Numerical Methods and Programming

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Lecture - 12

Cubic Spline Interpolation

Today we are going to discuss error in the interpolating polynomial. Remember, we were discussing a polynomial that interpolates a given a set of data points. So, today we will discuss, what will be the error in this polynomial that is, what is the difference between the value this polynomial gives and the actual function value that is what we will be looking at today. So, we saw this yesterday that the polynomial, p_n of x be a polynomial of degree n which interpolates a real valued function f of x at x_0, x_1, x_2, x_3 up to x_n.

That is the function this polynomial goes through all this points x_0 x_1 x_2 up to x_n. So now, what we are interested in is actually getting a value in between x_0 and x_n, any value which is not tabulated between x_0 and x n. So let us say, x is such a point and f of x is the actual value of the function at that point x between x_0 and x_n then p_n of x is the value that the polynomial gives for that x. So p_n of x is our interpolating polynomial that is an approximate curve that passes through all these points and x is the value somewhere in between which is not tabulated and then p_n of x is the value of the polynomial at that x value and f of x is the actual value of that function which we do not know.

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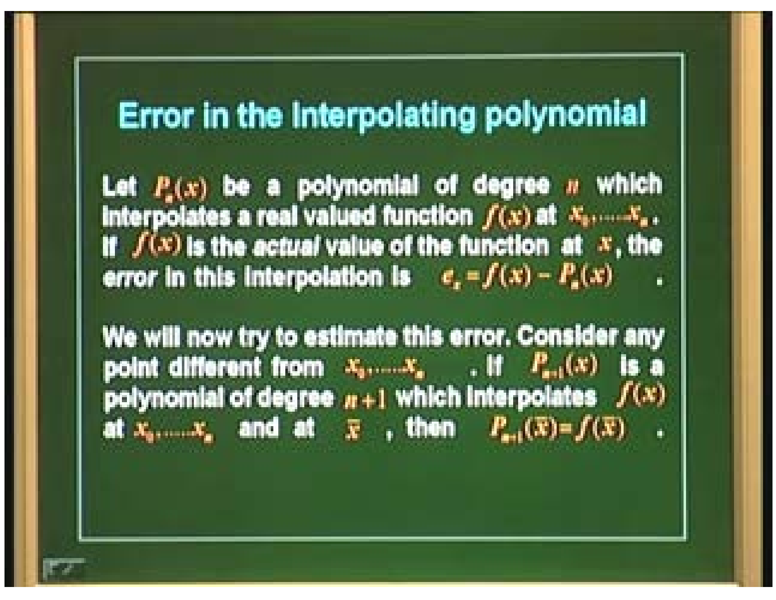

So the error is f of x minus p_n of x. So now, we should try to estimate this error, why is an estimate because, we do not know f of x, that is why we need a p_n of x. So we do not know f of x, so we need to make an estimate for this error f of x minus p_n of x. The way we do that is to say that, let us consider a point x bar, let us consider come point x bar between x_0 and x_n and we say, we assume that we know the function value at x bar that is, we know f of x bar we assume for the time being and then we can construct a polynomial of degree n plus 1 now, which goes through x bar and we call that now as p $_n$ plus 1, x bar and that will be equal to f of x bar and this is exact because, we define the n plus one th order polynomial as one which goes through x bar that is it takes the function value f of x bar at that point.

Remember, the way we construct a polynomial of degree n which goes through n plus one points is to make sure, by making sure that this polynomial takes the function values at all the tabulated points now, we added one more point to that tabulated set of points which is x bar and we construct a polynomial p_n plus 1 of x bar and that goes through that f of x bar. So that is, and then we can define the polynomial p_n plus 1 of x as p_n of x which was going through all the n plus 1 points.

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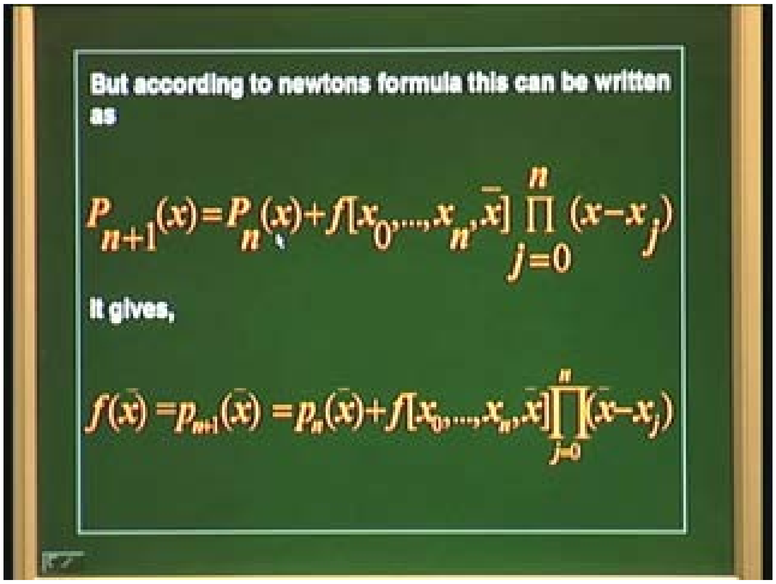

So remember, p $_n$ of x plus an additional term which will be now the n plus 1 th term, the n plus 1 th term will be the j equal to 0 to n product x minus x_j and the coefficient of that will now contain x bar. And then, we know that this polynomial p_n of x bar, p_n plus 1 will take the function value f of x bar there and then this will be p_n of x bar plus f_x 0 x_n x bar product j going to 0 to n, x bar minus x_j. So now we see, what we were interested in is actually f of x bar minus p_n of x bar, so now from this we can see that the error f of x bar minus p $_n$ of x bar is actually given by this term.

So it is given by this term, that is the next term in the polynomial of order p_n plus 1. So that, gives us an estimate of the error, the error is now like the next term in the polynomial. So what we had done, we had a polynomial of degree n which went through n plus 1 points and we wanted to know what is the error in this n polynomial and the way construct we define this or estimate this error is by saying that. Let us say, there is a point x bar which is in between this x_0 and x_1 and then we construct a polynomial of order n plus 1 which goes through this point and then we see that the error is actually the next term in the polynomial. So therefore, for all x bar which is not equal to any of these n plus 1 tabulated points the error is this.

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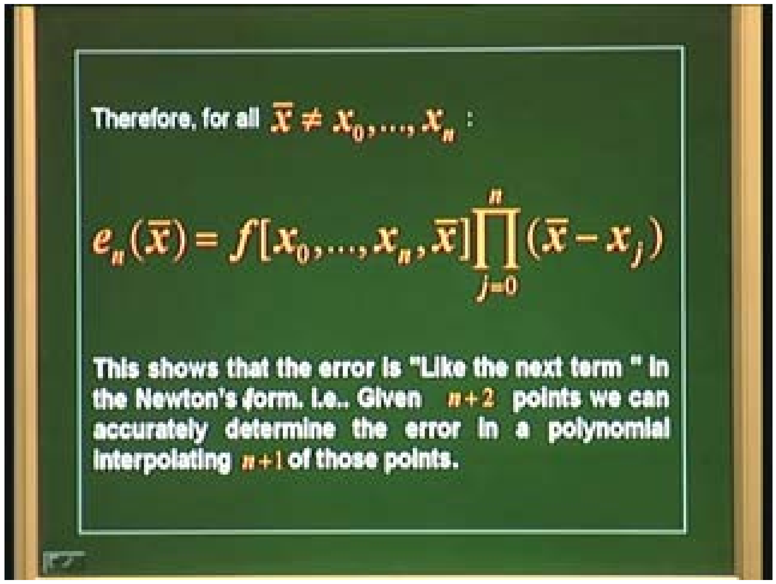

So given this n plus 2 points, we can accurately determine the error in the interpolating polynomial, of interpolating n plus 1 of those points that is let us say, you have a set of n plus 2 points, and then you construct a polynomial of degree n and then it is going through this n plus 1 points of the n plus 2 points given to you. Then, we can accurately estimate what will be the error because, we have an additional point here to estimate which allow us to estimate error because we can use that term to construct the next term in the polynomial and we can estimate the error, so that is the idea here. But, what happen, if you do not know f of x bar at all, we do not have an additional point, so then can we actually estimate the error still.

It turns out you can still do that by using this, what is called a mean value theorem which tells us that given a function f of x defined in the interval a to b and if that is k times differentiable then, if x_0 to x_k are the k plus 1 distinct points inside a and b within the interval a and b then there is a value of x equal to "i" call zeta here, which is between a and b, which is between a and b, such that this function this f of x_0 to x_k to x bar, that is this coefficient of the n plus 1 th term is actually equal to the k th derivative of the function evaluated at zeta divided by k factorial. That is what we are saying, trying to say is that this is n plus 1th term is coefficient is actually equal to the k th derivative of the function at some value zeta.

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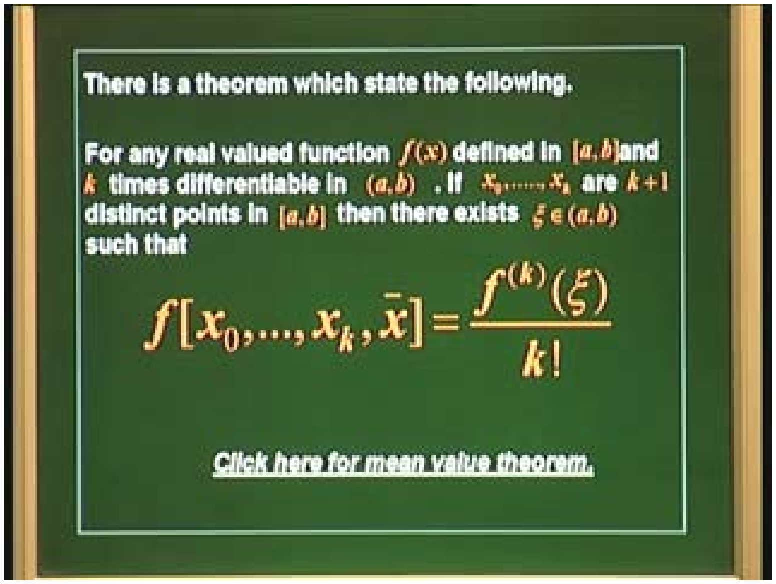

We are saying that there is a value zeta, which is between a and b such that this coefficient is equal to the k th derivative of the function evaluated at that point divided by k factorial. Again, we do not know what the function is so we do not know what the k th derivative is but we have an idea that this error will be of the order of the k th derivative so that gives us something.

