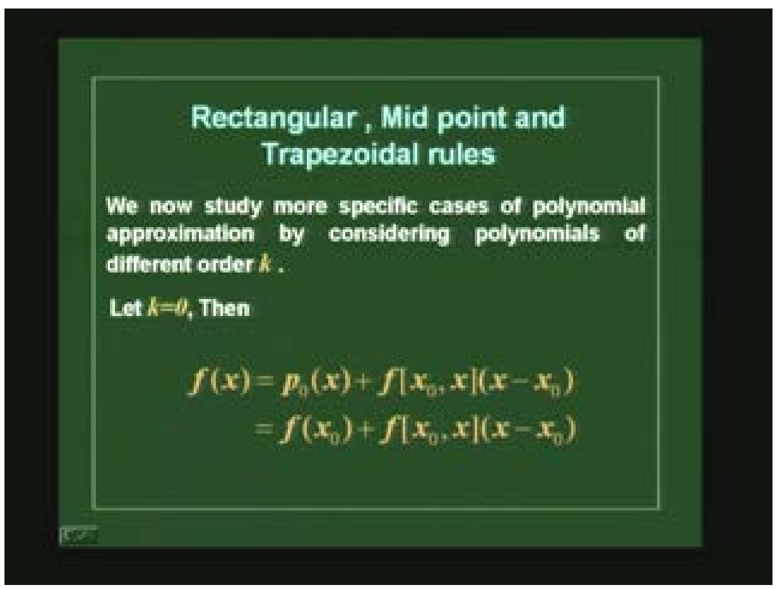

So what does it mean it means that we want to compute the area under this curve in a one dimensional case this one. So we just have simply had to compute the area under this scale. Okay that is what means by computing the integral of this function right so we can look at various schemes which I all here as a rectangular midpoint and trapezoidal rules and we can write down we can draw the area computed by these rules and see what the error in our calculation is. So a simple function like this would help us understand the error in these schemes.

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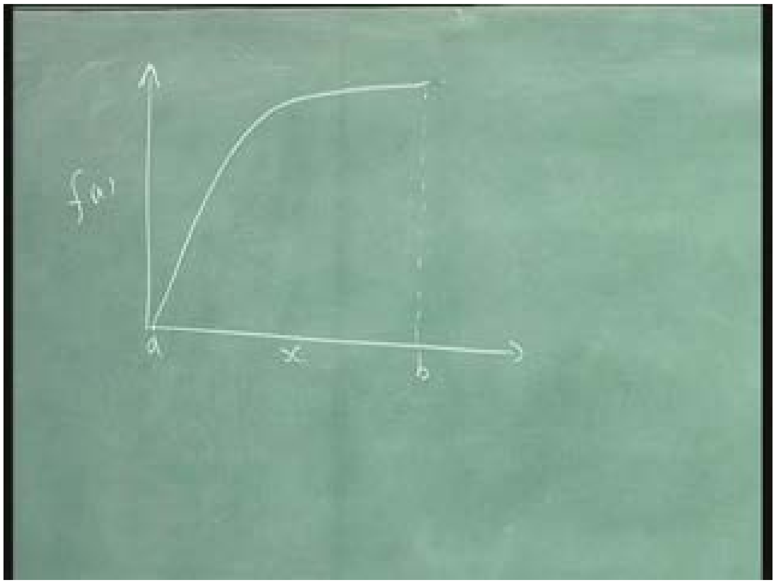

So let us start with a very simple case where we would approximate the function by a polynomial of order k, of order k that is k equal to 0 to start with, okay that is a simplest case that is what we are receiving is that we will approximate f of x by a polynomial of order 0. So what does it mean it means that we will it as f of 0 rights x_0 being the 1point at which the function is tabulated?

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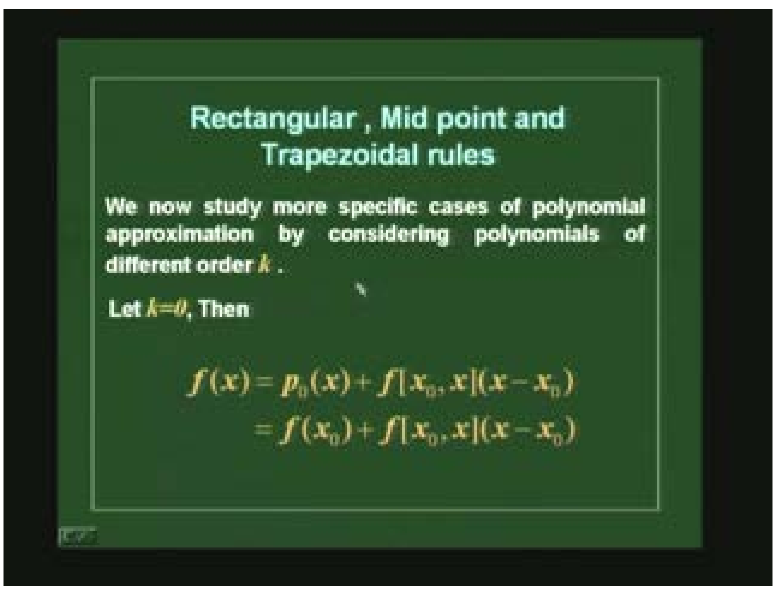

So look at graphically what this is mean that we are going to say that this whole curve is we are going to take just 1 point on this curve, okay some value x_0 we just choose some value x_0 and we say that is the point at which we are going to evaluate the function.

