Classical Physics
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Lecture No. # 06

Let me start by answering this question at course last time, namely the statement, the matrix L which has a set of Eigen values, lambda 1, and lambda 2, etcetera. In the two by two cases, we discovered that these Eigen values which control the behavior of linear dynamical system were determined by just two combinations, the trace and determinant of L. And the physical reason was that the Eigen values are independent of similarity, they are invariant under similarity transformations.

And I will explain what this has to do with the way the dynamical system looks like as a function of time, but before I do that, I would like to ask answer this question at course the last time, what would be the corresponding analog for a (N by N) matrix?

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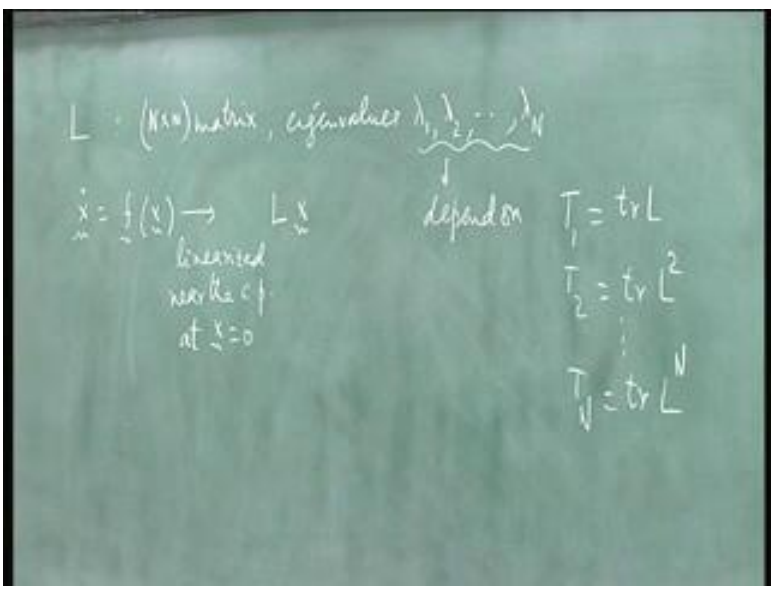

If you have a N by N matrix, and the dynamical system is x dot is f of x, and this is linearised in the vicinity of a critical point, which let me take through the origin for example, this is linearised near the C P at x equal to 0, and this goes to L times x, where L is the Jacobian matrix. And the question is, if L has Eigen values lambda 1, lambda 2, up to lambda N, then the solution to this equation here, is generally each component of x is going to be a linear combination of e to the power lambda i t, where i runs from 1 to N.

And depending on whether the lambda i's have positive real parts or negative real parts, the flow is going to be away from the critical point or towards the critical point, and we can start defining stability, asymptotic stability and instability. So, the question is what does it depend on, what do the Eigen values depend on, what invariant combinations do the Eigen values depend, for the N by N matrix, for the two by two we saw that it depended only on the trace and the determinant, what does it do for (N by N), any guesses, what should it be?

Well, cofactors not quite not quite remember, these Eigen values have to be independent of making a similarity transformation on L, so if you change L to L prime which is S L S inverse, where S is a non singular matrix, the Eigen values should not change.

The trace yes, it will certainly depend on the trace, because the trace does not change under this, nor does the determinant, but what else you need n of these combinations

The trace of L square and then

L Q in the that is true, that is the answer because, these are invariant combinations, because suppose you can pretend for a minute you can diagonaliz this matrix, and once you diagonaliz this the matrix looks like lambda 1, lambda 2, and lambda 3 and so on. And the matrix square looks like lambda 1 square, lambda 2 square and so on, cube looks like lambda 1 cube and lambda 2 cubes, and the sums of all these combinations are independent of making a similarity transformation.

So, the linear combination the invariant objects, these things depend on T 1 equal to trace l, T 2 equal to trace L square, T N equal to trace L N, those are the combinations; I leave it to you as a trivial exercise to show that in the two by two case, you can always write the determinant in terms of trace L square, and trace L the whole square, always write in a d minus b c in that form. So, the determinant these are in fact the invariant combinations; now what is the meaning of saying that it is invariant under similarity transformations, let us look at the two by two case.

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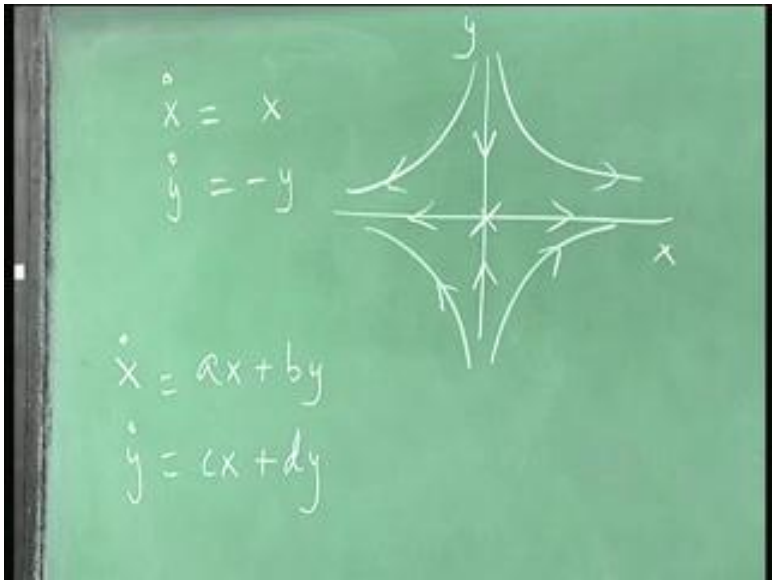

We saw for instance let me take a simple example, we took an example where you had x dot equal to x, y dot equal to minus y for instance, this was saddle point. Because, its already in a diagonal form and the critical point at the origin is a saddle point, because the Eigen values are plus 1and minus 1 to real Eigen values of opposite sign, and you immediately have a saddle point. Now, what does the face trajectories look like, here is y and here is y things are going to flow in the y direction, but flow out in the x direction, and therefore this is what the trajectories look like; this was the saddle point. Now, of course in the more general case, you would not have a case where x and y are decoupled in this fashion, but you really would have something looks like x dot equal to a x plus b y, y dot is c x plus d y, and we assume that the matrix a b c d, if it has one positive Eigen value and one negative Eigen value, you have a saddle point very similar to this picture.

But, the flow is not going to look like this, it is not going to look like this, at all its going to look a little more complicated, and my assertion is that it is just going to be distorted version of this, because you see in principle I can only change variables, I could call u equal to a x plus b y and v equal to c x plus d y and u and v are linear combinations of x and y. And therefore, in the u v co-ordinate system you would have very simple equations of motion, but what does what do u and v look like in the x, y co-ordinate, system two straight lines.

