Classical Physics
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Lecture No. # 05

Let us resume where we left off, remember that our general N-dimensional autonomous dynamical system or specified by set up first order equations, the vector X stands for a set up of N variables, and f is a vector field in the same dimensional phase space.

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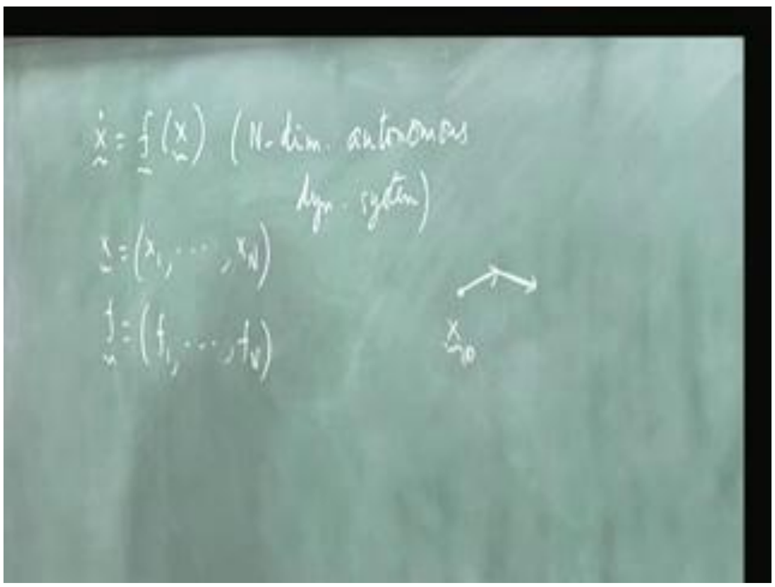

And at each point, the significance of f is that it tells you the velocity of the point in phase space; the dot of the X dot, X is the position of a point in phase space, and X dot is it is velocity change with time. So, this vector field f specifies locally at each point, in what direction the X is going to change, it is going to increase, so in some part of phase space this point X, if this is the direction of f it means in the next instant of time it moves along here. And at this point, if this is the direction of f in next instance of time it moves along there and so on, and you continuously join these little segments, and you can get the phase trajectory.

So, this vector filed f has the significance of being the phase space velocity, velocity of a point in phase space, of the representative point in phase space. And our task is to solve this equation, I pointed out that in general this is not an easy task, because f is a very complicated non linear function, which combines all these variables and intricately mixes them up. But if you look at any typical point local solvability is always possible, and the reason is if considered some arbitrary point X naught to start with, then in the immediate vicinity of X naught, I can do Taylor expansion of this function about the point X naught.

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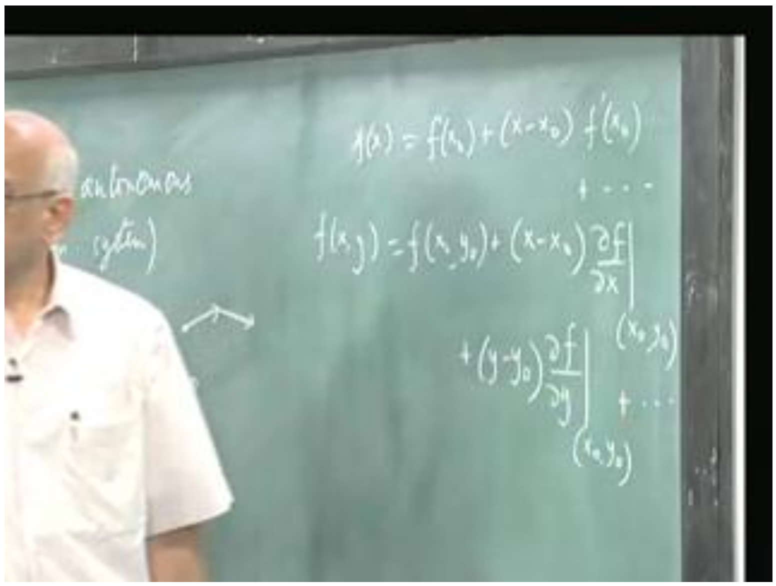

And what would that look like well, if it were single function if you simply had f of x then of course, I write this is f of x naught plus x minus x naught f prime at x naught plus etcetera, this is what I normally do, if I had as a simple point single variable x. If I have a set up variables x 1, x 2, etcetera; then again I do a Taylor expansion, but this is a joint expansion in all the variables.

And for instant if I had two variables, f of x, y I would write this f of x naught, y naught plus there is a x minus x naught multiplied by the partial derivative delta f over delta x at x naught y naught plus a term which is prepositional to y minus y naught delta f over delta y at x naught y naught plus higher order terms. The higher order terms would typically have x minus x naught whole squared y minus y naught the whole squared, and a cross term x minus x naught times y minus y naught, and then the partial derivative, the cross partial derivative and so on. So, all you need is to generalize the concept here to N variables, but in the leading approximation very close to x naught, it is clear you could throw away all terms except the leading term.

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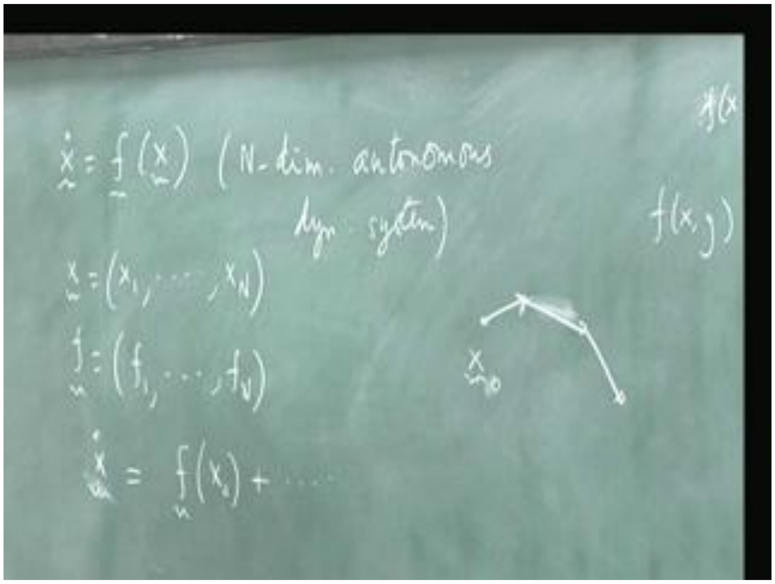

And if you did that then off course, you realize that this equation becomes x dot equal to f at the point x naught plus terms proportional to x minus x naught itself. And therefore, if you are sufficiently close to x naught it suffices to keep this; if you are arbitrary close, very close to x naught; then of course, this is a constant this matrix here, there is no depends on any of the variables, because it is set up a particular point constant matrix.

And your back to the old situation, where you have x dot equal to constant on this side, as to we have to solve, because the solution is simply x equal to the constant times T plus at the higher order terms and so on, so this very, very trivial to solve. And what is it imply it is a mathematical way of saying, if I am here then the leading term all have to do is to take this matrix, multiply take this quantity here, this vector multiplied by T and that tells me what x is at T time delta T, so here at this point.

