Classical Physics

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Lecture No. # 07

We have seen, so far a few general properties of dynamical systems. I would like to backtrack a little bit and go back to a formalism called the Lagrangian formalism, for dynamical systems. Specifically, created in order to reproduce the equations of motion that we know of the systems we already have, familiar with and to give new incite into other kinds of dynamical systems.

This is a very general formalism and it kind of explains why Newton's equations arise? So, you might have wondered from your elementary courses on physics, why do we have Newton's equation of motion? Is there something deeper some underlined structure from which you get Newton's equation of motion. And the Lagrangian formalism answers this question. So, let me go back and talk about mechanical systems for a while, then we come back and try to put it in the general framework of dynamical systems. In particular, I would like to introduce the Hamiltonian formalism which is going to play important role for us in statistical mechanics and quantum mechanics.

So, as a backdrop let us ask, where do Newton's equations come from? For instance the force is equal to the mass times acceleration for a single particle, moving in space, where does this come from? Was it just written down by Newton and we require, it was based on experience, based on experimental evidence, but is there a deeper principle. It turns out all equations of motion in nature, both classical and quantum mechanical can actually be obtained from certain principles fundamental principles called action principles. And the extremization of the action which, I am going to define leads to the equivalent of Newton's equations of motion, for the simple systems we have used.

Let me give an analogy, if you have a particle, system in static equilibrium then, we know from elementary physics, that the position of static equilibrium is given by the minimum of the potential energy; system tends to minimize its potential energy. That is an example of an extremal principle, something is minimized is that an extreme. Similarly, if you have in thermodynamics, if you have a isolated system thermal equilibrium, then we know the entropy is maximized. And again you have a extreme points, if you have a same similar thermodynamic system, at constant volume and at constant temperature; then the Helmoltz free energy is minimized in a state of thermal equilibrium.

If this were at a state of constant pressure and constant temperature, then the Gibb's free energy is minimized as a function as a function of variables on which it depends such as pressure and temperature. So, in all cases we have an extremal principle and the interesting thing is, we discovered that even in dynamics even when things are moving there is such a minimization or extremization principle in operation. And I am going to introduce this without much further do state what the lagrangian is and when we will come back and show that it yields the equation of motion in the cases we want; and goes further and helps you to find the equations of motion in more general cases.

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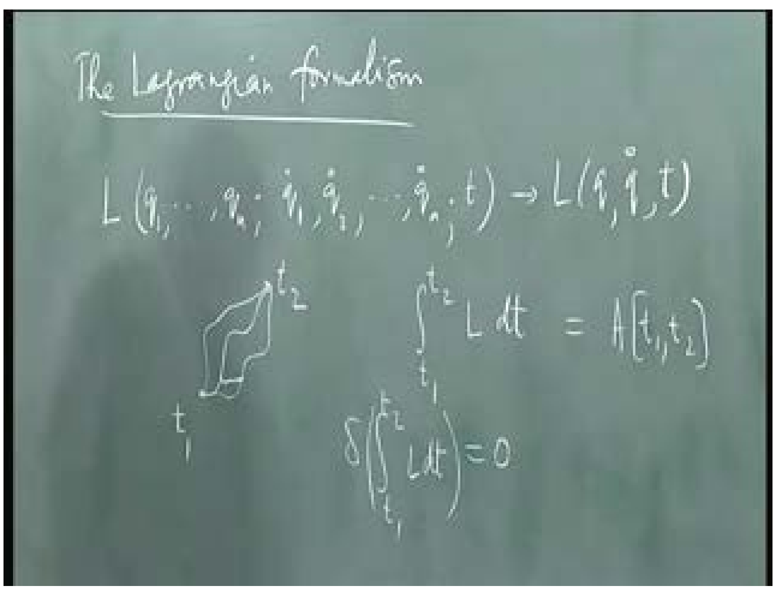

So, this is going to be the Lagrangian formalism, and we have in mind a system described by certain general coordinates q q 1, q 2 etcetera q n; and generalized velocities q 1 dot, q 2 dot, q n dot, and possibly the time itself, you have a function L which I call the Lagrangian, which I will write down as we go along.

This function L is supposed to be given to you, for a given system; you are supposed to discover what the independent generalized coordinates are, corresponding generalized velocities and then write down the Lagrangian and obtain the equations of motion for the system from the Lagrangian.

This is the program and the important question of course, is where does this Lagrangian come from, where do you get it from. We are going to a backwards guess the Lagrangian and the cases for which, we know the equations of motion and then generalize from there. There are a general principles which helps you to write this Lagrangian down and this is a very cumbersome thing to write down. So, let me just write this as L q q dot for short.

But always I have in mind the system with an arbitrarily large number of degrees of freedom 1 to n. We will not look in this course at systems which have an infinite number of degrees of freedom; we will come to that when we do statistical mechanics. This n is some finite number and then the Lagrangian is functions of q’s q dots which are independent dynamic variables; and possibly if the forces on the system are time dependent, if it is non autonomous, there could be an explicit dependence on t as well. The statement is when, that if an free space for an instance in q q dot space you had some point here at time t 1, the system is at this point in free space, that is some initial time q one arbitrary initial time t 1 and it reaches this point at time t 2.

It moves along a phase trajectory clearly, this could be the phase trajectories systematically along which it moves in this multidimensional free space. And the statement is of all the possible motions from t 1 and between t 1 and t 2, the system chooses that particular part, along which this quantity t 1 to t 2 L d t call the action; this is defined as the action and of course, it is a function of t 1 and t 2. It chooses that path for which the action is at the extreme, not necessarily a minimum in all cases could be a maximum also in certain cases.

But we will call this the principle of least action, because in most of the cases in which you are really interested it is going to be a minimum. So, it could have chosen this path for instance or it could have chosen that path; but it chooses that path that trajectory phase trajectory along which the action is at a minimum, is extremized which means that the path is determined by saying delta of t 1 t 2 L dt is equal to 0 where this stands for the variation of this quantity.