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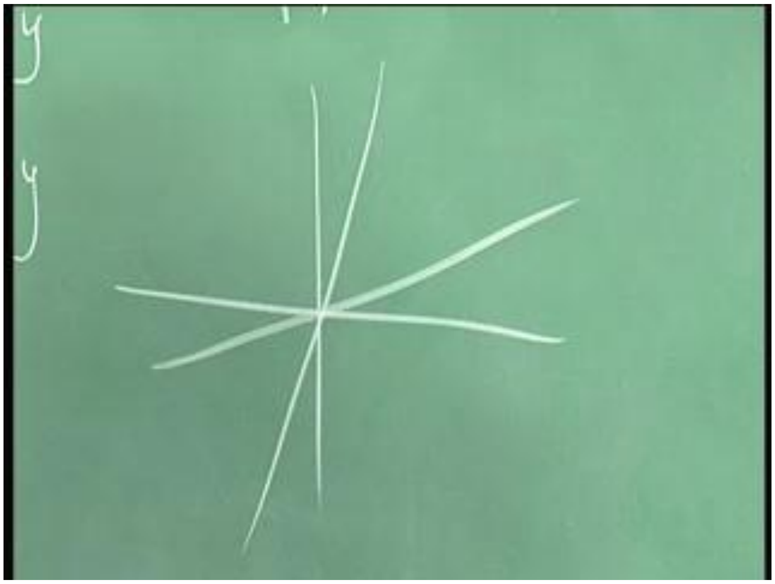

So, in the x, y plane this could be the straight line u equal to 0, and this could be the straight line v equal to 0, so it is just going to oblique coordinates nothing more than that, a linear transformation which keeps the origin unchanged, and then you have oblique coordinates. Now, how do oblique co-ordinates come up from rectilinear coordinates, obviously you should share the axis. And now my statement is any two by two matrix of the form a b c d can always be written as a combination of three kinds of transformations, one of them is rotation the plain rotation, so let us pictorially draw this how I get to these axis from the Cartesian axis.

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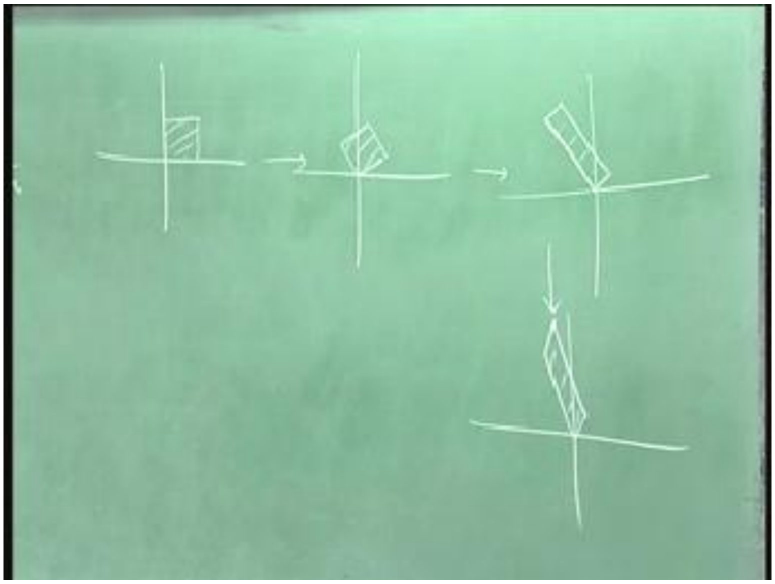

Well to start here, and you take as little square here, and you rotate the square rotation matrix, which is an orthogonal matrix, two by two orthogonal matrix with determinant plus one; then this shape is going to look like, this is what it is going to look like in the rotative coordinate system. And now you could also dilate, you could also magnify one direction, and contract the other directions for , you can always the change the scale of x and y. If you did that this picture would perhaps look like, after you change the scale in the x direction and new co-ordinate directions, it would look like this.

And finally you could shear the whole thing, and then this would look like, this is the most general distortion that you can inflict on a small square here to start of, so if I start with the x square in the x y plane and change co-ordinates to a x plus b y, c x plus b y this square would get distorted in to a shape at this kind. There is no reason why, you should have the same area, because there is no reason why a b c d this matrix should have determinant 1, if it had determinant 1 then the area is preserved, so in general the determinant is some number, which you take to be non-zero and it would be in general form like this.

So, my statement is a general linear transformation in the plane, which leaves the origin unchanged a linear homogeneous transformation, can be though of being made of rotation a dilation and a shear, we must fix parameters, we has make sure we have the right number of parameters.

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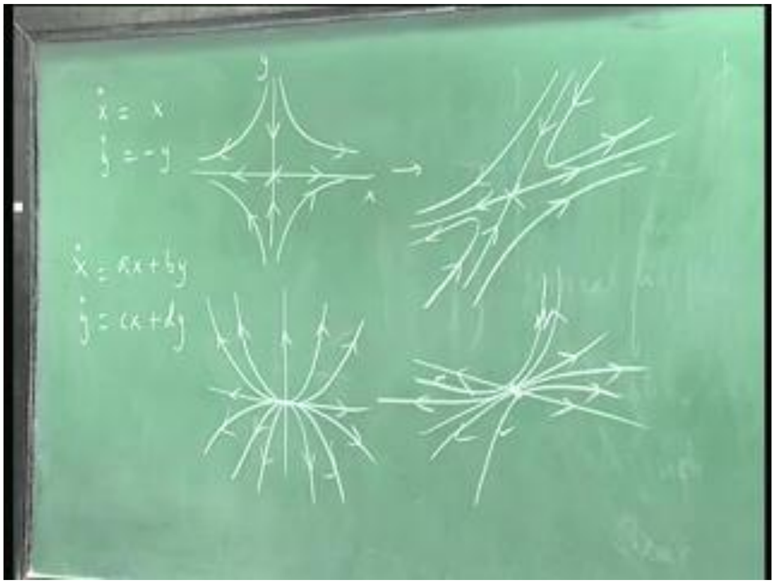

A general matrix looks like this, so I start with x y and I go to a x (a b c d) applied on x y which gives me these two combinations, so how many parameters are here, four parameters four real parameters, and we have assumed (a d minus b c) is not equal to 0 . So, whatever we do, however we write this matrix as a succession of transformations, we must make sure that the number of parameters has not changed, so it should be four; how many parameters do you need to specify a rotation on the plane, one, just an angle some angle.

How many do you need to specify a dilation of this kind to contract one direction expand the other arbitraries, so that the two more parameters, if three have gone there and finally you need one more to tell you to what angle do you shear, you shear one of these directions, so fix this hole and shear and then the question is through what angle. So, there is one more parameter and therefore, the same four parameters, this is not a unique decomposition, you can do this in many, many ways, you can do this in different orders and so on, but the number is fixed.

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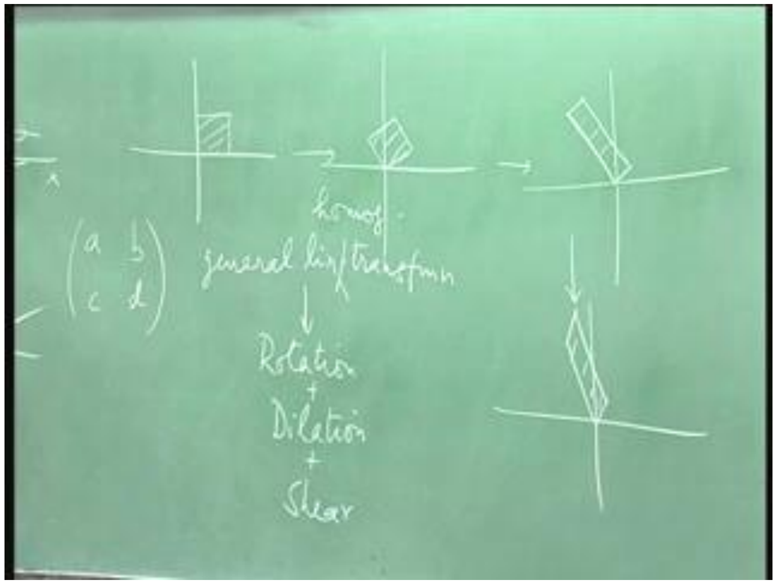

So, let me write this a side general linear transformation, linear homogeneous transformation, because I should be change the origin, I am not adding constants to the whole thing, I am not saying you have the new co-ordinates a x plus b y plus alpha and c x plus d y plus that would mean even the origin is shifted. I am not taking about the transformation of that kind, that is called an affine transformation, but I am talking about something where the point is fixed, the origin is fixed, and then I do the most general thing.