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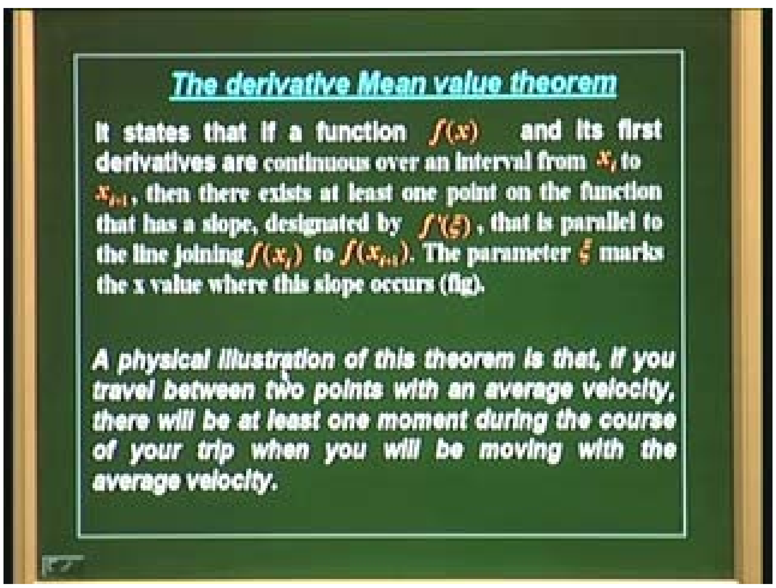

So here is just a illustration of the mean value theorem, which is explained a little bit more for the first derivative there we are looking for the k th derivative, we are looking for the first derivative. So the mean value theorem basically states the following that, if a function f of x and its first derivative so we are looking for the first derivative mean value theorem and it is continuous over an interval from x_i to x_i plus 1, then there exists at least 1 point on the function that has the slope designated by f prime of zeta such, that is parallel to the line joining f x_i to x_i plus 1.

What we are saying, we are saying that, if there are 2 points x_i and x_i plus 1 and there is a curve going through this point x_i and x_i plus 1, then along the curve at least at 1 point its slope should be equal, at least at 1 point its slope should be equal to the line joining x_i to x_i plus 1. That is what basically we are saying here. So that is, this point, this particular point zeta marks the value of x at which, where the slope is. So basically, physically this means that, if you travel between 2 points with an average velocity, so we travel between 2 points with some velocity there you can compute, what is the average velocity between these 2 points then there is along that travel at least 1 point you were traveling actually with that average velocity.

So that looks trivial, if you say this way that if you were traveling between two points at in some variable varying velocities and then you compute the average velocity then this theorem says that at least at 1 point along your travel you were traveling with the average velocity. So that is, the statement of this theorem which is basically illustrated here. So that is illustrated here, that is you are going from this point to this point and you are traveling along this curve and the average velocity will be marked by this curve.

So this curve connecting these 2 points and you see that there are 2 points here, what the theorem states is that there is at least one, where this slope is parallel to this one and here, there are cases where you have 2 points, where you have this slope parallel. And then, given this theorem you can actually estimate what that slope would be, you could just take that as r_0 divided by h, that is the idea.

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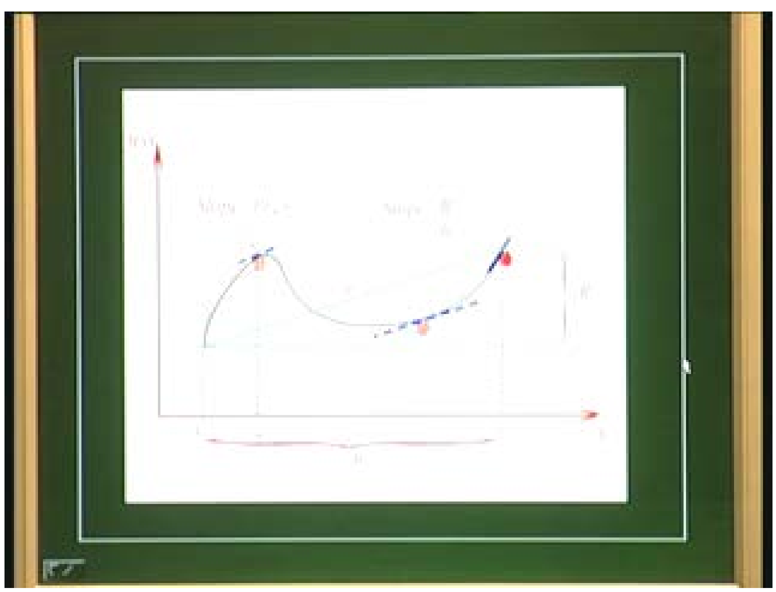

So, we can say that that slope is actually given by r_0 divided by h or you could say what that r_0 is by doing this. So that is, so now this is juts a simple proof for the derivative mean value theorem for the first derivative and this can be extended to the n th derivative we do not go through the proof of here, but it is an extension of this arguments which takes it to the n th derivative.

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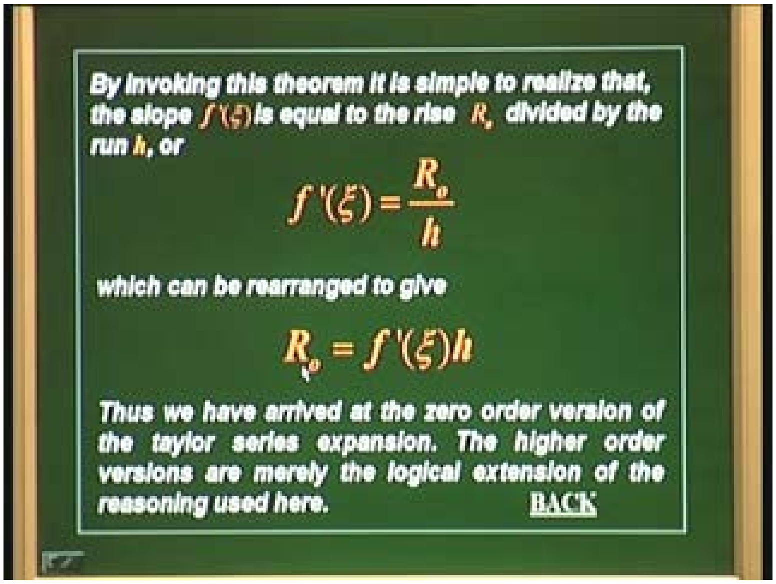

So back into this problem, we were saying that the estimate of the error is actually given by or this coefficient is f of f this k th derivative of f and divided by k factorial. So that means, what is the error term the e n f, which I call the error in the n th order polynomial at x bar. So now, that will be given by f of x bar minus p_n of x bar which we just saw is f of n plus 1, there are k points that is the k th derivative there will be n plus 1 points going from 0 to n which in this function.

So, here it will be n plus 1 divided by n plus 1 factorial with j going from 0 to n the product into x bar minus x_j there are say it is a product of 2 quantities, one is the k th derivative divided by n plus 1 derivative divided by n plus 1 factorial and then a product of x bar minus x_j that gives us some idea of how we could reduce the error in the polynomial, one is that, it goes as one by n plus 1 factorial and the error is of the order of n plus 1 derivative.

So, if you think the higher order derivatives are negligible in the, along the curve which you are interested in then, we could construct a polynomial of order a larger n so we increase the degree of the polynomial to just to reduce the error in this thing. So that is one way, but that is not very convenient because that also increases this product, so we will have more terms in this product and that is as we saw earlier is not a very good thing to do always because you could have round off errors or other errors due to floating point operation coming in. So increasing the degree of the polynomial though might help, but not always good because this term might work against us.

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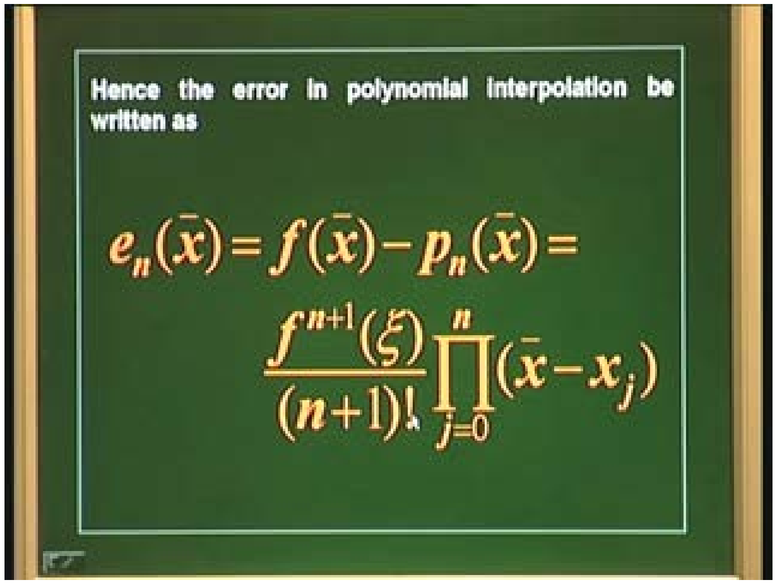

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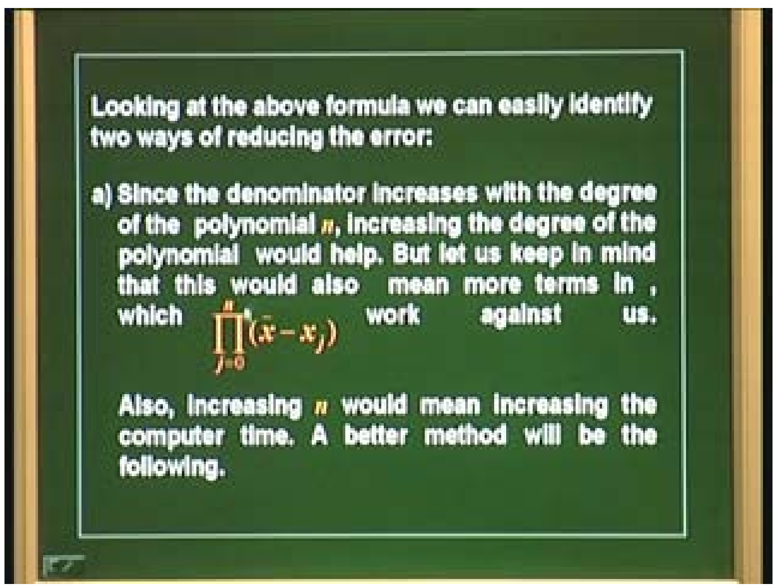

That is the idea and also increasing the n would mean more computer time. So a better method would be to actually to try to reduce the 2nd term that is this term, why do not we reduce the value of this term, that is remember we had two terms a product of two terms here, which is this and this and we can increase the degree of the polynomial and reduce this term may be, but that might cost more computer time and may be more errors in this term. So another way to do this is just to take a finite small n but try to reduce this value.

So what does that mean, so that would mean that, we will choose the points x_j in the product close the x bar of interest to us. So that is what, we would do, what we would do is, we will choose x bar at which the polynomial is to be evaluated, that is this x_j's, so x bar is the point at which we want to evaluate the polynomial. So we will choose the x_j is close to that x bar, so that is one way of doing this, then we will reduce this error x bar minus x $_j$ but then, if you have a set of points going from 0 to n, then it is not possible to choose all the points close to x bar because these points are given to you, this x_j is are given to you and then x bar is the point at which you want to evaluate. so what is the idea here is saying that, let us not take all, when you construct a polynomial there is no need to take all the n points, all the n plus one points to construct a polynomial instead what do we do is, we will choose a set of points which are close to x bar from the given set of tabulated x_j values, we choose a set of values which are close to x bar and construct a polynomial of degree let us say m, which is much smaller than n.

So that is the idea behind piecewise interpolation, polynomial interpolation that is, for a given set of points we keep constructing different polynomials for different intervals for every interval you construct a polynomial and we will use that polynomial to evaluate x bar. So that is the idea behind piecewise polynomial interpolation and that is what we would be looking at.

