Classical Physics

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Lecture No. # 9

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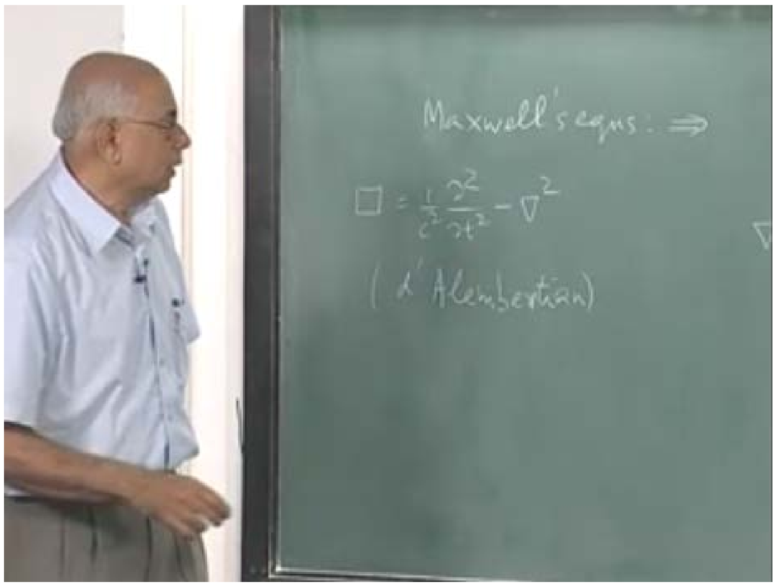

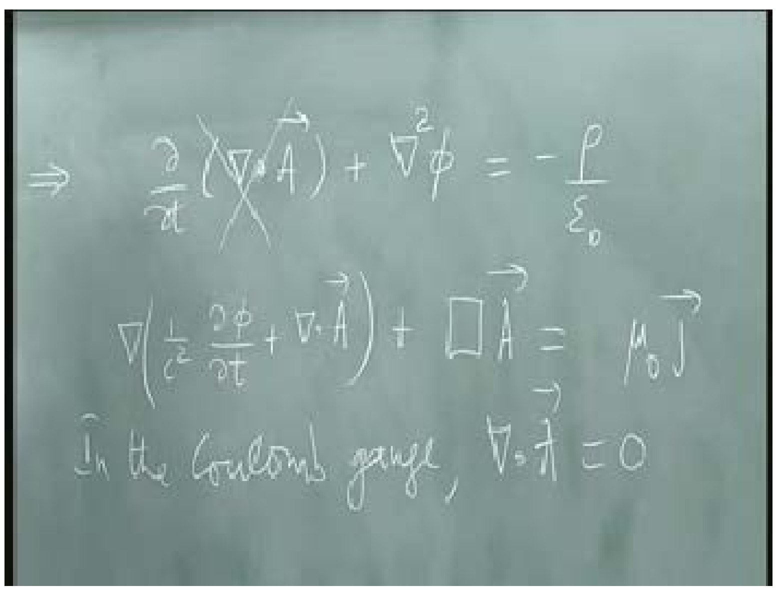

So, let us recapitulate where we ended last time. The Maxwell equations implied that you had two equations for scalar and vector potentials del dot A plus del square phi equal to minus row over epsilon naught; this is the first equation. And the second one was one over c square delta phi over delta t plus del dot A, was there a gradient of this outside, gradient of this plus instead of del square let me write box A equal to the right hand side mu naught j; is that a plus sign or a minus sign, plus sign, where this box is l one over c square d 2 minus del square. It is the four dimensional analog, it is the analog and the space time as we will see when we do relativity of the Laplacian operator. This operator which I just call a box is more properly called the d'Alembertian analogous to the Laplacian and as I said it is the wave equation essentially the operator that governs the wave equation.

So, if you did not have this it would just be the wave operator on A equal to mu naught j. These were the two equations and then I pointed out that this set of equations is true for every pair of potentials A and phi. So, the same equation is true if you had A prime phi prime everywhere and so on and you could choose the A and phi in such a way that del dot A was equal to zero, could always do this and that was gauge invariance. So, in the Coulomb gauge del dot A is equal to zero. Of course that does not fix uniquely A itself because you could always add to this A something once again something which satisfied del dot that new vector field equal to zero and of course you will still be in the Coulomb gauge. So, it is a family of gauges really.

In the Coulomb gauge this thing goes away and then you are left with Poisson's equation. Now how do you solve Poisson's equation? The only point you have to remember here is that this will imply del square phi of r comma t equal to minus rho of r t minus rho the function of r and t over epsilon naught and there is t dependence but then I am going to argue that this t dependence is just a spectator because whatever t appears on the left appears on the right as well and this is true for every t. You could treat this t as a parameter in this equation and really what you have to worry about is the r dependence the fact that this is a partial derivative with respect to space variables and that is the leading a function of r on the right hand side.

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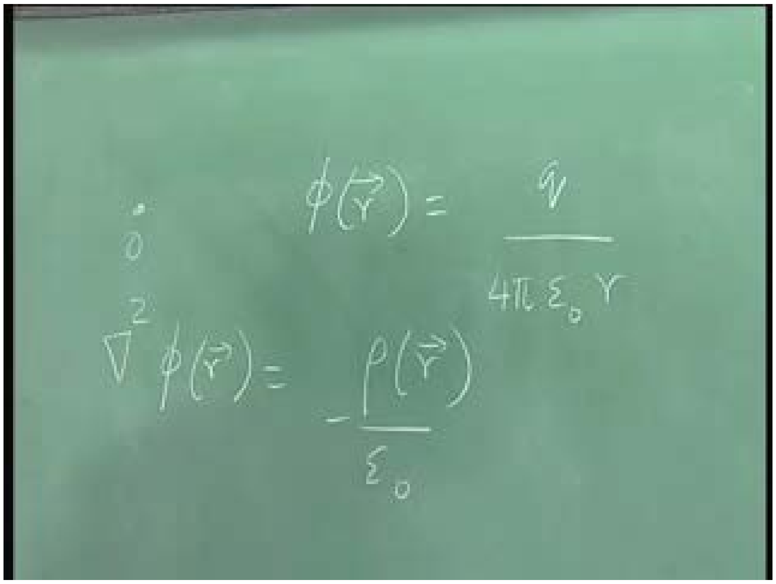

Now let us go right back and see what we know from Coulomb's law for a single point charge. I know that if you have a single point charge at the origin and I ask what the potential is at any point r in space, this is the scalar function, this is the electrostatic solution we are looking at just Coulomb's law and what is this equal to, what is the potential at any point? It is q divided by 4 pi epsilon naught r. That is the potential due to a single point charge at the origin and that is Coulomb's law. Now of course if you had many charges you would superpose the potential due to each of these charges and that would be the answer. So, once again I have for example some charge density on the right hand side, I superpose that each of these objects and then an equation of the form del square phi of r is equal to rho of r divided by epsilon naught minus.