And this transformation can always be written as a combination of rotation plus a dilation plus a shear, you need not dilate both you could say I am going to keep one of them fixed, but then you could shear about both the axis, both directions; so you could put two shear angles, and one magnification or demagnification factor. It does not matter the number of parameters is still four, the fancy way of saying is that the general linear group in two-dimensions it is called g l 2 r, r for real this has four parameters. And this group can always be written in terms of rotations, dilations and shear. But what does it tell us physically, what does it mean, it means that once I solve this problem in which x and y have decoupled, and the general saddle point figure is going to be just a distortion of this figure here.

And that distortion would mean, that the most general thing I have would look like this, this would really in the most general case look like this, this is what a saddle point will look like in general, some distorted figure. And the rest is detailed, what the action shape is and so on is relevant, but those two in some sense topologically equivalent, in the sense that I can deform one to the other, in a smooth way and therefore, once I studied this, I do not have to study each saddle points separately, it is just a distortion of what the original simple case was.

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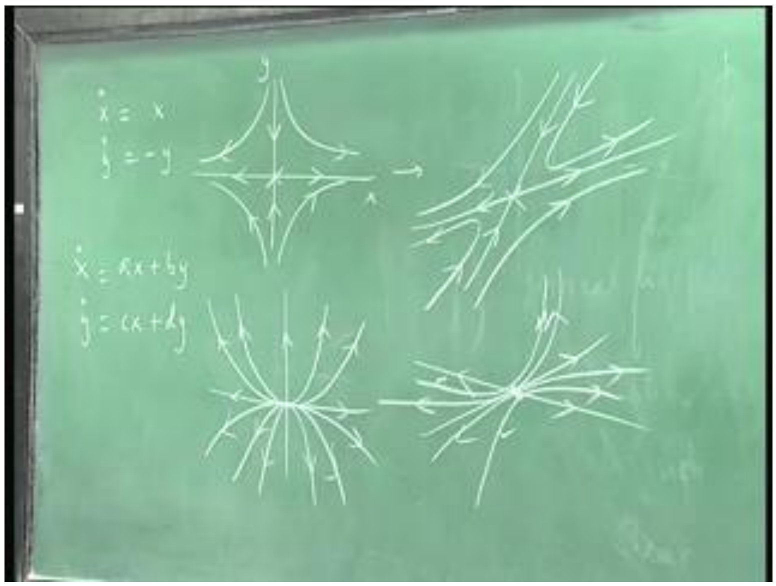

So, that is the reason I said, let us look at the simple cases and then the general case actually automatically handle, what happens if you had for example a center, well in a center you have going to follow this going around, you could distort it, so it would look like a strange kind of distorted oval, but it would look topologically more or less the same. In the original case what would the node look like, what was originally very, very neat and unstable node for instance look like this, and then similarly on this side, this picture here, again imagine taking these axis sharing them etcetera in some crazy fashion.

So, you could have this may be this is the only direction, and everything else flows from there etcetera, must be all meeting at that point just be the distorted distortion of the original shape. So, we wo n0t spend more time on this depending on the problem, you can recognize whether the critical point in the two by two case is a node, stable or a unstable node, stable or a unstable spiral point, a center or a unstable saddle point, these are the only possibilities that you have. Of these the only one that leads to periodic motion, that corresponds to periodic motion is the center, everything else does not have this behavioral at all, but there is something more there is something special about centers and saddle points, which we saw by example.

And that was whenever we looked at potentials, whenever we looked at particle moving in a potential, we ended up as critical with critical points, which were either saddle points corresponding to maxima of the potential unstable equilibrium or centers corresponding to minima of the potential. This is a general characteristic to turn out that, what we are going to call conservative dynamical systems specifically, Hamiltonian systems I will explain what that means, the only critical points possible are centers and saddle points, you cannot get the other varieties, which you get or what you are going to call dissipated systems.

So, now we are going to start making the distinction between conservative and dissipated systems, so let us do that right away in a general context and then come back problem of particle in a potential. I should point out, when I am doing that right away that if you had a three by three case, the kind of critical points that you could have could be very, very different, could be quite its actually enlarged, when you go to four dimensions five dimensions etcetera, four or five variables it can get quite .

Let us look at just three, look at the possibilities

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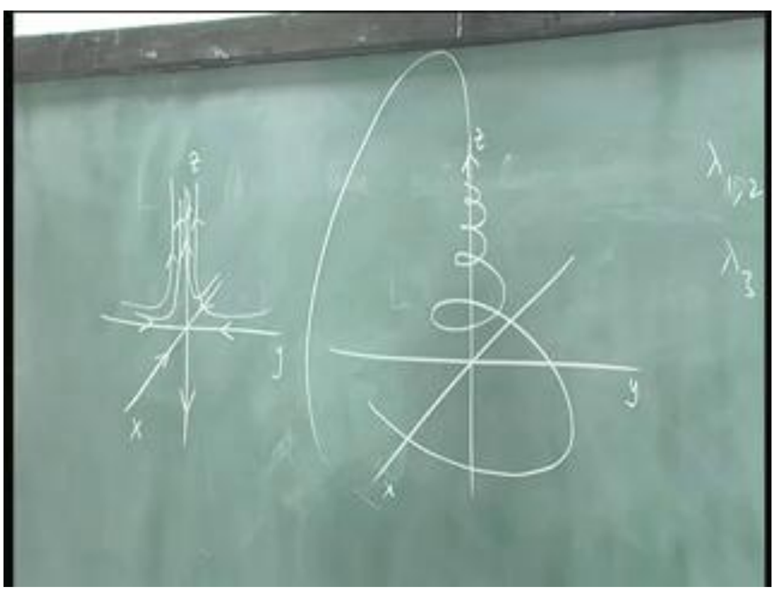

Suppose in three variables, the origin is a critical point then you could have a situation which is unstable as a node, that means the flow is outwards from this point everywhere or a stable node everything falls in. But you could also have this crazy possibilities, if this is the x axis, y axis, that is the z axis, you could have a saddle point in which in which two directions flow in and in one direction it flows out; there will still be a saddle point, so it could have this, and in this direction things flows out.

So, on this plane you would have flow like that, in this fashion and then from the other direction also you would have flow like this etcetera that would still be a saddle point; it could have even crazier possibilities, and this happens very often. You could have a situation in which two of the Eigen values form a complex conjugate pair, so you have lambda 1, 2 equal to some lambda plus or minus i mu and let us say this is less than 0 just to give an illustration. And the third Eigen value lambda 3 could be the positive Eigen value greater than 0, remember in the three by three with real coefficients, you can have only one complex conjugate pair of Eigen values, the third have to be real, you cannot have three complex real values, because they have to occur in complex conjugate pairs.

What happens in a situation like this, well if you did not have this direction the three directions at all, then it is clear that you have a stable spiral point things are flowing into the origin, in the x y plane, but in the z plane they are going off. So, it could have this, you start off like this and it flows in, but in the z direction it flows off; so only they spiral into this critical point, and the z direction takes off becomes unstable.