So as a problem, probably one could look at this, that this construct a 6th order polynomial interpolating, polynomial in the lagrange and newton's form to estimate the function x, e to the power of minus x between 0 to 10. so you take sample values at 1,2,3,34,5,6 and 7, so there are seven points here so you could construct a sixth order polynomial with it.

So you have 1,2,3,4,5,6,7, so 6th order polynomial and then you construct a 5th order polynomial by just taking any 6 points and then you could construct, estimate the error. So there are 2 ways of estimating the error in this particular problem which you probably should try out that is, one is you know the functional form here, so then you construct a 6th order polynomial going through the 7 points and evaluate at some value x which is not tabulated here and then compare with the actual function and see what the error you get and compare that with, the error you can estimate by making a 5th order polynomial to go through this by choosing any 6 points of the 7 values here, choose any 6 points and then make a 5th order polynomial and then find the difference between the 6th, the 5th order and the 6th order polynomial at that value that will be an interesting exercise to do.

So coming back to the piecewise polynomial approximation. So what we do is, if you are given a set of points say, n points here 1 to n so and we have the function values between all these points. So what we do is, for every interval x_1 to x_2, x_2 to x_3, x_3 to x_4 etcetera, we will construct a polynomial. So we know that, these polynomials have to be kind of continuous across this otherwise, it does not look, it is not correct.

So we have to make sure that this polynomials are continuous between the intervals, continuous means that it should satisfy the function values, the polynomial which you construct for the interval one to two and the polynomial you construct for the interval 2 to 3 should match at .2 and if, possible it is derivatives should also match at .2 then, we will make sure that it is continuous that is what we are going to do here.

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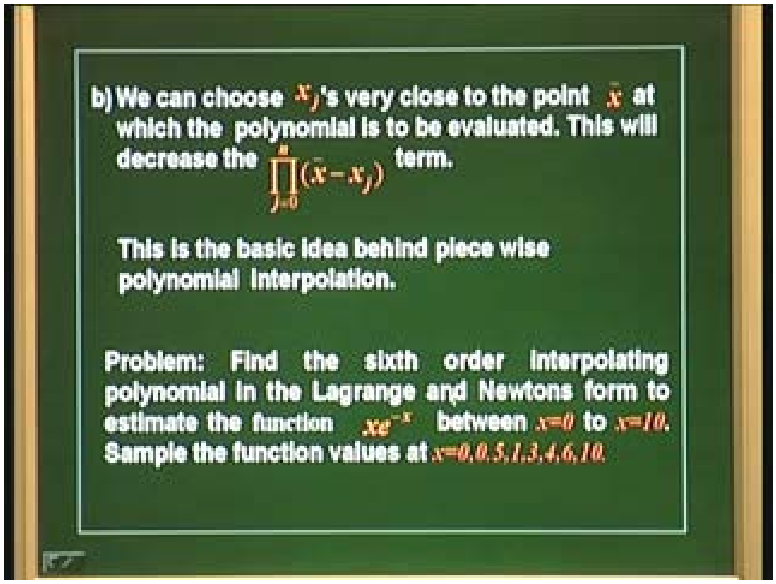

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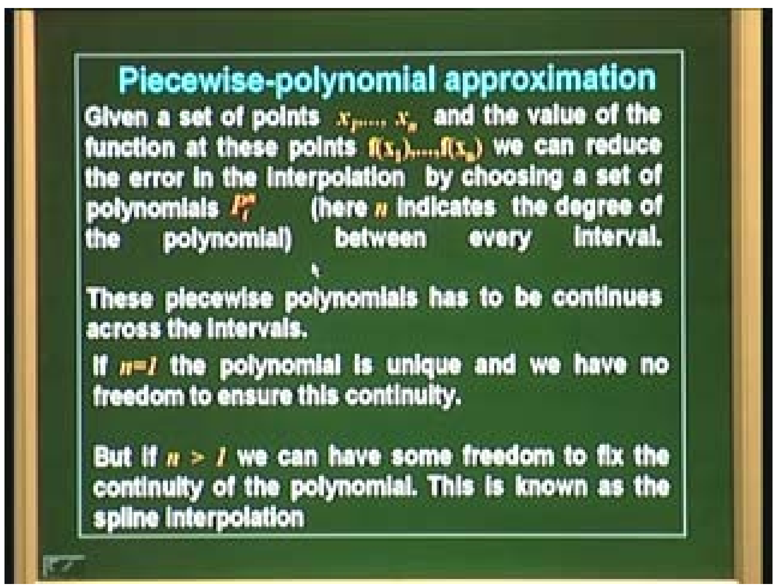

So if, you take the polynomial of degree one, if you say degree one, which for every interval, so we construct a linear function which interpolates between 1 to 2 and 2 to 3 then we do not have much of freedom, only thing we can do is to make sure that they match at the intermediate points. And if, you take a linear function to interpolate between every interval let us say, 1 to 2 to 3 then, what we can make sure is that this polynomials match at the .2 but its derivative we cannot do anything about it but, if it is more than n and more than 1 then we have some freedom to adjust some parameters such that it is continuous. So, that is the idea between the spline interpolation that is, you construct piecewise polynomials to go through all the intervals and then every interval you have a polynomial and every interval you have a polynomial of degree higher than 1.

So we have some freedom to fix the derivatives, a derivative continuity at the points where these 2 polynomials join. So that is, the idea behind spline interpolation, so we would match the function values and its derivatives at in between points we will see that now. So for example, let us take the simplest case of a quadratic polynomial just before that I just want to show you a graphical illustration of this and here, is a case here, is set of points here given by blue curves and you have a linear curve, a linear piecewise interpolating polynomial going through and the red curve, here is a cubic polynomial. So, you have a linear function which interpolates between these two and you can see that there are things at these points because, I cannot match the derivatives, you can only make them go through all these points but when you have a cubic function which interpolates all these points you have a more continuous kind of line.

So, we have chosen for every point a polynomial of order 3 for this red curve a polynomial of order 1 for the black curve that is, what we have seen.

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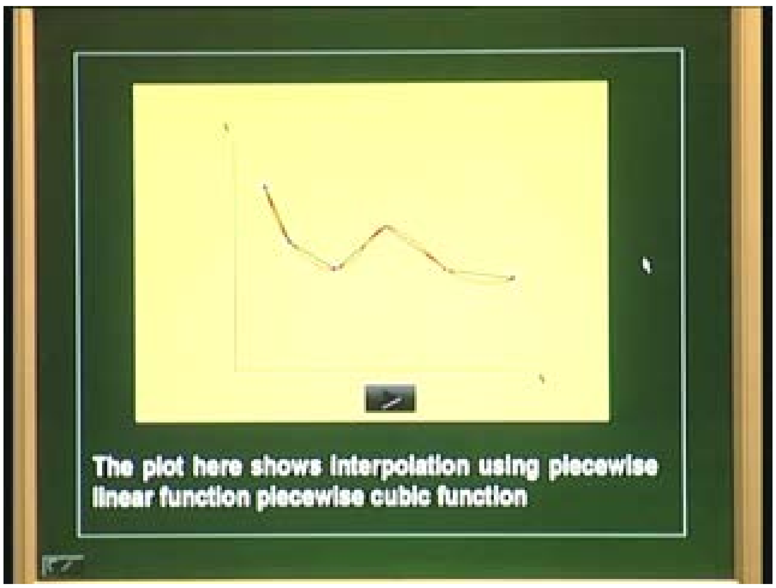

So to get an idea about, how do we construct this thing, the easiest may be to think about is just look at a quadratic polynomial, so slightly better than n equal to 1 would be n equal to 2, so typical quadratic polynomial which goes through x_i is just written here. So, this is a quadratic polynomial, so we will just see what, how do we construct these polynomial piecewise so, i here would mean something which goes from i minus 1 to i. So that would be the polynomial i or i to i plus1, one of this things we have to take.

So p_i, i denotes the interval, which is either i minus 1to i or i plus 1 to i depending up on what you fix it as. So we will see how do we construct a polynomial of this form. So we have three coefficients a_i, b_i and c_i for each interval. So we will have the set of coefficients, for every interval we have a set of coefficients, we have every interval in the polynomial is different so for, every interval we have a set of coefficients. So how do we determine these coefficients, first of course the polynomial i at x_i. So we have constructed here the polynomial which goes from i to i plus one as p_i. So p_i is a polynomial which goes from i to i plus 1, then we would say that p_i at x_i should be the function x_i and the p_i at x_i plus 1 should be function x_i plus 1 because, the polynomial has to go through, those 2 points because it is interpolating between i to i plus 1, p_i is interpolating between i to i plus 1.

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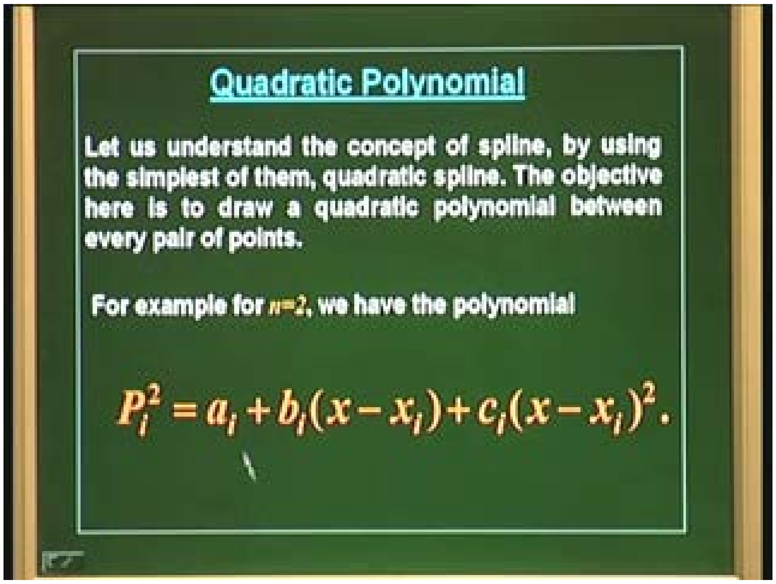

So it should match the function value at i and i plus 1.And then, we have also we can also demand that the derivative of the polynomial, at I, so this goes from i to i plus 1. So the polynomial, the i th polynomial which goes from i to i plus 1. So this its value at I, that is derivative at i should be the same as the i minus 1th polynomial derivative at i remember the i minus 1th polynomial will go from i minus 1 to i. So this goes to i and this also goes to i, so its derivatives at this point should match. So we have the 3 conditions at every interval and this are the set of conditions which we are going to use to determine this polynomial a_i, b_i and c_i and this coefficients of the polynomial a_i, b_i and c_i we will see that now.

So we have, let us look at the simpler quadratic spline of this form. So f of i of x which goes from i to i plus 1 as a_i x square, b_i x plus c_i and then, that is the polynomial at every interval. So we have n plus 1 data points going from 0 to n let us say, and that means, there are n intervals. So for each interval, we have a polynomial. So that means, we have 3 n unknown quantities here, to evaluate that is the, a's, b's and c's, 3 n unknown quantities. For every interval there are 3 and there are n intervals, so there will be 3 n unknown quantities. So now, we use the conditions that the function value at the boundaries of each of this interval should match the polynomial value and its derivatives should match the 2 polynomials and the two adjacent intervals should match the derivatives at the intersection point so that is what we do.