For instance, if this is the q direction and between this point and this point this is d cube delta cube variation. So, all possible variations, that we take into account and then of all those variations the system will choose the path along with the variation of these action between p 1 and p 2 is at an extreme. This is the principle of least action, I am not going to try to justify here and to prove it, but this is what experience tells us that, this is what leads to the equations of motion. It is very satisfying because, just as in all static situations you have a certain minimizations something is minimized; the potential energy is minimized, the minimum of the potential energy. Similarly, you have a minimum of something else even when you are moving, and it is the action.

Now, this leads to equations of motion, because it is really a statement which tells you how the system is going to propagate from this line to this line along this path. Therefore, you could expect that even though this thing here looks like variation of some integral between t 1 and t 2 some integral quantity is at 0, the first order variation is 0 even then it will turn out that since t 1 and t 2 are arbitrary. At every point in the path you can apply this principle and you should expect you will end up getting a differential equation for the motion.

Because I could choose this point to be t 1 and that point to be t 2 and its still true between those two points. So, the action is actually extremized all the time and therefore, we are going to end up with a differential equation and how do you do that? Well, the algebra is quite simple, I will write this down and I am going to keep t 1 fixed t 2 fixed for the moment just to get the equations of motion.

And once you are here or here, definitely there is no delta q here, there is no variation here everything all the parts join up here. So, you are looking at all parts you start here and which end there. So, there is no delta q at the two end points.

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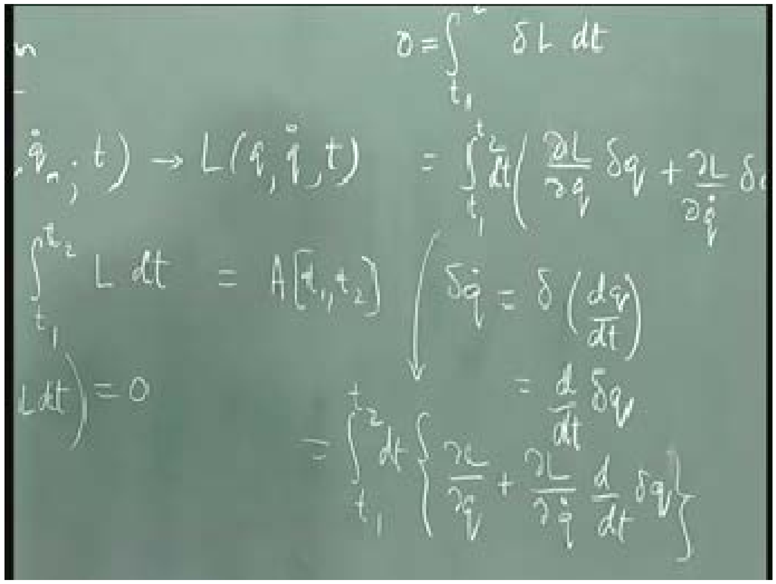

Therefore this principle reduces to saying t 1 to t 2, 0 is equal to this and times delta L, inside all possible shifts in L induced by a shift in the q.

So, if you arbitrarily set instead of q, q plus delta q, L changes something and all such delta L possibilities and then you equate as 0, but this is easily written down as t 1 to t 2. L depends on these variables therefore, this is delta L over delta q by the way I have n degrees of freedom, but just for simplicity of notation. I am going to write this thing as q, so it is understood that there is a q i here and all the i's are independent vary independently I will call them collectively this. This stands for summation over i delta L over delta q i plus delta L over delta q dot; remember these are independent variables that was our big lesson motion in free space and delta q dot times dt.

Time is not a dynamical variable time is the arena in which the dynamical variables vary in this way of looking at things. So, there is no question of varying time, there is no delta t and the point is that you assume you are looking at only those variations, where this delta q here and at this point as 0 in between you allow all possibilities.

So, this is the equation I will get, but then remember this delta q dot equal to delta of d q over d t; and the operation of differentiation with time has nothing to do with shifting this q.

The fiducially shifts and the q those are not dynamic variations those are just arbitrary shifts, arbitrary variations. These two operations have nothing to do with each other, therefore this is the same as d over d t of delta q. These two quantities commute then nothing to do with each other. And once you put that in this equation of motion reduces to t 1 t 2 d t and inside the bracket you have delta L over delta q plus delta L by delta q dot d over d t this is it.

The next step is obvious, I would like to pull out sorry there is a delta q here, I would like to take this out as a common factor, but there is a d over d t sitting there. So, the obvious thing is to do is integrate by parts.

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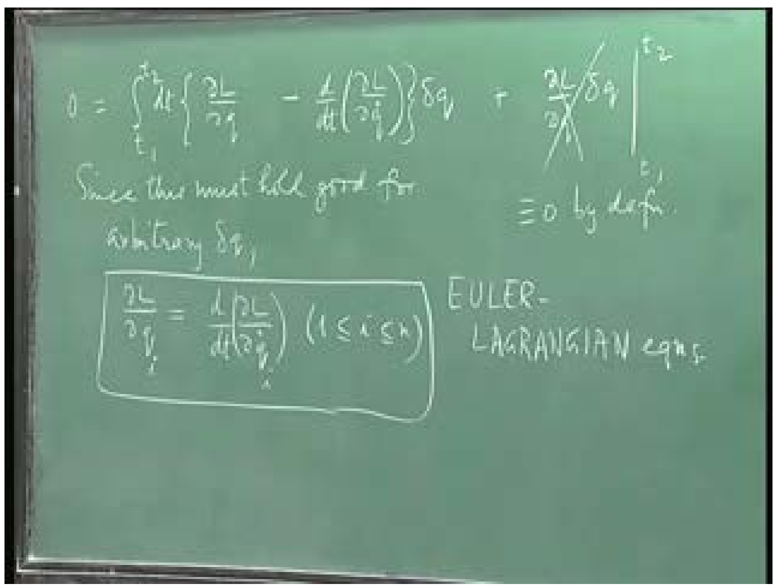

Therefore, say 0 equal to t 1 to t 2 dt delta L by delta q minus d over d t delta L by delta q dot. Now, I can take out a delta q, because I integrated this term by parts and put the derivative on this, that is the term I have written here and you get a minus sign here of course; plus the boundary term and this boundary term is of course, plus delta L over delta q dot delta q evaluated at t 1 and t 2.