If the system is non linear then you could have the following picture this goes off a long distance, and then flows out and then gets re-injected into this plane and repeatedly keeps doing this, this mechanism leads to something called homoclinic chaos. So, this is typical mechanism by which you have high dimensional flows, in very complex dynamic behavior, so things of this kind can happen, we will come back some of these.

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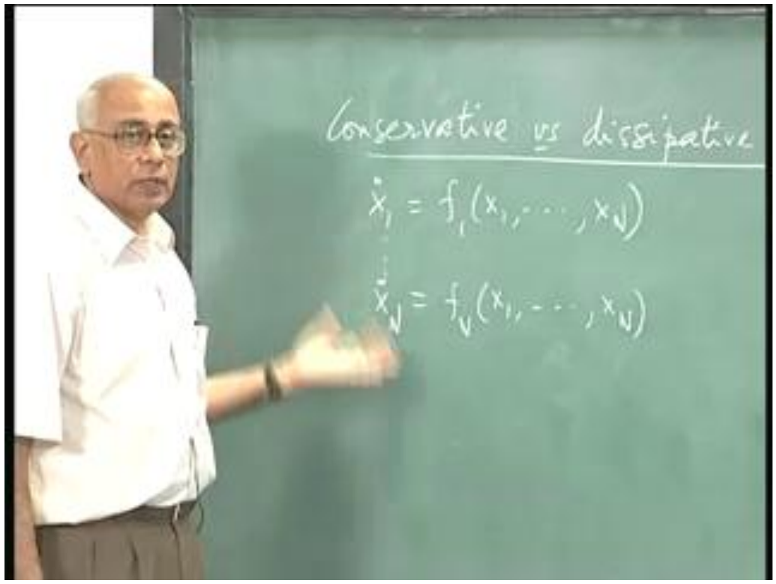

So, let us define now, what we mean by conservative versus dissipated dynamical systems, it is a useful classification not a very significant one in certain respects, but the one that I am going to make now, is going to be applicable directly to Hamiltonian clause, which we would look at. What do I mean by conservative dissipated systems, let us look at our general case, we have in mind a situation where an N dimensional system is given by a set of equations of this kind, this is not necessarily a problem of a particle in a potential or a set of particles or rigid body anything, just a general dynamical system autonomous.

Because, there is no explicit time dependence, the rules here and coupled first order N dimensional, N first order differential equations; and our statement was that if you specify initial conditions, the future is uniquely determined in principle. Phase trajectories in the system do not intersect each other, and if a phase trajectory closes on itself then it is a periodic motion, so this much we know. Now, what we would like to ask is the very crucial point, which I am going to very preoccupied with this, if you start at some point in phase space here, and you are on a phase trajectory of this kind.

And you start of a neighboring point here, perhaps due to initial resolution errors, I am really never going to be able to specify the initial point with absolute position, then what is the future of this point here. Common sense would tell us that in regular systems or nice systems, these two do not go too really go very far apart make a small error in the beginning, you stay with this error gets multiplied by a factor of two, but it stays negative.

On the other hand, most dynamical systems do not behave in this fashion, normally in dynamical systems; do not behave in this fashion at all. What really happens is, if you start here you end up there, and if you start here you may end up there. And the future can be very, very different the error in initial state or the resolution finite resolution it shows that two neighboring trajectories, can actually separate as a function of time, and the question is how fast do they separate, well if they separate exponentially fast in time, then in general there is no way you can compute these things.

Because, if I start with some initial point to some error, some accuracy and I would like to predict the future, I would certainly like to know, whether I am in that trajectory up there or down here, and if the error amplifies exponentially in time, and this is not going to be computable after certain amount of time. If at every time step the error doubles for example, right then after n time steps its 2 to the power n of the original error, which you can write as either the n log 2 times to the original error.

So, exponentially grows with time, then in binary for instance if in the million decimal place, if you made a initial mistake, after million iterations you are in the first decimal place; this is what chaos’s.

And this is why it is a serious problem, and we have to deal with it, it exists its very real and most systems r chaotic in the sense, and we have to deal with it, so we will see what should do about it. But for the moment the reason I brought that was, because we have preoccupied with finding out what happens to not a single initial condition, but a set of initial conditions, I start with a set of initial conditions, some neighboring initial conditions, some volume element in phase space.

And each of these points goes off into a trajectory, and the question is what happens after some time to this little volume element, now this is the question we would like to ask, well let us do that in the following way.

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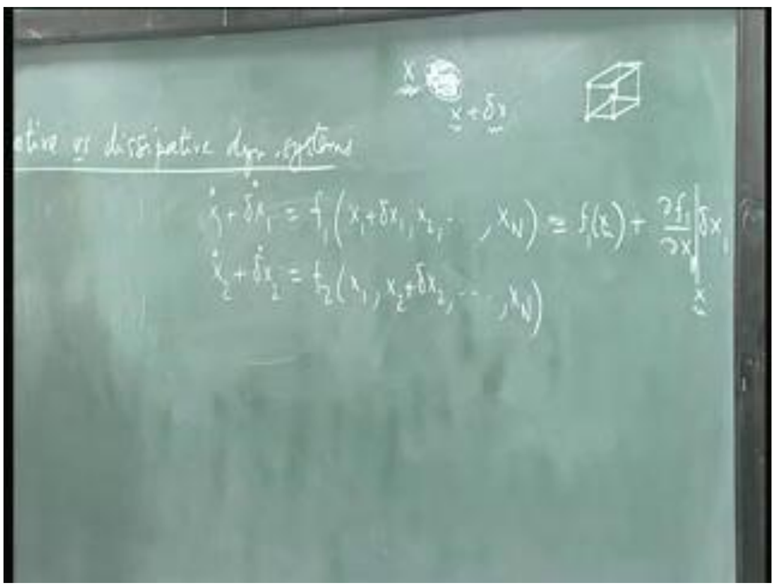

So, this is what x 1 dot is at a given point x, and there could be a neighboring point here, let me call that x plus delta x. And I am interested in knowing what happens, if you have a whole volume element here what happens to this volume element as a function of time, then I could also write x plus delta x 1, the dot of that this is f 1 of x 1 plus delta x 1, x 2, up to x N, so instead of x x 1 I start with x 1 plus delta x 1. Now, I say here, I have x here, I start with x 2 instead of x 2 plus delta x 2, the next variable and so on. So, I imagine that I am here, in three dimensions for example I can do this very easily, for all n cube of that kind and I call this the one direction, that the two direction and inside the three direction.

This is x, x 1, x 2, x 3, this is x 1 plus delta x 1 and x 2, x 3 as they are, this is x 1 x 2 plus delta x 2 and x 3 as it is, and this point here in 5 corresponds to change in x 3 by delta x 3 leading x 1, x 2 and x 3. So, look at each of these and write down the evolution equation, but you see this could also be written the first order in delta x 1, by expanding this function and if I tell you expand it, this is the original f 1 of x plus a partial derivative of this function with respect to x 1 evaluated at the point x multiplied by delta x 1; so this is delta f 1 by delta x 1 evaluated at the point x my original point and so for each of these.

Therefore, I could ask what is d over d t of the product delta x 1 delta x 2 delta x N, that is nothing but, d over d t of delta v the volume element in phase space. That is just the product of all these delta x, and ask what is the rate of change of value that, equal to this and this is equal to well by the chain rule its d over d t delta x 1 right, times all these plus delta x 1 times d over d t delta x 2 and so on.

