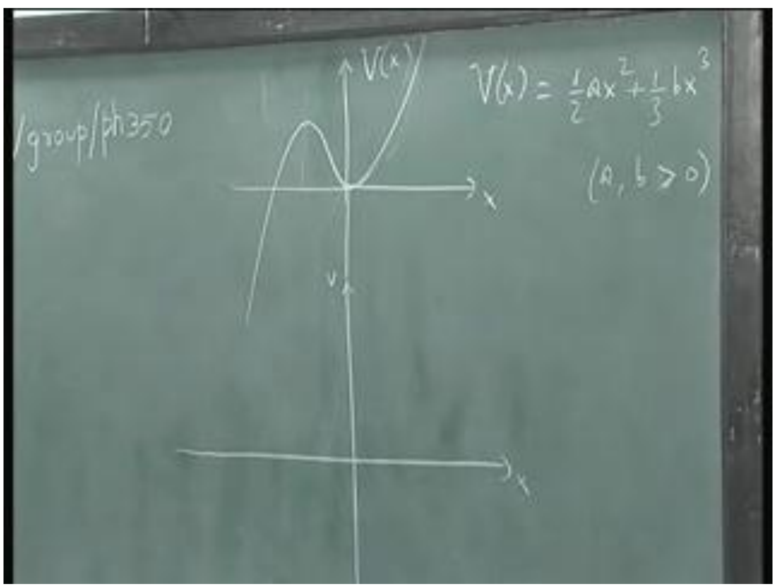
Prof. V. B. Balakrishnan
Department of Physics
Classical Physics
Indian Institute of Technology, Madras

Lecture No. # 04

You would like to continue with our analysis of the oscillator and related problems. But, before I do that, let us do one more problem of general potential, where I like to introduce the concept of stable as well as unstable fix point or critical point in the same potential. So, I am going to look at potential, where you have a maximum as well as a minimum and you know the maximum is going to be unstable equilibrium point. And the minimum will be stable equilibrium point and the question is what kind of phase portrait you get when you have both is equilibrium points present in same potential.

(Refer Slide Time: 01:45)

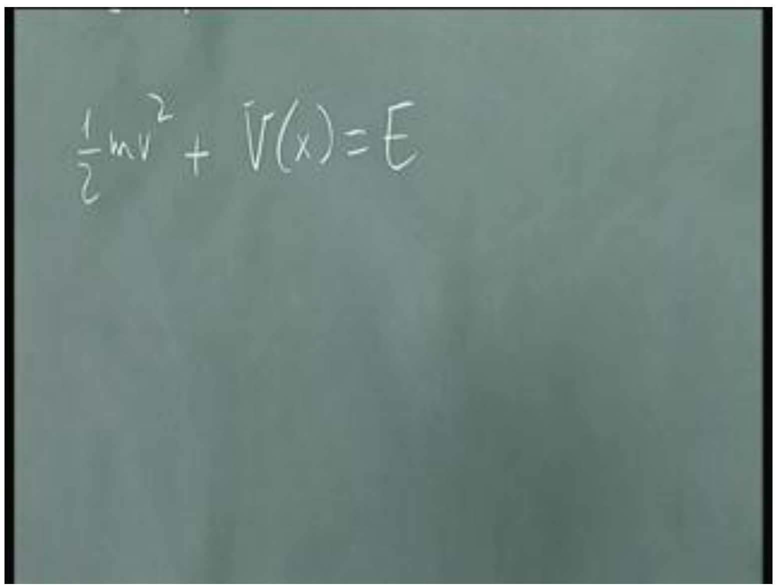


So, let us simple take an orbiter shape potential V of x like to have minimum as well as a maximum, so perhaps something of this kind. So, let us say you have maximum and minimum and goes off in this fashion. Of course, this point here is a unstable point and this point here is stable equilibrium point and we would like to know what the phase portrait is going to look like, you do not really need to know the equation for this potential, the actual equation nor to draw the phase portrait at least qualitatively.

But, let see you can do this you can write a formula down for the V of x. The simplest formula would be well; it is got a simple minimum here it is roughly parabolic in this region, so it is quadratic at that point. So, let us say 1 half a x squared and then it goes of to plus infinity when x goes to plus infinity minus infinity on this sides you need an odd term therefore, the next logical term is cubic term.

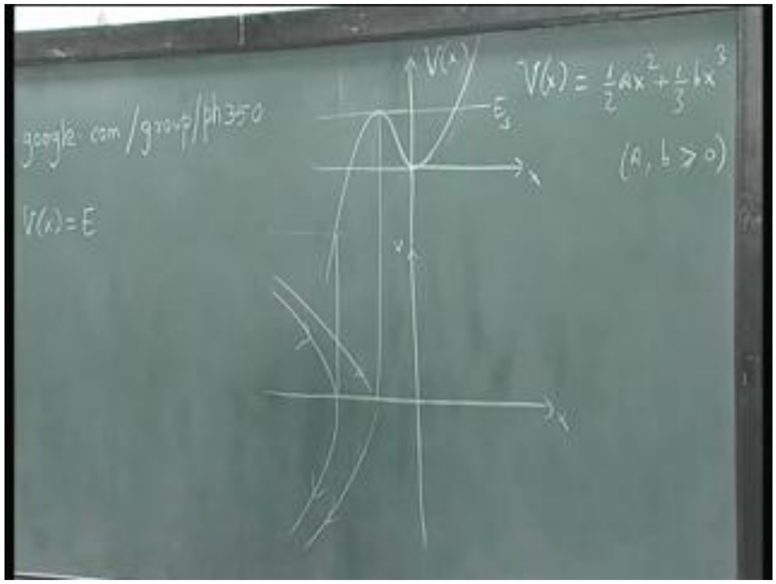
Let us put plus 1 3rd b x cube where a and b are positive, that would be a kind of simple formula for a potential with this qualitative shape. And the question is what happens to a particle moving in such of potential, what is the phase trajectory look like. Let us draw the phase trajectory right here on this place itself, so here is x and here is the velocity v.