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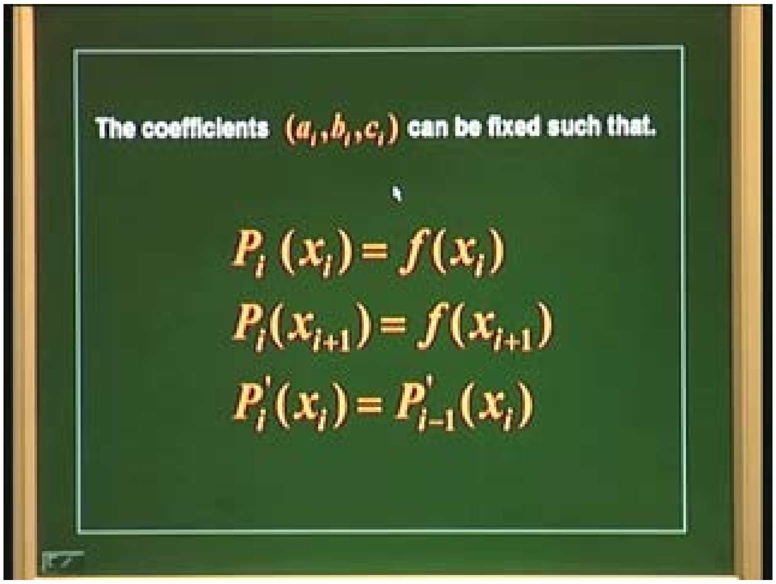

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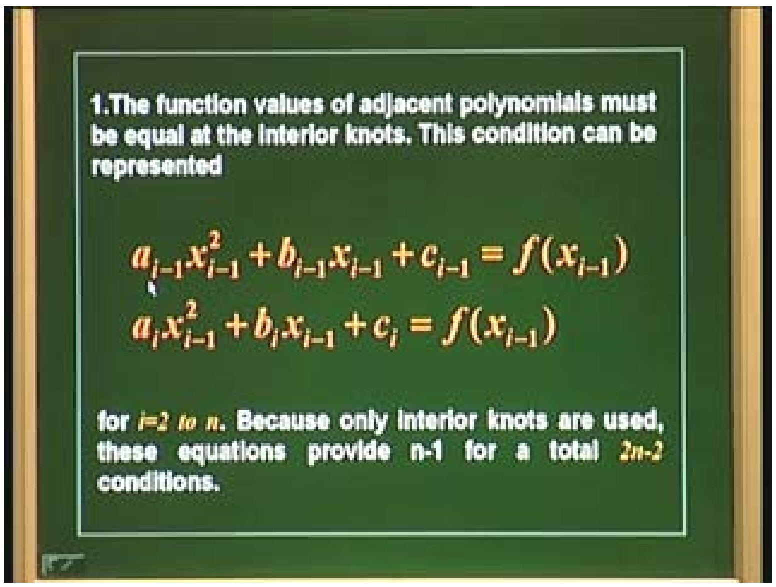

So we have these conditions that you know, the function the a $_i$ the i minus 1th polynomial should take at i minus 1, the function value i minus 1 and the i th polynomial also at i minus 1 should take the function value i minus 1, given by i minus 1 and so this will-, so this is matching the 2 polynomials and then that would give us, 2 into n minus 1 conditions that is 2n minus 2 conditions for each interval one, so there are n minus 1 intervals, so we get 2 into n minus 1 conditions. Remember, we have three n conditions to solve we have 3 n unknowns to solve. So we will come back to this again.

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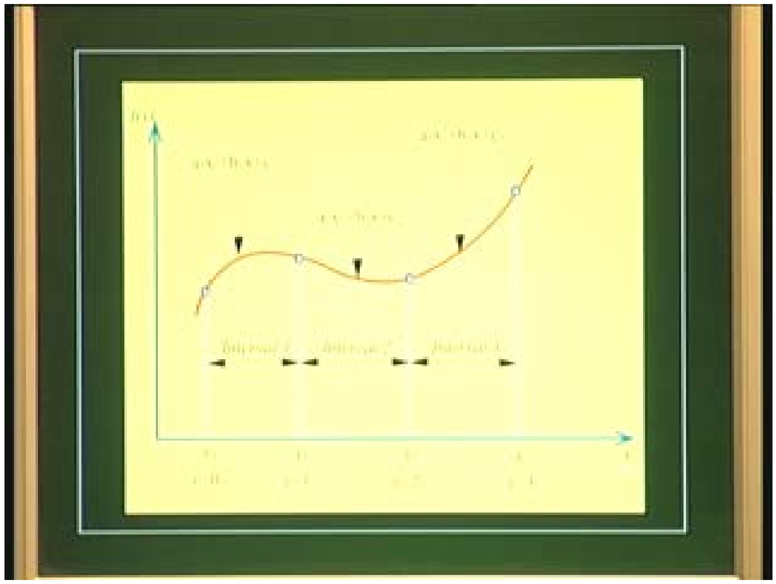

We have taken only the intervals and the in between interior points, there are n minus 1 interior points, 2 conditions and we have 2 boundary points, that we have, that is 2, that is plus 2, we have 2 n conditions. Remember again, that we have 3 n unknowns to satisfy to get.

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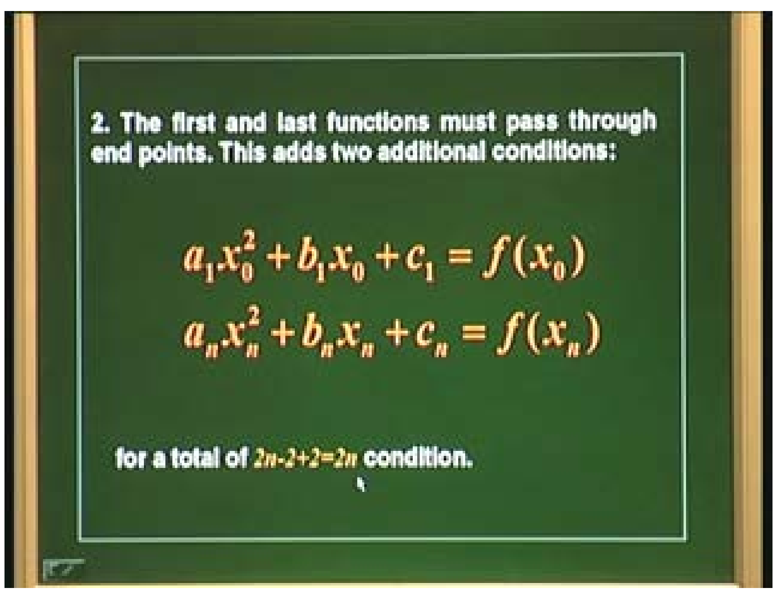

So then we use the derivative, so we match the derivatives at that points, so we have the derivative in the i minus 1th interval and, i minus 1th polynomial and derivative of the I th polynomial matching at i minus 1. So the derivative of the i minus 1th polynomial and the derivative of the i th polynomial will match at i, at i minus 1. So, that gives us n minus 1 conditions again, because this is again interior points, all the interior points we can match th derivatives of this polynomial, so we have n minus one conditions. So we had now two n conditions before and we had n minus 1 conditions to it, so we have 3 n minus 1conditions, but we still have, we have 3 n unknowns.

So we are short of 1 condition, so we need to put in some extra information to decide this 3 n coefficients and that is normally chosen as the derivative, the 2nd derivative at one of the end points also we could just say that we could take a derivative of one of the points and we would fix the derivative to be 0. So the functional values and that will give us additional constraints and that has to be substituted into this thing.

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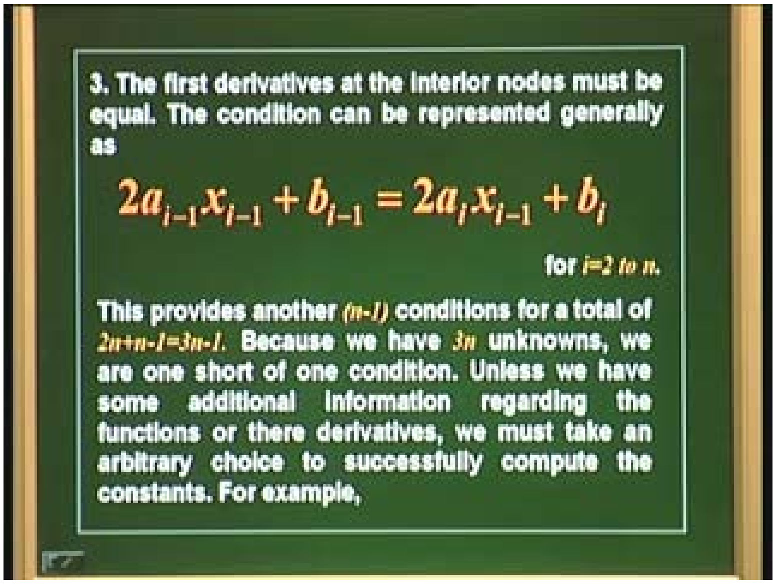

So then, once we have that condition, then what we have is, so what we have is this polynomial, here given by a polynomial of this form that f_i x is equal to a_i x square plus b $_i$ x plus c $_i$ and we will have all the function values here, so we could fix-, we have to have this extra condition which tells us what the second derivative at the boundary is to fix this extra-, at one extra coefficient which we cannot determine from the fact by demanding that, the derivative of the piecewise polynomials match at all the interior points and the polynomial themselves match at all the interior points. So the polynomials match at interior points gives us 2 into n minus 1 conditions and the derivatives match at the interior points give us n minus 1 conditions and the 2n points values for the polynomials gives us 2 conditions so we get 3n minus 1 condition we need 3 n conditions and we have to put in one extra condition.

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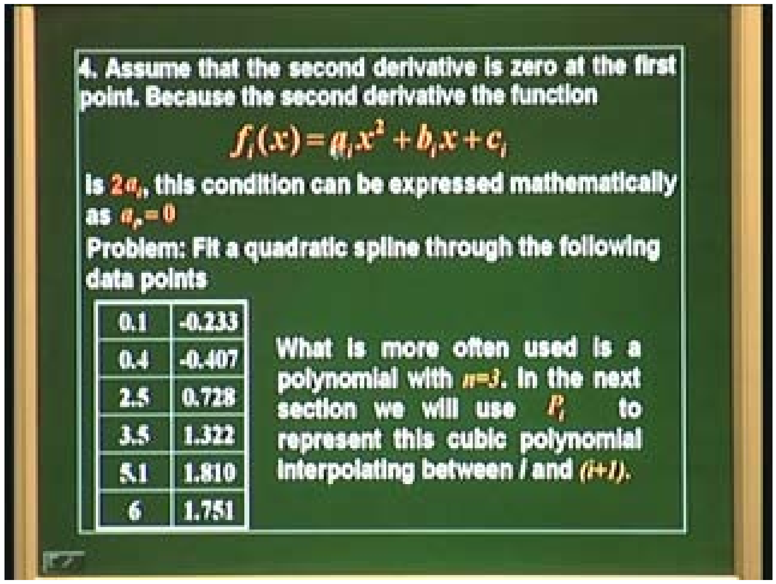

And once, we have that condition and then we can actually determine the polynomial by, we can determine the polynomial and then we will have a subject like this, so we will have for each of this intervals a separate polynomial and it is graphically marked at here, so intervals 1, 2 and 3, for interval 1,we have polynomial with coefficients a_1, b_1 and c_1 and interval 2, we have a_2, b_2 and c_2 and then interval 3, a_3 b_3 and c_3. So that is what, we would have. So this is just a illustration of what the polynomial a piecewise polynomial would mean, so that is we can actually express this we can try to construct therefore this, given set of, such a polynomial for a given set of data points given here so you could try that as an exercise and then we would have 4 points 1, 2, 3, 4, 5, 6 points, so that means we have 1,2,3,4 interior points and 2 boundary points.