That we can find by subtracting from this equation, the original equation x 1 dot is f 1 of x 1 x 2 etcetera, and that immediately says this goes away, this function here is cancelled and you get just is, the first order in delta x 1 says this, because the original f 1 at this point is cancelled out and so on, for each of these therefore, we end up with delta f 1 over delta x 1 times delta x 1 multiplied by delta x 2 delta x 3 etcetera.

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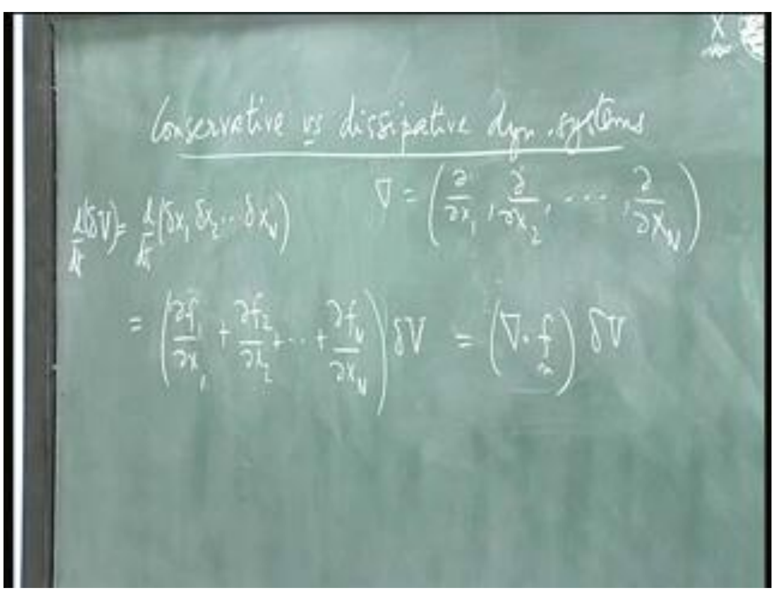

So, each time the corresponding delta x obligingly comes out, you can take that out common this becomes delta f 2 over delta x 2 times delta v itself, I have rewritten delta x 1 through delta x N, as delta x. But what is this, what does this combination remind you all remember, f 1 f 2 are the components of a vector; and x 1, x 2 are the coordinates in phase space. And let us take the first component differentiate with respect to the first coordinate etcetera, and sum all these and what is this called?

It is the divergence of this vector field, so this quantity is nothing but, the divergence del dot f remember, the del operator is just a vector with components delta over delta x 1, delta over delta x 2, up to delta over delta x N, as you would expect this is the geometry of the divergence. So, when is this 0, when can you assert that an arbitrary element in a phase space for a given flow really does not change in volume, that volume remains the same.

Then the divergence of x is identically 0, so if this vector field f has 0 divergence, then you can assert that the volume element does not change, and I would call it as conservative dynamical system.

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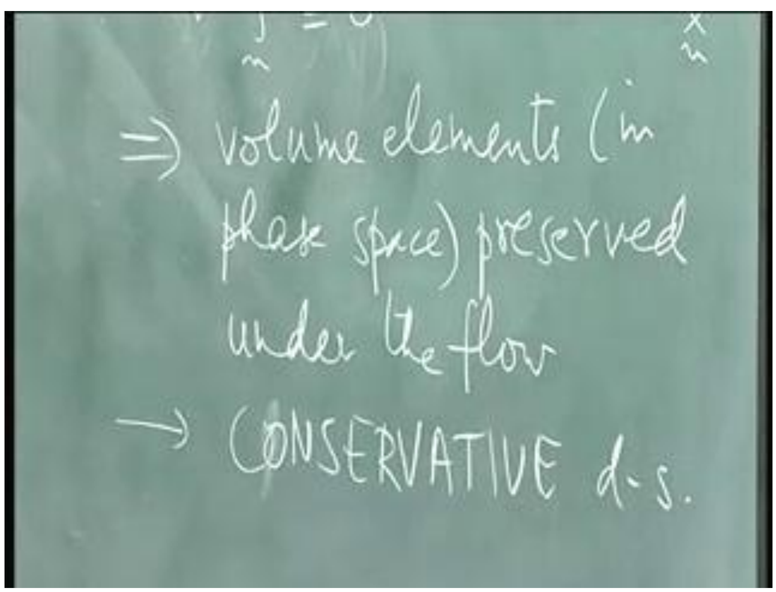

It preserves the volume in phase space, in phase space always preserved; and I would call this conservative that is my definition. Of course, in mechanical context you are more used to say that the energy is conserved, we will see how that is connected to this a little later, it is connected in a little intricate way.

That is just one kind of conservation, mechanical systems could have other conservation laws angular momentum could be conserved, linear momentum could be conserved and so on, the energy is very, very special as we will see. And the conservation of energy in Hamiltonian systems, it is directly linked to the fact that volume elements in phase space do not change, we will see that; but, this is in a general context we have not even talked about mechanical systems.

I would like to define a conservative system as one for which this vector field f has 0 diversions, what happens if you look at the simple harmonic oscillator for instance, would you think that conserved, we have to check this, we have to specifically see whether this is true or not. I would expect that in problems things are conserved, so let us see, if this is true in phase space, I have not defined my phase space too well, but purpose but let us see what happens.

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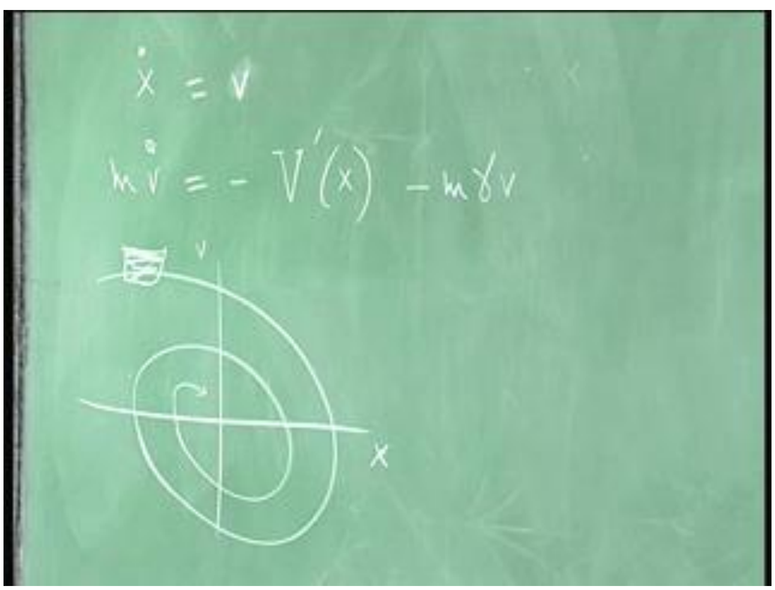

You have x dot equal to v and m v dot equal to f of x, which is minus v prime of x or a potential problem, and the question is is the volume element conserved in this case or not, what would you say, this problem x 1 is just say x. And x 2 is v this is f 1, and you have to differentiate it with respect to x, does it have x dependence no, there is no x dependence its v, it is a independent dynamical variable, does this have any v dependence; so each of these terms is 0, this is 0, that is 0.

Of course its conservative this is trivially so, its trivially conservative, I put a little bit of damping what happens then, remember that damped harmonic oscillator was written in this form, what happens now, to this divergence of this term, the second term minus m gamma v has v dependent, and you have to take its partial derivative with respect to v. What happens then, minus very important, minus m gamma minus, you pluck that in here and what does it say it says d over d t of any volume is negative a negative number constant times delta v.