(Refer Slide Time: 03:23)



And we have as always in these conservative systems 1 half m v squared plus V of x equal to the energy of system. And this real number E in this problem as you can see can take values of minus infinity to plus infinity, because V of x does so.

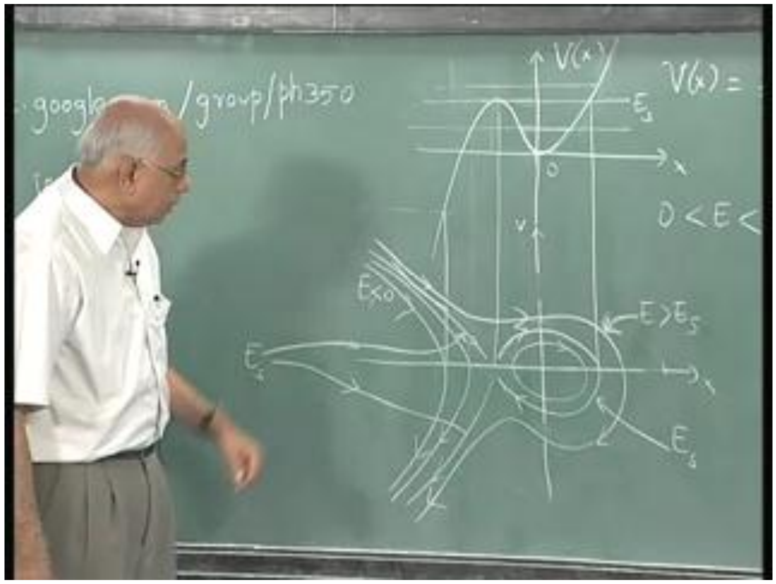
(Refer Slide Time: 03:34)



What do the phase trajectory's look like well it is evident, from what we said yesterday, that if energy is sufficiently low if this is your total energy negative value. Then, the system cannot penetrate into this region at all and must restricted to region the classically accessible region to the left of this point. And as you know if you come all the way from the minus infinity you can go up till here than you can roll down and the trajectory would perhaps look something like this.

And that is about it the full trajectory as we increase the energy you penetrate further and further to the right till of course this point here corresponding to the maximum of the potential. Then you would have a trajectory, which does this asymptotically reaches that point or as asymptotically flows away from that point. But at this critical value of the energy, let me call it E subscript s, s for separate x because it is going to separate two different kind of motion. It is also possible that you have an oscillator motion in this region here.

(Refer Slide Time: 05:08)



And of course if you start here, it is going take you a often an long time to fall down in this potential well go up till here and then crawl back up here. Because, as you come back the restoring force goes to 0 therefore, it is going to takes longer and longer to reach this point. And this would correspond to trajectory which does this, in this direction at intimidate values of the energy.

So, this for E less than 0 at an intermediate value of the energy say between 0 and E sub s. There are two kinds of motion you could have motion in this region, so that could be trajectory which does this but, could also have oscillator motion in this region here, which would corresponds something like this. I made a mistake, I exaggerated this, because you are here at this point you could never go be beyond that point. So, it is clear that I need to draw this more carefully, here is that point and come back.

So, notice that for 0 less than E less than E sub s you have both unbounded motion here, on the left hand side as well as periodic motion in this well. In the moment you reach E sub s you reached separate tricks, you reach the end point here. Little great energy, little higher than E sub s would imply that you can actually potential come down, come up of all the way cross this barrier go all the way up till there and then you fall down.

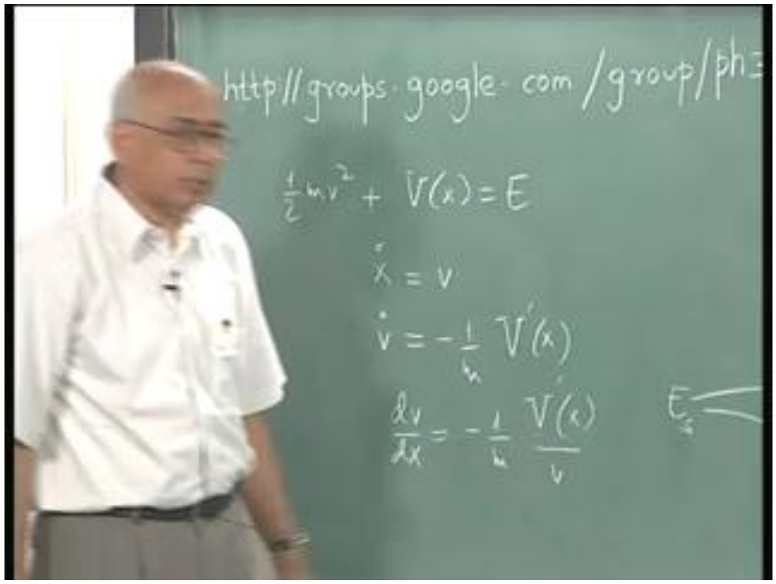
Because when you come back here, you still have that much energy you zip point fall of here. That would mean the trajectory does this does this, it comes here it goes all the way round that point and then close of to infinity.

And this for E greater than E sub s this trajectory, this thing here corresponds exactly to E sub s as do this 2 trajectories. This trajectory as well as this trajectory these 2 trajectory corresponds to these E sub s precisely. So, now the separate x separates unbounded motion from another set of trajectories, one part which could be unbounded and the other part could correspond periodic motion.

And it is clear from this figure that this set of trajectories this separates; now comparison is really three different trajectories. One of them asymptotically flowing in towards this unstable fix point the saddle point. The other one flowing away from it, and the third one starting their close to it as asymptotically tending back to it. Unlike the previous case where we have four asymptotes quite separate. Here one of them as curve back on itself and it starts of as part of unstable direction here. And I will explain by what I mean by unstable direction things are flowing away from it but, then when it comes back here flows towards it again.

And it forms kind a close loop here, asymptotically because this point these things never touch this point as you can realize and this kind of orbit is called homoclinic orbit or homoclinic cycle and it is going to big role. So, this is the qualitative phase portrait of this for this potential of course, you can justify this you solve the equations of motion, so on. But I want you to notice to things in particular one of them is that all these trajectories, when the vertical axis when horizontal axis is intersected the trajectory appears do.