And once you are here, you repeat the procedure and you here and then you here and so on, so local solvability is not an issue, no matter how to complicate this system is locally you can always solve this. Here is a rigorous mathematical way of saying it, and under goes by the name of the Rectification theorem; what it says is, if you have such a function, then locally at every point you can find a change of variables such that, the equation is trivial to solve, that is exactly what you done here. It is the mathematical theorem, I might as well write this down it is called Rectification theorem.

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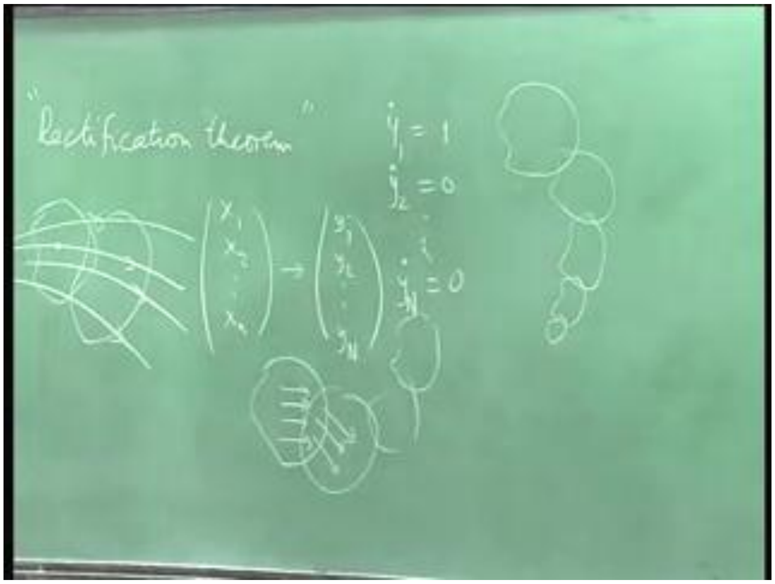

And in pictures it say the following, if this vector field f, if the phase trajectory look like this for the system, so in initial point here travels along this trajectory point, here travels along that and so on. Then some neighborhood some place here, some arbitrarily small neighborhood of my initial point x naught, the vector field points in this direction, and you go little bit further way, points some other direction and so on.

And what the rectification theorem says is, in a sufficiently small neighborhood of any given point, normal point you can find the change of the variables from the set of a variables x 1, x 2 up to x n, to a new set of variables y 1, y 2, y n, you can change variables where the y is a function of the axes. In such a way, that this same neighborhood in y space looks like this, that is all these vector fields have been rectified in other words, in the y variables this set of equations x 1 dot is f of f 1 of x 1 f x 2 up to f x n. would start look like y 1 dot equal to 1, y 2 dot equal to 0 etcetera, up to y N dot equal to 0.

And the solution of this is simply that y 2, y 3 etcetera remain constant, and y 1 alone increases linearly with T, and if this is the one direction it says only the point 1, only the first coordinate increases, nothing else changes; for this is rigorously establishable, and that's essentially what we you done here by saying to leading approximation this is constant matrix. But of course you could ask what happens next, well point is if you go to this neighborhood, then that is same change of variables would not work anymore, you need a new change of variables.

So, from y have to go z or something like that, that neighborhood here look like this, typically the vector field would start looking like this, and this is how the entire thing gets curved as you move along. And the difficult with this program to implement it for all time is to fold, typically what happens is these changes of variables would applying certain neighbor hoods; and it could so happen, that is keep going you first start here this neighborhood, when you go to this point, then you go to this, then you go to this and so on. It is possible that this change of variables applies in a smaller and smaller neighborhood, in therefore finally pitters out to a point, and the program does not work anymore, this is one possibility.

The other possibility is that you go along, and make these changes of variables and various neighborhoods, so both these thing occur, both these phenomena actually occur quiet often, and that is the reason why, local solvability is not the same as global integrability. If I could solve this set of equations explicitly, and write everything down as a function of time then of course, I can make T as large as I like, plug it in and get a formula for the x, and that is global integrability.

But this is local solvability, locally I solve, but it cannot be extend to global integrability, and this is the problem we will face this problem as we come along, there is really no in general of course, systems are not integrable, that is the way it is and we have look for alternative ways of handling this whole situation. But, what is interesting is that, in general this program would go through at every point with one exception, when would this program not happen, when would this not work, when would it not be possible to say, this guy here is just a constant vector C, this thing here is some C.

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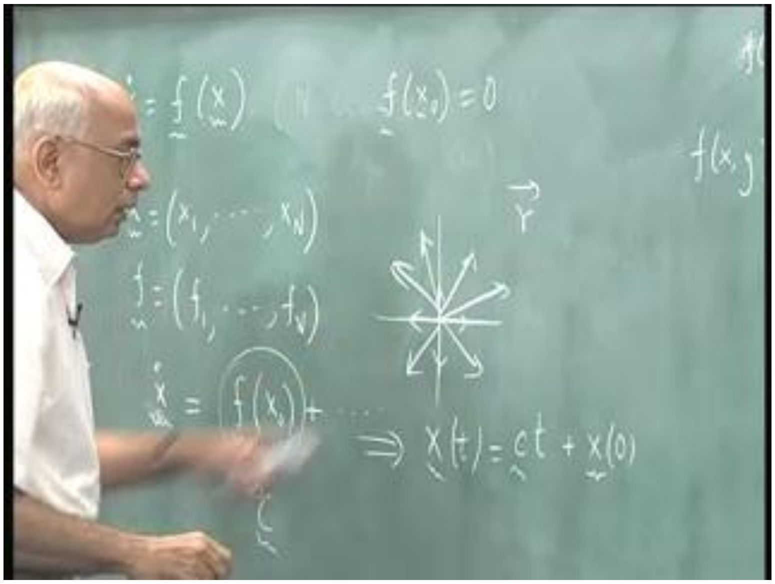

And this would imply immediately that x of t is equal to c t plus x of 0, when would this not work?

If f is 0, if f of x naught is 0, then this program is not true any more, so if the leading term in the Taylor expansion is 0, then you have to get the next order term, and what is the meaning of saying that the leading term a 0, what is the meaning of saying that f at some point x naught equal to 0, what is this mean, it is a vector. And therefore, it means every component is 0, and it means if you are that point, the phase space point does not move at all; therefore, it is equilibrium point or a critical point, more generally a critical point.

So, at a critical point when this vanishes, when this vector field vanishes, then of course you have singularity of this vector filed, incidentally unlike scalar functions which would say singular, when they become infinite or something like that. For a vector field you have singularity, even if the field vanished at some point, because when you do not know in which direction it points completely.

So, even the ordinary vector, the vector r, the position vector r, at the origin is singular; where it is not defined at that point, at the origin this field is not defined its direction is not defined. And if you draw the field lines of this vector of course, to realize immediately y because, the field lines look like this, at any point things look like this, in this radial fashion and of course, of the origin its indeterminate. So, vector field become singular even when they vanish, and that exactly what happens at a critical point.