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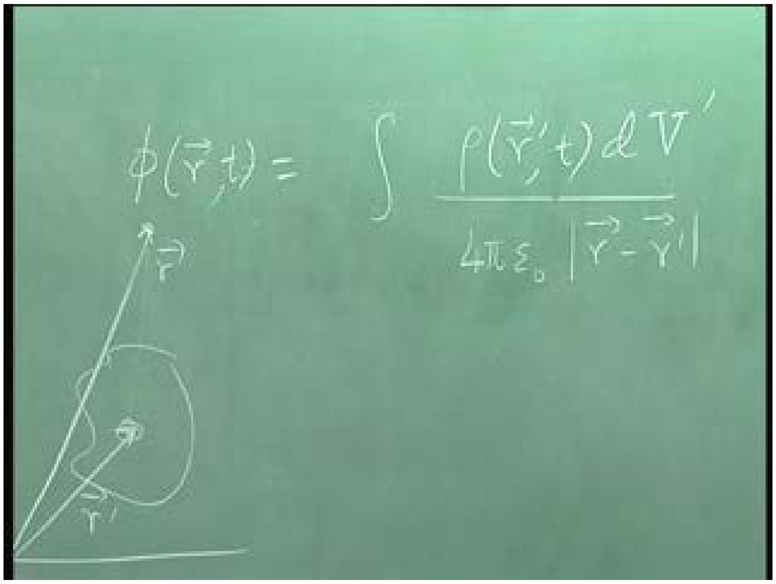

This kind of thing is solved by saying that phi of r at any point r is equal to an integral over all the charges that you have and if the charge is at some point r prime then it is rho of r prime d v prime volume element at the point r prime divided by 4 pi epsilon naught modulus r minus r prime that is the solution. If you have a distribution of charges in space somewhere and this is a typical point r prime and you are asking for the potential at a point r, then all that matters is the distance between r and r prime multiplied by the volume element of charge here that is rho of r prime d v prime integrated over all r prime. So, the entire extent of the charge distribution and divided by this kernel mod r minus r prime. So, I know that the solution to this equation is that that is the free solution and the solution is the one that is appropriate of saying phi of r vanishes at r equal to infinity.

So, boundary conditions have been put in, in writing that solution down and the boundary condition that has been put in is that phi vanishes when r tends to infinity when the distance tends to infinity. So, with that boundary condition, I know that this Poisson equation has that solution. Of course, if you specify finite boundaries and the potentials on the finite boundaries and so on, you must change the solution to Poisson's equation, but this is a free solution, and I do the same thing here. Now the fact that this is a mathematical equation which has phi of r comma t equal to rho of r comma t is completely irrelevant as far as this free solution is concerned and it is quite clear that for every t this equation is true. So, I could certainly write this in the right hand side

Sir second order equation, that is right, how many boundary conditions do you need? Good point, the second order differential equation how many boundary conditions do you need? Well, in the theory of partial differential equations all second order partial differential equations have been classified and such equations you have to specify you have to find out first whether it is well posed or not, whether the solution exists or not, whether its unique or not. It turns out that in this case this kind of equation is called an elliptic equation in this theory. It suffices to specify phi at all points at infinity, so really it is an infinite number of points and what I am saying is that the boundary conditions specified is that phi of r tends to zero as mod r tends to infinity. And of course that is such spatial infinity in all directions that is like a big boundary. So, imagine the surface all these charges are enclosed inside some finite sphere and then you let the radius of the sphere go to infinity impose the boundary condition that on the surface of the sphere the potential is zero, then it is a well posed problem.

What if the boundary is finite? Pardon me. That is what I said if the boundary is finite if for example somewhere you have a plate and that plate is earthed and the potential on that is zero then that is not the solution to Poisson's equation. You have to ensure that you add extra terms to it in order to make the potential zero at this point but we are talking about free space and we are talking about free boundary conditions at infinity. These are natural boundary conditions at infinity the potential is zero. Then this is the solution but this is only for illustration. The point I am making is that, given a set of boundary conditions appropriate set you can solve Poisson's equation; that is all I am trying to say and I wrote the solution down without actually trying to solve this equation. The simplest way of solving this would be to go to Fourier transforms and then this del square operator would just get multiplied by k square and then you invert the Fourier transform and so on. Since that would be a course on Green's functions in partial differential equations I do not want to do that right now.

Sir, is there a retarded potential? Pardon me. Since it is a referent of time, now he has brought up an interesting point which is what I was going to do, he has anticipated me. You see I have convinced you that if you did not have this t dependence electrostatics Coulomb's law we know is actually valid and therefore that is the solution and now I argue that for every t this is true because t just acts as a spectator and then let me paraphrase the question which I think is going to bother him its bothering him and that is the following. This potential leads to the field because the physical field E depends on this potential. Now the rho of r prime t is over the entire charge distribution at every point in space. Suppose I have a charge distribution on the other side of environment or galaxy and somebody there changes it. The potential here is affected immediately because you see no matter how far r prime is from r, if this potential is changed a little bit from the other side of the galaxy, then the potential here is affected immediately and this seems to violate relativity.

What would you say? Pardon me, but this is what the equation says and that is the solution. It is very easy to verify that that is the solution to this equation. Exactly, E does not depend on phi alone E also depends on the partial derivative with respect to time of the vector potential. So, remember this formula; remember that E is equal to minus delta A over delta t minus grad phi and we have not yet solved for A. So, when you solve for A you will discover that these A causal effects are cancelled out. So, this is another illustration of the fact that phi itself is not physical. It contributes gradient of phi contributes to the electric field but it is not the whole story. You also have this piece which you have to now solve for. So, that is a good point good observation. In any case, once that solved then the idea is that you put in the solution for phi here and move this to the right hand side and del dot A is zero in the Coulomb gauge and then you have to solve the wave function.

Now the next question you are going to ask is what is the solution to the wave equation? For this I have to write down the Green’s function, may be we will do it later in the course but right now I do not want to get into this because it is going to take me a while to write down the solution to this equation. Laplace equation was not so bad Poisson equation because we already know Coulomb’s law and I just convince to do that; that is the solution that is it, but this is going to take a little more doing, solving an equation of this kind it is not a elliptic partial differential equation, this is called a hyperbolic partial differential equation. This minus sign here is very crucial and things are a little mess here but you have to take it from me as an article of faith that there is a well-defined technique to solve this equation as well and the problem is solved in principle.

But Maxwell's equation pi is not electrostatic potential? No, it is not phi; phi is the scalar potential. Pardon me. What are you trying to do with the electrostatic potential? No no no, I said let us go back instead of solving Laplace Poisson's equation here, I have an issue I said let us go back and recall our knowledge of Coulomb's law and electrostatics where I switch off all the time dependences and I just have del square phi is equal to minus rho of r and I said I know the solution I know Coulomb's law. I put that down and then my superposition that integral turned out to be the solution and then I said t is just a parameter in the exact equation here the general case and I just put that down there. So, that was just to show you that that is the solution can be written down and in this case I am not even doing that. I am just saying that the solution can be written down. We will not do that at the moment.

So, in the Coulomb gauge we know in principle how to solve the Maxwell's equations. There is the other gauge which is the Lawrence gauge. In that Lawrence gauge this combination is zero then you start by writing down the solution to the wave equation with j as the source term for the vector potential; put back in here, this will no longer be zero here and then del square phi is equal to the given function rho plus a known function and therefore once again you solve Poisson's equation and the problem is solved in principle. Now this is not my primary concern. We will come back to this when we do relativity because we would like to find out what is the meaning of this gauge and I might as well say it here right now.

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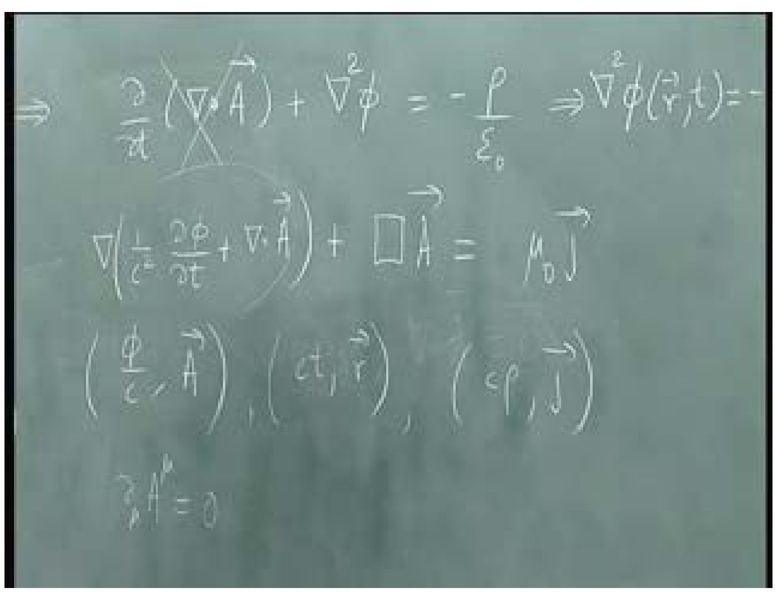

It turns out that this quantity this set of quantities phi over c and A together form what is called a four vector a four dimensional vector just as a vector that you have used to in three dimensions is defined as a quantity a set of three quantities which transforms exactly like the coordinates themselves do under rotations of the coordinate axis. In exactly the same way this set of four quantities will transform like the space time coordinates do like time and the free space coordinates do under Lawrence transformations and then this would be called a Lawrence vector or a four vector. So, this combination itself has a specific transformation property just as ct and r have a specific transformation property under Lawrence transformations. I put a c here because I would like to have all the components of a given vector to have the same physical dimensions and once I multiply t by velocity fundamental velocity this has dimensions of length.