Because, you have the surface term because of the integration being finite integration here you have the surface term, evaluated at these two points. But by definition delta q is 0 at the two end points; two end points there is no variation therefore, this is 0 by definition. And went up like this and this is supposed to be true for any given t 1 and t 2 for arbitrary delta q chosen as you please that is only possible if what is inside the curly bracket itself vanishes.

So, since this is true for all, since this for arbitrary delta q, we must have delta L over delta q sorry I have to put outside, I do not need that here minus or equal to d over d t d L by d q dot. Let us restore the i that is the equation of motion that you get. Now of course, there is many, many questions we would arise the first one is why should I take the Lagrangian what ever it be. I am going to give you subsequently, why should I take it to be a function of q ‘s and the q dots why not the q double dots q triple dots and so on.

Now, my statement is that, we are going to assume that the dynamics is described occurs in free space and the independent dynamical variables for any system are the generalized coordinates and the generalized velocities; and everything else is derived from it. Newton’s equation already tells us this is true because, Newton’s equation says something about the acceleration and that is determined by the force we apply on a system. On the other hand the position and the velocity can be specified by you independent of what force you apply.

Therefore, they are the independent variables, independent variables are always which you are free to specify; everything else is determined by the rule of the game. Therefore, that experience has been taken over, generalized to many many other situations and it seems to be true in most cases that you can think of and in those cases where it is not true, you can answer why it is not true, because you have not included the right number of variables. There are some few exceptional cases, where double dot also as an independent variable, but in most cases it is just q’s q dots.

When we study Hamiltonian dynamics we will see this in a little more deep fashion, why its just q’s and q dots positions and momentum not anything more that is the way it turns out to be. Of course, mathematically you could allow for q dots and q double dots and triple dots and so on. If I had a q double dot for example, L was q q double dot then there would be a term delta of q double dot. Again I would remove the time dependence by integrating by parts I would have to do it twice; and if you did it twice you would end up with a plus sign and plus sign would lead to a plus d 2 over d t 2 delta L over delta q double dot and that would sit in the equations of motion.

You could go up to n derivatives, any number of derivatives and they would alternate in sign. This does not happen whole lot of times, so we are not going to look at that there are some classical problems, where it does happen where there is dependence on the acceleration as well. But as I said before the majority of systems do not have this behavior. These equations these are the equations of motion if you are going to deal with very very important they are called the Euler Lagrange equations.

Next thing is to tell you what is L that is million dollar question, what is L?

Now, as I said we have to write down L, this is the whole point for a given system I have to write down L, I will do this by experience, but we will use this we will use a trick. The cases which, we already know the equations of motion Newton's equations I am going to tailor an L to produce those equations of motion.

Then, I will generalize after that and it turns out that in a whole lot of problems, conservative problems without dissipation, L turns out to be the difference between the kinetic energy and the potential energy. This is not a formula for L; it is just one possible expression for L which yields the correct equations of motion. And there are other cases where it is not true and we are going to do one very such important example.

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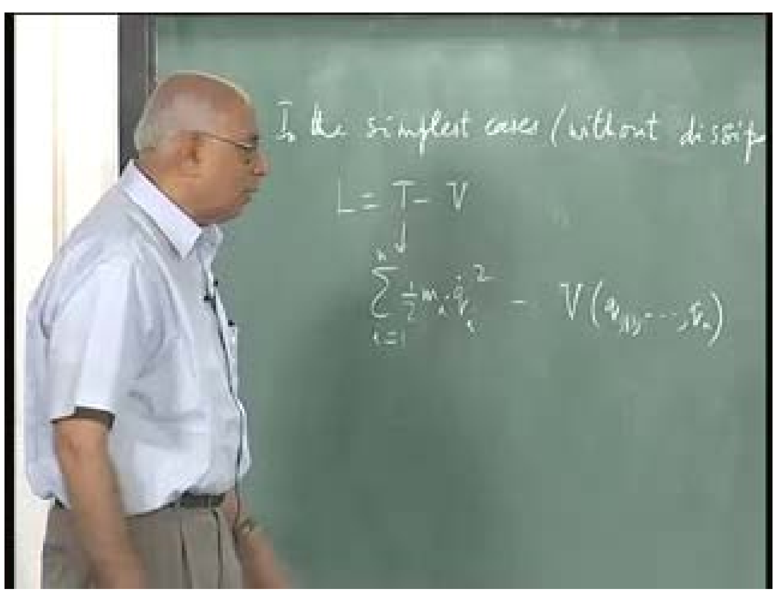

So, let me write down what L is in such cases will check out this is really true or not, so in the simplest cases, without friction without dissipation. L turns out to be T minus V just check this out whether this is true or not. So, we imagine a collection of particles moving in some potential there is mutual interaction between them, there could be an applied external force and so on do not care.

But a certain L is T minus V in this case now what is T normally lets look at a case of a set of particles whose coordinates are specified by q 1 q 2 q n. The kinetic energy is easy to write down, so what would T be in such a case lets take a very simple case where we assume that the q's are actually the Cartesian coordinates of a set of particles moving around. Then, this would be equal to one half m i q i dot square summed over i that would be the kinetic energy, set of particles minus potential in general would depend on q 1.

Good question, L is Euler function, L is a scalar function, it is not a vector.

Pardon

I am saying it, I am asserting L is a scalar function, the kinetic energy is a scalar function right.

It could have had a vector function absolutely, but L is a scalar value. So, I should have defined it L is a scalar value. It has the physical dimensions of energy back to I state. We will generalize it as we go along we will see what is going to happen and we will also see how we are going to write down L in more general cases; that is really the trick when more complicated situations what is the possible what are the generalizations where does this get us this is a crucial question we will answer that.

So, in the simplest of instances if I simply assume all the q's are the Cartesian coordinates of various particles moving around, then this is what the kinetic energy looks like and this is what the potential energy looks like. What is the Euler Lagrange equation give for you it immediately gives you exactly what you want? Because, delta L over delta q dot since this potential does not have any q dot dependence we are assuming potential without any velocity dependence here.

Delta L over delta q dot is minus delta v over delta q i, that is the left hand side of Euler Lagrange equation and what is the right hand equal to I have to differentiate this and these q i's are all independent variables, the q dot dependence is sitting here. So, delta L over delta q i dot delta L over delta q i dot is equal to the half cancels out it is just m i q i dot no summation for each i this is true and what is d over d t of this.