So and then what will be the integral of this function. So i of f between the limits a to b. So we say between the limits integral a to b f of p of x right p of x will be p_0 of x dx. So p_0 of x is just f of x_0. So that is integral a to b f of x_0 dx, so that will be simply f of x_0 into b minus a that is what it is.

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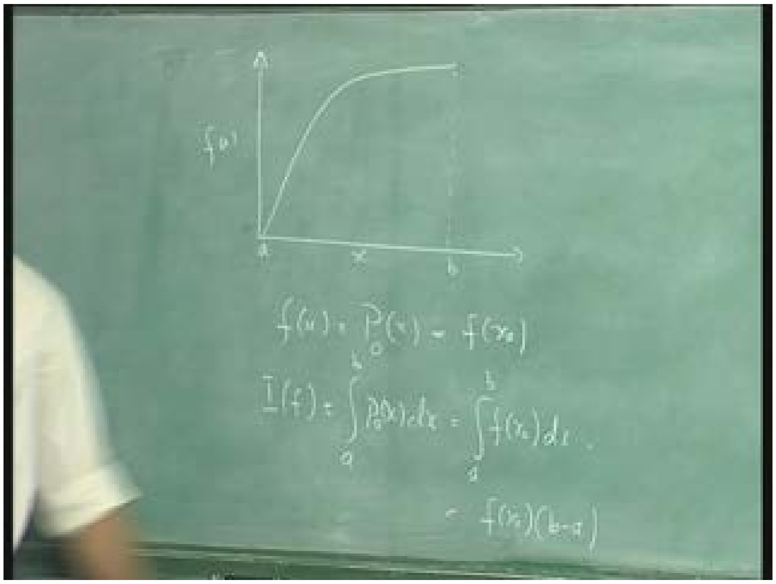

Okay so we have approximated the integral by a polynomial of order degree 0 that is k equal to 0 which means that the integral would be just f of x_0 into b minus a. we approximated the polynomial by the function by a polynomial of order x_0 that is integral by f of x_0 into b minus a so that now it depends upon how x_0 to be. Okay if we choose the x_0 to be let us say a. okay that will become f of a into b minus a or if we choose f_0 to be x_0 to be a plus b by 2 then it is x minus a into f of x, a plus b by 2 into b minus a etcetera.

So that is the schemes which we are going to look at so let us choose lets now choose f of a to b here or somewhere here we will just say a is this. Okay so now if I choose x_0 to be a and then this saying that I take the function value at their. Okay that is my f of a and I am saying that I will take the function value there and multiplied by b minus a.

So what am I getting I am basically computing the area under this curve right, area under this this box. Okay, that is what I am going to compute. So I am going to approximate this whole integral area under this thing by area under this box if I choose my so I actually had to compute the area under this whole curve starting from here to here.

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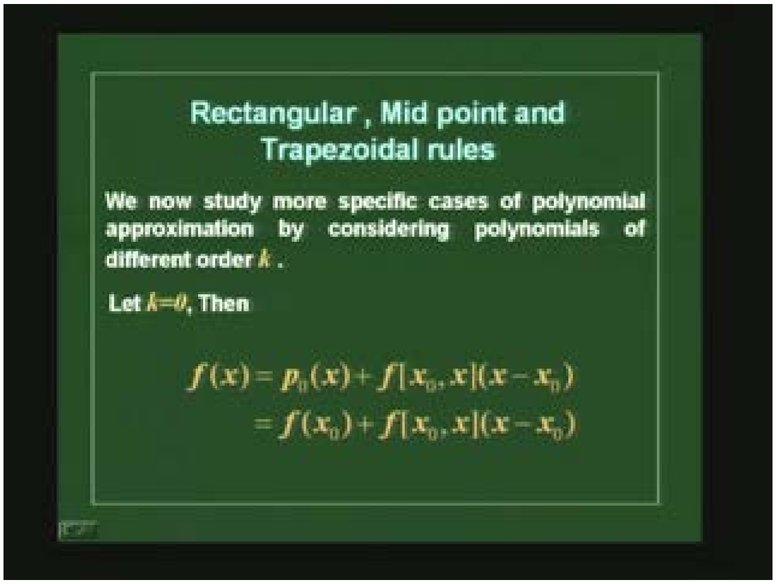

Okay I should have computed this whole area instead of that by this scheme I am computing it as the area under this this box here. So my error is obviously all this unshaded area here is my error so that is what we in this scheme we are just taken to be f of x_0.