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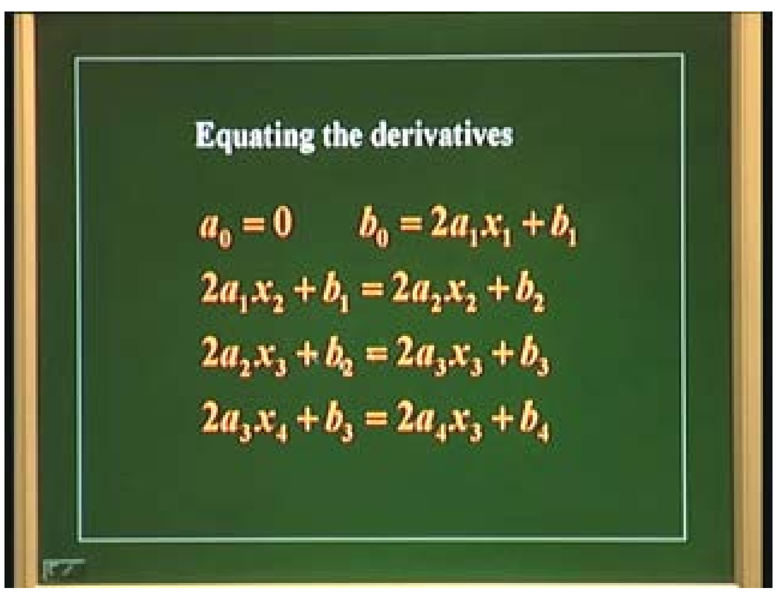

So we will have, so we can equate the derivatives at this interior points and so that will give us 1,2,3,4 conditions and this is the initial condition, 1 boundary condition which I had put in and so we have this 5 equations here and then we could have the function values for the 1 polynomial in the interval 1,2,3,4 to satisfy the function values at 1,2and 3,4 as this and then we know that we have the next polynomial also to satisfy the same function values at x_1, so the interval the polynomial going from 1 to 2 and the polynomial from going from 2 to 3 should satisfy the function value at 2. So that is another set of 4 function values. So that will be this so that is polynomial going from 1 to 2 satisfying the function value at 2.

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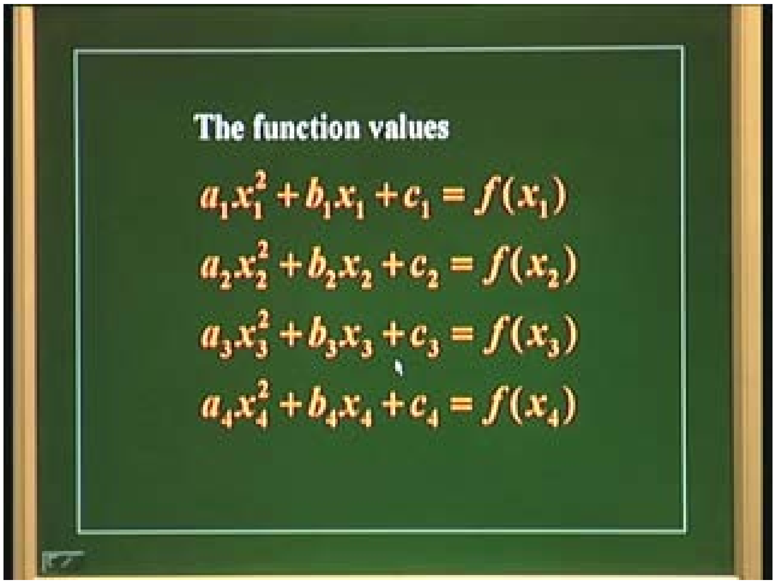

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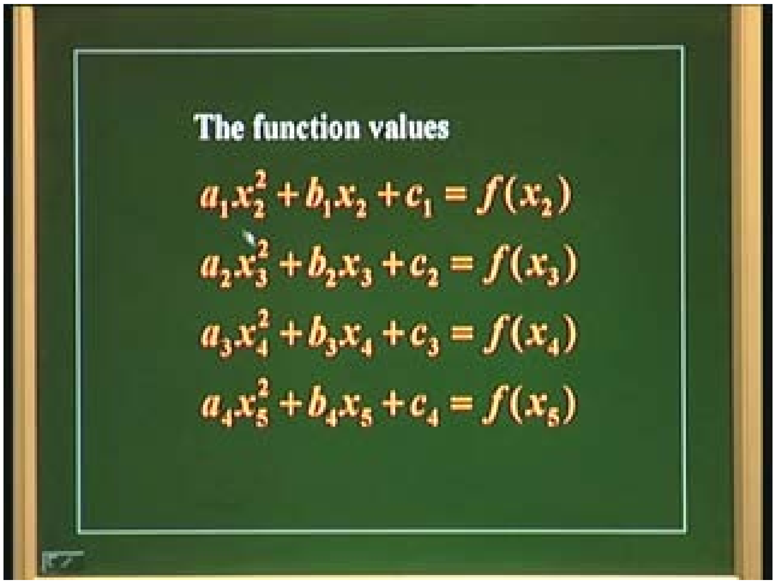

So you remember, the previous one was the polynomial going from 1 to 2 satisfying the function value at 1 and going from 2 to 3 satisfying the function value at 2 etcetera and this is the polynomial going from 1 to 2 satisfying the function value at 2. So we have 4 interior points here and then 4 function values and we saw 4, 4 into 4 8 function values and then, we had one extra boundary condition and then, we had 4 derivatives to be satisfied and then, we have 2 boundary values to satisfy, that gives us the full set of points 15 points which we want to evaluate, 15 coefficients which we want to evaluate.

So what is more commonly used in the spline interpolation is instead of radix spline there will be cubic spline that is, what more commonly used and that is what is used always and this, we find that this is much more easier to evaluate too and the function now, we have 2 conditions, so that is of course the function the polynomial going from i to i plus 1 would satisfy the, continuity of the functions that is p_i into x_i plus 1 should be equal to p $_i$ plus 1at x_i plus 1.

We know that whether it will be continuous across or other way of saying is p_i of x_i should be f of x_i and p_i of x_i plus 1should be f of x_i plus 1 same as what we saw before. And then, we have the derivative continuity this is also we just now saw and now we also have the 2nd derivative continuity, so we say that p_i double prime the 2nd derivative of the function should satisfy all the polynomial in the interval i to i plus 1 should match with that between i plus 1to i plus 2 at i plus 1.So we have an additional condition here and because we have a cubic polynomial now.

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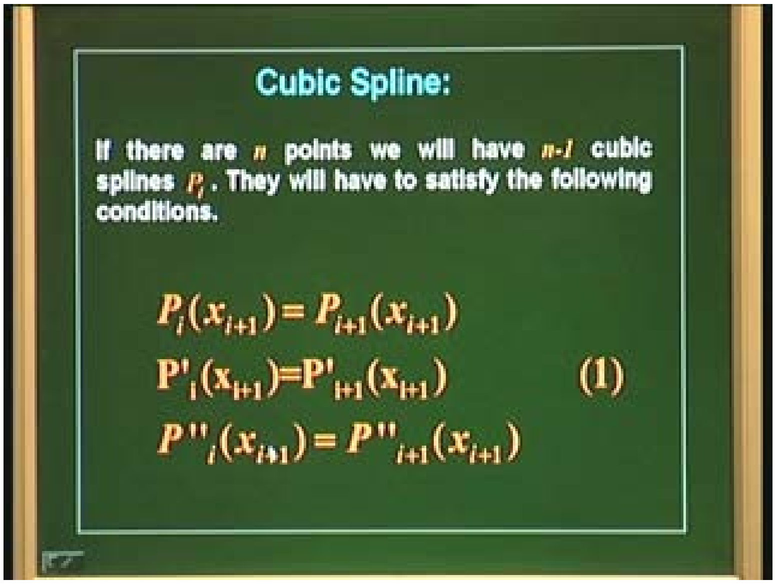

So now, since the p_i’s are cubic and their second derivatives are linear so the, we construct this in much more cleverer way, this polynomial it is much easier to construct this way, that we say that, the polynomial is cubic so its 2nd derivatives are linear, and then such a linear polynomial the second derivative of the cubic polynomial can be written down in this fashion.

So now, you can see that this polynomial, that is the interval from i to i plus 1 and the 2nd derivative of that polynomial is given by f double prime of x, that is the 2nd derivative of the function value at x_i, some quantity which we do not know because, we do not know what the 2nd derivative of the function value at x_i is, we only know the function value at x_i but let us write it like this. So this is an unknown quantity into x minus x_i plus 1 by x_i minus x_i plus 1 that is the lagrange's form and f double prime at x_i plus 1, x minus x_i plus 1, x_i plus 1 minus x_i. Now, you can see that, let us evaluate this polynomial the 2nd derivative polynomial at x_i plus 1. So then, what do we get is this at x_i plus 1, we would get that as this goes to 0, x_i plus 1 minus x_i plus 1, this term goes to 0 and we would say that p $_i$ double prime at x_i plus 1 is equal to f double prime at x $_i$ plus 1.

So the 2nd derivative of the polynomial going from i to i plus 1has f double prime at x_i plus 1as its value. Now, what will be the p $_i$ plus 1double prime of x p_i plus 1 double prime would be x_i plus 1minus x_i plus 2 x minus x $_i$ plus 2 x minus x_i plus 1 divided by x_i plus 2 minus x_i plus 1 and this will be x_i plus 1 this will be x_i plus 2 and we can very easily see that p_i plus 1 at x_i plus 1 will again give us f double prime at x_i plus 1. So by this construction, we have already satisfied that the 2nd derivatives are continuous by this construction of the 2nd derivative, we have made sure that the 2nd derivative of the polynomial at i plus 1, x_i plus 1 polynomial i, at x_i plus 1 and polynomial i plus 1 at x_i plus 1 are the same.