So, it says the mass times the acceleration is equal to the force this is the force. So, it reproduces Newton's equations. It will do a lot more than that we will see, but it reproduces in the simple instance Newton's equations. Therefore, to use this formalism successfully, you are going to have to write down the kinetic energy and the potential energy even in those cases, where L is T minus V and that itself can get quite intricate as we shall see.

But, in the very simplest of instances it does this. Now, let me show you immediately the next advantage of using the Lagrangian formalism. You know there are very many problems in which you have constraints, the coordinates are independent of each other there are constraints and then the standard way which you studied in high school was to use constraint forces like normal reactions.

If I want to study the motion of this object in this table, I better include the normal reaction of the table on this. Now, this is a bugbear for generation of students, because they really do not know what this normal reaction does its only purpose is to keep this object on the table and if you ignore it you are in deep trouble whereas, at the same time you do not really know it is a physical force or not certainly is, but this is a pain.

Keep these constraint forces in or if you put a particle on a bead or a string and move it around its always on the bead, on the hook, the bead is always on the hook and there is a normal reaction, reaction force, this is difficult to keep track of. Let us look at the simple problem where you have a constraint of this kind and I will show you the Lagrangian formalism actually present involve the constraint forces explicitly.

Once you eliminate coordinates it takes constraints into account automatically and this is important. Let me go back and the risk of causing that little sinking feeling in the stomach take you back to JEE and will look at airports machine.

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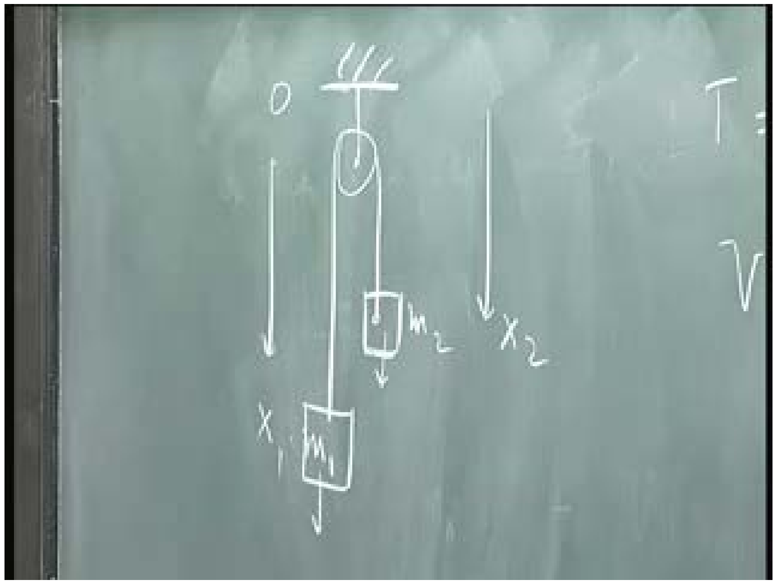

This is something all of you have studied and its just a simple pulley of this kind and there is a mass here and there is another mass here M two and these guys are pulled down by gravity; and this is supposed to be a pulley which is frictionless and so on and you are asked to find what is the acceleration of the mass M 1.

Hold these two guys and let the heavier one will sink and you are asked to find its acceleration, what is the formula?

Pardon me you have to assume a tension in the string clearly its been a while since you answer because three years junior to you would write the answer down instantaneously.

Pardon me G divided by M 1 plus M 2 well the acceleration downwards of the heavier mass is the same as the acceleration of the upward mass, we need to know what it is?

Difference over sum multiplied by G that is a good way of remembering, let us write the Lagrangian down for this. We need to write some coordinates we are all interested in vertical coordinate. So, let us put the origin here and let us say the position of this is at x 1 and the position of this is at x 2.

I measure x downwards and I take the potential energy to be 0 here. So, that this potential energy is minus m g x 1 and this is minus m g x 2.

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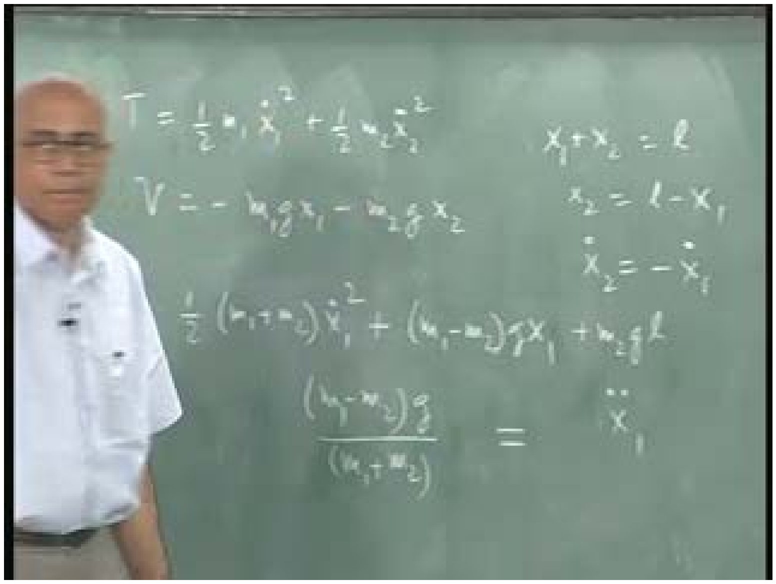

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So, the Lagrangian as you would normally write down T equal to one half m 1 x 1 dot square and the potential energy v is minus m g x 1. So, L is T minus V, but now I have a constraint the constraint says that there is a tension in the string right at every point you assume that there is a T upwards and a T downwards. Now, what is the purpose of this constraint, what does it really tell you.

Pardon me the length of the string is constant just say the string is inextensible that is it, in that sense incidentally I am going to give you as an homework problem what happens if the string is not inextensible, but actually can extend then of course, this constraint is not used in the way is simple fashion.