(Refer Slide Time: 09:34)



So, at right angles and this is indeed, so because you look at the equations motion you have x dot equal to v and v dot equal to minus 1 over m V prime of x those were the equations of motion. And therefore, the slope of phase trajectory given by d v over the d x you just divided one by the other and this is equal to minus 1 over n V prime of x divided by v. And whenever it intersect x axis little v is 0 and therefore, the intersection is at right angles the slope is infinite unless V prime x happens to vanish also then of course, the slope is indeterminate you have to a take limit. And that precisely what happens at these points notice that these are not at right angles and the reason is at this point V prime of x is also 0.

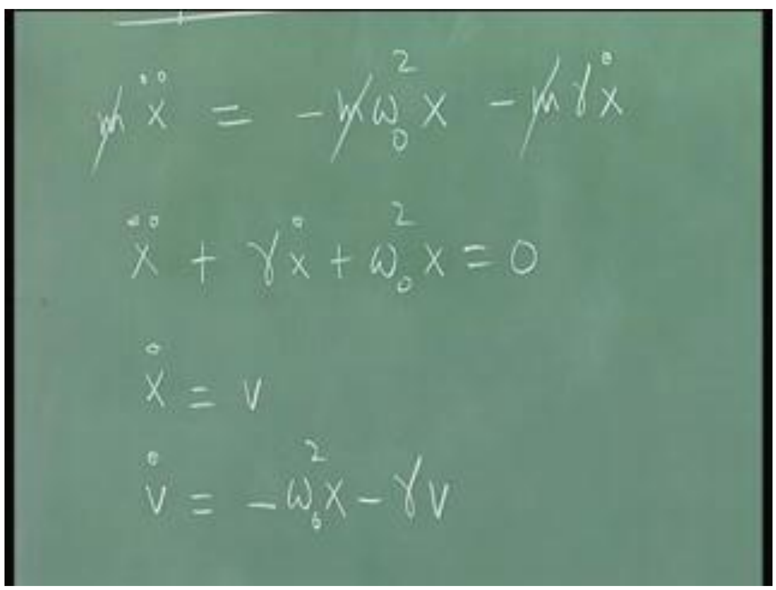
Therefore, you have to calculate the limit as you tend towards this point of what the slopes are and typically they would not be right angles. These would not intersecting right angles but, all these other points here the intersection is actually is at right angles except the separatrix trajectory. And again I would call this a centre, because there are small oscillations about it on this in this point I would call a hyperbolic point or a saddle point, because locally the whole thing looks like a set of hyperbolas around it.

So, here is a problem where you have a stable and as well as unstable point. And you can see you can generalize this you can have many more these, we will look at some problems were you do. It is immediately obvious that in these simple problems the maximum must be followed by a minimum you can have two maximum of a curve without minimum in between.

And therefore, stable and unstable equilibrium points must alternate it is abundantly clear; because it is very difficult to see, how could draw trajectory if we have two centers next to each with no other single point in between it is not possible. So, already this gives us some in inkling of what a general situation would look like, having done this, let us now ask look at the case of two equilibrium points.

One stable and another unstable and the same problem, let us look at some more cases let us look at a case where we include dissipation, we have not done. So, so far at all and we would not for a quite while in the formal development but, we may as well look at it in very simple example the damped simple harmonic oscillator. So, let us look at what happens if I take just a ordinary simple harmonic oscillator and I put damping in there or friction in the problem, what does this analysis do for us.

(Refer Slide Time: 12:19)



Well the damped harmonic oscillator, simple harmonic oscillator looks this all of you know that the equation is going to be x double dot, m x double dot force is equal to mass times the acceleration is equal to minus m omega squared x. Let me call them natural frequency oscillator omega not in absence of friction just, so that I keep notation straight and then I would like to include the effect of friction.

I pretend that this oscillator is moving in viscous fluid for instance, many ways of modeling friction. For instance dry friction should correspond to what happen to this object says if I push on it, it is not go move till certain critical stress is reached after the threshold it is going to start moving. But, then in a fluid this is not the way friction operates, in a fluid if the friction would be typically propositional to the velocity, instantaneous velocity and directed opposite to it.

It is like a particle moving in a, person moving in crowd the faster you try to move in crowd the more you get buffeted in front. And therefore, you have retarding force on you, this is exactly the way you have a force propositional to your velocity, the faster you try go the great the retarding force. And therefore, it is reasonable to assume the simplest instant, that this damping force is of the from minus m and it is propositional to x dot itself. So, let us the write to come proportionality as gamma and put an x dot.

I do this because I know that this quantity and cancels out by the by the equation and I know this is length divided by time squared, this is 1 over time squared here. So, this insures that gamma as dimension of time inverse. So, I put that is way I put m in there and now if I write this way were use to well you know the solution to this equation this is ordinary second order differential equations. So, x double dot plus gamma x dot plus omega naught square x equal to 0.

Well this is not quite, so trivial to solve as the simple harmonic oscillator were the solutions were cos or sine omega naught t, what is the solution look like; it is got an exponential damping. There is damping which is whose coefficient is propositional to gamma here we could write this solution down. But you really need to know how big gamma is relative to omega naught the both have dimension of one over time.

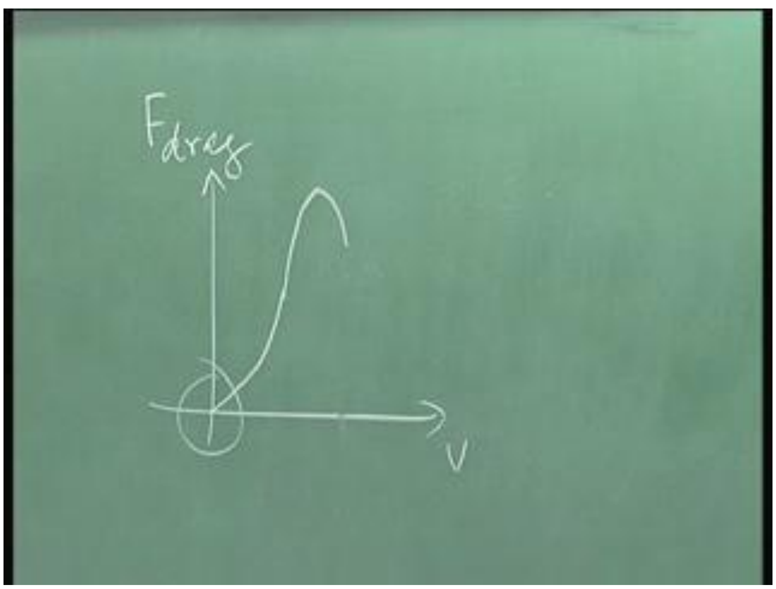
And you need know which is bigger, if gamma exceeds certain critical value in this case it is going to be something like to omega naught. When we know the damping dominates over the oscillations where as mega naught is bigger than half gamma, then the situation is that of an under damped oscillator.