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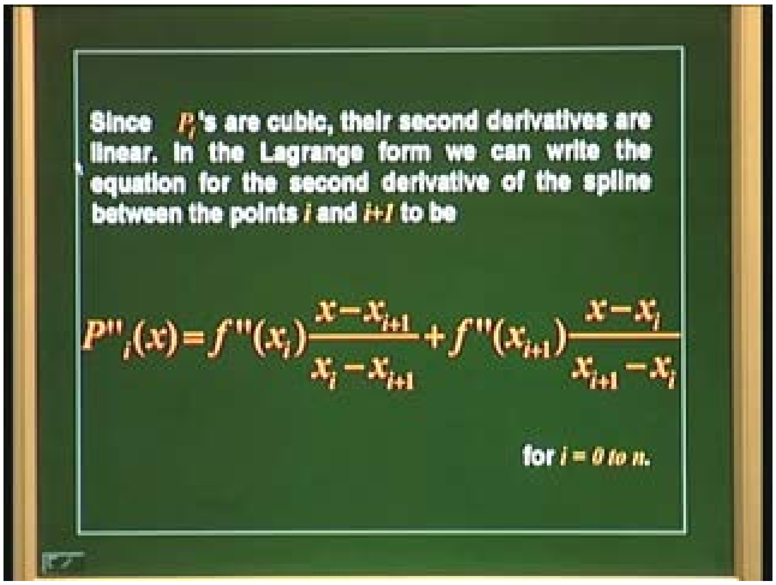

So we know that. So, now if you integrate that linear polynomial twice we get, a cubic polynomial and that is what I have done here. So we have a cubic polynomial obtained by integrating that polynomial twice, so we have now x_i minus x_i plus 1 cube by 3. So, if you differentiate this twice you get back that polynomial again. And then, we have x_i minus x_i to the power 3 divided by 6 here and we have this additional terms linear terms a into x minus x_i and b into x minus x_i plus 1. So we have this polynomial. Now, this is our cubic polynomial, so we have a cubic polynomial which goes through which is interpolating between i to i plus1 whose 2nd derivative is continuous. So we have already made sure that.

And now, we can get this a and b values by saying that p_i at x_i plus 1 is equal to f of-, p_i at x_i is equal to f of x_i. So a is, b is f of x_i it gets p_i at x_i p_i at x_i would mean that this term will go to 0 and then we can construct this value from that. So, p_i at x_i is equal to f of x_i. we can construct from that the b function from this because this goes to 0 and we will have this goes to 0 and we will have only these and these terms left and from that we can actually construct the b term and then, we have p_i at x_i plus 1 going as f of x_i plus 1 and from that, we can construct the a because this will go to 0 this will go to 0 and we will have x_i plus 1 minus x_i, x_i plus 1 minus x_i square by 6 into f double prime of x_i. So we can construct that polynomial.

So having done that so if you put in those boundary condition that the function value so we have constructed this cubic polynomial already in such a way that its 2nd derivative continuity is guaranteed and then we fix this some coefficients by saying that its function values, its values at the boundaries of those interval should be equal to the function values, then what we have to fix now. So now, what we have to fix is this coefficients double prime x_i and double prime x_i plus 1 that is the only thing which is unknown in this now. We fixed a and b by demanding that this goes through p_i at x_i plus 1 is equal to f of x_i plus 1 and p_i at x_i is equal to f of x_i . So we have fixed that and now we need to fix these quantities.

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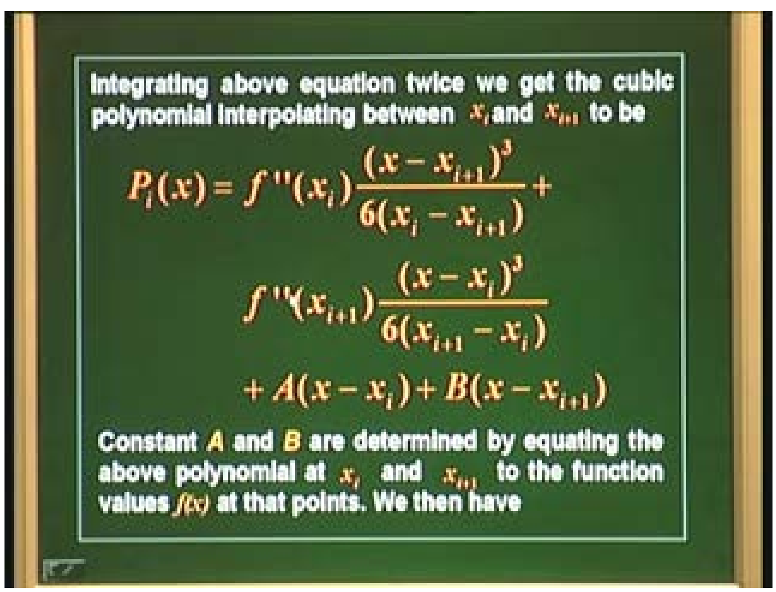

So for that we have this, if you substitute all that we got a polynomial of this form. So we have the complete polynomial now, the only unknown in the polynomial now is f double prime at x_i, f double prime at x_i plus 1, all the double primes are unknowns, we do not know the 2nd derivative of the function because, we do not know the function itself now we need to compute this f double prime of x_i so what we are going to use is that 1 more condition which is left to us.

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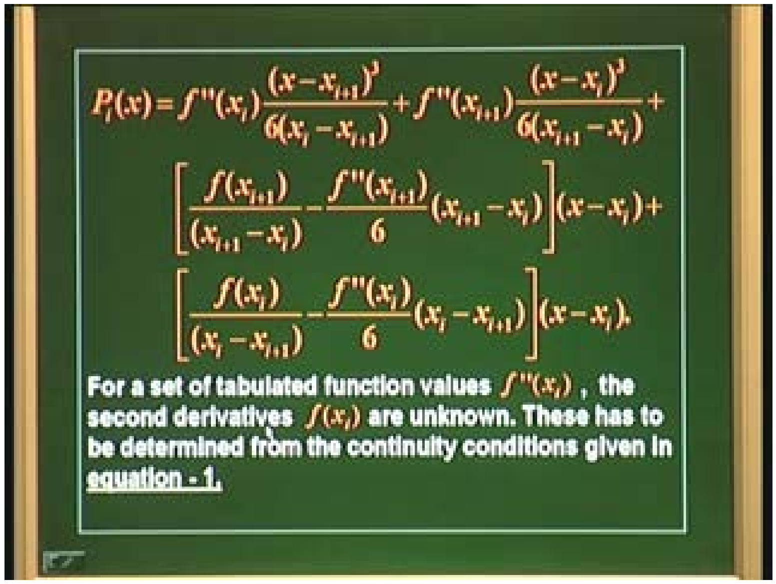

So that is remember in this equation we had this condition left that is p $_i$ at x $_i$ plus 1 and p prime at i plus 1 at x_i plus 1 should be the same that is the 1st derivatives has to be continuous that is, what we are going to use. So we can use that condition to get this coefficients which are unknown in this polynomial and then we have fixed the cubic polynomial.

So we will go ahead and look at that value now. So, this is the condition which says that the 1st derivative I have just taken the previous function phi of x and finds it is derivative p_i at x_i plus 1 and p_i plus 1 at x_i plus 1, the derivatives of this 2. I just match them here. So I get this equation, so that is an equation this is coming from p_i at x p_i prime at x_i plus 1 and this comes from p_i plus 1 at x_i plus 1. So these two conditions, these 2 derivatives have been equated and I have rearranged the terms into this form. You can see the right hand side of the equation is all known functions and the left hand side, we have the unknowns f double prime of x_i f double prime of x_i plus 1 f double prime of x_i plus 2. And this equation has to be written for all n minus 1 interior points and you would notice that this, so you will have n minus 1 such equations to solve and the right hand side of this equation is completely known.

we know the function values we know the x_i values so right hand side of the equation is known, these coefficients x_i plus1 minus x_i, x_i plus 2 minus x_i, x_i plus 2 minus x_i plus 1 are known. So what is not known is the derivative at x_i, i plus 1 and i plus 2. So each equation such n minus equations each of these equations will have 3 unknowns and they are the 2nd derivatives at i, i plus1 and i plus 2. So, if you write down this equation from i going from 0 to n minus 2, if you write down this whole equation and then you will get a tri diagonal matrix form.

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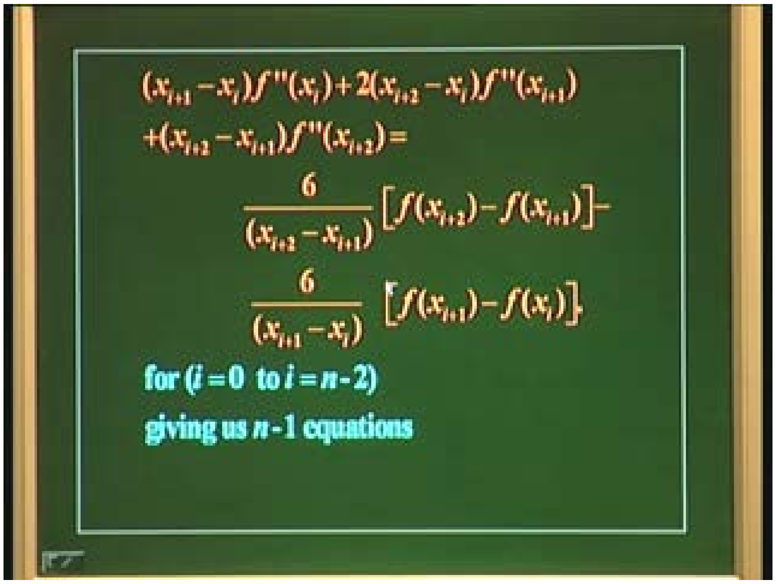

So that is, what you would get so we can use some shorter notations, we will use z, 2nd derivatives are represented by z_i and we will call in this equation we call this as d_i, x_i plus 1 minus x_i as d_i, x_i plus 2 minus x_i as e_i and x_i plus 2 minus x_i plus 1 as h_i and this whole right hand side as r_i. So if I use this short notation r_i is the right hand side and I use this notations 2 x_i plus 2 minus x_i as e_i and i plus 1 minus i as d_i, i plus 2 minus i plus 1 as h_i and then I can write down this in a very short form d_i z_i minus 1 plus e_i z_i plus h_i z_i plus 1 is equal to r_i. So in this form I go from I going from 1 to i should go from 1 to n minus 1 that is what you will have. So you will have n minus such interior point equations.

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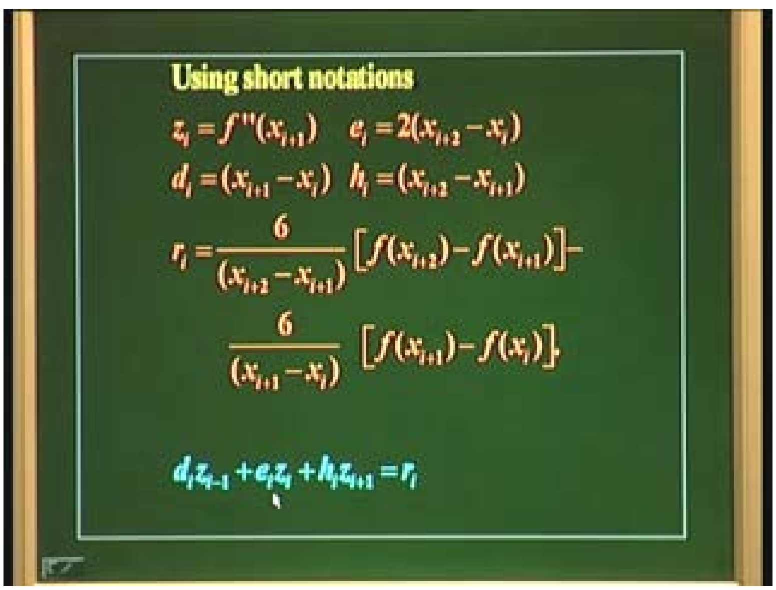

So you will have a tri diagonal form. So there will be but we have n minus 1 equations and we have n plus 1 unknowns because, we have n plus 1 unknowns but we have only n minus 1 equations. So we need to fix this two remaining conditions remember what are our unknowns, our unknowns are our 2nd derivatives at every point, 2nd derivative at every point and we have n plus 1 points, we have n plus 1 points so we have n plus 1 unknowns.