So, this length and those three lengths here x y z transform under Lawrence transformations by the rules of Lawrence transformations and my assertion is that this set of quantities transforms in exactly the same way. They form the four components of a four vector, the time like component and the three special components but you can find other such combinations. For example c rho and j itself, so it turns out that the charge density multiplied by c and the current density have exactly the same physical dimensions and that combination also transforms like a four dimensional vector. So, that is the reason why electromagnetism looks so complicated when you write it in terms of three dimensional vectors as we have done here.

Once you write it in the four vector language it is very very simple, it is very straight forward and the equation were very elegant indeed. You could also ask what the gradient operator is; what is the analog of the gradient operator four dimensions and just as we had in d'Alembertian operator here, one would like to define 1 over c delta over delta t and of course the del operator but for a reason it should become clear later it is minus the del operator. This quantity would transform like a four dimensional vector. That is the appropriate four dimensional divergence four dimensional del operator and it is not hard to see that if you take the del operator and dot it with itself you would get this d'Alembertian and the dot product has to be defined in a specific manner which we will do when we do relativity.

So, the point I want to make is that this combination here turns out to be the four dimensional divergence of this four dimensional vector and it would be written in the form del mu A mu where mu runs over the values this index runs over the values zero one two three, zero for time, one two three for the space variables and therefore setting this equal to zero is equivalent to setting this equal to zero and this is a scalar. So, this means under Lawrence transformations the numerical value of this quantity is the same in all frames of reference all inertial references. Therefore, once you set it equal to zero in one frame its zero in all the frames. In other words once you are in the Lawrence gauge in one initial frame you are also in the Lawrence gauge in all initial frames; that is the great advantage of the Lawrence gauge. That advantage is lost here because del dot A does not remain unchanged when you go to another inertial frame, it changes from frame to frame.

So, if it is zero in one frame it is non-zero in another inertial frame and you can make it zero once again by making a subsequent gauge transformations but it could get tedious. So depending on the problem that you have you have to choose either the Lawrence gauge or the Coulomb's gauge or any other gauge; find your own gauge and solve your own problems. So, its infinite amount of freedom that is available; you have to observe some other rules but otherwise the choice of gauge is extremely useful and you saw here how useful it was but I emphasize once again that physical quantity measurable quantities in classical electromagnetism would depend on the fields on E and B.

If they depend on A and phi you must make sure that they do not change under gauge principles; so that is important as we saw. So, you can use this freedom quite a lot but then the fields should not change and indeed if you recall what the energy density of the electromagnetic field is it is epsilon naught e square plus 1 over mu naught B square multiplied by half and it does not depend on A and phi; directly it depends A and B and that does not change. Similarly the point in vector which tells you the momentum density is E cross B apart from some constant and again depends only on the physical fields. So, this is important to remember and with this let us do the following. Let us go back and ask what could possibly be the Lagrangian for a charged particle in an electromagnetic field. I am going to write the Lagrangian down and then explain where it came from. So, let me just first write it down; I will just assert that this is a Lagrangian and later we will justify, I will justify it post factor based on the invariance considerations.

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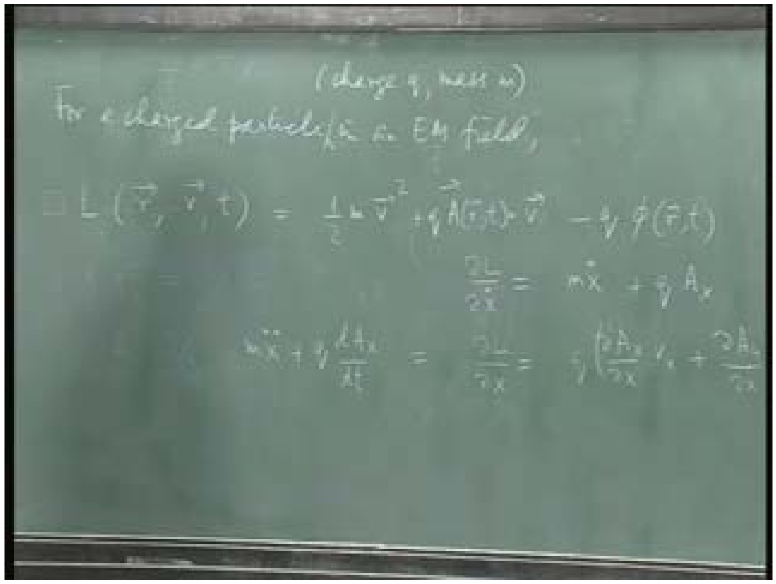

So, for a charged particle in an electromagnetic field the L for the particle and for the moment I am focused only on the particle; I am not worried about the field. The field is applied from outside could change with space and time but I am interested in the equation of the motion of the particle. So, what can the Lagrangian depend on? It depends on the coordinates of the particle and the velocity of the particle q and q dot but instead of q I really have the three coordinates L r and the velocity is v. It could depend on time explicitly because the fields that I apply could change with time. I could switch on an alternating current or something like that an alternating voltage in which case I must include this t explicitly and this is equal to again the kinetic energy one-half m v square and it is not t minus v and the reason is the magnetic field produces a force which is velocity dependent q v cross b and this is not within the purview of the normal t minus v kind of thing slightly different and I will write the Lagrangian down and we will see where it leads us.

This Lagrangian is q; this is the charge in the particle. So, charge q and the particle are assumed to be non-relativistic. So, this entire formalism is valid non-relativistically. So, even though the fields would obey Maxwell’s equations which have speed of light buried in them, the particle itself is assumed to move always at a speed much slower than the speed of light; that is an assumption. Of course we could write it down in the relativistic case but we will do that after we study relativity plus q times A dot v minus q times phi. That happens to be the Lagrangian for a charged particle in an electromagnetic field very strange combination of this kind. Now let me just try post factor to justify slowly where it comes from. The Lagrangian is a scalar as was pointed out. So, all terms on this are scalars. We are trying to describe the interaction of the particle charges charged particle with the applied field. So, it is reasonable that the some property of the charged particle and some property of the field getting coupled to each other.

This is the charge and that is the potential scalar potential q times v is the current due to this charge and the current is getting dotted with the vector potential. You can already see this is looking like starting to look like a dot product of some property of the particle and the dot product with some property of the field and indeed if you recall that j mu is zero and j current density and the vector potential A mu is phi over c A, the four dimensional analog of the current density and the four dimensional analog of the vector potential here the four vector potential, then this combination here is nothing but j mu A mu. It is the four dimensional dot product of the current with the vector potential; that is the reason for this Lagrangian.