But let us write down what it is when the length is constant if the length is constant then I know that x 1 plus x 2 equal to some l or x 2 equal to l minus x 1, which immediately says x 2 dot is minus x 1 dot that we know. We know that the speed of 1 is minus the speed of origin. So, we remove the constraint by putting into this, then L becomes 1 half m 1 x 1 dot square plus 1 half m 2 so, we add up minus V.

So, that becomes a plus m 1 g x 1 plus m 2 g x 2, but x 2 is L minus x 2. So, you permit me write this as m 1 minus m 2 g x 1 and then there is a minus plus m 2 g l. This is irrelevant, because it is a constant you can always add a constant to the Lagrangian and since the Euler Lagrange equations do not involve derivatives.

If you add a constant to the Lagrangian it does not get differentiated on either side. So, the first lesson the Lagrangian is not unique. You can add constants to it its not going to change any physics, but this you already know; the potential energy V of q you can arbitrarily shift its reference level and it does not matter. You can do all these tricks non relativistically, but you cannot do relativistically.

Because, when there is something called absolute zero of energy then you cant shift things around, but in this kind of non relativistic physics you are not worried about rest masses and so on. Then the zero of energy can be shifted as we like you can even shift it by infinite amounts in the simple harmonic oscillator you take the minimum of the potential to be 0 at the origin and then the potential is half the x square going up.

But in the hydrogen atom problem or the Kepler problem, you take the potential to be minus 1 over r, and you take it to be 0, when the two particles is infinitely far away from the center of attraction. So, you really move the 0 to point at infinity. May be problem you have to specify before hand, what is your reference level is now we specified r as potential energy being 0 at this point.

But it is irrelevant and that is why it comes out as a constant and now we short order the equations of motion delta L over delta q i there is only one q lets write and it immediately says m 1 minus m 2 g is equal to d over d t of M 1 plus M 2, so that comes out i differentiate L with respect to x dot and I get a x one dot here two cancels and I do d over d t x 1 double dot, therefore the acceleration. So, I did not introduce a constrained force.

On the other hand I use the constraint to get rid of one of this coordinates we will see with further examples, this is one of the real advantages; the real power of the Lagrangian method you do not have to explicitly introduce constrained forces. You can actually use the constraints to eliminate coordinates, when ever you can and then automatically the equations of motion to the remaining independent coordinates emerge; of course x 2 double dot is the negative of this.

Now, the deep question what happens if you put in friction then of course, you need a model for the friction, imagine both these things are suspended in some liquid and then there is some friction which is proportional to the velocity you cannot write down such an easy Lagrangian.

We will see in a minute how to do this in some cases in all in a general case there is no prescription as such really is not really, do not know the dissipation either classically or quantum mechanically, simply because the nature of this is so intricate the nature of this dissipation. The mechanism has to be identified and you have to write a model for it. So, that is a very satisfactory answer at the moment we will look at examples.

That is again a deep question where does friction really come from all forces are conservative, where does friction let me give a quick answer of top of my head, where does this friction really come from ultimately if you have a surface here and another surface and you do this the friction comes due to molecular forces.

Surfaces are rough of course, at the molecular level they really are not uniform surfaces at all and then bonds break, so these are really at the molecular level. So, first we have to use quantum mechanics for that, but ignore that for the moment really is at the molecular level.

But, then this means the energy can be transferred from one system to another at the molecular level, what is molecular random molecular motion called it is called heat. Really it is entirely possible that even though in principle all the forces that you deal with are conservative forces; it may turn out that some of the work goes into heat into random molecular motion which you cannot recover.

The origin of irreversibility is very, very deep, when we do statistical mechanics I will make many more comments about it, where this comes from what is the way we understand it, where does macroscopic irreversibility come. Although the even though the microscopic atomic and molecular level you have reversible forces, this is a deep question one of the real unsolved unanswered questions; in it is full we have lot of answers for the last 150 years.

But, it is not clear that the question is completely settled.

Good the question is, is the Lagrangian formalism is just a tool for solving problems does it explain anything more than Newton's laws? The answer is yes indeed it does much more than Newton's laws, so let me list some of these things and then we will come back try to get this straitened out.

First as a problem solving tool, I have already mentioned that it takes constrained forces into account, that is the first advantage; the second advantage it has is that its possible to generalize it relativistic mechanics, when Newton's equations are not valid. So, the Lagrangian formalism will give you the equations of motion the right equations of motion in the relativistic regime as well.

Then, the Lagrangian formalism can be extended to the case of a continuous number of degrees of freedom. In other words fields which is not possible with Newton's equations of motion. Newton's equation of motion, do not look anything like Maxwell's equations of motion for the electromagnetic field. But, Maxwell's equation of motion for electro magnetic field, or the Euler Lagrange for the electromagnetic field written down from Lagrange.

So, again that unification exists, so its possible to relativistic physics its possible to do fields and its possible to include constraints. These are among the advantages that go well beyond Newtonian framework. Disadvantages not drawbacks in the sense that cannot be applied or anything like that, but disadvantages not easy to quantize when you want to quantize systems.

Then, the Hamiltonian framework which is like another brand of tooth paste, if you like this works we will see the equivalence between these two . One of them can be pointed out right away, what is the order of this differential equation; second order it is d over d t and there are q dots here, second order whereas, we kept saying the dynamics is going to occur in free space and everything is going to be first order differential equation.

So, you should have taught me immediately that these are second order differential equations. We got over it by saying each time, I defined x dot this v n then I put v dot equal to the force divided by the mass, but really these are the Euler Lagrange equations and the second order dynamics

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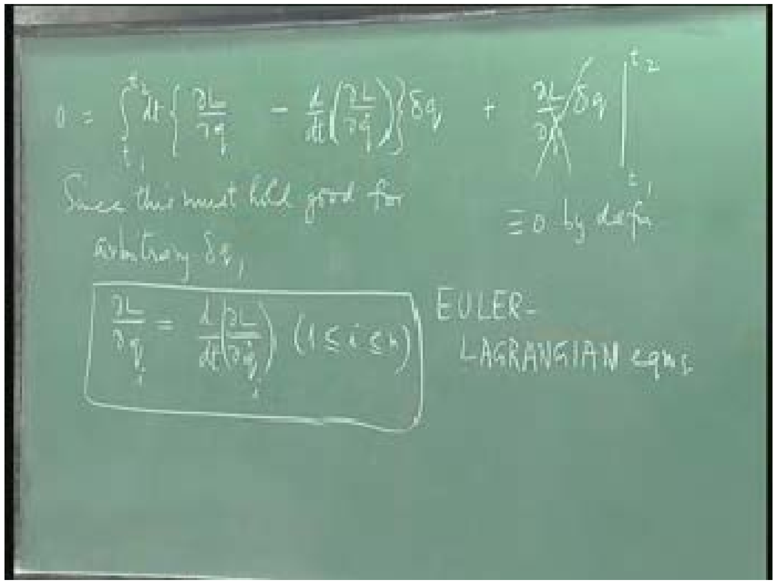

Little more complicated then the first order dynamics, we used to that is why we will shift to the Hamiltonian formalism which is first order dynamics and these are equivalent. So, it has that extra advantage.