So in the cubic spline interpolation with the unknowns are the 2nd derivatives at every point and we have n plus 1 points, we have n plus1unknowns but by this equation, we are getting only n minus 1 such equations, so we have 2 points to be fixed we need two more equations or we need to fix these 2 points and these 2 points or these 2 values are fixed by demanding that f double prime of x_0 and f double prime of x_n are 0, that is the 2n's we say this their 2nd derivatives are 0 and this is known as natural spline.This is not the only way to fix it you could fix it by some other way but this is the standard practice and that is known as the natural spline.

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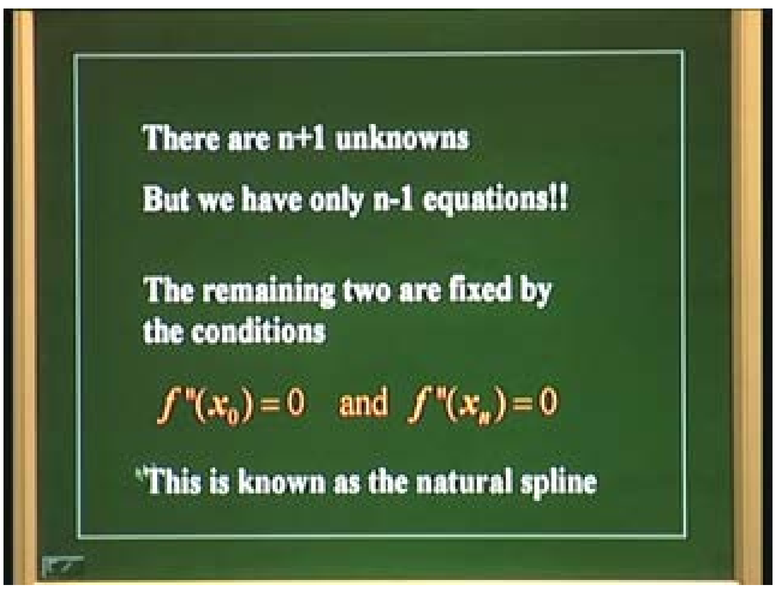

So when you see a standard algorithm a program which gives a polynomial interpolation, a cubic spline interpolation, it is most probably using this particular boundary condition and that will be natural spline interpolation. So now, let us say we have this problem which we discussed earlier, so we want to find out a spline interpolation through these set of points that is, we have 1 spline going from this to this that is 1 and then 2 and 3,4,5. So we have to find out 5 splines, 5 cubic splines, so 5 cubic polynomials which we have to find out.so then, as I said the i is 0,1,2,3,4,0 that is the interval.

Now, we can construct the e_i's d_i's and h_i's and r_i's at this point I have computed it for this points, this values, so e_i remember is 2 times x $_i$ plus 2 minus x_i and d_i is, x_i plus 1 minus x_i and h_i is, x_i plus 2 minus x_i. So I constructed all that here and r_i is the right hand side which was come known function, in terms of the function values that comes on the right hand side. So in terms of this equation, this is set of equations I showed you here these equations now, can be actually written down for this particular data for the set of data in this form so for every internal point interval, I can write down this equation and then I will get this n minus two equations

So, there are 6 values here 1, 2, 3, 4, 5, 6. So I have n is n minus 2 such interior points and I have this n minus 2 equations here. And I can solve this n minus 2 equations, you can solve them by elimination method you can multiply as you said you can multiply this equation by d_1 and divide it by e_0 and the whole equation and then, and subtract it from this equation and multiply this by d_2 and divide this by e_0 and subtract from here and that way, you can eliminate this row and similarly by doing such elimination method you can actually solve for all the values z_1, z_2, $z_3$and z_4 here again, we have used $z_0$and z_5 as 0. So we have used that condition z_0 is 0 and z_5 is 0.

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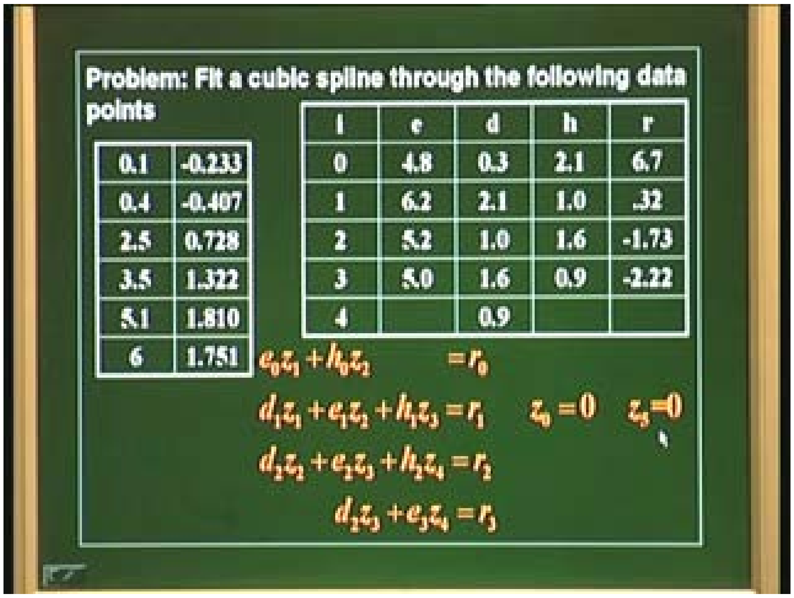

So I can do that and then, that means we have actually used this boundary conditions for also for this, so we can do this and for this case and actually solve this polynomial and I can quickly show you a program which implements this. So here is a program which would actually implement this method, so I have this tab again, I have this x f e. So I have the arrays declared here as x f e d h and r and this z, all this arrays declared here and then I have put in all this function values into this arrays x f and x and f and then I have 1st constructed this e h r d i have this e, evaluated here as x_i plus 2 minus x_i into 2 for all the intervals and h_i as, x_i plus 2 minus x_i plus 1 and then d i as, x_i plus 1 minus x_i and then I computed r_i.

So I have got all the known values from the data. Basically, I made this table inside this of the program, so that is simple enough, so you can see, it is very simple to construct this and then I have done this elimination as I said, the eliminating this row.

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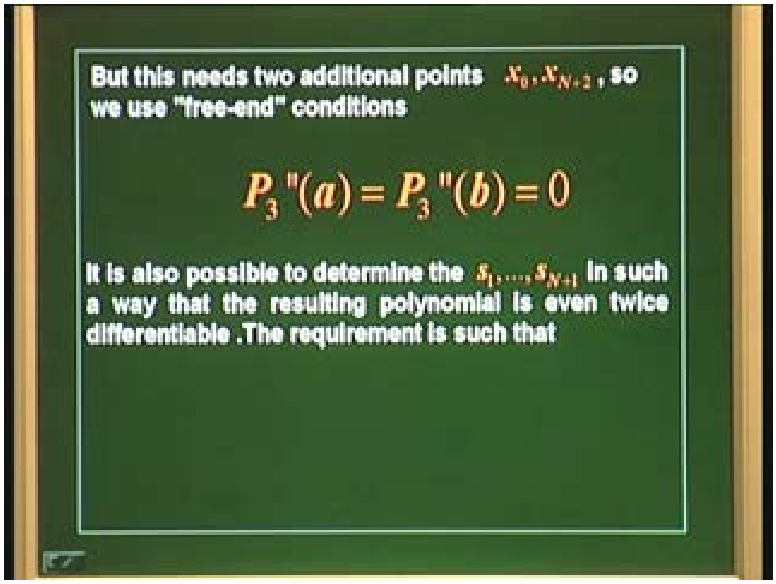

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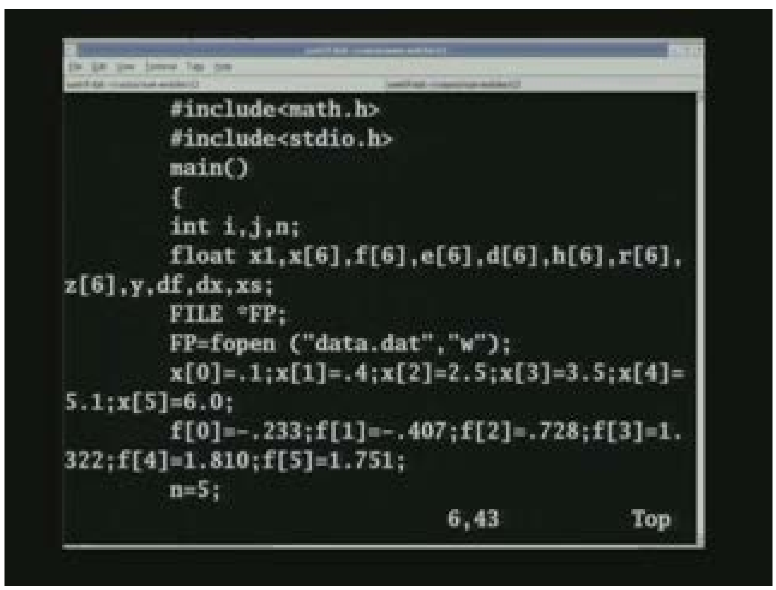

So I had taken the first row, I will take the first row divide that by e_0 and then multiply it by d_1 and then subtract it from this and then divide this by e_0 and multiply it by d_2 and then subtract from this, that way I can eliminate this row. So that is what is done here, So I goes from 0 to n minus 3, I have taken d_i and divided by the e_i and then I have subtracted it from the rest of the thing that way, I eliminate all that thing and then I can back substitute this. Once, I have eliminated all the d's actually I should actually write this this is d_3 so I have to eliminate d_1, d_2 and d_3.

So I have there is a small mistake here, this should be d_3. So this should be d_3 here, so if I eliminate all this rows and then I have a simple equation here, from this I can solve for z_4.So, once I solve for z_4 because this whole row is eliminated I can solve for z_4, once I solve for z_4, I can substitute that here and solve for z_3 and then, once I have $z_4$and z_3 I can substitute that here and solve for z_2 and then z_1.

So I can solve all this by back substitution as it is known. So once first task is to eliminate this row, this whole column that is the d column, I eliminate this d column and then I have a modified value for e_3 and then a modified value for r_3 because, I am going to multiply this and subtract from this and once I have done with that value and then I can compute z_4 from that and once I have this z_4 in this there is no d_2 here I have modified e_2 and I have modified h_2 and I know z_4 and I modified r_2.

So from that I can compute z_3, so I can go back like this and that is, what the next loop is implementing. So here is, I have eliminated the d function values and here I am doing the back substitution I start from the bottom that is from n minus 3 and go all the way to 1 and evaluate all the z values. So, z equal to the first 2 values have been put in as z equal to 0 and z equal to 1 as 0 that is been put in here z equal to 0 and z equal to 1 as 0 and then I can actually construct my cubic polynomial, that is simple for every interval, I am evaluating what I am trying to do is for every interval, I am trying to evaluate.