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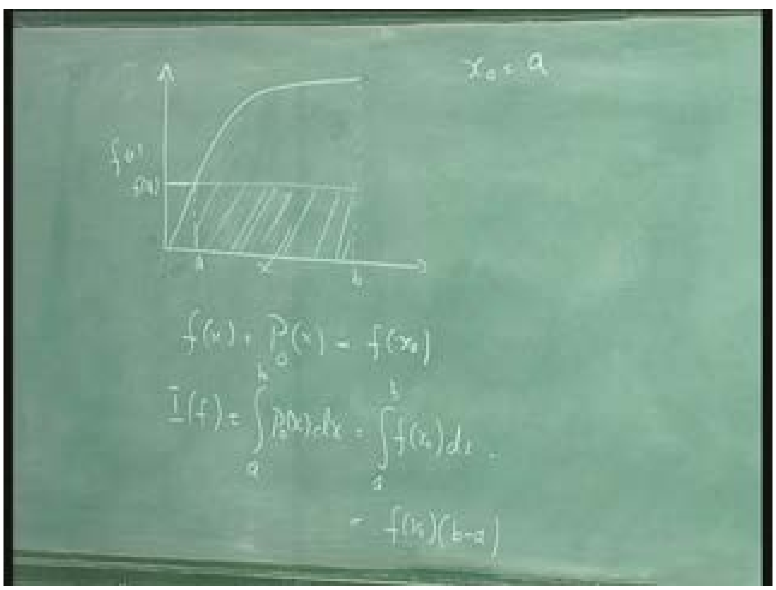

So I choose x_0 to be a and then this approximation becomes am simply b minus b of b minus a into f of a that is obviously called rectangular rule because of as we see graphically here we just using a rectangle for instead of computing this whole area under this we choose just a rectangle of this this shape here and we compute the area under that is the theorem that is the integral.

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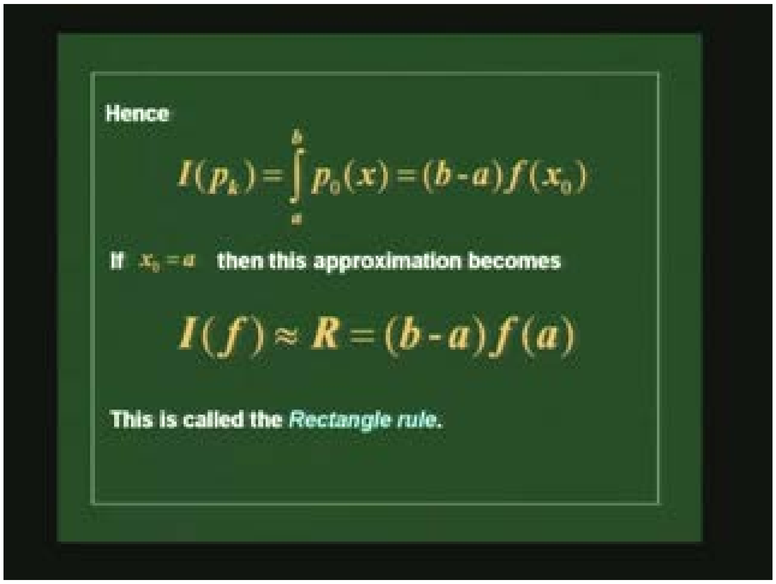

So in the polynomial approximation we know the error in this polynomial in this computation would be the next term in the integral of the next term in the polynomial okay which will be p_0 p_f of the error would be then e would be integral f of x_0 x_1 right, into x minus x_0 dx right, that is the next term in the polynomial.

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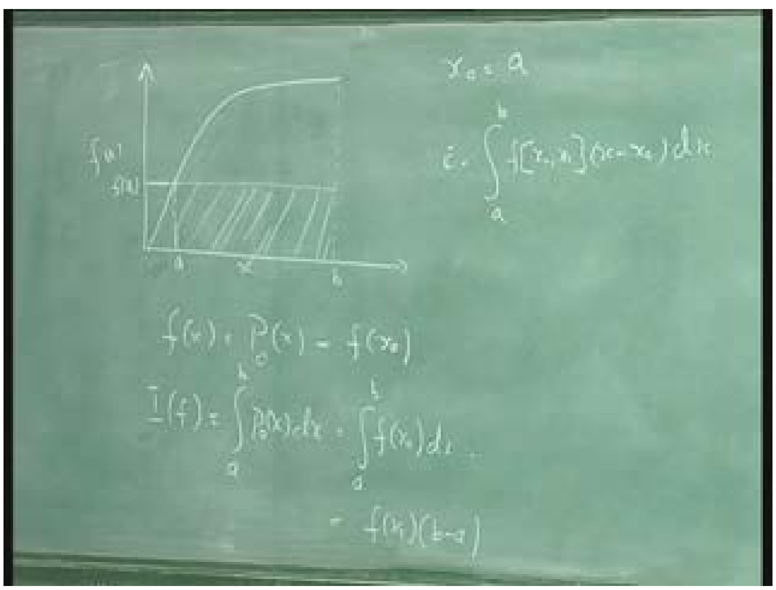

So that is we are choosen our new point x 1 okay and you say that f of x_0 x_1, f x_0 x_1 this is the coefficient of the next term in the polynomial into x minus x_0 dx. So that is the next term in the polynomial. So, the integral of the Okay so I will write that error as that and now if I chose my x_0 I have chosen my x_0 to be a right that is what I have chosen here I chose my x_0 to be a and said that, in the case x_0 is a. next term in the polynomial is my error here okay so that is what we will look at now.