So, we have three situation depending on weather under damped over damped and critically damped oscillator. We would like analyze all of them in one shot, it is simply like to know what qualitative behavior the phase trajectories. So, let us do that, let us rewrite this as x dot equal to v and v dot equal to minus omega naught square x minus gamma v that is a set of, yes please.

It does not need have a mass dependence I put this this some constant here multiplying the velocity, I extracted an m and call the rest of it gamma. So, this gamma has dimensions of time inverse the same as omega not otherwise the algebra gets messy. So, it is for just dimensional reasons without any loss of generality.

We are not talking about more complicated questions like, why should be linear in the velocity, why not a square of the velocity or cube of the velocity and so, on. This entirely possible and in fact, ask if you ask, what the drag force is on an object moving in air like an airplane. Then the drag force is propositional to the velocity only for sufficiently small velocities and as the velocities increase, it becomes highly non-linear right and it is much more complicated problem.

(Refer Slide Time: 16:40)

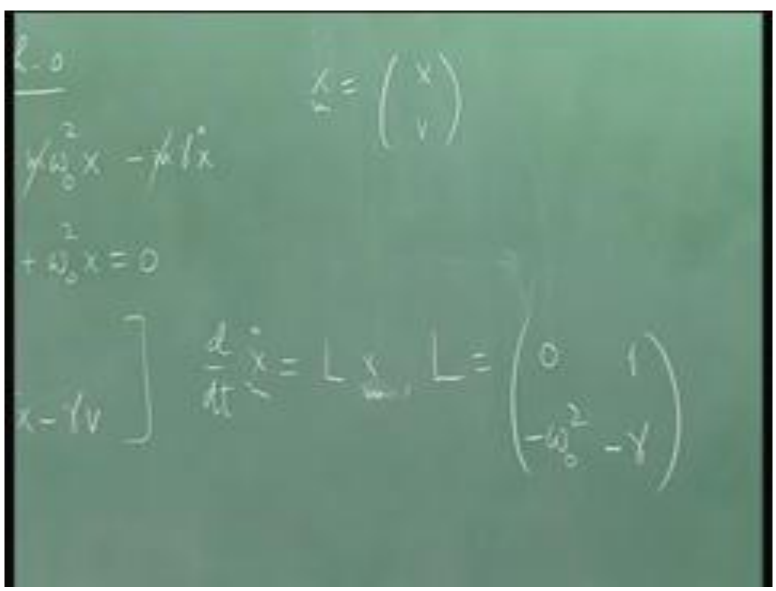


So, if I plot for example, this speed here versus F drag what is this curve typically look like well it starts of linearly. But pretty soon becomes quadratic higher powers and so, on; then increase very steeply and then it comes down once you break the sonic barrier. So, as soon as you hit the speed of sound after that drag force comes down dramatically but, before that it is increase extremely rapidly. And that is a highly non-linear situation and we are not going to look at that just going to look at this little region, where the damping is propositional to the velocity first power.

So, here our set of equations and I prefer to work with this set of equations for several reasons. One of them is that it is set of first order differential equation as a post to second order differential equation which is always hard at solve. And the advantage of the physical incite you get in to set off a first order differential equations is that, to specify a solution uniquely you need to specify initial conditions that is it.

I just have to tell you what x and v is at t is equal to 0, I do not have tell you what v dot was and so, on etcetera, etcetera Just have to tell you two initial conditions, two couple linear equations and problem is state forward to solve. So, it looks a little formidable in this language, we think that better to element and write it in this form and then try to solve it. But it is actually easier to solve in this set in this format we will see. But, before the solve the equation, since you are lazy would like ask in physical terms, what do I except, what can I expect of this.

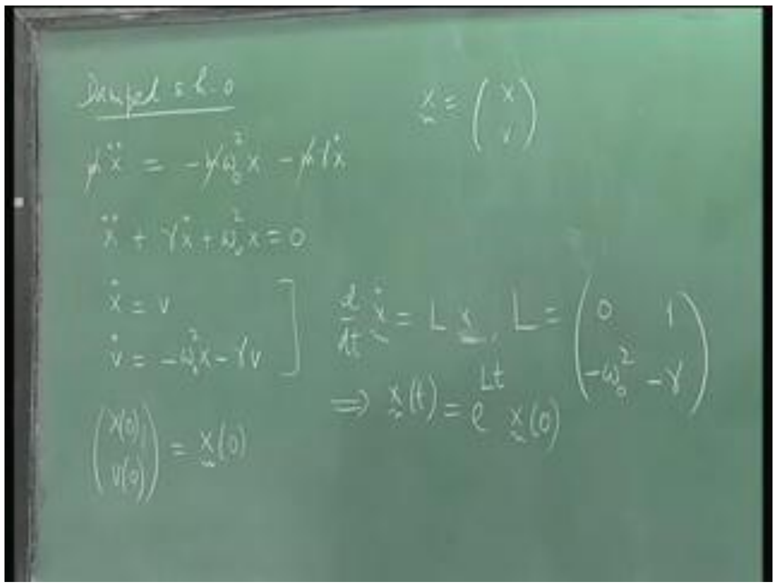
(Refer Slide Time: 18:22)



So, let me do that part of general frame word, I would like to write following, I would like to write to x as x v. Let me just define a vector a two dimensional object with components x and v they have different physical dimensions better take are care of that and be careful, another call underscore x that is not co-ordinate, that is point in phase space in x v.