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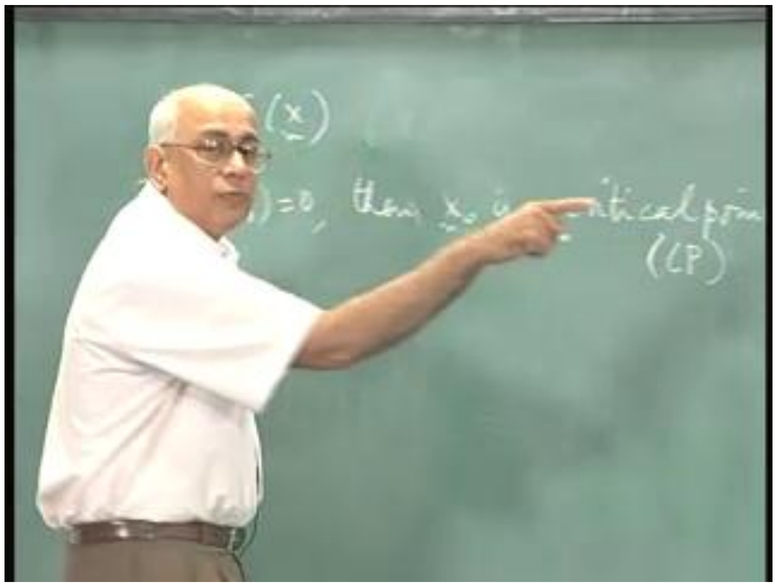

What would you do then, and let me therefore define this now, let me write this, and I abbreviate it with CP. Once you are at a critical point, and in our mechanical examples, critical points correspondence to equilibrium points, in a more general context, it is just the singularity of some kind with various interpretations. Once you are at a critical point, when this program of linear, is this program of rectification does not work anymore, and you have to go to the next term.

And what would that be, so let us suppose x naught is a critical point, and might a critical point and without loss of generality, let me take the origin to be at a critical point, because I could always shift the origin, in x without too much trouble in these problems, so let me assume that I focus on one particular critical point, and let me call it for a movement the origin. So, f of 0 is 0, what should I then write, when I do a Taylor expansion, what should I write.

What should I write, how do I do a Taylor expansion in variables.

I have x minus x naught delta f over delta x plus y minus y naught delta f by delta y plus etcetera, etcetera, this is the first order term. So, how should I write this in general, what should I write it general as…

Well no no, I would like have us compact formula for the N dimensional case.

I should something with something, so what should I what should I .

x dot gradient, but f is a vector f is a vector right, so what should I do, you are right, so what should I do?

Are you more comfortable with index notation, with index notation I mean which ever you like, we have x 1, x 2, x 3, so let me call it x of i.

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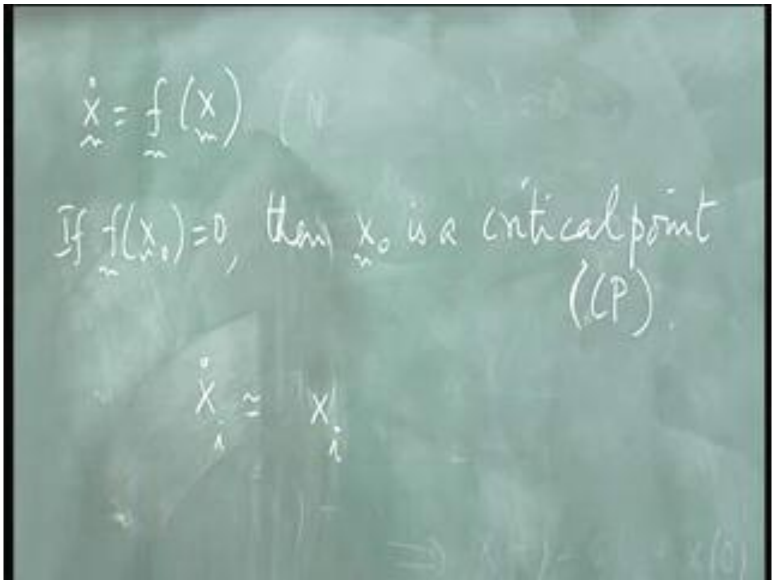

And then off course you have x i x i dot x i dot x i dot equal to x i multiplied by…

Delta, which f…

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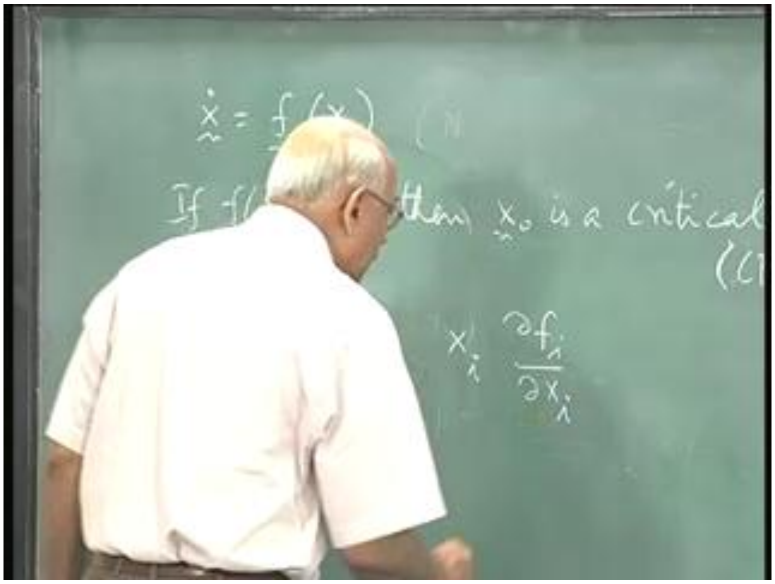

This you have to be little careful about, I mean there are too many repeated indices, the index notation says, that if you have an index i on the left hand side, it should appear on the right hand side, as well if you have index j on the left, j should appear on the right as well. Every free index which appears on the left must appear every time on the right as well, and every repeated index is summed over, so the notation is very simple and self correcting. Free indices will appear only ones on either side of the equation, a repeated index will appear twice and it is a dummy index it summed over; if it appears thrice you made a mistake, as simple as that. So, this does not look very good to me, so what should I do?

Well, look at the following you have x 1 dot is approximately equal to x.

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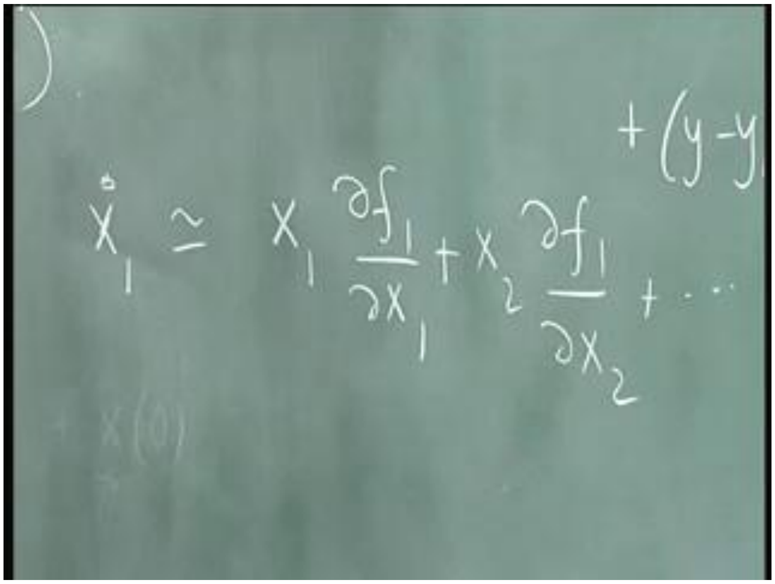

We put everything at the origin, so it is equal to x 1 delta f 1 over delta x 1 plus x 2 delta f 1 over delta x 2 plus etcetera is it not, the equation for x 1 dot involves only f 1 on the right hand side. I am Taylor expanding that function about the origin therefore, you have first order terms in x 1, x 2, x 3 etcetera, so this gives you a hint what should I write here?