To understand where Newton's equations came it was just written of the top of Newton's head right, so you wanted to understand where did it come from it turned out there is a unifying principle, all the principle of least action a little bit of history many people worked on this.

Many, many people worked on this, all the big names are Hamilton, Jacobean, many many people worked on this before they got the correct equations.

I agree, but then I have to tell you i am going to give you an answer which may not be very satisfactory, but that is the only answer there is, this is not an axiomatic subject we haven't reached a stage where something some absolute truth is written at some level and everything is derived axiomatically from that physics does not work like that.

It starts by simple examples and generalizes and once you put whole lot of things together and the same formula applies for all of them instead of individual formulas; you take that as granted and then you ask what are the limits of its applicability and generalize it and further and further and further and so on.

This is the way it goes and that is how we found that all these equations of motion can all be subsumed by one principle, principle of extremal action. So, we believe it is true till somebody comes with a better alternative. So, believe me this has been tested for several 100 years and this is at this level it is completely established that where are you going to write the Lagrangian from.

Especially, when you go to quantum mechanics when you go to quantum fields where are you going to write the Lagrangian down from you do not have the experience of Newton's equations you do not have this. You are not in the non relativistic regime you are not in the classical regime then invariance principles play a big role. It turns out nature is guided to a large extent by symmetry and invariance principles which already manifest here which already are hidden here.

But, I bring them as we go along you would write the Lagrangian down based on considerations of simplicity; simplest possible choices subject to the invariances one of them is he already pointed out the Lagrangian must be a scalar; but, not a scalar in the sense that you and I understand the scalar in this course, whereby scalar means something which is not a vector.

Something does not change under rotations of a coordinate system, but by scalar something that does not change not only under rotations of the coordinate system, but also Lawrence transformations shifts from one initial frame to another. I will use what is called a four-dimensional scale or a Lawrence scale, scalar under special relativism that would guide me in writing down the equations of motion in more complicated sequences.

So, this is what we are going to do start with the simplest and go on to the more complicated problems. The very next problem I am going to look at several of them I want to do, but let me do one of them at least is the problem of the simple pendulum and this is again something we will return to just to give you a inkling of how you should handle coordinates other than Cartesian coordinates.

So, let us look at the example of a simple pendulum.

Pardon me

The least action principle

Yes, yes.

Well that is that is also an interesting question we derive the equations of motion by looking at those variations, where delta q vanished at the two end points; you could ask suppose I do not do that I look at other variations would I get a different set of equations. The answer is no you get exactly the same set of equations, so, I chose the simplest set of principle to derive the equations you can generalize it.

You can look at arbitrary variations and Euler Lagrange equations do not change. So, that is it?

Yeah yeah

I think what he is trying to say is a very very interesting question once again. It looks like time t 1 here time t 1 tomorrow and now integral from t 1 to t 2 is what is being minimized. On the other hand the system does not know what is going to happen tomorrow right now.

So, how does it do it and that is the reason we end up with a differential equation. So, it is really a local object its going from one instance to another. Remember the action principle was true between any t 1 and any t 2. So, you could slice up any time between t 1 and t 2 into intermediate intervals and for any two of them the motion would again be along the action path. So, that is the way the differential equation. So, let us get a few more examples under our belt before we discuss further formalism.

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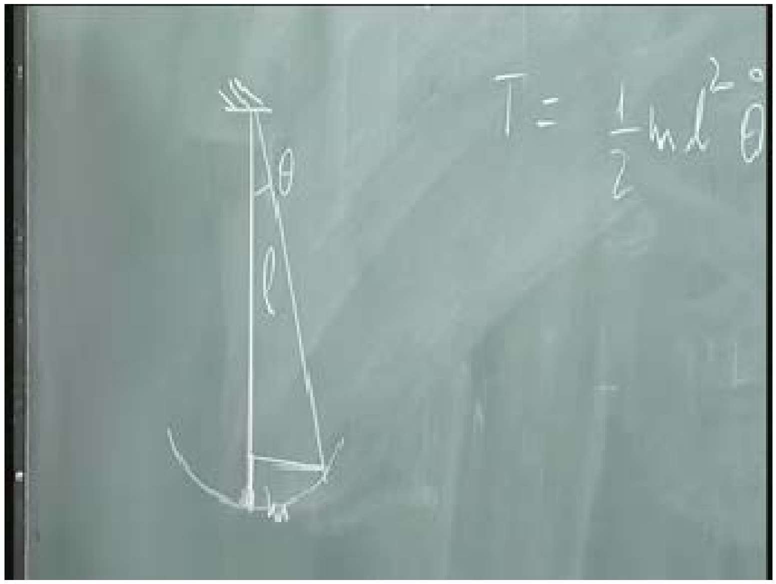

So, let us look at the pendulum what does this do I have in mind a light mass less rod of length l, a bob of mass m and this bob executes oscillations in this fashion. We will assume that this is the given plane and it is frictionless and the oscillations occur with some angular displacement theta and I will take this to be 0 theta is in the vertical position.

So, what is T in this problem this is a light mass less rod. So, the only mass is here the kinetic energy is just the kinetic energy of this bob. And what is that equal to l theta dot is the linear velocity, so half m l theta dot square is the kinetic energy. And what is the potential energy? Again let us assume the potential energy under gravity to be 0, when it is at the lowest position here; and then at a given rate this is the amount by which it is raised. And therefore, it is m g times the height there and v therefore, is m g l into 1 minus cos theta right. So, the Lagrangian is equal to one half m l square theta dot square minus m g l plus cos theta.