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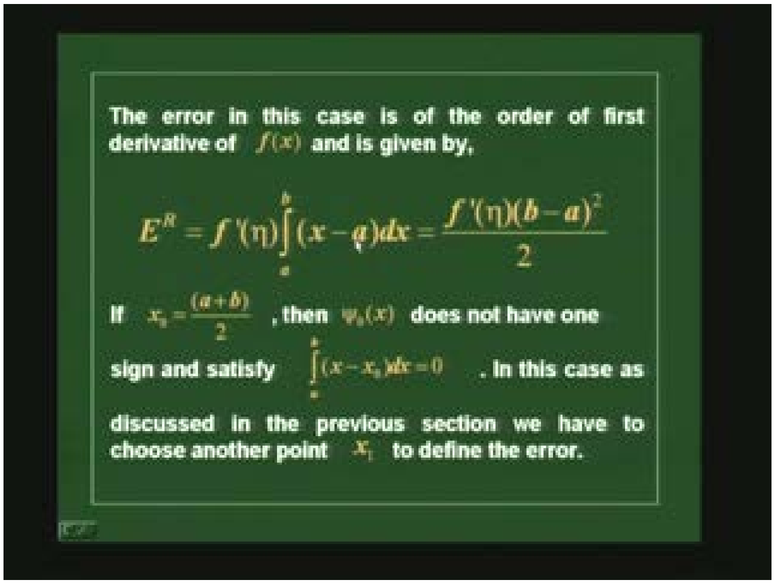

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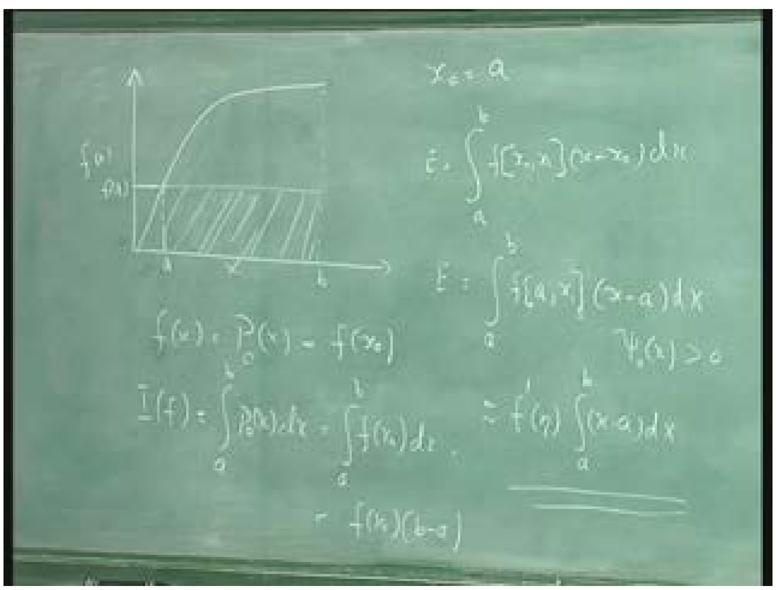

So this integral was f of a into b minus a and the error would be integral a to b, f of a times some point x_1 which we have to choose right into x minus is it here, x minus a dx now, here see we have choosen a to b as the x_0 we have choosen that is one of the limits. So all x values are higher than a, because we going a, x staring from a to b.

So all x values in this integral are greater than a, so x minus a is always positive. So this quantity which was our psy 0 of x is of one sign is always greater than 0 for all x values. So is of one sign, so I can pull this out. Okay and I can say that my error here is of the order of f first derivative because 0 is our order into ay some point eta and integral x minus a dx between the limits a to b that will be my error in this rectangular rule. Okay that is what I tried to write here.

So that is now I can just integrate x minus a dx as we said this becomes x minus a whole squared by 2 within the limits a to b. So I get my error in this rectangular rule which I call as er, as f prime that is the first derivative of the function multiplied by b minus a whole squared by 2. So If we want to get the maximum error we could approximate this by the maximum value of the derivative in the interval a to b multiplied by b minus a whole squared by 2. Okay so now that is one case of the rectangular rule which we just saw.

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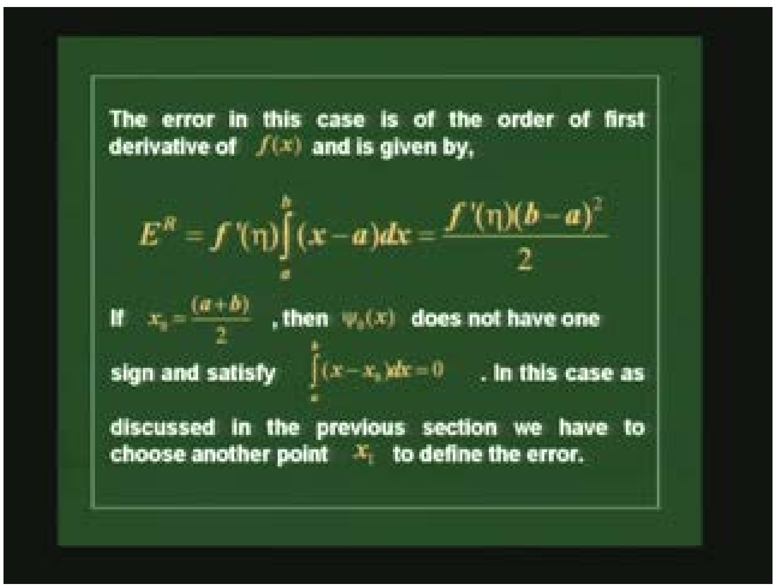

So now let us look at, so that is the rectangular rule and we call that as er. Okay to represent it as a rectangular rule now in this particular case, we have choosen x_0 to be a okay, so that psy 0 of x is greater than 0 all the throughout the interval greater than or equal to 0 throughout the interval. Okay so that is of one sign.