But again at the moment you have to take this Lagrangian from this given a matter of given quantity and later we will justify it a little more but I thought I should mention this here as to where this comes from, what is the rationale behind it and later on we will justify it more rigorously. So, here is the Lagrangian but this is not a very simple Lagrangian because this quantity could depend on space and time it could change. And similarly here plus q of A r, t dot v and remember r is the coordinate of the particle, v is the instantaneous velocity of the particle and this is the Lagrangian.

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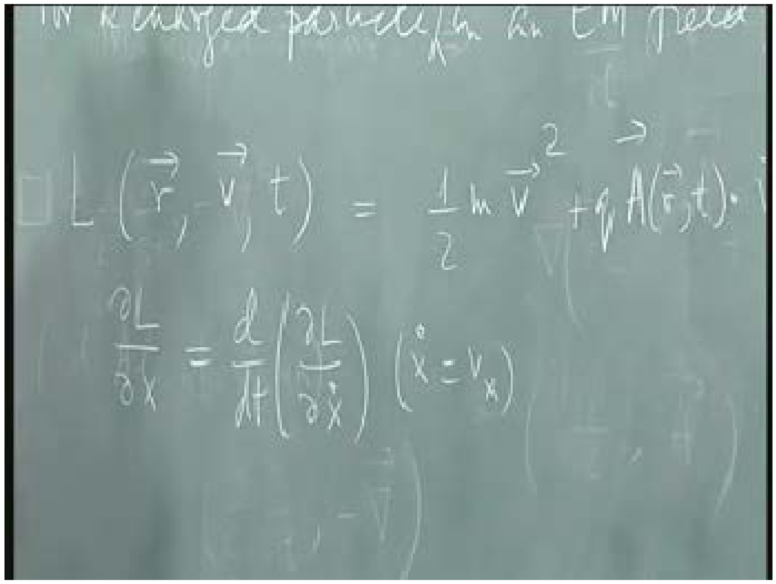

Now what is the set of equations of motion? Well, the Euler Lagrange equation say delta L over delta each coordinate x y z, etc delta L over delta x must be equal to d over dt delta L over delta x dot; x dot is v sub x by definition the x component of the velocity and similarly for the y and z coordinates. So, you have three equations of motion for each. How are we sure that it is d by dt rather than del by del dt; how are we sure that del L by del x dot does not depend on time? It would depend on time; it does depend. It does not depend on other coordinates. It does depend on other coordinates; that is why it is a total derivative that is the whole point. This is a total derivative; it is not differentiating just the partial derivative.

In fact if you did not have time dependence in the Lagrangian if you had an autonomous system then this thing here would be zero if you had a partial derivative and that is not true. It is a total derivative; absolutely it is very important. It is a good observation; it is the total derivative with respect to time. So, if the coordinate’s changes with time then you have to differentiate the time derivative would the non-zero; it is crucial, it is the total derivative. Now let us see where this gives us and so on and similarly for y and z. I could use the index notation and do this but let us just do this with x y z so you become familiar it looks familiar.

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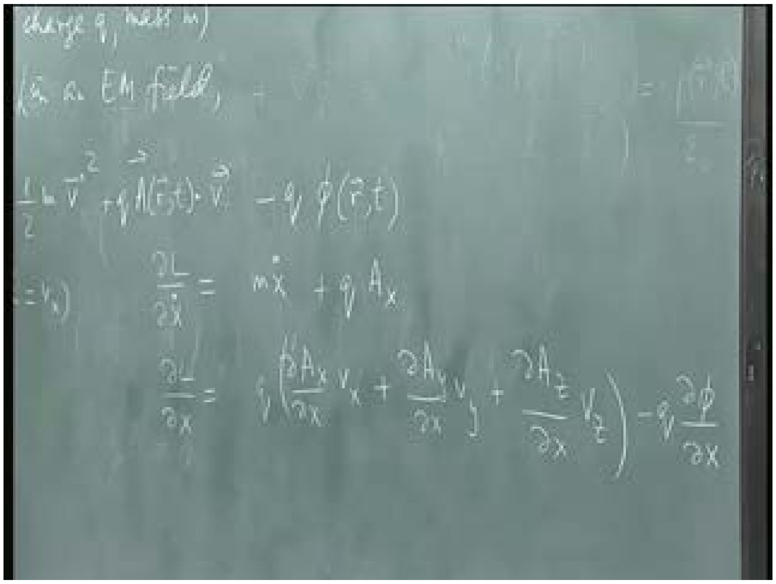

And let us compute what is delta L over delta x dot equal to, what is this going to be? Well, there is going to be a derivative here because it is half m v x square and if you differentiate it you get m x dot; that is from this point here. Remember v y v z are independent of v x as dynamical variables so you differentiate with respect to v x alone and you differentiate only half m v x square here which produces a factor two cancels this and gives you that but there is also dependency here and this is A x v x and if you differentiate it you get plus q A x. I will suppress the r and t dependence inside, it is understood and what is delta L over delta x equal to. There is nothing from here because these are velocities but there is dependence in here on x y as well as z of every component.

So if I write that down you get q delta A x over delta x v x plus delta A y over delta x v y plus delta A z over delta x v z; that is this term here if I differentiate with respect to x and then there is derivatives here; so you cannot forget that. So, it is minus q delta phi over delta x this expression. Now we plug it into the Euler Lagrange equation and it says this quantity is equal to on this side the time derivative of this the total time derivative. So that is equal to m x double dot plus q d A x over dt total derivative and this is the equation of motion. So, now permit me to erase this part then rewrite it.

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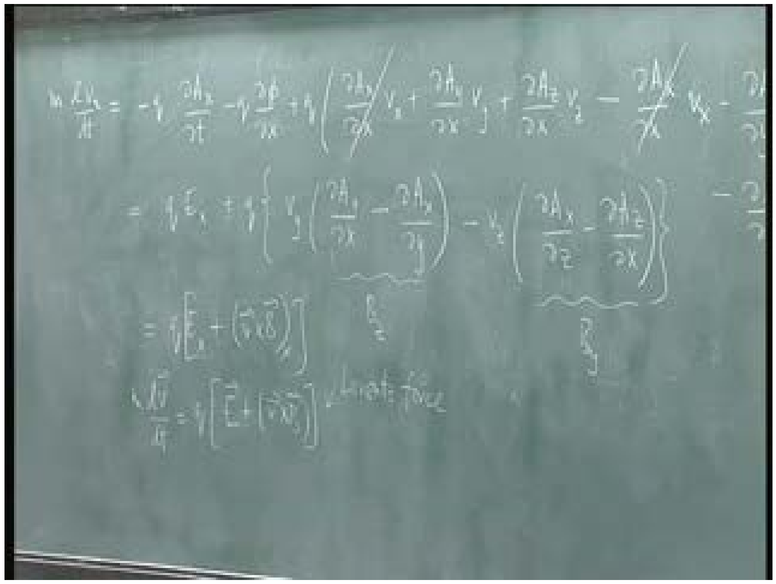

And you get mass times the acceleration m x double dot equal to or let us write m d v x over d t equal to, move this to the right hand side and let us see what happens. So, you have minus q d A x over d t but that can be written as minus q delta A x over delta t and then delta A x over delta x d x over d t and so on but let us bring that last term here first. So, that is minus q delta phi over delta x. So, that takes care of the last term and then I have to keep track of all the other terms. So, plus q times delta A x over delta x v x plus delta A y over delta x v y plus delta A z over delta x v z minus the terms that were left out from here the special derivatives and the first of these was delta A x over delta x d x over d t that is v x itself minus delta A x over delta y d y over d t that is v y and minus delta A x over delta z v z. So, that is the full set of terms; painful doing it components, easier to do it with index notation but this should illustrate what is going on and of course this term cancels and then you begin to see what is going on.