Write this as x j, x j this is fine.

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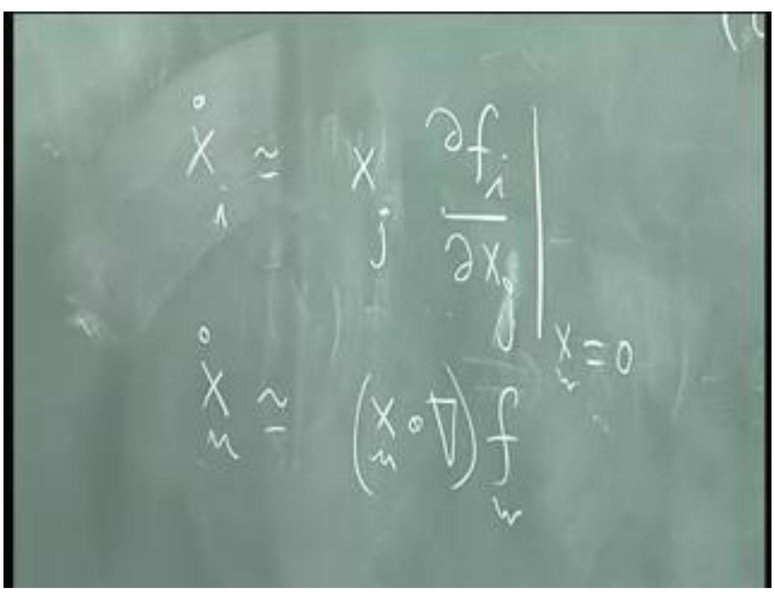

So, you have a free index f on the left, i on the left and 1 on the right, so its fine each has vector, and then index j summed over; in vector notation how would I write this, x dot approximately equal to what?

x dot del, x j 1.

x dot del.

x dot del yes.

Operating on f, you agree this yes, this is a scalar operator acts on a vector produces a vector, and this is a vector on the left hand side, so is this fine good, so you write it on the bottom of notation you like write the equations out if you like, but its fine either way, plus higher order terms, which require more indecision to write down and so on. But, now let us make an assumption that the critical point where involved with is a simple critical point, by that I mean that this matrix, this set of number evaluated at the origin, by the way all these things are to be evaluated at x equal to 0, because you want everything at the critical point.

So, this matrix this is a matrix, because it got two indices now, that Jacobean matrix you should assume is not singular; that a singular, then you have higher order critical point and things becomes more complicated. So, the initial assumption is that this Jacobean matrix is not very singular, what then do the actual equations looks like at this point.

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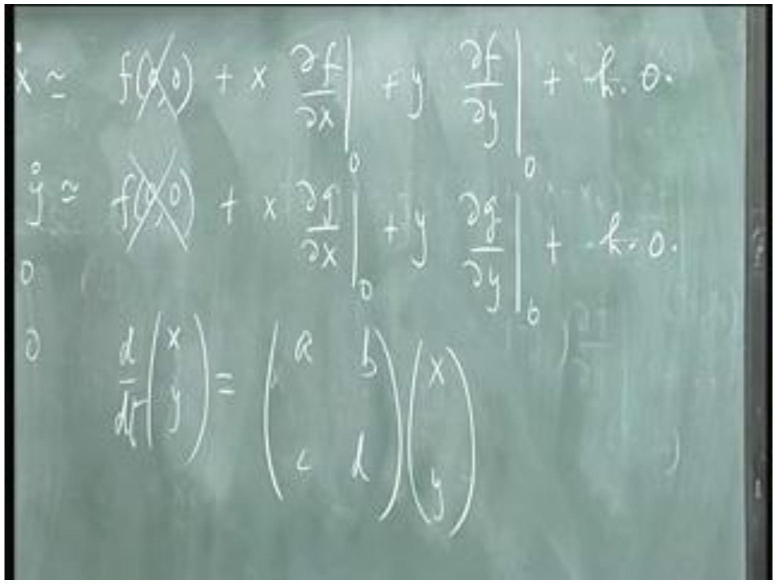

Let us write it to the two by two cases, the two-dimensional case, then you see the generalization to the N dimensional cases obvious. So, in the two-dimensional case, let me just write x dot equal to f of x, y and y dot is equal to g of x, y, just so that we have two different functions, instead of f 1 and f 2 I call them f and g, and set of x 1 and x 2 I just call it as x and y, and look at what happens. Then this is N equal to 2 critical point given by f of x, y equal to 0 g of x, y equal to 0; that is, where the vector field vanishes.

Now, typically f of x, y is equal to 0 is a curve in the x, y plane and so is g equal to 0 and generically two curves in the plane would intersect at isolated point; and those points are your critical points. Then take any one of those critical points, let us call it the origin, this critical point is origin and we would like to see what the system does near the origin, and the statement is I write near the origin x dot equal to f of 0, 0.

But that 0 by definition, because it is a critical point plus x times delta f over delta x at the origin, let me just denoted by 0 plus y times delta x over delta y at 0 plus higher order terms proportional to x square x, y and y squared and then the cubic terms and so on. And similarly y dot is f of 0, 0 that 0 by definition plus x delta g over delta x as origin plus y delta g over delta y origin plus higher order terms plus higher order terms.

So, what have be succeeded in making a system look like, it is evident now immediately that in the close vicinity of the origin, this system is looks like x y d over d t equal to a b c d x y, where a b c d are these four partial derivatives evaluated at the origin. And their numbers, and the assumption this is not a singular matrix; what has happened, then what sort of system has this become a linear system, it is become a linear system. So, vicinity immediate vicinity of a simple critical point, this system has been linearised.

And now, we know how to solve linear equations, because we know that the solution with as exponential an exponential in time, and this is our trick, this is going to be our basic trick; so this linear matrix L this is nothing but, the Jacobean f, g over delta x, y evaluated at the origin, this is a short hand notation for this set of partial derivatives, all evaluated at that point. And you put them in a matrix and then the equation looks like, d over d t f preposition of the represented point in phase space; it is position in phase space is a matrix, constant matrix multiplied by the vector itself, so the system has been linearised.

It will become intrinsically non linear exactly, if its singular well, we will come to this if these partial derivatives vanish for example two cases, one is all the partial derivatives vanish, in which case it is become a non linear system. The other possibility is does not vanish, but the determinant of this matrix is 0, then of course you have situation which is singular and we will see what happens, when it is singular. We will see that it is become degenerate in certain case in a certain sense; we will look at that case in some detail.