And then this set of equations is nothing but, d over d t x dot is equal to some 2 by 2 matrix acting on x v and this 2 by 2 matrix. Let me call it L and it acts on x of once again and the matrix L is 0 1 minus omega naught square minus gamma. So, it is advantages, when you have set off couple linear equations differential or otherwise to write everything down in the matrix form then of course it is very easy to solve, such equations.

(Refer Slide Time: 19:32)

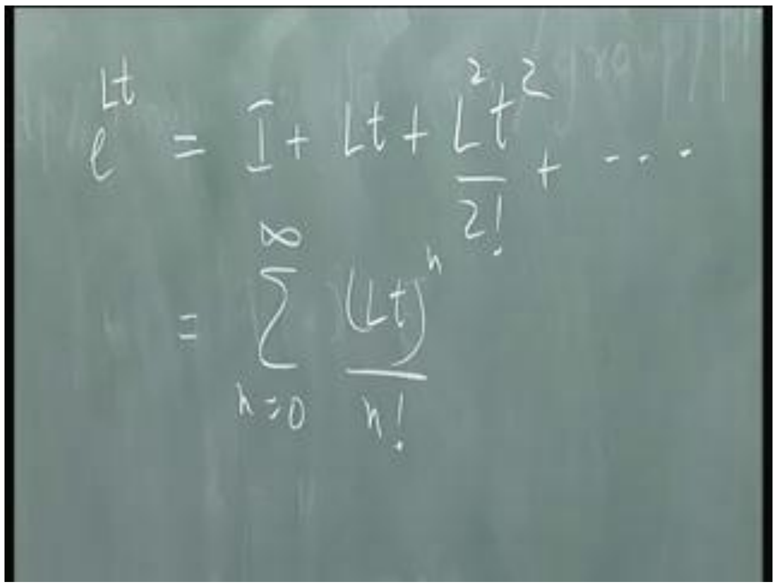


So, this equation looks very, very simple as it signs and what is the formula solution of this equation. Well suppose x was just a number I mean it was an ordinary quantity not not a matrix, not a column vector, then off course it will be e to the power L t. So, the same goes through goes through when this is a constant matrix there is no x depends there no t depends there. So, this implies that x of t is in fact, equal to e to the L t but, I have of to impose initial conditions I have to tell you what x of 0 is. So, x of 0, v of 0 has to be given you has to be specified and let me call this x (0) by definition.

So, where does as x vector of zero appear there, should it be on the right hand side or left hand side. Should be on the right hand side because, we would like have 2 by 2 matrix act on a column vector to give another column vector. So, we have to be careful you write this because remember this is column vector that is a matrix 2 by 2 matrix to be have to careful you have to write this thing here because remember this is a column vector and that is matrix 2 by 2 matrix.

So, we have to careful we cannot write it in other order as you could for ordinary functions. That is the formula solution but, now I ask what is the meaning of this e to power L t what is what is the meaning exponential of a matrix, what is that mean? I should, I define this exponential by it is power series, I define it by it is power series. And since we are going to play around these things might as well have them properly defined.

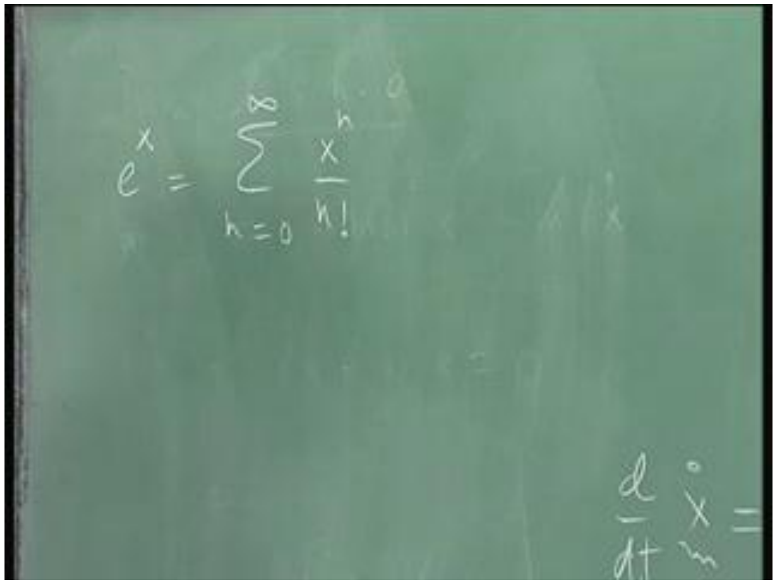
(Refer Slide Time: 21:20)



So, e to power L t, t is just scalar parameter here running from zero to infinity just a real number, but L is 2 by 2 matrix and definition of this is equal to power series. What is the first time in the power series, the identity matrix plus of course, as usual L square t square over 2 factorial plus etcetera all the way infinity.

So, let us write this summation n and equal to 0 to infinity L t to power n over n factorial L to the power n is also by 2 by 2 matrix. And of course we have faced with the apparently formidable task, which is to find all powers of L and then add them up and sum it. But you do not have to do that but, even before I do that I need to know whether the series makes sense do you think this converges after all it is infinite series even it converges. Well let us go back, we have our we have channel two here let us go back and ask.

(Refer Slide Time: 22:29)



What is the definition of e to the power x this is summation n equal to 0 to infinity x to the n over n factorial. Of course when does it converse to all values of x all values less than 1 in magnitude. All values of x of course it converges of all real values of x all finite values, the reason is x to power n increase like x is large may be it increases million. Then it will increase like million to the n but, n factorial in the denominator increases like n to the power n faster than e power.

So, this dominates in process series converges all of x the write with look at the all its ask when does e to power z converse where z is complex number that is way look at power series always the arguments complex number. When does that converge for all z all z infinite, infinite will it converge z is infinite no of course not.