So let us choose the next case that is we choose we want to now look at the example, where this integral would be 0 that is integral psy 0 of x dx is 0. Psy 0 of x is not of one sign but integral psy 0 of x dx is 0. So we can choose for that the case x_0 equal to a plus b by 2. So what are we saying that okay we are saying the following that now we will have the function evaluated at.

So we take x_0 x_0 to be a plus b by 2. Okay that is the case where that is somewhere here that is my, a plus b by 2 point and that is the function value at that point and I multiply that by a minus b. So my integral now become if I choose x_0 to be a plus b by 2. Okay my integral i of f which is approximated by I of p_0 of x this will become what we get is integral a to b right we will get that as f of a plus b by 2 into dx and that will be f of a plus b by 2 into b minus a.

So that is the area of the box area of the follow this this box that is, okay so now we have a shaded area all the way like this all the way up to that. Okay so I have a shaded area now all the way this is not correct so that is my function and then I evaluated at this box okay that is f, a plus b by 2 that is f of a plus b by 2multiplied by a minus b. So that will be the area under this shaded curve here okay.

So that is my next approximation to this function and that is my polynomial. So it is the same one polynomial of the order degree 0 but I choose x here to be now a plus b by 2 and then I have a different integral or integral approximation or approximation of the integral.

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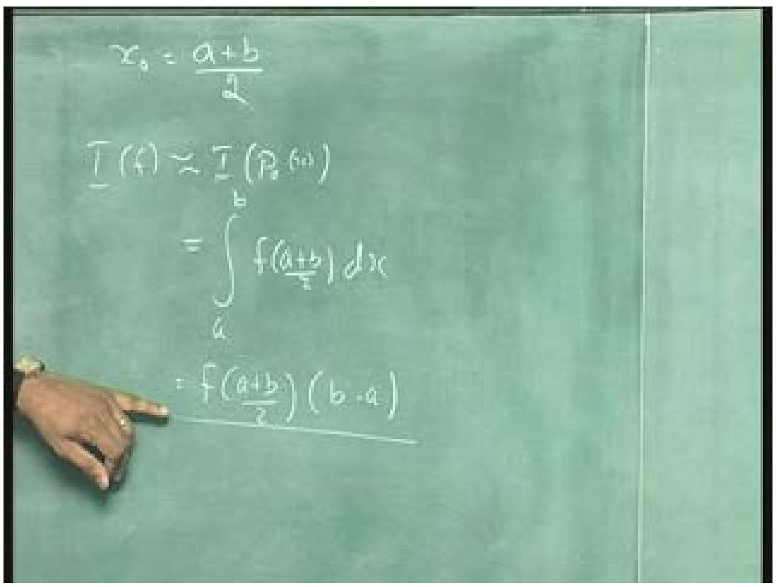

Now I choose my x_0 to be a plus b by 2, now look at that we have this integral now here this psy 0 of x right, psy 0 of x is now x minus a plus b by 2 in the limit a to b. Now that is 0, so now we have to choose a new point x_1 such that we can define the error. So that is the second case first case where psy of x had the same sign and the same 0 th polynomial 0 th order polynomial approximation. We can choose x_0 differently now, we choose the mid point that is a plus b by 2 mid point of the interval such that the integral of x minus x_0 dx is 0 and then we can now diff the error as the next term in the polynomial or next order term that is now we will have a second order derivative right.

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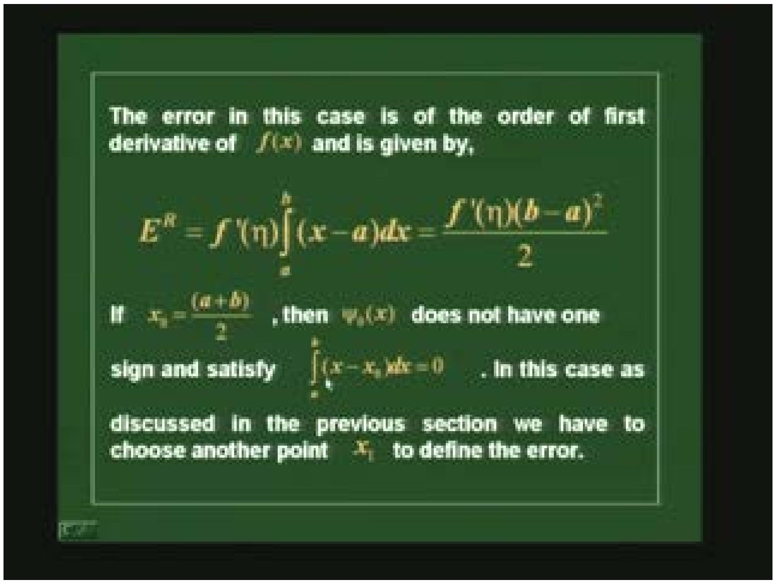

So we will have a second order derivative into b minus a by 2 in the case where we has psy k of x is of one sign right is of one sign okay and then I said that the error would be of the order the error in the polynomial error in the integral will be of the order f k plus 1 there is a derivative k plus one th derivative by k plus one factorial right integral psy k of x dx between the limits a to b now we had seen that we have to choose we had seen that we were choosing x_0 to a plus x_0 to be a plus b by 2 this integral is 0. So now we choose a new point k plus 1 th point right and write this as a next term right.