So there is in this program, so I have now for every interval a cubic spline and I am saying that you take each interval going from i equal to 0 to n minus 1 for each interval and within each interval you construct you take 5 points and try to evaluate this spline and that particular spline at that point. So that is what this is doing and I will show you that, so here is the polynomial then.

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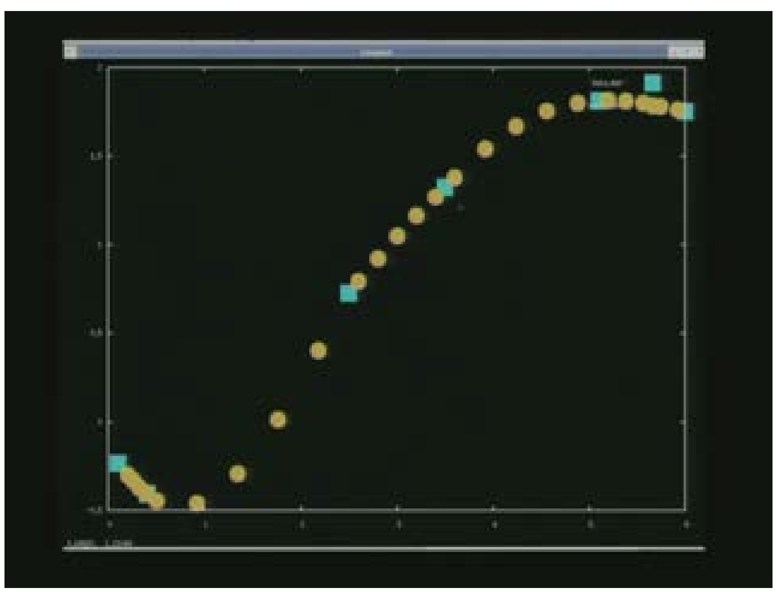

So again these points these are the points tabulated these square points the square symbols are the points which are tabulated and this circular discs are the points which are evaluated using the cubic spline for each interval 1 and you can see that this is pretty continuous and I can get this function values in it every interval satisfying. So again to summarize this point this program, so let me again go back and then say that so what we are doing here is, we have the function values tabulated here and I first construct these numbers that is e_i's, d_i's, h_i's and r_i's which are as you saw has been given in this from here as here, so this can be easily programmed and this r_i's can also be programmed because all of them are known in this interval and then we have this set of equations and that is what is shown here.

So we have this set of equations and then in this equations, I can eliminate this whole first array of points for the d_i's, I can eliminate by, so if I write this all the z_1 coefficients because, this z_1 coefficients can be eliminated by multiplying this dividing this row by e_0 and multiplying $d_1$and this z_2 coefficient can then be evaluated by, can be eliminated by dividing this row which now does not have a d_1 by $e_1$and multiplying by d_2 and the z_3 coefficient can be eliminated by multiplying this row dividing this row which now does not have d_2 and dividing it by e_2 and multiplying by d_2 etcetera.

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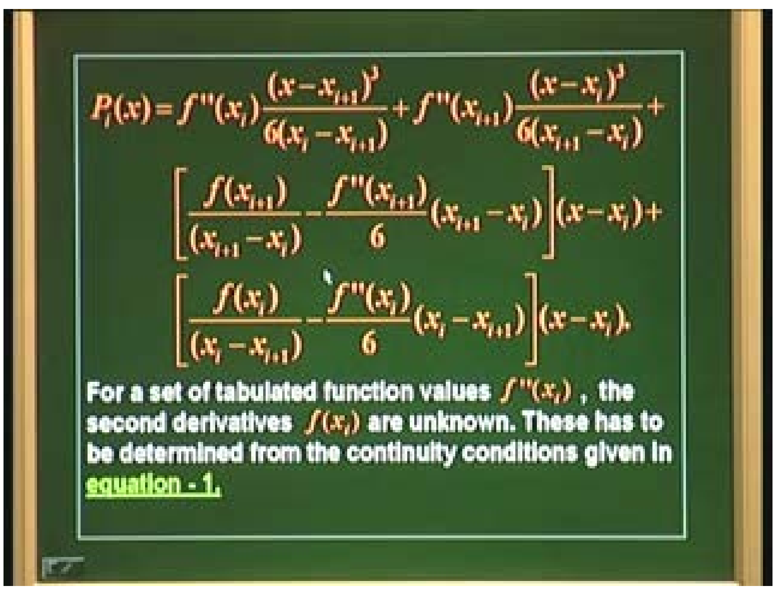

So you can continuously you can do this and then you can eliminate all this rows all this columns and then you will have z_3, e_4 is equal to r_3 as the equation all modified values of this coefficients e_3 and r_4 r_3 and from that you can compute z_4 and then you can continue back substitution and then get the answers which we wanted to get all the coefficients we wanted to get and then once you have that coefficients, we can construct this polynomial, once we have the coefficients we can construct the polynomial so we have this polynomial and all the function values all the coefficients are then known and then we construct this polynomial and evaluate the values in between every interval using a polynomial for that particular interval.

So, when you actually write a program, we need to be given some value x, we need to know what polynomial to choose because, we have now n minus one um polynomials so we have to choose one of the n minus one polynomials. So we have to know which interval it fits into and then choose the appropriate polynomial and then evaluate the polynomial at that value.

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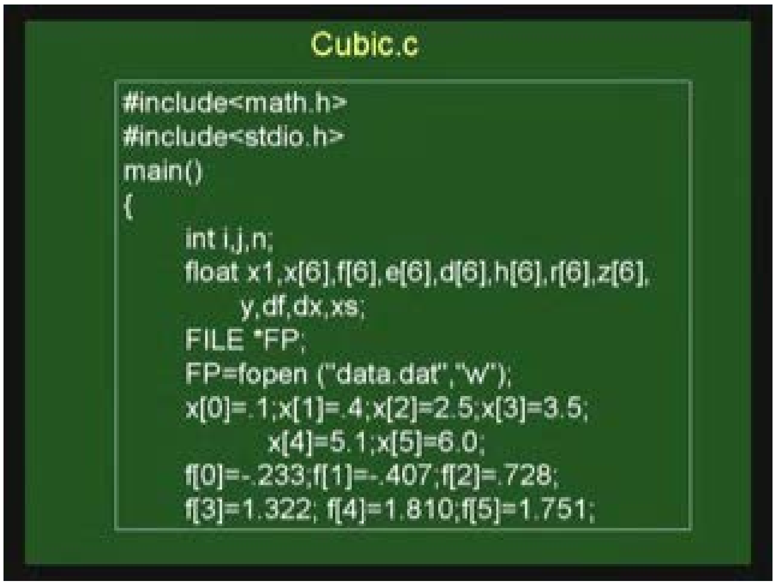

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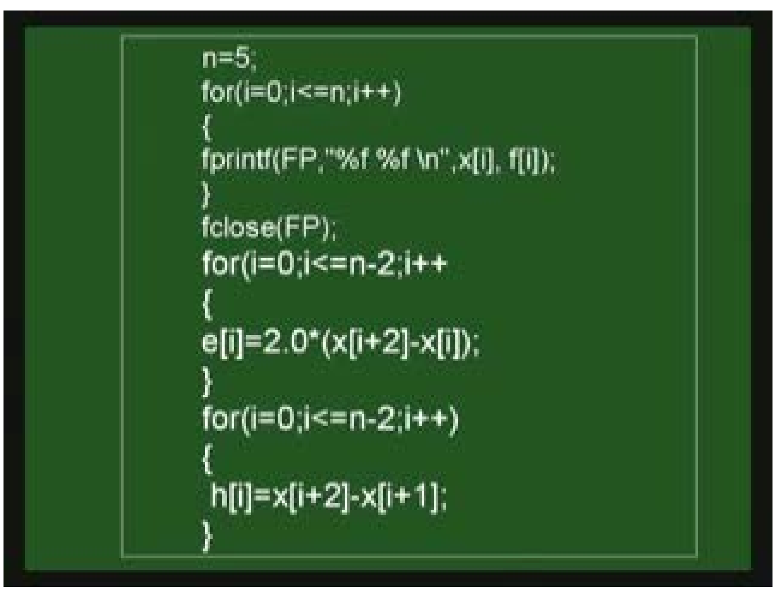

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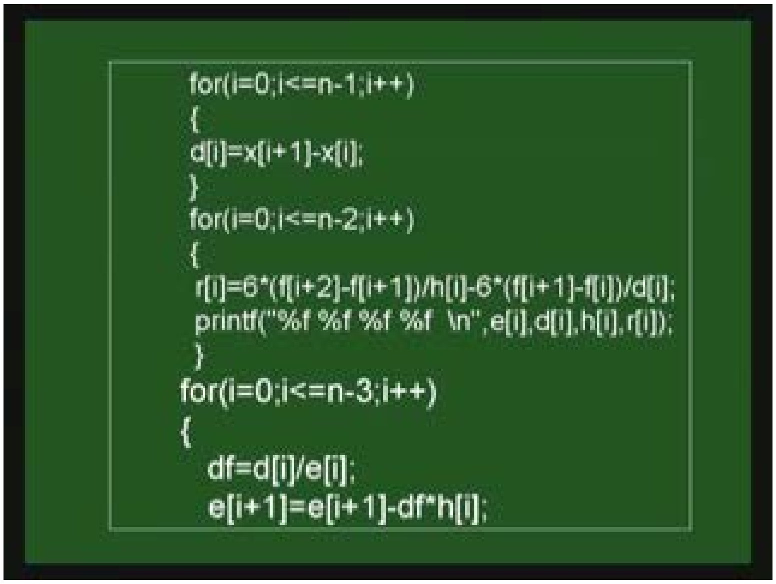

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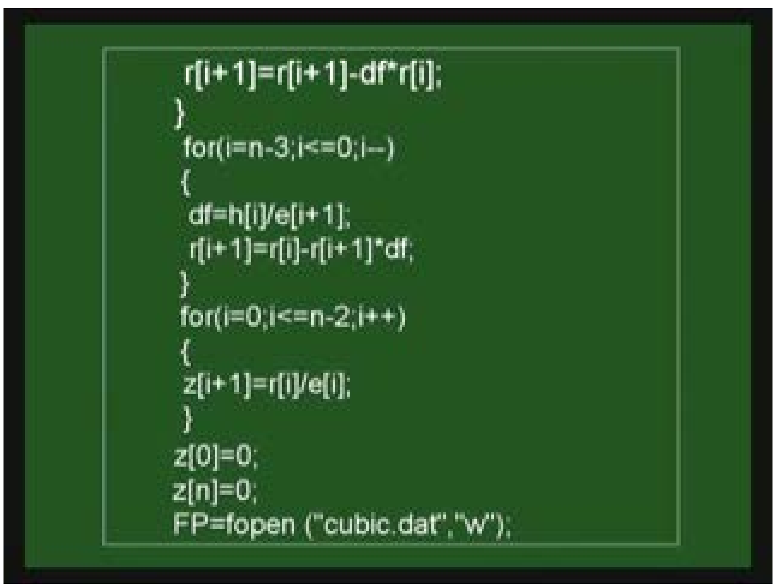

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