This is equal to minus q rather plus q. What is minus delta A x over delta t minus delta phi over delta x. Remember that E was equal to minus delta A over delta t minus grad phi; so the x component of E is minus delta A x over delta t minus the x component of the grad of phi which is minus delta phi over delta x. So, you see this is immediately emerged. This is q times E sub x it has immediately emerged plus q times and now let us take terms together. So let us take the v y terms here and you have v y times delta A y over delta x minus delta A x over delta y minus v z times delta z minus delta A z over delta x, so that takes care of these two terms here. But what is this equal to? Well remember B equal to curl A. So B y B z is equal to delta over delta x A y minus delta over delta y A x in cyclic order of one two three.

So, this quantity is equal to B z and similarly this quantity equal to B y. So, therefore this says this is equal to q E x plus v, v y B z minus v z B y that is equal to v cross B the x component of it. So, it says the x component of m d v vector over d t is the x component of q E plus the x component of q times v cross B and now it is true for every component because I choose the x component arbitrarily and its true component by component. So, this immediately tells you that the equation of motion is m d v over d t equal to q E plus v cross B as vectors, this is the Lawrence force. So, the Lagrangian has given us the correct equation of motion. We work backwards I mean shamelessly wrote this Lagrangian down in order to produce the right equation of motion but this is the way it is derived and as I pointed out the Lagrangian itself is obtained by invariance considerations.

So, there is something fundamental about the whole thing, the fact that the four dimensional dot product of the current with vector potential is the Lagrangian. It is called minimal coupling, it is the least you can do and it is been experimentally verified to incredible accuracy, ten decimal places or something like that in quantum electrodynamics. So, we know this is true and we have good reason to believe that this is exact but the fact is that we work backwards we really said we have the Lawrence force as an experimental observation and what is the Lagrangian that gives it, well that complicated velocity dependent Lagrangian happens to give it but now with your perceptive you should ask the obvious question. I wrote this Lagrangian down and that is a very very deep point.

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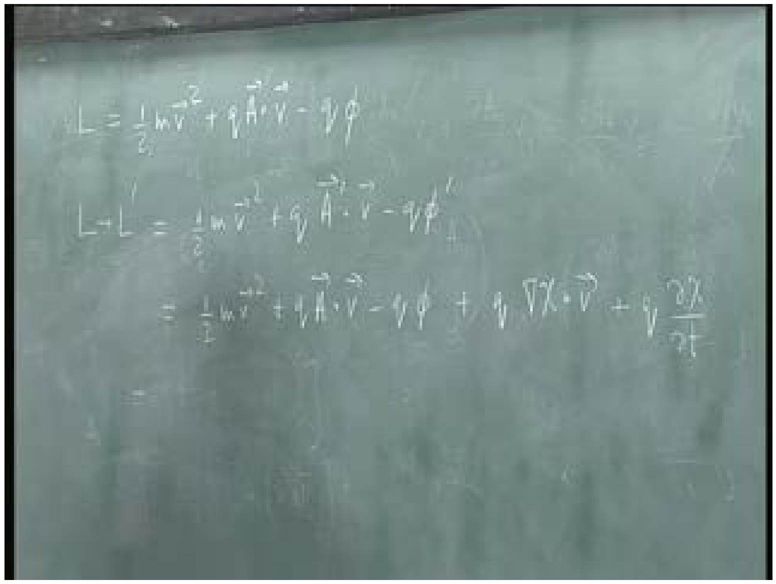

Remember that the Lagrangian that I wrote down was one-half m v square plus q A dot v minus q phi; two questions arise. You could ask what about the electromagnetic field itself that could change and it should also come from Lagrangian. I should write a Lagrangian density for the electromagnetic field; yes indeed you can but we are not finding the equations of motion of the electromagnetic field. We assume we know them already, what are they; the Maxwell equations. So, I have not derived the Maxwell equations from a Lagrangian density; that is a little more complicated exercise. It requires classical field theory; we have not done that. But the fact is for the particle non-relativistic particle this is the Lagrangian; does something strike you funny about this Lagrangian, does it make you uneasy a little bit? Pardon me. It depends on the auxiliary variables, it depends on A and phi; does not seem to depend on E and B directly and yet we are producing a physical equation here.

Please note the equation here is in terms of E and B; these are physical quantities measurable quantities on both sides. We started with these auxiliary quantities which are not unique. So it should bother you that the Lagrangian is not unique not in the trivial sense of adding a constant which does not get differentiated but serious non-uniqueness because it is in terms of A and phi which I can change by gauge transformations and the question is how are we sure that this is going to remain as it is eventually. Well the answer of course is trivial because if I put any other

A and phi exactly the same thing would happen because E and B are gauge invariant quantities and we know that under gauge transformations E and B do not change.

So, just B as B is curl A, B is also curl A prime and E is minus delta A over delta t minus grad phi also minus delta A prime over delta t minus grad phi prime. So, it does not matter. On the other hand you could ask what happens to the Lagrangian itself if I make a gauge transformation. Now that is a serious question, so what happens to this L? And this leads us to the non-uniqueness of the Lagrangian. Under a gauge transformation L goes to L prime is equal to one-half m v square plus q A prime dot v minus q phi prime where A prime is related to A by addition of a gradient of chi. So, this is equal to one-half m v square plus q A dot v minus q phi plus q grad chi dot v because that is A prime plus q delta chi over delta t because phi prime was phi minus delta chi over delta t and this is the extra piece in the Lagrangian.

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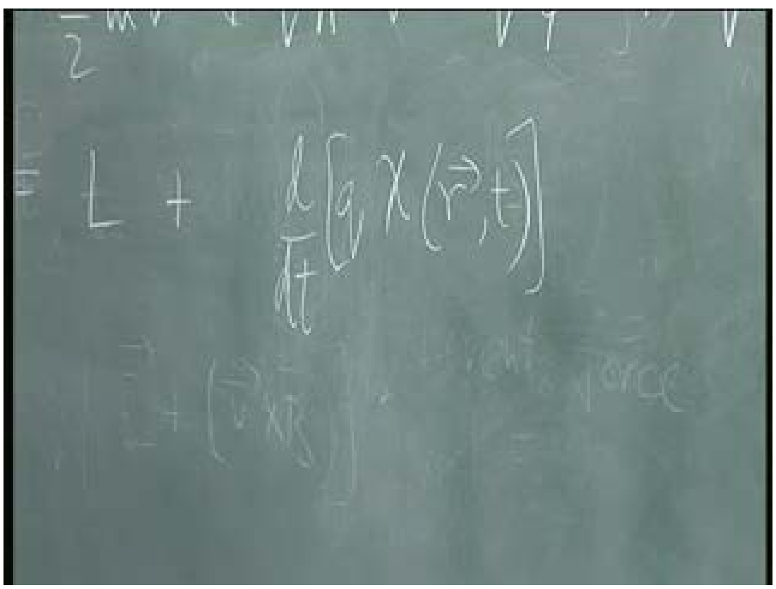

So, this is equal to one-half equal to L the original L plus this quantity here now what does one make of it. Well, please look at it and realize that it is equal to q times d over d t of chi because the total derivative of chi which is a function by the way of r and t in general as we know is equal to delta chi over delta t plus delta chi over delta x d x over d t and so on and that is just this.