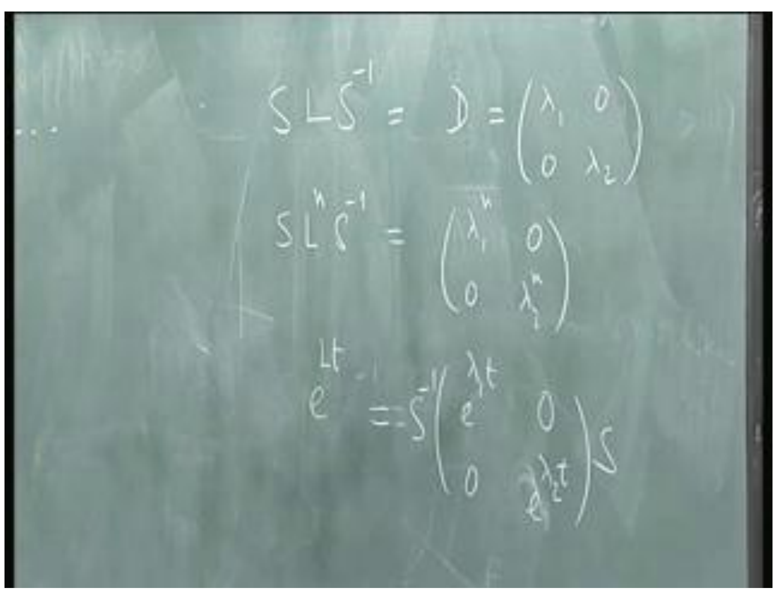
So, all finite z converge all finite z, as long as the magnitude of z is finite everywhere in the complex plane, this series converges absolutely it converges. So, well that I can differential it term by term integrate it term by term I can do all kinds of things to it. But, now we are phase with matrix know and I have to decide whether it converges or not for the complex number. I just said the magnitude is less than infinite that is fine, but what about a matrix, how do I measure the size of a matrix.

One way to do this would be to looks at Eigen values and ask for all if it is Eigen values are finite this would certainly be true. The other way to look at it would be define a size of this matrix size for this matrix called the norm of the matrix.

You would not going to the it is technicality but as long as let me just state that as long as all the elements of elder finite, this converges is no problem at all to you guaranteed that all the Eigen values are finite and so on. And there is no difficulty this is one of the important properties of the exponential series, plays a very fundamental role in all of analysis. And in this case there is no difficulty with convergence, you can close our eyes and go ahead and do this.

Of course the next job is to find this number, you have to find this quantity here, then sum it, but we have spared this problem. Because, actually what happens finally, is that what would happen I take this L, I raise it all powers and do this summation and finally, I get an answer, which would depend on the Eigen values ultimately. You could imagine for instants that I diagonlised this L by x similarity transmission if I could do this we will come to when you can do this, if I could do this.

(Refer Slide Time: 25:30)

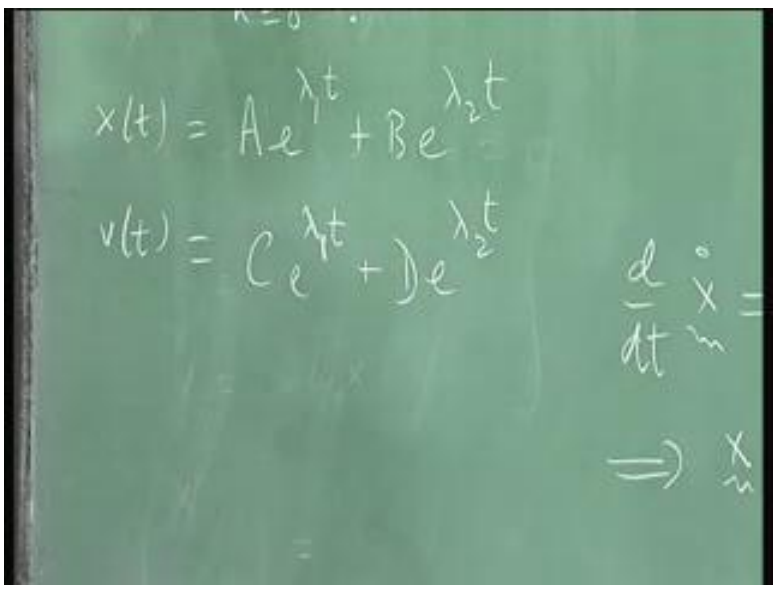


So, I take this matrix L and I apply a transformation on either side yes and it gets return in a diagonal form. So, it becomes D and the elements of diagonal form are of course, two Eigen values of L which I called lambda 1 and lambda 2. If I could do this, then it is immediately clear that S L to the power n S inverse is equal to D to the power n which is, in fact, lambda 1 to the n 0.

So, you immediately get this result, because I write this L square for example, L S inverse S and then L again and so on. So, this immediately obvious and in fact you go little further and you discover that S e to the L t S inverse equal to e to the lambda 1 t 0 0 e to the you discover that. Any function of L once you diagonalise it becomes just a diagonal matrix with a corresponding function written at in which of the diagonal elements. And then to find e to the L t itself, all you have to do is to undo this. So, I put S inverse on this side and get rid of that and I put an s on this side and get rid of that.

And what is final outcome it is says e to the power L t acting on x of 0 does not do anything very much. It says that both x of (t) and v of (t) are just liner combinations of e to the lambda 1 t and e to the lambda 2 t some suitable liner combinations.