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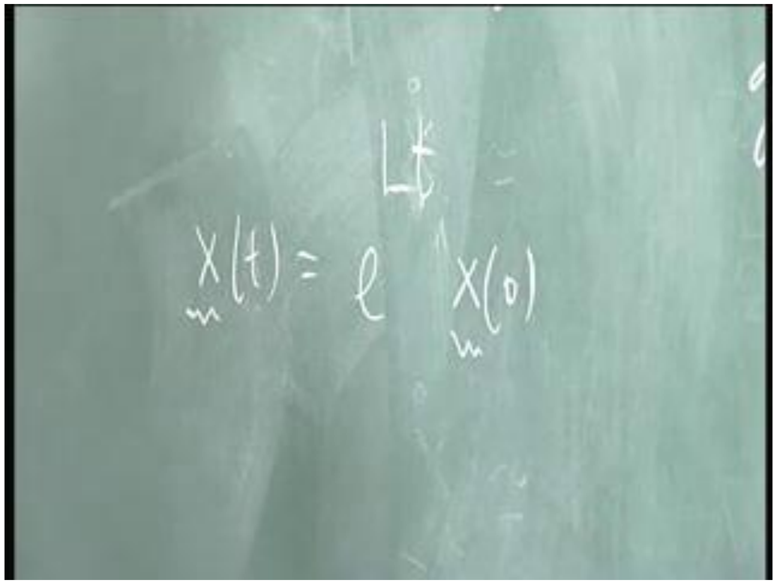

But generically typically you are all expert that behavior you expert these function f and g, would have first derivatives typically at every point and therefore, if the slope is define slopes are defined at that point, you have matrix L. And you have to deal with this linear problem, which is of course well know to us, and what is that solution look like, we know that x when equal to e to the L t times x at 0, this is x at time t, sufficiently close to this point. Therefore, immediate vicinity of this critical point, the trajectory would behave its time dependents will be given by this exponential here; not at the critical point at the critical point, if you start with an initial condition at the critical point you remain there, because that is an equilibrium point.

But, in its neighborhood if this therefore, is your critical point in its neighborhood, wherever you start where you are in move out, and move in will depend on into the point L t, and an what property is this L would this depend, on the Eigen values; if the Eigen values have positive real parts, you would have behavior with grows with time.

If they have negative real parts things were fall in to this, and if they had 0 real parts you expect that, what would you expect?

Pure, you would expect oscillatory behavior, you would expect things would go around, this was the lesson we learnt from the harmonic oscillator; and this is exactly what we want to codifying, so everything is going to be depend on the set of Eigen values of this matrix here. Now, there are two ways of doing this one them is to just write down the general case, and the other thing is to do the two-dimensional case in a little more detail, that prepare to do in two d case in more details, because you can see geometrically what is going to happen.

So, for some time from now, let us continue focus on the two-dimensional case, and just I will continuous with this notation instead of x 1 to x 1 and x 2, I just use x and y let us continue this notation. Now what happens next, everything would depend on e to the power of L t, and I told you that the simple ways of finding out what e to the power L t is, by the way you do not need to again I emphasize, you do not need to be able to diagonaliz L, all you need its immediate Eigen or its Eigen values, that value easy to find.

And do you know how to exponentiation this matrix, how to explicitly find e to the power L t, for an arbitrary two by two matrix this happens to be possible, not so for an arbitrary three by three or four by four or n by n matrix. But in the case of two-dimensional matrix, just two-dimensional matrix arbitrary two by two matrices, this turns out to be felt easy and simple to do, and let me spend a few minutes and do this, I will look at some examples, and then you get some practice.

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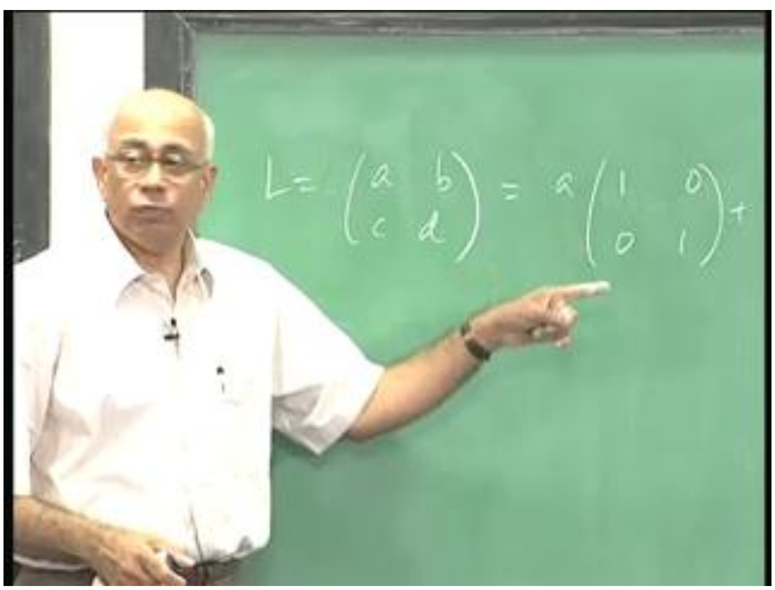

You see the difficulty is that if you took a matrix L a b c d and you want to find a L square L cube and so on, this can be extremely tedious and the reason it becomes tedious is because, this is short hand for a times 1 0 0 1 plus b times 0 1 0 0 plus c times 0 0 1 0 plus d times 0 0 0 1, this thing is hear is short time for that, and these are the four basis matrices the natural basis by two by two matrices. In the language of linear algebra that is the natural basis, and this is what you mean by matrix however, the problem arises sorry this should be a 0 here.

The problem arises, because these matrices there commutation properties are not a trivial, so if you take two of these matrices and ask what is the product and what is the product in reverse order, then not necessarily equal to each other, and you can see that this gets fairly complicated as you go along. So, what one would like to do is to change the basis to a better basis, where the mutual commutation properties between the basis vectors, basis matrices is little simpler, and these are the famous Pauli matrices.

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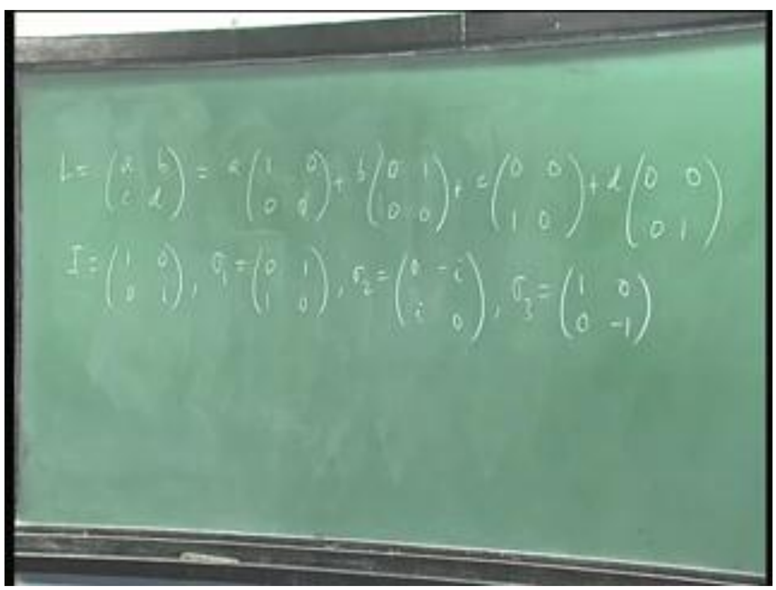

So, you have of course a unit matrix which is 1 0 0 1, you have first matrix sigma 1 equal to 0 1 1 0, sigma 2 equal to 0 minus i i 0, sigma 3 equal to 1 0 0 minus 1, these are call the Pauli matrices. Are you familiar with the matrices, yes some of you are, some of you are not never mind it is actually quite simple, these matrices have a very interesting properties, they call the Pauli matrices.