What does that mean, delta v.

It shrinks to a point, and indeed this is true for dissipated systems things will shrink because, remember that the phase trajectory is now would be spirals, for the under look like this and things would fall in.

So, if I started with a big volume element here, and ask what the set of points does, they all have to fall into this point here finally and therefore, the volume element shrinks as a function of time. The fact that del dot f is negative, if it takes care of that says globally everything shrinks everywhere of course in such cases, where there are situations in some parts of things could expand and other parts they could contract and so on, that would not be a conservative system by our definition.

If del dot f is identically 0, then I call it a conservative system and also see that if del dot f is indeed negative everywhere then things are really shrinking, this must be a very dissipated system, in shrinking of some point. Let us look at an example, very far from remote mechanical examples, just for a illustration see what happens here, this is a very, very simple model, but it will tell us why we should expand our definition a little bit. I should mention that del dot f not equal to zero I call a dissipated system, and in general I am going to be interested in del dot f negative meaning contraction.

Because these things expand if the phase space is finite they cannot go beyond the boundary in any case unless the phase space is infinite in which case things could expand forever, but that is not very interesting to me. We are going to always worry about cases where you have dissipation things would shrink, but I must warn you right away that in general the complicated system would have some regions of phase space where things could shrink other regions could expand, and yet other regions could move in a very complex way.

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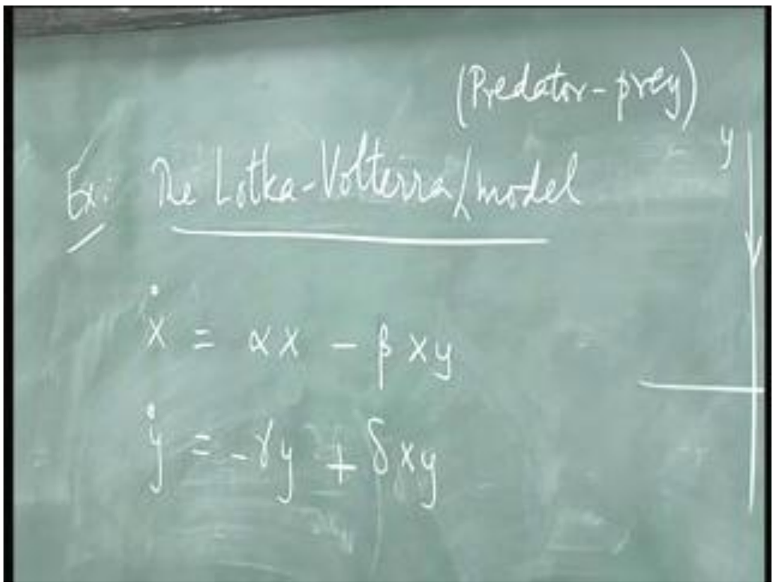

Let us look at another problem, another example this is called the Lotka-Volterra, it is an extremely simple model of population dynamics; and it has to do with two species of animals, which are supposed to have sufficiently large numbers that you can right on differential equations for that population, as supposed to be differential equations.

Because, the population is always an integer and non negative integer, but I am going to pretend that you have large numbers therefore, you could write down differential equations, for these problems. Let us call them typical case foxes and rabbits, so you have rabbits running around and as you know rabbits have a proclivity to breed, and let us call the population of rabbits at any instance time x, and x dot is than going to go, because there is plenty of grass and they eat grass there is plenty of grass and they would grow.

And they would grow at a rate, at a certain birth rate alpha depending on the current population x and this is exponential growth, so the solution is going to be e to the alpha t times x 0, then any positive x of 0 is going to be an explosive tone. On the other hand, suppose there are foxes and let us call them y, these foxes do not eat grass, they eat rabbits but, if there are no rabbits around then of course, they would just die and the death rate would be some gamma y.

So, we can draw phase diagram in this problem we really need only the first quadrant because, populations cannot be negative but, mathematically you can look at both positive and negative x and y, and it says it would just come down here, and these fellows would just grow off. And these already suggested that this is an unstable equilibrium point, because infinitely near it things flow out.

Now, we put a coupling between these two, and this say

Well I am saying that foxes, if you do not have any food would die out

The total number of the rate at which the population decreases would depend, if there is a death rate 2 per 100 and the population is 2000, then the death rate is going to be the number of deaths is going to be rate multiplied by the current population right, so this is the rate at which the total population is decreased. Therefore, it is proportional to the current population multiplied by some constant death rate, if there is no interaction at all, so this is what the phase diagram looks like.

But, now let us suppose now that the foxes are allowed to eat the rabbits, then clearly the rabbits are going to suffer a loss so minus sign, and it is going to depend on how many foxes there are, but the rate at which it is going to happen is going to depend on how many foxes there; the more foxes the faster the rate of depletion. Therefore, this is some beta times x y, the rate of depletion of foxes of rabbits here is going to be beta y, is going to depend on y in this fraction similarly, these are going to increase now, because they are able to eat.

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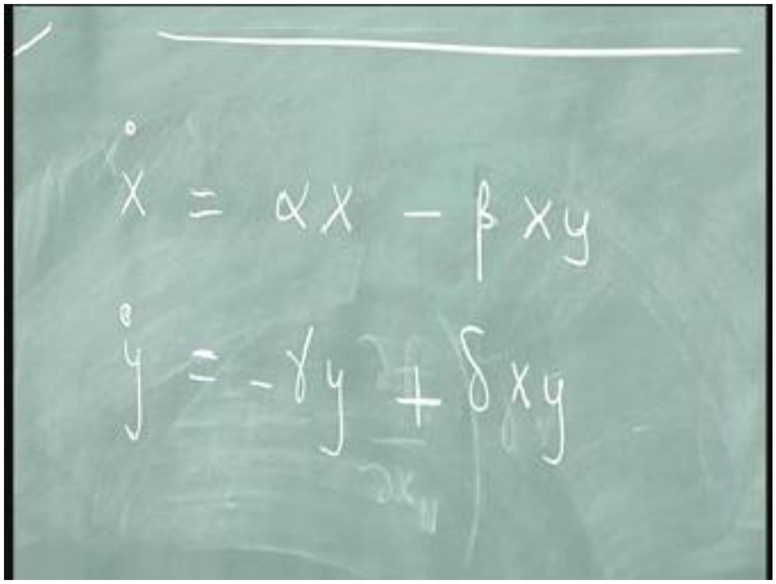

And this is going to be some delta x y, the rate at which they grow is delta x, so these are two species it is called the predator prey model, and now you can see a coupling has been introduced between them, and the question is where are the equilibrium points defining, 0 is certainly a equilibrium point. And now by our rule, if you linearise near 0 it means you throw away the non linear terms, keep only these terms, then since all these rates alpha, beta, gamma, delta, are positive by definition. It is clear you have one positive Eigen value, one negative Eigen value, therefore this must be a saddle point here, this is a saddle point.

The question is, is there any other critical point, so what is it like, what does it do, let us make life simpler, I am going to lead you to analyze the more complex problem, and put all these constants equal to 1, just for illustration put all these constants equal to 1; then you could write this as x times 1 minus y and you could write this as y times x minus 1. So, are there any other critical points, are there any other values of x and y where the whole thing vanishes, 1, 1 1, 1 is a critical point, so you have another critical point here. Now, would like to know what sort of critical point is it, what you should do is to actually linearise about that point.