(Refer Slide Time: 27:23)



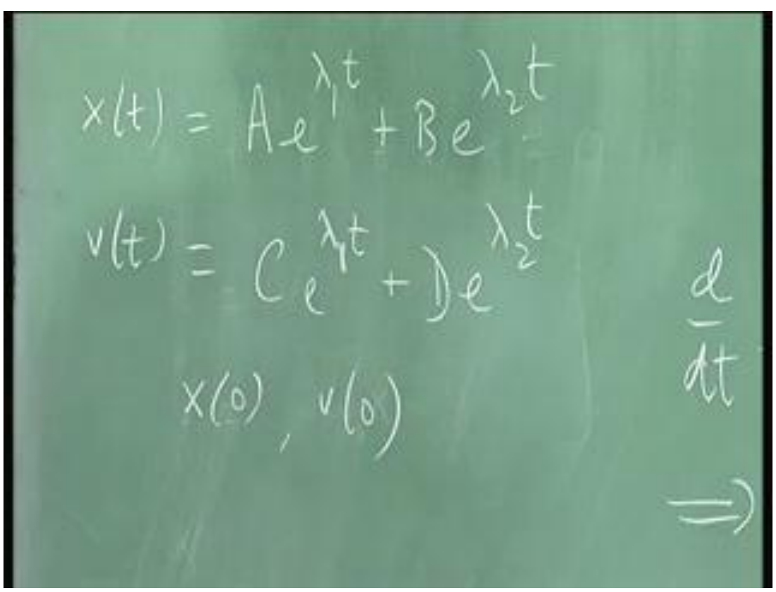
So, for example, in general this would look likes sum A e to the lambda 1 t plus B e to the lambda 2 t. This would look like whether combination with the constant A B C D would depend on x of 0 and e of 0 and numerical factors. But the time dependence after this hole rigmarole is essentially this and this is what you write down when take the second order differential equation for x, x double dot plus something. And you say let us assume a trial solution of form e to the lambda 1 t plus e to the lambda 2 t. Then you discover that lambda 1 or lambda 2 or Eigen value of this matrix, so exactly what we have done.

So, this is kind of thing justifies finally, you are going to get this behavior nothing more than that. There is one exception there is one exception, when would this not be the solution. Well no, we will come to that is separate good point, this point is you may not able to diagonalise this matrix but I do not need to do the diagonalization.

I just need to find the Eigen values of the matrix and that I can do whether the matrix is diagonlised or not it can be diagonlised or not. If the Eigen value is repeated, if you have just one Eigen value of lambda 2 is equal to lambda one then what you do what is solution look like. Yes the solution no longer a linear combination e to the lambda 1 t and e to the lambda 2 t but, it is linear combination of e to the lambda t and t e to the lambda t; if L is 3 by 3 matrix of n by n matrix.

And an Eigen values repeated r times, then for that Eigen value the linear independent linearly independent fundamental solutions are e to the lambda t, t to the lambda t, t squared e to the lambda t over 2 factorial for good luck up to the t r minus 1 over r minus 1 factorial e to the lambda. Well that is a general case look at the case of repeated roots later on nothing much happens. But this is what the thing looks like i have a little puzzle here that is we started by saying.

(Refer Slide Time: 29:50)

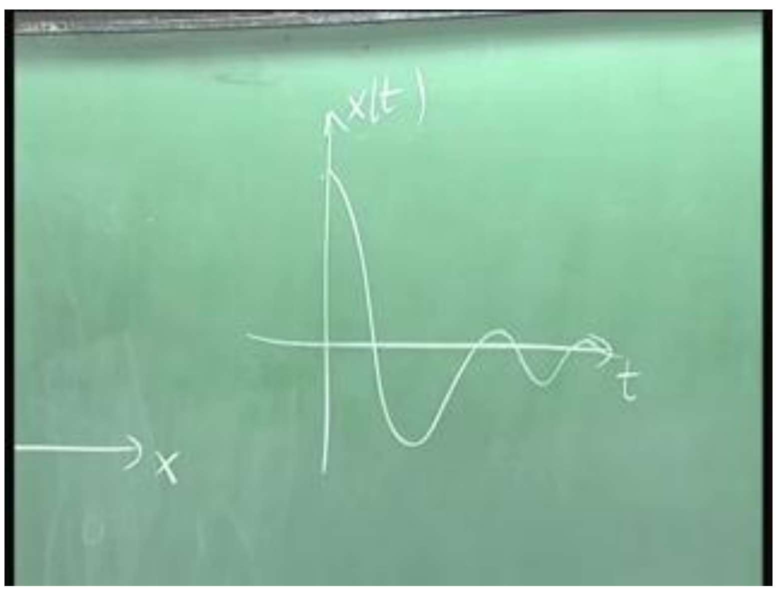


I give you x of 0 and v of 0 and end up with solutions which look like this but, there are four arbitrary constantans here A B C D. But, I give you only two pieces of information how may I going to determine 4 constant from 2 pieces of information. And I use the equation themselves and use the equation themselves the differential equations are also valid at t equal to 0. So, used the fact that x dot equal to v and v dot equal to minus omega naught squared x minus gamma v.

And if I look at this set of equations at t equal to 0 at t equal to 0 x dot of 0; of course v of 0 but, v dot of 0 is given by this and I plug in these in to the solutions. So, I take this equations and form these 4 equations I can find out all 4 constants provided I use the differentially equation themselves we look at the examples. So, there is no conflict here this is sufficient, you actually find all the constant of motions and the solution looks like that.

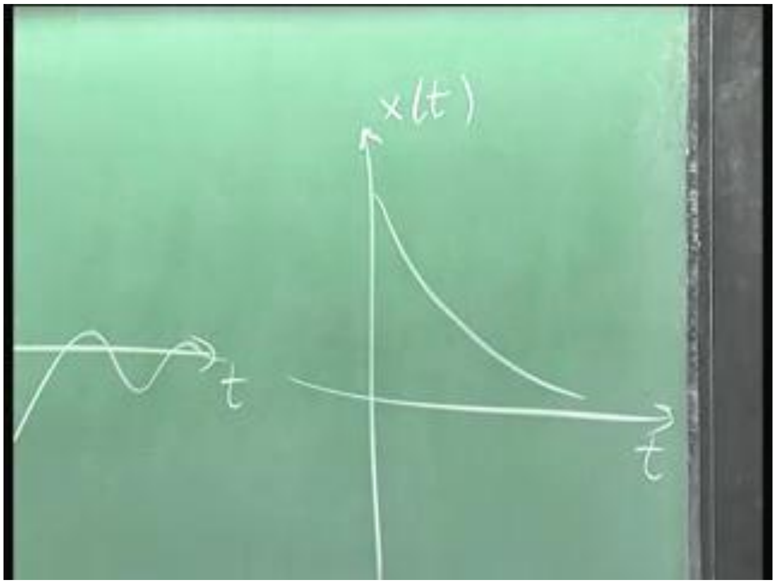
Now, if the solution looks like that, what does phase trajectory look like that, was aim after all our ambition was simply to draw the phase portrait, what is the phase trajectory look like. So, I still to solve that problem.