And they are linearly independent of each other and its elementary statement to say that any arbitrary two by two matrixes can be expanded uniquely as a linear combination, not of just these four basis say matrices, but of these four basis matrices. So, you could write any arbitrary two by two matrix as a linear combination of I sigma 1, sigma 2 and sigma 3, that is a simple exercise, I leave it you do that, because start with this a b c d; and insist on writing it in this form.

So, write this thing as the same matrix, write it as alpha naught I plus alpha 1 sigma 1 plus alpha 2 sigma 2 plus alpha 3 sigma 3, and you see immediately that alpha naught, alpha 1, alpha 2, alpha 3 are linear combinations of a b c d; and you can solve for terms of other and voice verse unique, so this immediately shows that you could expand a matrix in this basis as well. There is no restriction that they alpha has to be real or anything like that, because as you have see on of the basis matrices itself has got I.

The advantage of this basis is that these matrices are hermeation, they are equal to the complex transposes, that is not true here, on the other hand these matrices have that symmetric property. And it is also a simple exercise to show, I am going to gives this is a problem set look at all, work out all the properties of these.

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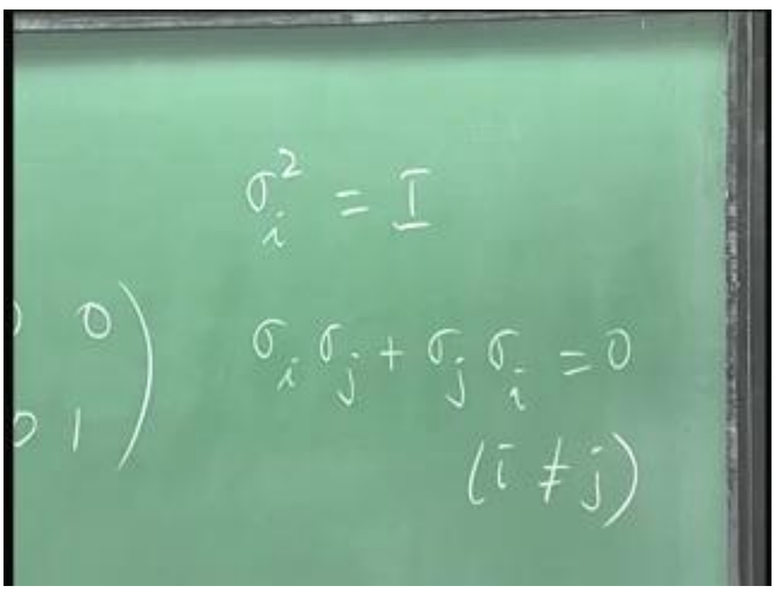

Sigma is square equal to the unique matrix, the square of the everyone of these matrices is equal to the identity matrix; moreover sigma i sigma j plus sigma j sigma i equal to 0 if i naught equal to j, the anti commute with each other, that is very, very powerful property for what we want to do. Because, if you want write e to the power L t, you write e to the L t equal to e to the alpha naught I plus, let me call it alpha i sigma i with the summation over the index i implied, looks almost like a vector alpha dot sigma, except you must remember that the components of alpha, alpha 1, 2, 3 need not be real numbers, could be complex number in general.

And we have task of finding this quantity here, what is this equal to well, you know e to the power a plus b is not equal to e to the a times e to the b in general where a and b are matrices. And the reason is e to the power a would have only powers of a, and e to the b would have only powers of b, so if I write e to the power a multiplied by e to the power b, all the a's are on the left and all the b's are on the right, that is not true when I have a plus b in the numerator.

So, you can see that e to the A e to the B is I plus A plus etcetera i plus B plus etcetera and if i work this out and write it out all the A stay on the left and all B stay on the right; but, if took e to the A plus B equal to i plus A plus B plus 1 over 2 factorial, and what is the next term, what is the quadratic term, its A square plus A B plus B A plus B square. So, you see there are terms which have B on the left and A on the right, and as you go to higher powers more and more of this will happen, that be combination like B A B, A B A and so on, that is not allowed for here.

So, this is not equal to this quantity is not equal that for matrices or for operators etcetera. On the other hand, if I look at this quantity here, and this alpha 1 sigma 1 plus alpha 2 sigma 2 plus alpha 3 sigma 3, and I expand it out this quantity commutes with everything, so you can actual move it out. You can write this as e to the alpha naught I multiplied by this thing here, but when you expend the exponential here, you get terms like sigma 1 squared which is a identity, sigma 2 squared the identity, sigma 3 square is the identity; when you get combinations like alpha 1 alpha 2, sigma 1 sigma 2 plus sigma 2 sigma 1 that 0, because of this anti commutation property.

Therefore, you can actually find the exponential, when you take any two of these in opposite orders and taken the output valences, completely valences, so I leave the rest of basis exercise to you, actually work out what e to the power alpha, alpha naught I plus these guess then write a very compact formula for it in terms of the alphas completely. What would you expect, what would you expect is the final answer after go through the entire, what would expect nothing very complicated, because this is also two by two matrix, and we just said that every two by two matrix could be uniquely expanded in terms of the identity matrix, and the Pauli matrices.

So, the final answer has to be something of the form, some beta naught I plus beta 1 sigma 1, it has to be of that form, and there is no option nothing but, that all that happens is the beta's are complicated functions of the alphas, in this case not very complicated. So, I leave you to work this out, and that will tell you how to exponentiate such a matrix, is a very simple formula which once you remember, the formula or work it out you can apply in every given case and this is the answer. Our interest however is not in finding the expressed solution, but in finding the nature of this solution, we like to find of what us to do, what kind of behavior you have in the neighborhood of the critical point, at the origin this is our target.

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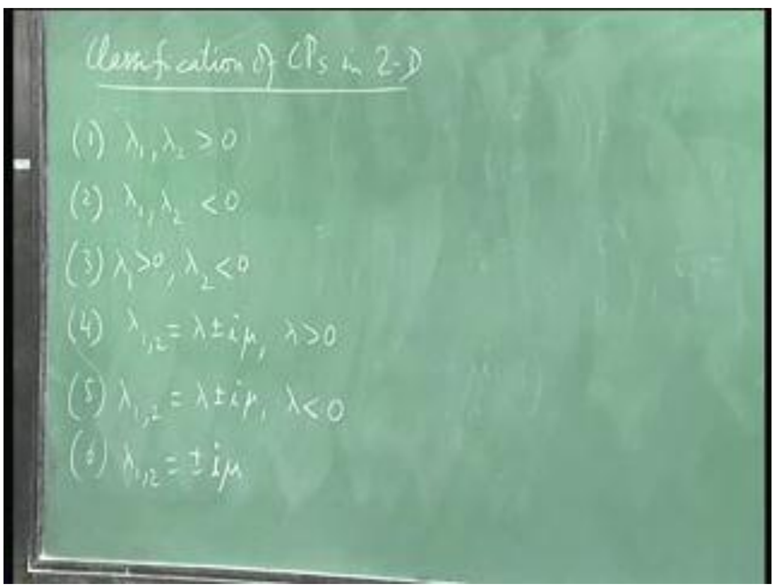

Let us start classifying it, let us start finding out what we should have, and what the possibilities are, let me call this classification, so I classify all the possible critical points in two-dimensions, in the x y plane. The first possibility is that lambda 1, lambda 2 are greater than 0, two real Eigen values both positive; let us list all the possibilities and then look at the pictures to see what happens. The second possibility is lambda 1, lambda 2 less than 0 incidentally, the situation where the Eigen values become equal to each other, and both positive or both negative is a special case that is we can something you can under quite easily. The third possibility is one of them positive, and the other negative this is possible two.