So, you should really say I am going to look to first order, I must find this matrix l, so the way Ii do it is by saying,

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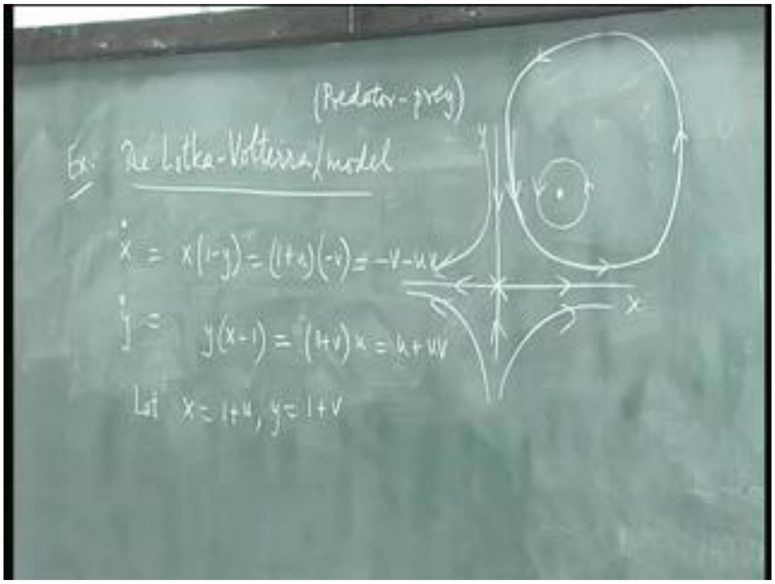

let us set let u equal to let x equal to 1 plus u and y equal 1 plus v, so I shift my origin to this point here, and then ask what do the equations look like in this case. What does it look like, that is right this is x minus x y, and this becomes 1 plus u leave it in the original form, x minus 1 minus y is equal to 1 plus u multiplied by 1 minus y which is minus v.

Therefore, this is equal to x is equal to plus u and 1 minus u is minus v, so u minus v and this is equal to this is minus y, so minus 1 minus v, y is 1 plus v multiplied by x minus one which is u sorry, u plus u v, and this is u minus v minus u v, it has to be non linear you cannot just make a linear change and then get a non linear set into a linear 1, so there are these higher order terms. Now, what does u equal to 0, v equal to 0, u v axis what does the flow look like near it, near this point in this immediate vicinity you have u dot, because x dot is the same as u dot its shifted by a constant, is minus v and v dot equal to plus u linearizing.

Now, the matrix L, so this says L equal to (0 minus 1 1 0), and what are the Eigen values of this matrix plus or minus i plus or minus i, that is a pure imaginary set of Eigen values so what kind of critical point is it, it is a center, it is a center. So, what do their trajectories look like in its vicinity, they look like little circles, because you can integrate the set of equations trivial to integrate, because you write d v over d u is u over v, and gives you u square plus v square is a constant or something.

They are circles in which direction do the circles go, would they clock wise or counter clock wise, they have to have continuity with this in between, it is true that you found this in the vicinity of that point, but there is nothing else in between, so it is clear that they must have continuity with this. Therefore, they have to go in this fashion, there is no option you cannot suddenly change directions of course, as you go further away from the vicinity of the critical point at 1, 1 the orbits no longer are circles, they would get distorted, but they cannot cross each other.

So, what its telling you is that no matter, where you start at this point I start with the very large population of foxes, and a very small population of rabbits, the foxes start dying there is no enough food, so they come down they die, but as more foxes die, the rabbits become they have get a impunity, they start growing and therefore, the rabbits expand for a while. But then they get so far away that the foxes start eating them once again, and goes up in this fashion then of course, eat too much and therefore, the rabbits are depleted and they come back.

So, the full phase portrait this is very irregular set of curves, but they are periodic is not trivial to show at this is true, but I leave it as an exercise, and we will see a little later, you can show this very elegantly, that this is indeed true, this is what the phase trajectories look like, with module of some distortions.

So, there is a center here, and there is a saddle point here, this oscillation was seen in the catch of two species of fish in the 1920s by fisherman in the early arctic sea, and then the mathematicians Pythagoras solved this problem; they realized this is what is going on they solved this sort of this equations, and showed that this is the phenomenon. That really you have the oscillatory behavior, the population of predator and the prey they are out of phase by a fixed amount here, you can see that one is large the other is small and so on they are out of phase and the reason for it is precisely buried in this equations. Now, is this is a conservative system by the way mathematically you can solve the set of equations for the remaining values of x and y also, nothing very interesting happens, this guy here just looks like this, just looks like a saddle point here and there interesting point is this 1, 1 that is a point of co existence. That is the point at which the critical values of foxes and rabbit populations, at which they can actually co exist.

They have the right number of being eaten, and the right number of foxes dying and things can exist.

Even though there is a conflict between them there is peaceful coexistence possible, we have not included many, many facts, this is the simplest possible model we have not included competition among the foxes for the same food, we have not included competition among the rabbits for the same food, for the same grass and so on.

So, if you include competition among the rabbits there must be a minus constant times x square at the very least, and similarly here too there would be competition, there would be minus something times y square, so that is predator prey interaction together with competition and so on. Then things get much worse, there is no instantaneous response, once the population of rabbits increases it takes a while for the fox population to catch up, so really the fox population growth rate would depend on the rabbit population at a slightly earlier time; and foxes have to eat them, they have to mature, they have to mature to the breeding stage and so on. So, the problem is actually much more complex much much more complex, but already it tells you in this simple instance that things can get quite interesting, is this a conservative system well, we need to find out, we need to take this.

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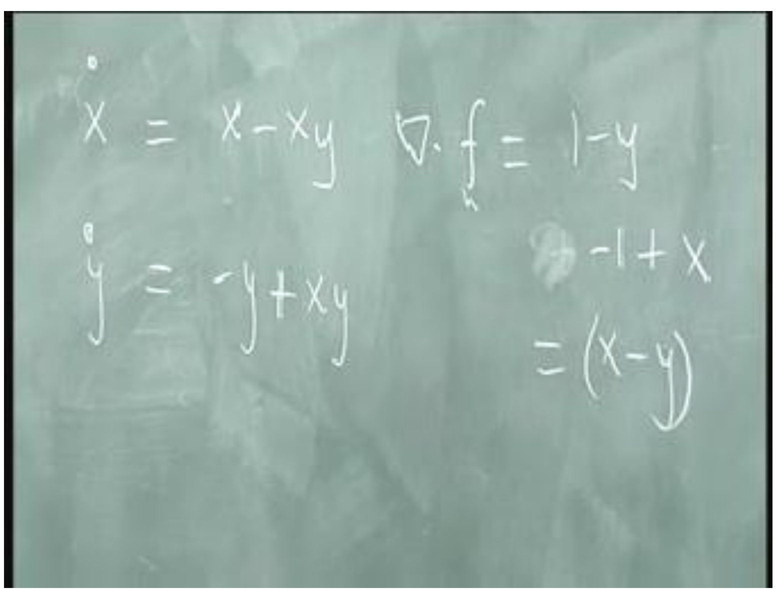