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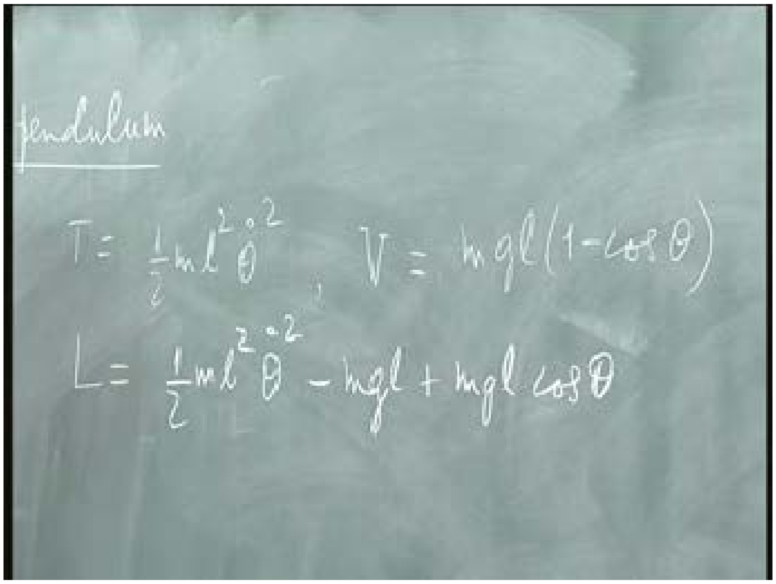

And all the Lagrange equation of motion says delta L over delta theta which appears only here is equal to d over d t of delta l over delta theta dot, which will give you m l square m l square d square theta is equal to the derivative of, which is equal to minus m g l sine theta or theta double dot is equal to minus, that is the equation of motion of a simple pendulum. That is the exact equation of motion are these harmonic oscillations are these harmonic oscillations is it simple harmonic motion?

No, only for sufficiently small theta when you can approximate sine theta by theta then of course, it is the harmonic oscillator problem with a natural frequency with square root of g which is square root of g over l and the time period is 2 pi root l over g. Otherwise it is not true is this a linear problem is this equation of motion linear highly non-linear, the sine theta here is all powers of theta from 1 to infinity.

How small should theta be in order that you make this liner approximation, how small should it be 4 degrees, 6 degrees, 0 of course, then of course, it is exactly true. He says 0 I think that is pretty funny right, so we are going to laugh at him for a while then we find that the laugh is on his side of friends, how small should theta be absolutely I am very happy to hear that this class does not give me the answer that should be 4 degrees some other state its

6 degrees because some guy wrote a text book saying that it should be less than 5 and a half degrees and so on.

This is meaningless it depends on your accuracy you are after all neglecting theta cubed over six compared to theta in a sine expansion and therefore, you can easily find out how small should data be in order to precise you specify an accuracy and I tell you how small theta should be. If you want 1 percent accuracy it has to be much smaller than if you want 10 percent accuracy and indeed if you want complete accuracy you have to do what he does, when we put theta equal to 0 only then these two are exactly equal. But, otherwise it is not a linear problem, but for sufficiently small theta some prescribed degree of accuracy this becomes a linear equation, then you can solve this.

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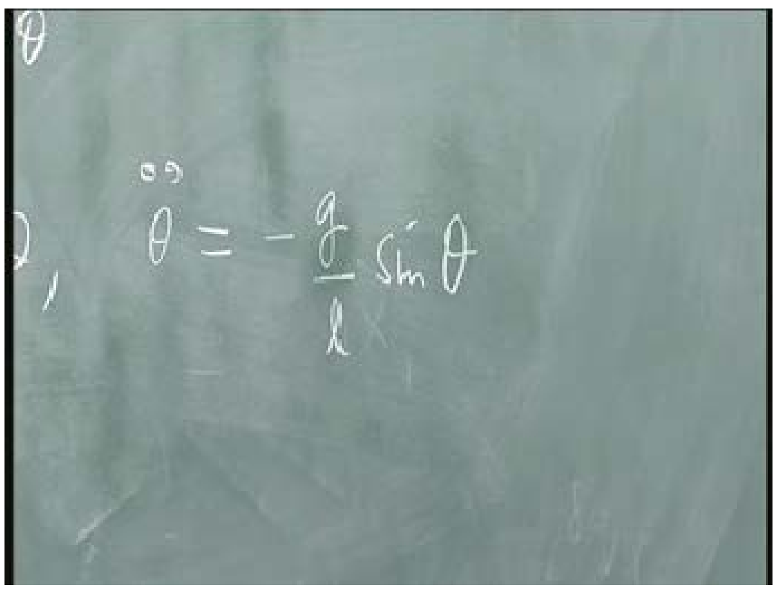

This equation as it is also is solvable in explicit closed form, but the solutions are called elliptic integrals; and the time period increases as the amplitude increases and like the cases of harmonic oscillations. This equation is very profound it occurs in many, many parts of physics not with g over l some constant here it is called the sine god in equation simplest equation.

But, do not look at it right now what is the phase portrait of a simple pendulum going to look like, what is that going to look like. And let us quickly do that because remember this is a rod its not a string it is a rod and I am going to assume that it can actually go around all the way. So, you have oscillations and then you could also have complete rotation,

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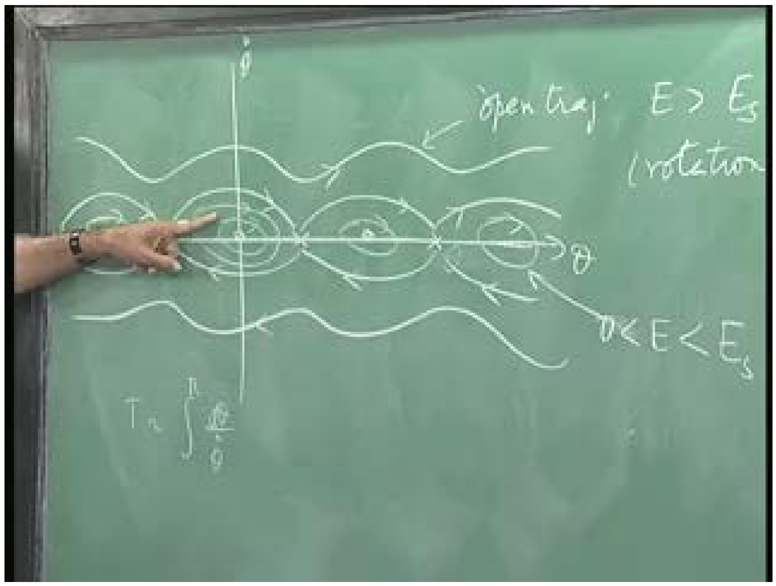