The fourth remember, you assume the matrix is not singular, so we do not allow for a 0 Eigen value at the moment, what is the next possibility they become complex, these things become complex; so certainly you would have lambda 1, 2 equal to lambda plus or minus i mu, where lambda and mu are real numbers and lambda is greater than 0 the real part is positive that would mean an exploding exponential.

The next one is lambda 1, 2 equal to lambda plus minus i mu and lambda is negative, there is one more possibility.

Lambda equal to 0 pure imaginary, but not 0 pure imaginary Eigen values, so the last of this lambda 1, 2 plus or minus and that is it, this exhausts everything. Now, we have to do it is look at each of the cases find out, what kind of critical point it is and you have various types of critical points, and them succeeded in classifying all those critical points two-dimension flows, where you do not have higher order critical point, or you do not have d generate map. Let us see what happens, a simple way to do this simply to look at an example, draw the picture and then argue that the general case is the distortion of this picture, so let us do that.

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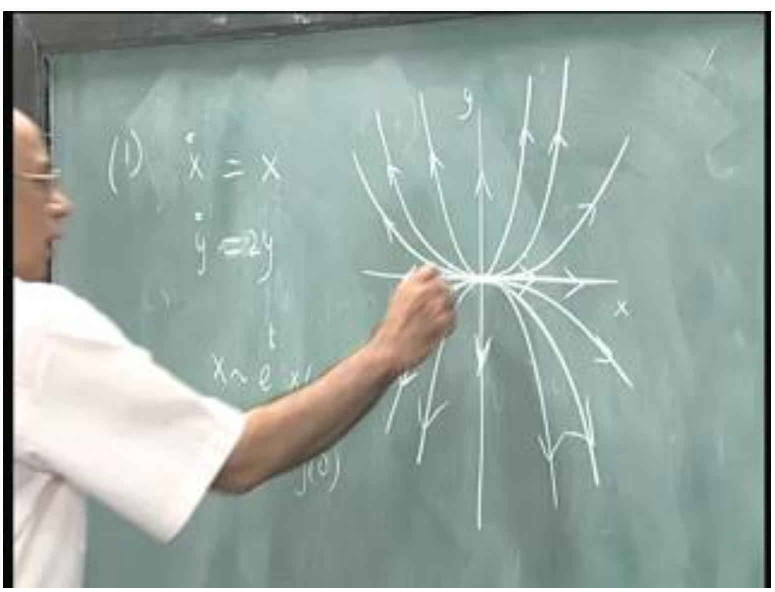

Let us look case one, which is two Eigen values positive and I look at example always, so let us look x dot equal to its got to be linear function of x, and the function has to vanish x is 0 and y is 0, because we are looking neighborhood of a critical point which we assume is that the origin. So, let us look at this that is certainly satisfies criterion, because in this case lambda 1 is equal to lambda 2 is equal to 1 both are positive, is do not want that kind of generously, so let us make this 2 y, what do the flow lines look like.

Let us try to draw the picture directly, so here is the x plus x axis, y axis and I like and this is the origin that is the critical point, and remember magnified this figures, so that I might very close to the critical point, I have just blown up this figure to see what the trajectory looks like and what would they look like, well good ideas always to start with special trajectories and then look at the more general case. So, suppose I start with an initial condition here very close, y is 0 and therefore, never takes of this guy does not take off, and x x dot x and say x is increasing, because x is positive.

And which direction is flow going to be, it is going to be outwards, that is the trajectory by itself, so wherever you are on this point on this line it’s going to move out words along that. So, you really do not care, it is hole family of initial conditions, you have taken care of had you started on this side then of course, it would move in the negative direction, remember we are not interested in how you are moving as function of time this says x goes like e to the t times x naught, so what about the x naught is just multiplied e to the power t and moves off. If you start with y then of course, you go there, and you go there we start now, away from the axis so we taken care of four axis, to start with an initial condition let us see here, what would it do well, it would tend to increase x, and tend to increase y but, remember x goes like e to the power t times x of 0 and y goes like e to the power 2 t times y of is 0.

And you could in fact eliminate t between these two, and you discovery y is propositional to x square, so from here what would you do, you move in a parabolic path, so really get off like this, in this passion and if you start at here move of like this, you start at here do this, if you start of here you do this. Except if you start on the axis you always going to be one of these curves, remember the actual flow once move sufficiently far away from the critical point, you would have to include non linear terms higher terms and then it would change, but in the infinite as mean finite of the critical point this is what a flow looks like.

And wherever you are, it would do the same, except for this point, so the entire neighborhood has this set of trajectories moving out and these two, so it is as if where you have, you are you a common tangent the x axis, and there is one exceptional direction the y axis.

Yes, but I could start with x naught here, that would also lie on its own trajectory, so whatever initial condition I start on, I am part of a parabola, so I have a family of parabola some pointing up, some pointing down in this passion. So, this is what the picture looks like, what would call this stable point or an unstable point, unstable because, every direction is bring thrown out therefore, we should really properly put a cross here, and immediately call this unstable; it is called an unstable node, the different terms for this but, I will use one set of terms once for all it is an unstable node.

Case two its trivial to solve, incidentally the case when the two node when the two Eigen values are equal, what would happen then, it was still be an unstable node but, what would happen if you did not have this two is that both of them would go like e to the t, in the ratio would there would be a constant which would mean, that the ratio that these are the straight lines.

So, the picture would really look like this, with get a kind of star pattern this would is what would happen it lambda 1 equal to lambda 2 right then see, that two is an unstable node. The difference between these two pictures is that here, there is a common tangent with one exceptional trajectory direction, but it has no such thing, every direction is allowed. So, this radially outward pattern immediately tells you that it is an unstable node, so does this what would happen, if both these things where negative.

Case 2 all I have to do put minus signs, what would happen to this picture all the arrows will be reversed, everything would flow in as asymptotically would you call this stable or unstable?

I call it as asymptotically stable, because I am going to distinguish between stability and as asymptotically stability and that different property all together. So, things would flow in be as asymptotically stable, so all arrows reversed everything reversed etcetera, you would not bother to rest of it, and this is an as asymptotically stable is asymptotically stable node.