So we have to make this into one sign, okay so now this is not of one sign but the integral is 0 now I had to make my psy k plus 1. So I want to write my psy k plus 1,e as psy k into x minus x_k plus 1 and I want to say that this is of one sign. Okay so that is I want to say x minus a plus b by 2 now I want to multiply it by something else such that that is of that is of one sign throughout the interval. So, that is x minus x_k plus 1 is of one sign throughout the interval and that I choose to be x_1 itself.

So that is we will just choose this x_k plus 1to be again a plus b by 2. So we will choose okay we will choose x_k plus 1 to be a plus b by two again such that our psy k plus 1is now x minus a plus b by 2the whole squared and which is of one sign right because if it is square its always positive. So that is the idea here okay.

So I choosen my next derive next term to be x_k plus 1 itself such that my error is now of the order x minus a plus b by 2 the whole squared integral x of this. So my error now is e is equal to f_2 that is a second order derivative x of eta divided by 2 factorial which is 2 integral x minus a plus b by 2 the whole squared dx between the limits a to b. Okay that is the idea so what will we do we this is an example of two choices of psy is. So we started with a polynomial approximation which is we choose to be a_0 th order polynomial that is function is evaluated only at x_0

So first we choose x_0 to be a and then the integral is simply f of a into right in this case the integral is approximated f of a into b minus a, so that we what we call it as rectangular rule and then we had in that rule, we had the error is of this form right this is the first derivative multiplied by x minus a dx that is that is we get it as x minus a the whole squared by 2 between the limits a to b that is b minus a whole squared by 2 that is the error the error would be here b minus f of f prime the first derivative multiplied by b minus a whole squared by 2.

Okay so that is the first rule and then we choose the next case we choose a different point as x_0 still the same polynomial p_0 of x. So we choose the integral to be integral of p_0 of x but x now chosen that is the tabulated point is now a plus b by 2 and the integral is simply f of a plus b by 2 that is f of a plus b by 2into b minus a but the error now is this coefficient multiplied by x minus a plus b by 2 dx but this integral is 0 because it is symmetric and its plus in 1 half and minus in the other half, it changes sign it is not of one sign It changes sign.

So it is 0, so we said okay then in that case what we can do is we can use this property to make psy of k of one sign and to which psy of k of one sign I add one more point that is x k plus 1now. Okay and then that is that we get psy of k plus 1 as x of x minus a plus b by 2 into x minus x_k plus 1 and then the integral of that would be error in this case and that turns out to be the next order f of 2 eta this a second derivative of a function multiplied by x minus a plus b by 2 whole squared dx. Okay that is the next error.

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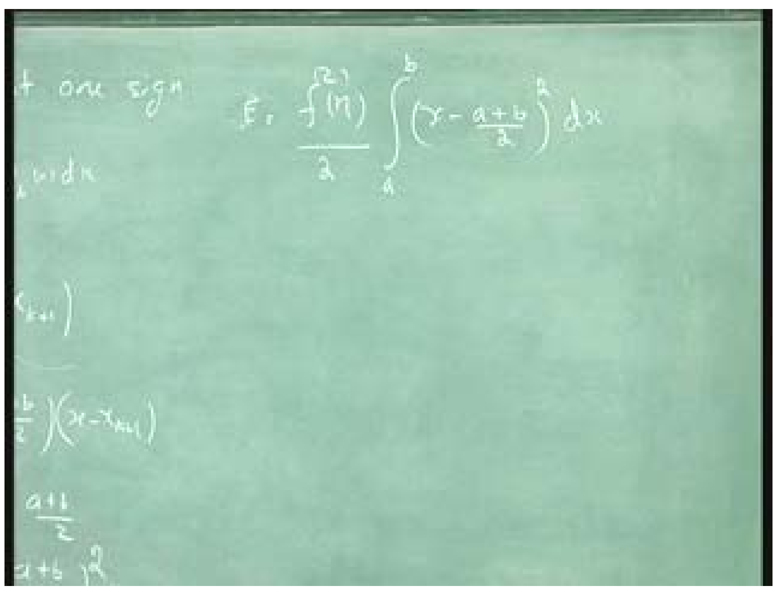

Okay that is we get it as b minus a whole cube by 24. So that is called the midpoint rule, so we call it as m. So that is the case, so midpoint rule is we can see the error here. So midpoint rule in those particular function case is initially better in the since that it is approximating the area in this case between this of the of this curve of this curve from this point to this point by a box of this shape that is it is better than using the case of rectangular rule where we had approximated it by a box here. So this is has much less error.