What is the phase portrait look like well at least theta dot plane; here is theta this is theta dot what would it look like. It depends on where the critical points of the system are, where are the critical points of the system? Let us write it as we should really write it in terms of two variables well I would not do that for a moment I will come back and do that in the Hamiltonian framework, but its clear that the speed theta dot angle of velocity is 0 at equilibrium and you are at the minimum or maximum of the potential.

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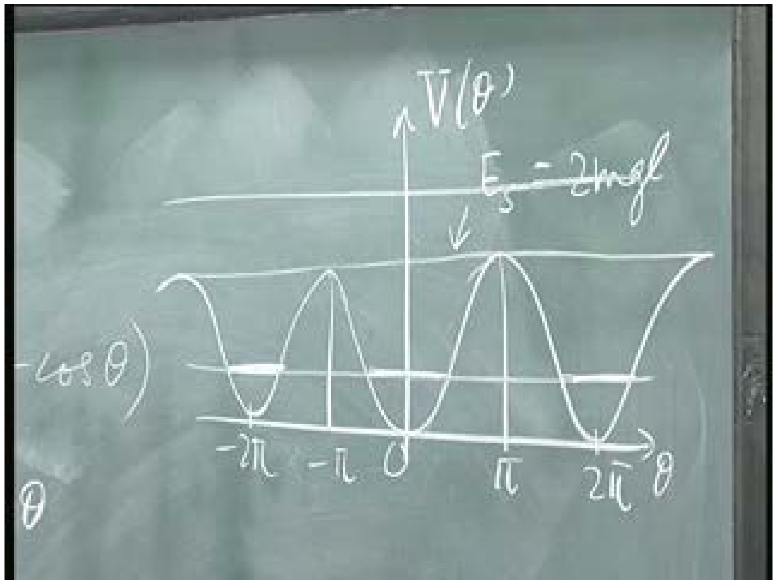

What does the potential look like? By definition it has the minimum at 0 and then it goes up and comes down. So, it is sine potential it looks like this, this is 0, this is pi, this is 2 pi, this is minus pi, this is minus 2 pi and so on. So, it all at all the odd multiples of pi it has maxima unstable equilibrium points and at the multiples even it has minimum. So, what do the critical points look like there at every even multiple what do they have a centre, a stable center.

At every odd multiple what would you have a saddle point. So, this thing would be very, very simple center, saddle, center, saddle and so on. And you would have small oscillations about this point or this point or this point. So, this pendulum could have small oscillations about 0; or it could have been rotated once and then about 2 pi it has small oscillations and so on. They are all equivalent points you could have an oscillation which starts at minus pi and goes all the way to plus pi, but does not quite till it over and then go back.

So, that oscillation would correspond to starting a little bit going all the way up there and then going back and of course, as the restoring force goes to 0, it is going to take longer and longer to crawl up to equilibrium. So, you really eventually would have larger oscillations amplitude oscillations; and then a separatrix and there would be a separatrix from this point to that and one from that to this and so on.

So, two guys going in two guys coming out, so as the energy of the pendulum is increased if this is the energy you could have periodic motion in this well or this well or this well and so on. Those are those closed orbits as you go up to the critical value what is the value here what is the critical value of the energy separatrix.

It is the maximum of this potential 2 m g l and that value you have these separatrix solutions this is again a cycle, but it is not a homo clinic cycle which has two saddle points, connects two saddle points what would you call this it is not homo clinic cycle it is hetero clinic cycle.

But, it is very similar to the kind of picture, we saw for a single maximum and minimum what happens for energy greater than 2 m g l what happens if I give this much energy. Well clear this fellow can escape over the barrier and make rotations; but of course, very time the potential energy is very high the kinetic is low every time the kinetic the potential is low the kinetic is high.

So, you have this kind on this, this side you have this kind what do these open orbits correspond to, what kind of motion? Rotational motion rotation and what do the closed orbits correspond to? Oscillatory motion both are periodic, but one of them is oscillatory and the other is rotational. Open trajectory E is greater than E sub s rotational motion and this corresponds to E less than oscillatory motion.

What happens to the time period as you go towards the separatrix, what would be the time period on the separatrix? Infinity absolutely right infinity, can we show this in a simple way, it is actually infinite we do this in a simple way. You have to write the formula down for the time period and you want to ask how long does it take say from here to here. And the answer is going to be infinity because it is going to crawl from this point up there and I am claiming that this has taken infinite amount of time.

Because, the time period T is going to be proportional up to pi, theta over theta dot, we have to hit the value pi and you have a theta here and a theta dot here and what does this theta dot look like. Well put e equal to this whole thing is equal to is the energy, so one half m l theta dot square plus the potential energy total energy and the total energy is two m g l or a separatrix and put that in and you end up with a d theta over which something blows up when not integrable over sine theta or something like that which blows up at this point, therefore, the answer is infinity as you expect it to be, so we will check that out.

What is very interesting in this problem the pendulum problem for deep reasons is that although you cannot write the actual function theta is the function of t in elementary terms, when the amplitudes when the oscillation is no longer simple harmonic, when you have to the full equation here I said these are elliptic integrals. On the separatrix you can once again, so if you put E equal to 2 m g l you can actually solve for theta as a function of t and it is related to what is called a Colton solution for the sine Gordon equation.

So, I am going to give this as a homework problem even the simple pendulum as you see has very deep physics buried in it this equation here in particular appears over and over again in applications huge number of applications in generalized form. We will try, we will see, we can encounter some of these.