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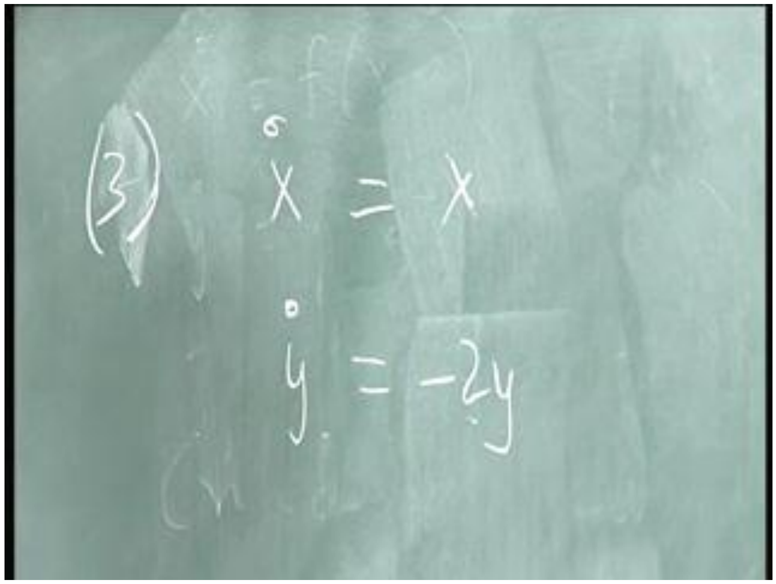

Let us go to case 3 which is interesting so case 3, the simplest very do the x dot is x and y dot is minus y a, let us put it as minus 2 y with 2 Eigen values a 1 and minus 2, in this case and things are flowing in the y direction that exploding in the x direction. And what would the picture look like, once again if I start at y equal to 0, then x just increases the time so this is suddenly a trajectory and so is this a trajectory, the moment you have things going out it means it unstable, so let us just put across there right away moment you have things slowing in some direction or the other, it means system is really unstable the across. And in y things come in and sort by start with it x exactly equal to 0 to start with in initial condition that of course things broken. But, if I do not do that and I start here what would happen?

Yes exactly, so the whole plain is tried by these, curves its look like hyperbolas, but of course they are not hyperbolas, the reason they not hyperbolas is because, x increases like e to the t, but y increases like e to the decreases like e to the minus 2 t, so you do not have x y equal to constant.

x squared y equal to constant, it does not matter it still got a shape which looks roughly like this may not hyperbolas, but its call it hyperbolic point or a saddle point; so this is a saddle point, unstable saddle point. Next case 4, lambdas positive on this case, what would it look like, this is a little trickier here little harder to do we could write on example which would do this, but we can argue this out even by without writing down a very simple example.

But mind you have to object at this stage on say well, how do we know this is true even if you had more complicated flows, I assumed extremely simple flows, I said x dot is x, y dot is y or minus y. So, I decoupled x and y and the question to be asked is, if you couple x and y, if you have general linear terms, which involve both x and y how do we now, this is still true I have to still convince you that this is exactly, what is going to happen will come that in a minute.

But let us finish the classification, when we go back and tell you what happens in the general case, what would happen there is that both x and y, would have terms like e to the power lambda t, e to the power i mu t plus or minus both x and y would have linear combinations of this. This quantity here is simply sign cosines, and signs it just osculate but, e to the lambda t when lambda is positive is explodes out words, so you really have osculated behavior such that, the sign of x and y would change between positive and negative. But, the magnitude would increase in this case, so to be an unstable spiral.

So, in this case it is clear, that the picture is going to look like, its going to outward which direction it moves an etcetera as a matter of detail, but it something which gets out. Therefore it is not too hard to see that this is an unstable, spiral point; there are other names given to this one of them is focus sometimes its called dot x whatever but, you just call it an unstable spiral point.

Yes, that is a good point, if mu goes to 0 this fellow this should tend to the case of the unstable node, we will see explicitly how this happens, will see distortion would occur an here, and we will see what happens. You can see that immediately, in in some sense, if mu goes to 0, the time of this period of this oscillation, the frequency of the oscillation is mu, so the time period of the oscillation is infinite. So, you can see that it never complete a spiral, it will just move out or move in immediately; but, we will see this explicit.

What would I call this, it would just be an asymptotically spiral point things would flow in, this is what happened in the case of the damp simple harmonic oscillator, so this is this as asymptotically stable spiral point. And finally the case of pure imaginary Eigen values, we have already studied one case

The simple harmonic oscillator the and damped oscillator, what was that critical point they were ellipsis, they will like ellipses, but it was a centre, this point is called the centre. So, let us write the oscillator case once again down is x dot equal to v, v dot equal to minus omega squared x, so in this case L the matrix was 0 1 minus omega squared 0 and lambda 1, 2 is equal to what, what are the Eigen values of this matrices?

Plus or minus I omega, that is why the solutions like cos omega t, and sin mega t this is you know that right nil, went in nor went out because, there is now exponential fact to pre multiplying the cosine or sine, and this was the centre, would you call its stable or unstable?

It is certainly not unstable, but it is not as asymptotically stable either, it does not follow into this point I call stable, that will tell us how to distinguish between stability, and asymptotic stability; notice this is the only place where I have used the words stable everywhere else its either unstable or asymptotically stable, that point is just stable.

And what was peculiar about this problem, which is not there everywhere else

Periodic

Periodic periodic this corresponded to periodic motion a moment you have a centre you has periodic motion, none of these motions is periodic, they either damp periodic motions or there motions without any such periodicity, but this is the only one which is periodic. And it is delicately balanced its periodic, because the linearise matrices around the critical point has exactly 0, real part and you see how accidental result.

So, we will see the periodic motion is quite exception, things are be just right a little bit friction here and its gone completely, a little bit positive feedback and it explodes give the direction. So, maintaining periodicity is not so easy, really very delicately poised and we will see it is going to give us lot of headaches periodic, then the general case. Now, let us come back and answer this question of why did not I look at the general case, why did I do this, my answer is following.

My answer is that the Eigen values they determine, whether thing is stable or unstable or asymptotically stable; and these Eigen values are independent of linear transformations that I make on these matrices. If I took a matrices, and then a similarity transformation on it then the

Eigen values do not change, that is easy to see because, what are the Eigen values of L, L is a b c d, and then secular equation is lambda minus a lambda minus d minus b minus c 0 equal to this, equal to lambda square minus lambda times (a plus d) plus a d minus b c equal to 0.

And the Eigen values therefore, lambda 1, 2 equal to a plus d plus or minus the square root of a plus d the whole square minus 4 times a d minus b c over 2, but do these combinations remind you of anything, what is a plus d?

It is the trace of the matrices, it is the sum of the diagonal elements, its call the trace of the matrices, so I have trace equal to T equal to a plus d and what is a d minus b c, its determinant, the determinant L equal to D. The Eigen values can therefore be written, as equal to T plus or minus square root of T square minus 4 D, this tells you that the Eigen values of two by two of the matrices, do not really depend on all the elements in some independent fashion but only on two combinations; the trace and the determinant.

And both these are independent are in variant, when you make a similarity transformation, so what will do is to start at this point and explode these invariance to argue, that we do not have to study the case in generality in each case, explicitly what we have here based on our simple example suffices; this already tells us this is the general case, and I exploit this in variance here to talk about this by the way this should be 2 of 3 by 3 or 4 by 4, so I leave it as home work exercise for you.

Next time will talk about it what happens in the three-dimensional case, what would the Eigen values depend on, they should also depend on combinations, it would not change when you make similarity transformation on the matrices; you have trace is one of them, determinant one of them but you need a third. Because, there are three Eigen values in that case in the N dimension case, you need N of them, so think about this and you came back because, it has some significance, we will stop here.