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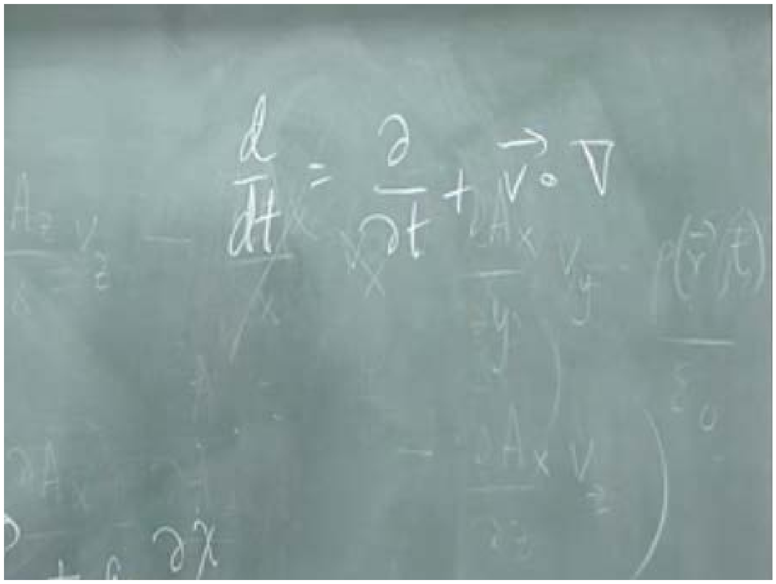

And I have written this simple thing down that d over d t of a scalar function is plus v dot del. When you act on a scalar function on the right hand side this is exactly what you get and you are familiar with the second term in fluid dynamics, it is called the convective derivative. So, the total time derivative is the partial derivative with respect to time plus the convective derivative. It is just a rule of calculus here. Well let us take this q in and write it as d over dt of q chi. So, the lesson we have learnt over here is that if you make a gauge transformation the Lawrence force equation would not change.

It is in terms of physical quantities but the Lagrangian changes by the total derivative of a function of the coordinates and time in general; that does not change your equations of motion. This is a general statement not restricted to electromagnetic fields or anything like that. In general the Lagrangian is non-unique through the extent that you can add to it an arbitrary total time derivative of an arbitrary function of the coordinates in time; that is true for any system and if you go back here and look at how we found the equations of motion it will become obvious.

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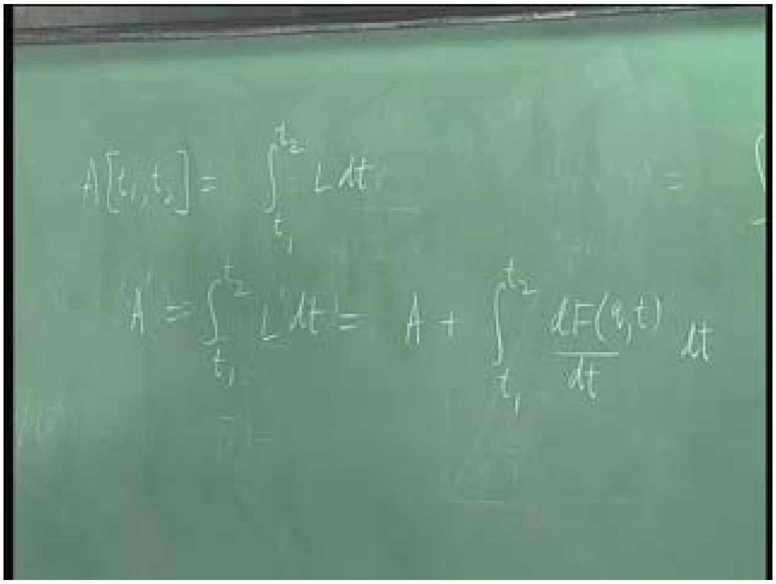

We started by saying I define an action t 1 and t 2 equal to integral t1 to t 2 L dt and L was a function of q q dot and t in general. I go back to the notation q for the coordinate; I do not want to use it here because I used it for the charge and now if I write A prime is t 1 to t 2 L prime d t this is equal to the original A plus an integral t 1 to t 2 d F of q and t dt. So, the original Lagrangian to that if I add a d F over dt I get L prime as L plus d F over dt for an arbitrary function of the coordinates and time and that is what A prime becomes. In the equations of motion the Euler Lagrange equations were found by varying A and claiming that the first order variation is zero the principle of extremal action. So, I varied it between time t 1 and time t 2 by any path whatsoever no variation of the q's at these points, delta q was zero here and zero here.

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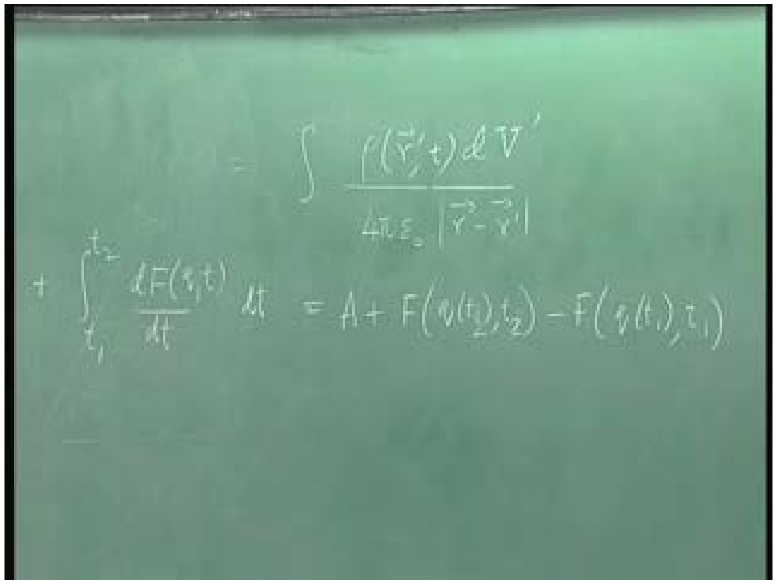

But then what does this give you? This is equal to A plus F of q of t 1 t 2 rather minus F q t 1 t 1. I can integrate this out the precisely because it is a total derivative term and then it is the two points at the ends. And now our variation to derive the equations of motion permit all possible variations inside with the end points kept fixed and at the end points there is no delta q and since this extra piece depends only on the q's at the end points, these two terms do not contribute to the variation at all. Therefore, the equations of motion do not change. Now of course you should ask what kind of transformation is this, what does this mean, what kind of transformation is this? It will turn out; I will explain this as we go along, it will turn out that this electromagnetic example is a very, very useful one because it is very instructive, it tells us what is going on in the general case. Well, it was a gauge transformation here.

So I could in some sense continue to call that a gauge transformation. By that I mean to the Lagrangian of a system, I add the total time derivative of an arbitrary function of the coordinates and time and the equations do not change, not velocities also please notice. I have not allowed for you to add a function of the q dots here. This could be done but then I have to start putting conditions on this f but without any conditions this is the general case. It is called profound implications because when we do the Hamiltonian framework it will turn out that this same freedom of adding to the Lagrangian the total derivative of such a function would be called a canonical transformation in Hamiltonian mechanics.

So, there is a close link between what was on here and what we are going to study a little later and then it will become much more natural in that framework, but already you can see that the Lagrangian is not unique, you can add the total derivative and in the electromagnetic case in this case physically a gauge transformation is equivalent to adding the total time derivative of an arbitrary gauge function to the Lagrangian; does not change the equations of motion. That is a lesson worth keeping in mind. Is there somewhere we could have expected this? Well you can put a q dot but you will have to put further conditions on it. It would have to be a suitable symmetric function and so on but not getting into that right now, but this is the simplest case. I am not sure if I have answered your question.

So, in this language it is obvious; I mean in this language once I say there is no variation here, it is quite clear you can add something which depends only on these end points and then the equations would not change. So, in that sense we could have predicted this, but it is going to have physical implications. The implication of that in this context were precisely gauge transformations but that would come about from other transformations as well. So, we are going to look at that. There could be other symmetry transformations on the Lagrangian which would leave the equations of motion unchanged and the necessary condition for that would be that the Lagrangian should change only by the total time derivative of a function.

So, it will get linked to various other quantities; we will see we will get back to this. This brings me. Does Lagrangian exist for every dynamical system? No, no, I am looking at only that class of dynamical systems for which you can write down Lagrangians for which the equations of motion arise as Euler Lagrange equations. So, that is a restricted class of dynamical system. So, in between I also went and did general dynamical systems for which you cannot write any Lagrangian or anything out. So, you can push that to some extent but once you have a Lagrangian structure you can push things much much more, so certainly not all dynamical systems. This brings me to a very very important observation and that will take us back to general dynamical systems and that is the following.