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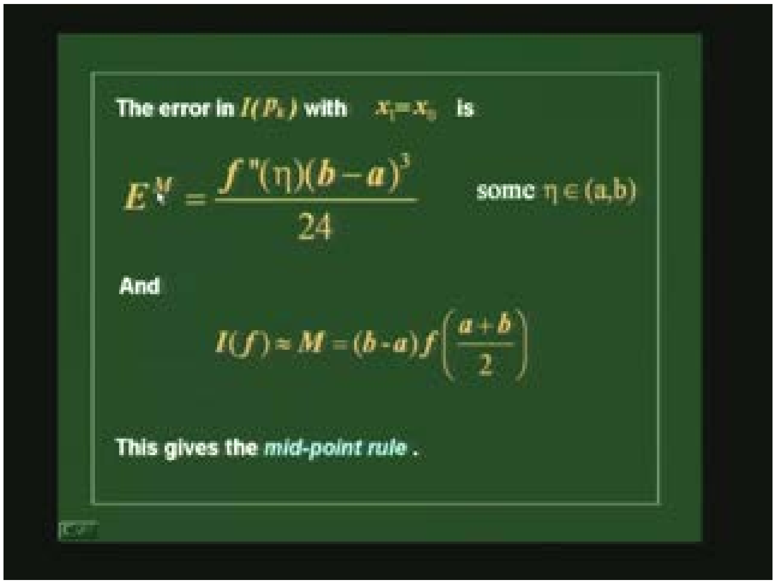

Okay and that will do if we have obviously depend on the derivative of this function and the type of this function etcetera so it is not general scheme but in this particular example, we see that the mid point rule is better than the rectangular rule. So the other cases which we should look at.

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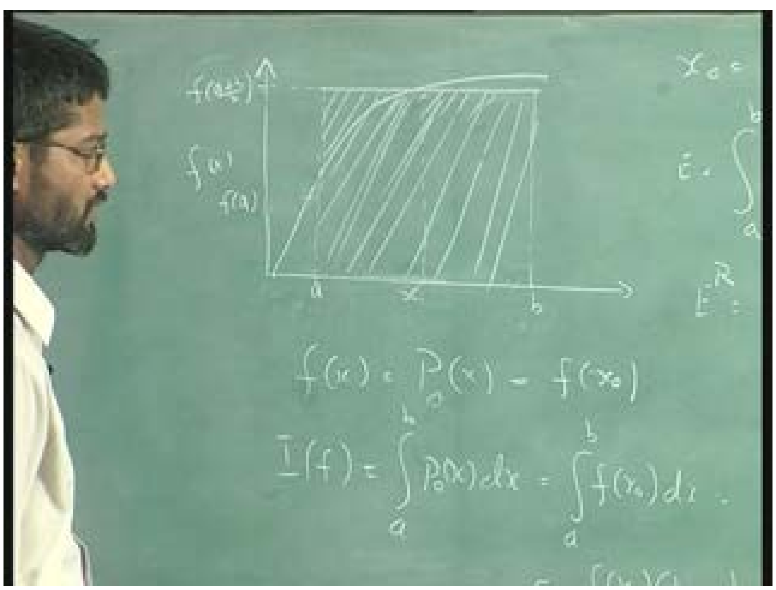

So we could now consider a slightly higher order polynomial. So far we had taken p equal to 0, so no we look at p equal to 1, k equal to one case. So we will then approximate the function by p_1 of x and then the error is x minus x_0 into x minus x_1 right that will be the error.

So the first order polynomial here the psy one of x now will be x minus x_0 into x minus x one okay so then the p one of x is now f of x_0 plus f of x_0, x minus x_0 and this error here x_0 x_1 x into psy 1 of x. So that is what we are going to use of this of this um define this error so now again we will choose x_0 to be a and x_1 equal to b and then we can compute that. Okay so now we are doing a slightly better job of this polynomial. We are doing the polynomial approximation here.

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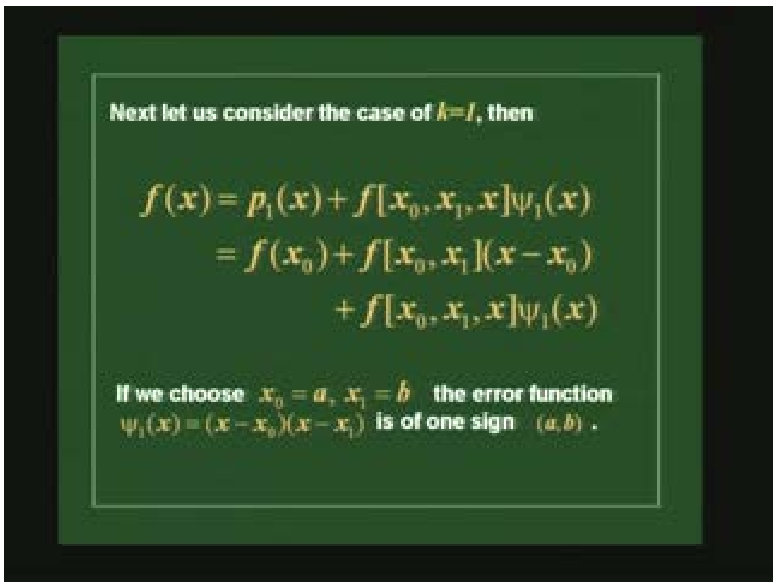

So we will, so far only p_0 of x okay now we are going to do p one of x and then write the integral in this form. So let us look at that. So we are going to write our integral i of f is now as approximated by i of p_1 of x right and then so that is, so and we are going to write p one of x as f of x_0 plus f of x_0 x_1 into x minus x_0 that is the p of 1 of x. So that is the integral would be now i of f would be then integral a to b f of x_0 plus f of x_0 f square bracket x_0 x_1 x minus x_0 dx. Okay that is what we are our aim would be and the error then would be in this particular case.