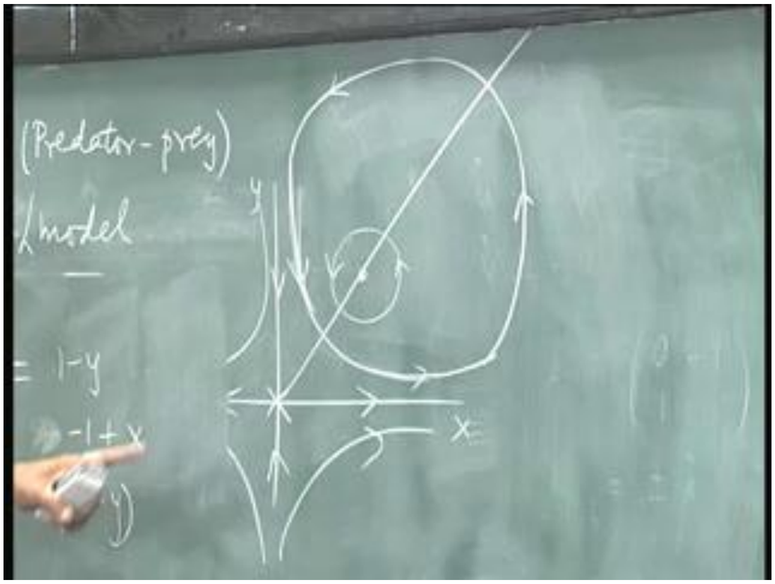

So, x minus x y and we have minus y plus x y and what is del dot f equal to, it is the derivative of this with respect to x, that is 1 minus y plus the derivative of this with respect to y, that is 1 minus 1 plus x, that is equal to x minus y. So, is it conservative, not in the strict sense of the word it is not, but if x is equal to y may be on this line, it is certainly true, that the divergence vanishes, but that is not enough.

Because the trajectory intersects that line only twice, but then on the average it is conservative, because for every volume element in which x is bigger than y, it grows del dot f is positive here, there is a corresponding volume element there, where its negative. And therefore, over a full cycle if you start with this little volume element and followed it as along here, on the average when you integrate it out, it would be 0. So, we have a slightly more general definition of a conservative system, if its periodic motion in this case is then you should really look at the volume element, as an average over the full cycle, then if it is 0 you will say it is conservative, not in the rigorous sense of the word but, certainly for all practical purposes, you could regard it as conservative. Well later I will show that this system by changing variables actually is a Hamiltonian system, and the fact that you have a center and a saddle, really tell you that this is writable as simple as Hamiltonian system, not a conventional one, but its writable in that form.

You could now ask what happens if I have three species, and I generalize this model then the matter is not so simple at all, also some possibilities are open up, yes.

No I do not think so, no not to start with, it is just that this problem with these constants is so simple, it is so symmetric it has this wonderful symmetry about x, about the 45 degree line, that is the reason for the special feature of this, problem here. So, it is not very significant that that sense, but its significant in the following sense, if a system is a Hamiltonian system which I have not defined, then by definition conservative.

And you might wonder why this system which is closely related to Hamiltonian system turns out to be non conservative, and the answer is it is due to the changes of the variables.

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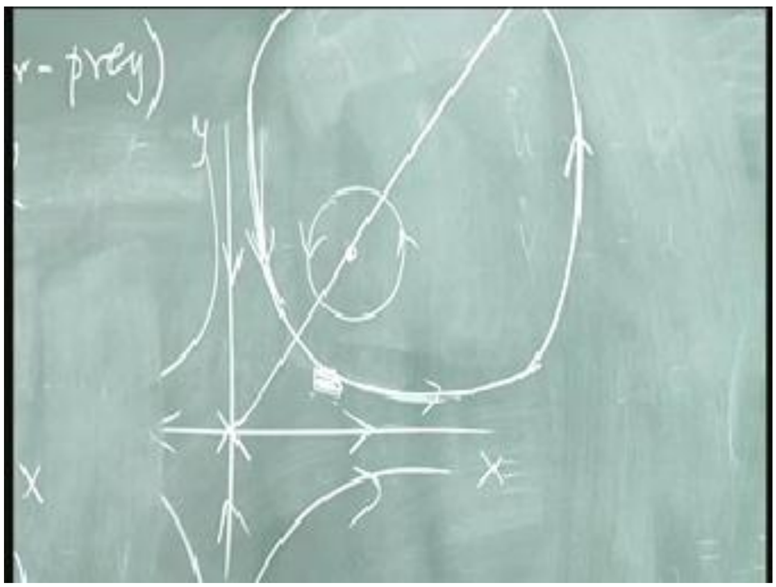

So, effectively what ever you start with every point here, every point in the phase plane in the positive quadrant, lies on a periodic orbit, and across a full cycle the volume element does not change; there may be some reasons, where one of them is winning the fox is the winning, the another region rabbits are winning, but overall nothing much happens. So, it is in that sense, I specified this, so I pointed out this because, this distinction of conservative and dissipative systems is a matter of convenience matter of convenience sometimes it has deep physical significance, and we will see why that is so.

I should point out that the fact that del dot f is equal to 0, tells you that the flow in free space is that like of a incompressible fluid, and this is worth appreciating, because a fluid flows according to the Navier-Stokes equations. Now, if you write it down in the simplest case, if you write down the equation of continuity for this fluid, so let us do that in half a second.

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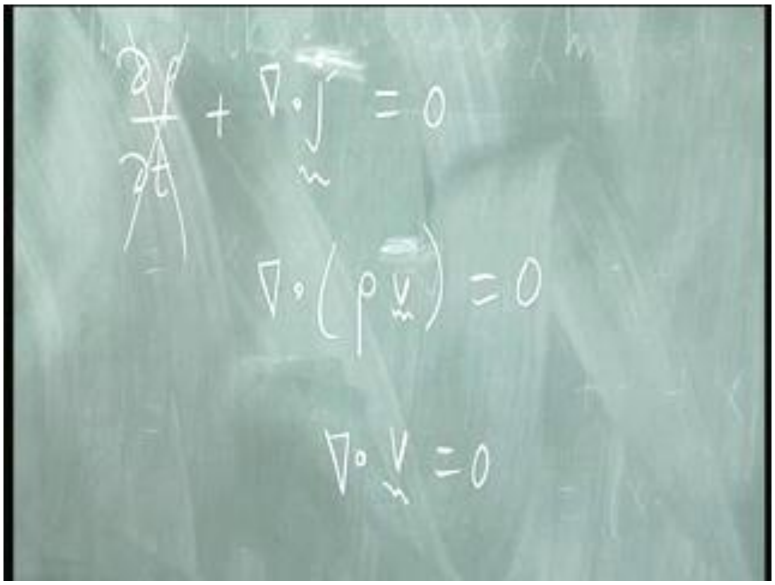

In real space a fluid flows with no sources and shrinks in this fashion, the divergence of the current is 0, this is the equation of continuity for a fluid, it says in any volume element, whatever comes in must go out or else the density changes, this whole thing is just a balance equation, nothing more than that. Now, suppose you have an incompressible fluid, rho does not change its constant, in space and time then this term is 0 identically, and del dot j is 0, but what is j which is del dot its rho times the velocity v equal to 0. But, rho is a constant, so this implies that del dot v equal to 0 right and our f was just the velocity of points in free space; the phase space velocity for the flow in phase space.

So, this is the reason why one says, that a conservative dynamical system the flow in phase space is similar to that of an incompressible fluid in real space, for which del dot v must be 0, that is the analogy. So, very often you will see this written in text books that the flow looks like, this state what is called Leo wells theorem in Hamiltonian dynamics, and say this looks like an incompressible fluid or the reason is this. But, remember that the flow that we are talking of about is in phase space, it is not a physical flow of a fluid, but this has its uses, and we will see what they are.