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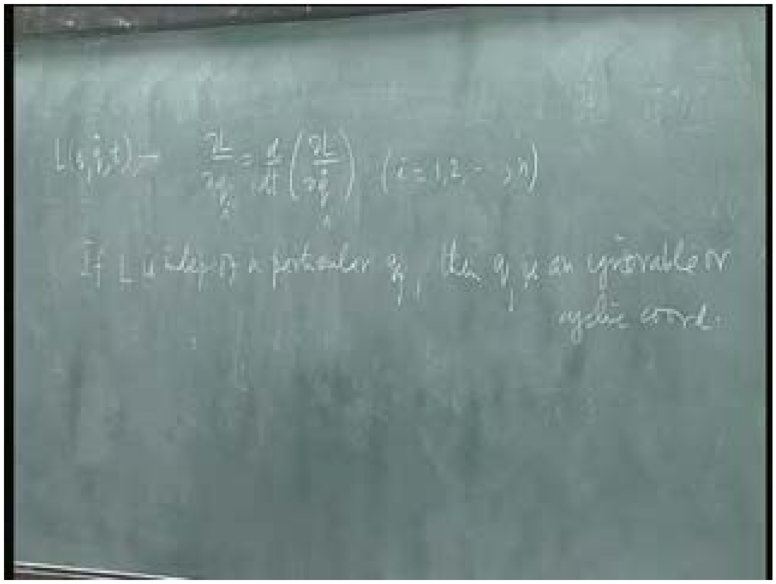

When I wrote the Euler Lagrange equations down let us rewrite this thins. So I have this L of q q dot and t and from this I had equations of the form delta L over delta q i equal to d over dt delta L over delta q i dot and i was equal to 1, 2. We looked at systems with a finite number of degrees of freedom. So, these were Euler Lagrange equations. And now one of the most crucial things in solving general dynamical equation is to find the constant of the motion. The moment you find the constant of the motion, as we saw in the case of harmonic oscillator, we had just the energy as the constant of motion and immediately that gave us the phase trajectory as ellipses. So, the important lesson is every time you have a constant of the motion you have solved part of the problem. You would like to find more and more constants of the motion.

Now in this problem the face space is two n dimensional, the trajectory is a two n dimensional is a line in this 2 n-dimensional space. It is like a piece of thread moving around in this two n-dimensional space. To specify this line in a two n-dimensional space, how many constants of the motion should I have? I should have 2 n minus 1 because if I have one constant of the motion some function of the q q dot equal to constant that gives me a surface of 2 n minus 1 dimension. If I have another independent constant of the motion another 2 n minus 1 dimensional surface, the mutual intersection of these is generically 2 n minus 2 dimension. Just as in ordinary three dimensions I have a sphere surface of a sphere that is two dimensional; I have a plane that is two dimensional. These two cut each other and the result is a circle in general which is one dimension. So, every time you have two surfaces intersect the dimensionality of the intersection is lowered.

Now you want to go from two n-dimensional face space to a one dimensional object you need to have 2 n minus 1 constants of the motion. So, every time you have a constant of the motion you solve the problem a little bit. Sometimes you can use and they are going to be closely linked to symmetries and that is one of the things I want to emphasize in this course. We would like to find the constants of the motion; this is very very important. In fact in a sense all dynamical systems of this kind are governed by their constants of the motion. Once you have done that job is done. One of the things that happen in Lagrangian mechanics is that sometimes you may have a situation where the Lagrangian does not depend on a coordinate on a particular coordinate such a coordinate is called cyclic coordinate; for reasons i would not go into it right now. So, if L is independent of a particular q j, let us say so I avoid confusion with I, some particular q j then q j is an ignorable or cyclic coordinate. What does that imply ones from this equation?

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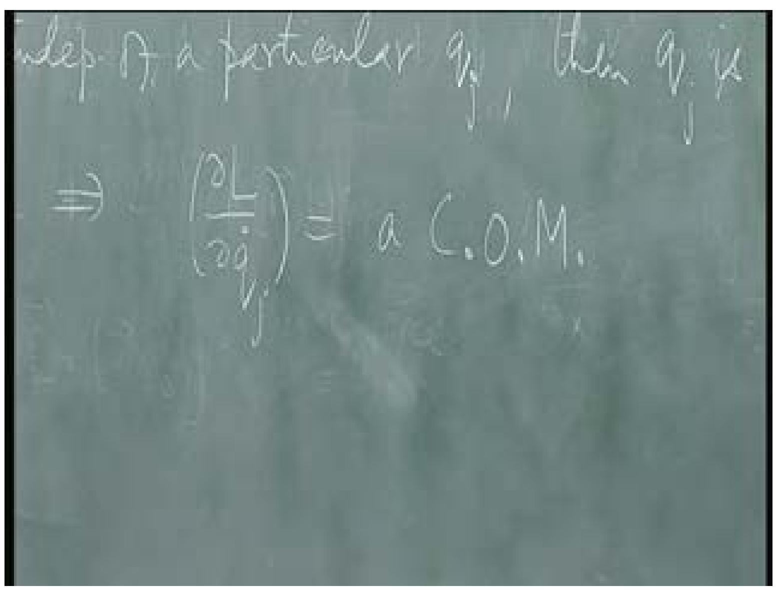

This would imply d over dt delta l over delta q j dot is equal to zero but that equation is solved immediately and what does that imply? I am going to use this C.O.M for constant of the motion. So, it says delta L over delta q dot that particular q dot is a constant of the motion, does not change. Its numerical value would be the same for a given initial condition would remain constant in time. The numerical value would depend on the initial conditions and once you specify that it remains exactly the same.

Therefore, a very very important thing is to find cyclic coordinates because then you find constants of the motion and every time you do that you won the battle a little more, you have gone a little further. Let us look at motion of a particle in a plane two dimensions and let us assume for example that the force the particle sees is a conservative force and that the force is always directed to the center of the attraction or the repulsion the central force, what the kinetic energy would look like.

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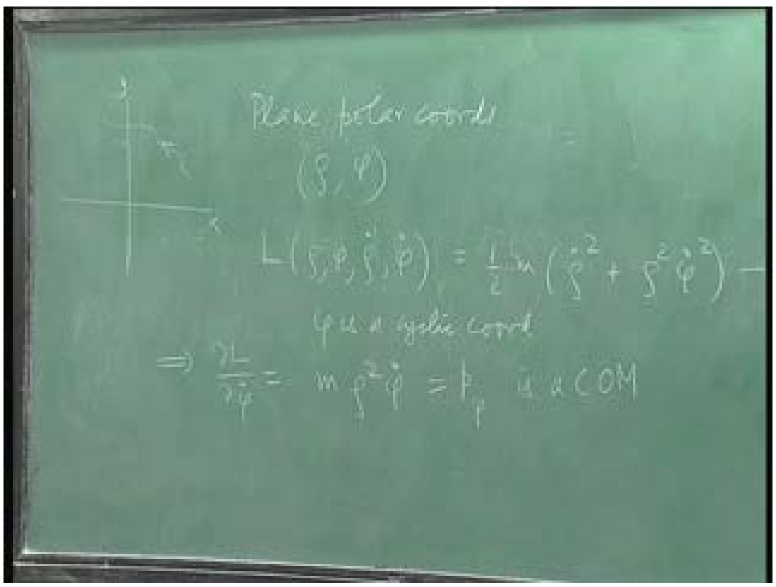

So, I have motion of a particle in a plane and these are the x and y axis and I look at polar coordinates plane polar coordinates rho and phi. This is the square root of x square plus y square and that is tan inverse y over x. I use this rather than r and theta because I want to reserve that for spherical polar coordinates. The Lagrangian of this particle is a function of the coordinates x y, x dot, y dot but I could also write it in terms of rho phi rho dot phi dot and let us assume that it is a constant potential in the sense that it is time independent and always acts in a central fashion. It is a function only of the distance from the origin of rho. So, this is equal to the kinetic energy half times m times v x square plus v y square but I should write this in plane polar coordinates. What is v square in plane polar coordinates? Rho dot square because there is a radial velocity rho dot square plus rho square phi dot square; that is the kinetic energy minus the potential energy v but now I am going to say this is a function of rho alone central potential. Is there a cyclic coordinate? Phi is the cyclic coordinate immediately it follows that phi is a cyclic coordinate.