So we are saying that we will now have to compute, so the error will be the next term so we will now write the approximated error by integral f of x_0 x_1 x into x minus x_0 into x minus $x_1$dx between the limits a to b that is what the error is and that is what we call psy 1of x. So that is so and we will have to look at the cases, so a to b f of x_0 x_1 x into psy one of x dx. Now we are going to we can we can do that thing, we are going to choose x_0 to be a and x_1 to be b. So we have two points, so now we are going to evaluate the function values at both a and b.

Okay 2 values okay we are going to evaluate the function values there okay that is f of b and f of a and then we are going to compute this this integral of this. Okay that is the next point and that is what we would called the trapezoidal rule, okay and we will see this but we can compute this and see what the error would be and what is the nature of psy 1 of x is what is the integral psy 1 of x is, so psy 1 of x is now remember psy 1 of x in this particular case is x minus a into x minus b okay the integral of that between the limits a to b is what we have to find.

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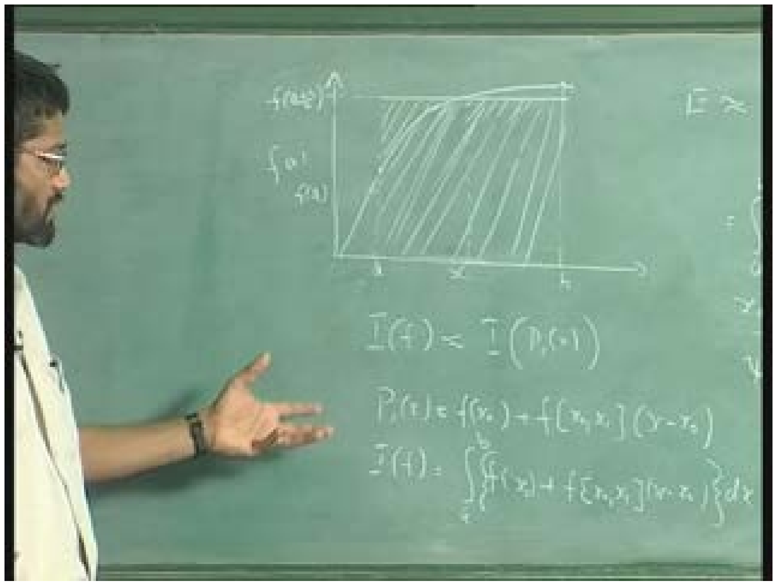

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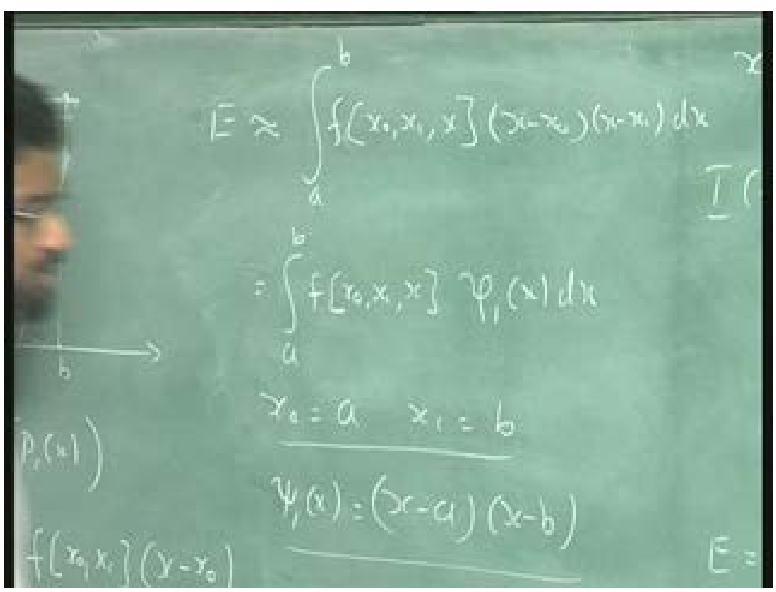

So we will look at in the next class this particular case and also the graphical representation of this particular rule which we are going to see that is by choosing a polynomial of order one now and choosing x_0 to be a and x_1 to be b.

Okay 2 points and then look at this case in all these cases which we had discussed so far we have made clever choice of x_0 and x_1. We are not choosing x_0 and x_0 arbitrarily anywhere in between we have chosen the case where x_0 is of 0 th order polynomial and we choose x_0 to be a and x_0 to be a plus b by 2 the 2 cases and now we are looking at the first order polynomial x_0 is a and x_1 is b.

Okay, now we will evaluate the error in this and look at the and compare this 3 methods in the in the next class.