So, you are absolutely guaranteed phi is a cyclic coordinate and that implies that delta L over delta phi dot is a constant of the motion and what is delta L over delta phi dot. Now we are coming to that, it is angular momentum. So, if I differentiate it here I get a phi dot here and if I differentiate the two goes away. So, it is equal to m rho square phi dot and of course you recognize that this is the moment of inertia of the particle about the origin and that is the angular speed angular velocity. So, this is equal to the angular momentum of the particle about the origin and we guaranteed it is constant. Of course we know that in a central force there is no torque and therefore the angular momentum is constant but it is nice to see that it comes out automatically as a trivial statement. Phi is a cyclic coordinates, so the angular momentum is constant. Incidentally let me call this in anticipation of the notation p sub phi to show that it is the momentum conjugate to phi. You are going to use this word conjugate repeatedly; I will explain what it means but let us anticipate this a little bit.

So, it says this is a constant of the motion; that is a big help. In this problem how many degrees of freedom are there; two. What is the dimensionality of the face space; four. Now you like this problem to be solvable. How many constants of the motion do you need? You need three. How many do you have? We have found one but we also know in the back of our minds that the total energy is going to be constant, actually the total Hamiltonian is going to be a constant. So, we have a second one and we need one more. We need one more and that is not always easy to find in this problem but let me anticipate things again by saying that in problems of this kind if you have a two n-dimensional face space sometimes if you have n constants of the motion that is enough, you can find the rest. They are called integral but I am jumping ahead little bit but this is an example of such a system. These two are enough; they are independent, the angular momentum and the total energy are independent. You can specify each of them independently and therefore, this problem is in some sense solvable.

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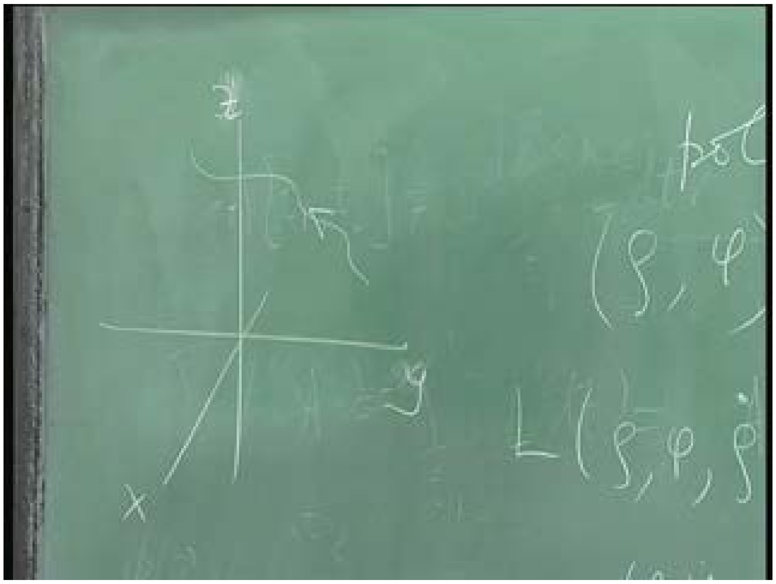

Let us do the same thing for three dimensional motion we will look at what happens. In three dimensions I have to use spherical polar coordinates and I am again going to assume why do I always put x here y there and z there; we could have done it the other way. Why do I do that? Pardon me. Well, why do not I choose x from right hand side and why outside the board and put setup there. I would like to choose the right handed coordinate system, like to choose it just. No special reason but I would like to make sure that when I twist from x to y I go upwards and there is no special reason. It is a left-handed coordinate systems are just as good and this reminds me of this famous TV scene about George Bush because every time in class I explain right handed coordinates I always use the left hand because the right hand is chalk but then George Bush does this when he says well the right hand does not know what the left hand is doing; this is very, very famous scene.

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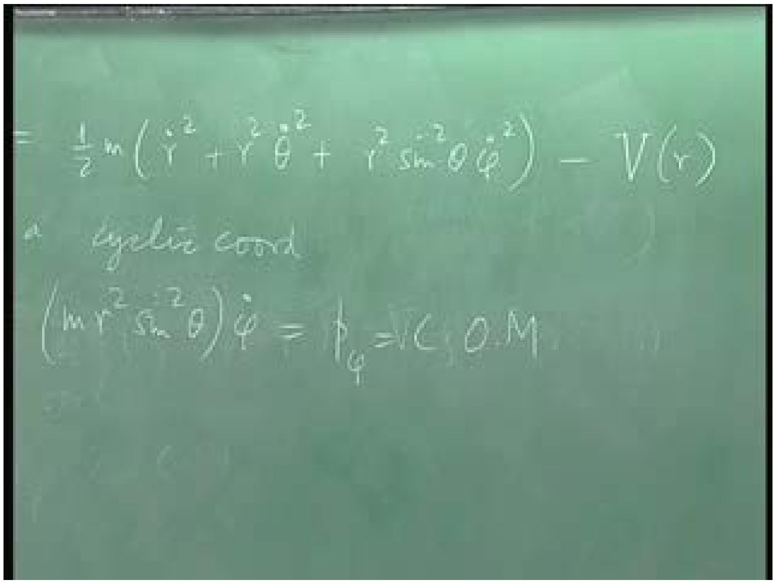

So, here we are going to have L of r theta phi r dot theta dot phi dot and that is equal to one-half m and now there is a r dot square plus r square theta dot square plus r square sine square theta phi dot square minus v of r alone; that is a central potential, is there a cyclic coordinate? Yes, it is a cyclic coordinate. So, once again the azimuthal angle phi is a cyclic coordinate. This implies that the derivative with respect to phi dot is a constant. So, it implies that m r square sine square theta phi dot equal to p phi equal to a constant of the motion. What is that quantity? It is in fact the angular momentum but in three dimensions angular momentum has three components unlike in two dimensions where it has just one component

In three dimensions angular momentum has three components and therefore this is just one component the azimuthal component of the angular momentum and buried here although we do not see it as cyclic coordinates, there are other constants of motion. There is actually the total angular momentum that too is a constant but that is going to involve theta, it is going to involve other components as well. So, this is just one component and in this problem by the way how many constants of the motion do you need to solve this problem? You need five but like I said these are systems where three would do. We have the total energy the Hamiltonian, we have this P phi and we need one more to solve it and it will turn out to be the square of the total angular momentum.

Now once you have that the problem is solved. Incidentally that is the reason why when you go to quantum mechanics the hydrogen atom is specified by a principle quantum number and arbitral angular momentum quantum number and a magnetic quantum number because constants of the motion translate into quantum numbers in quantum mechanics. This is the connection between these two which we will see. So, once again I will stop here. We see that the existence of cyclic coordinates is extremely helpful. We discovered the cyclic coordinate here by changing to spherical polar coordinates; had we looked at it in Cartesian coordinates, this would not have been apparent. So, you also begin to see the advantage of using symmetry to choose coordinate systems appropriately. So, let me stop here and then we take it up from this point; we are ready to do the Hamiltonian problem.