(Refer Slide Time: 31:13)



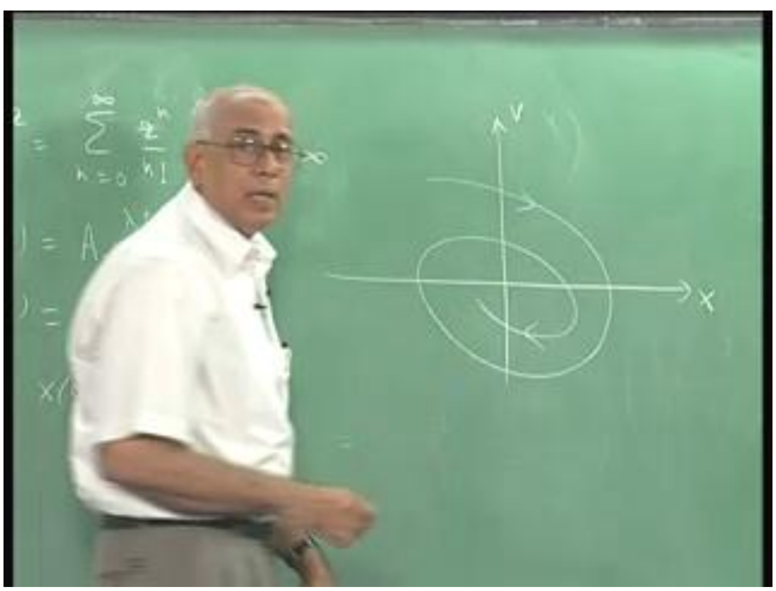
Here is x, here is v and we have a mess of this kind and you really have eliminate t before you can draw the phase trajectory but in practice we can do this much more simple way what would the phase trajectories look like. I think physically and I say look I know that if this an under damped oscillator. Then x of t as a function of t x of t would start at some point and it would oscillate and there is oscillation amplitude decrease to 0.

(Refer Slide Time: 31:47)



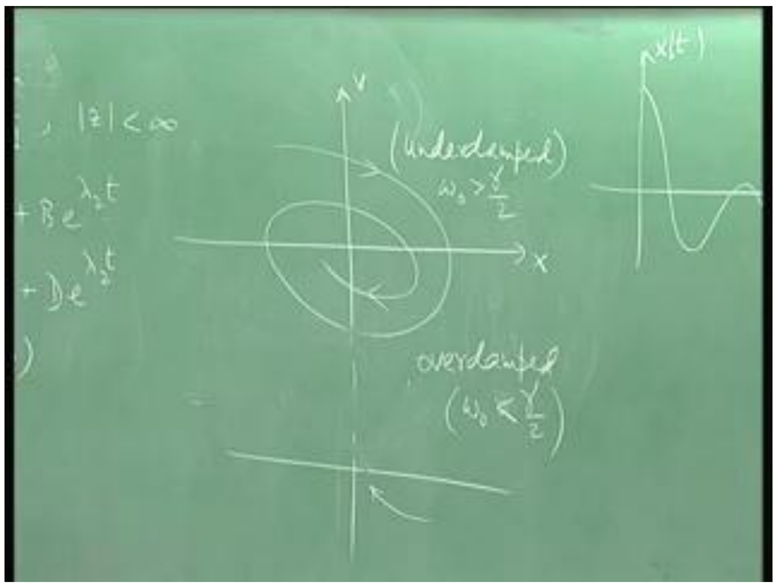
If it is over damped oscillator then, here is t, here is x of t it would start damp of in this fashion and if x of t is oscillates about it is central point with decreasing amplitudes. So, does v of t it is after all the derivative and if it monotonically decreases to 0 in the over damped case, so does v of t.

(Refer Slide Time: 32:07)



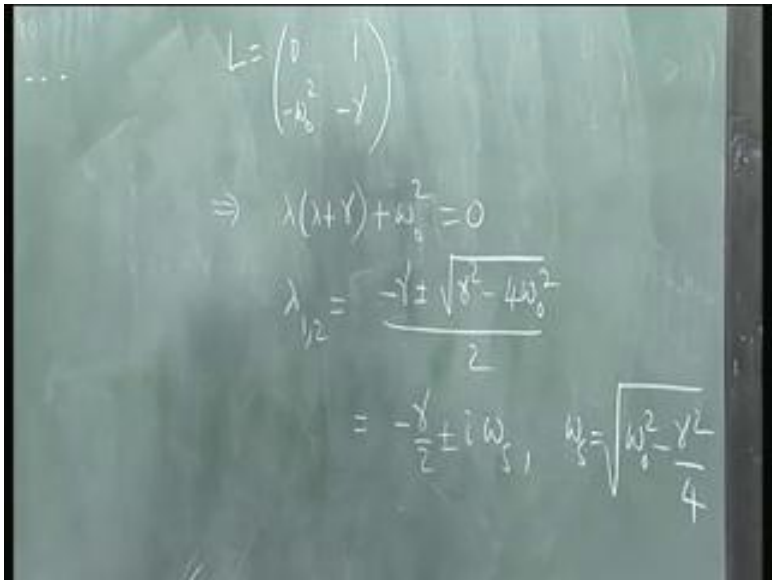
So, immediately suggest that in general if I start at of some here point here this thing would spiral in towards the origin. Unlike the old case where you had ellipses you do not have conservation of energy in this problem and phase trajectories would simply be some kind of spirals in the under damped case.

(Refer Slide Time: 32:28)



And in the over damped case in the over damped case you would have something starting here and essentially going off asymptotically to 0 without really oscillating. In fact, would not even change sign it is over damped. So, it is really start of somewhere here and simply fall in in this fashion.

(Refer Slide Time: 33:04)



We need to determine, when it is under damped, when it is over damped for which we need to know the Eigen values lambda 1 and lambda 2 that is trivial to do. So, let us do that or matrix L was 0, 1 minus omega naught square minus gamma, that implies that lambda times lambda plus gamma, plus omega naught squared equal to 0.

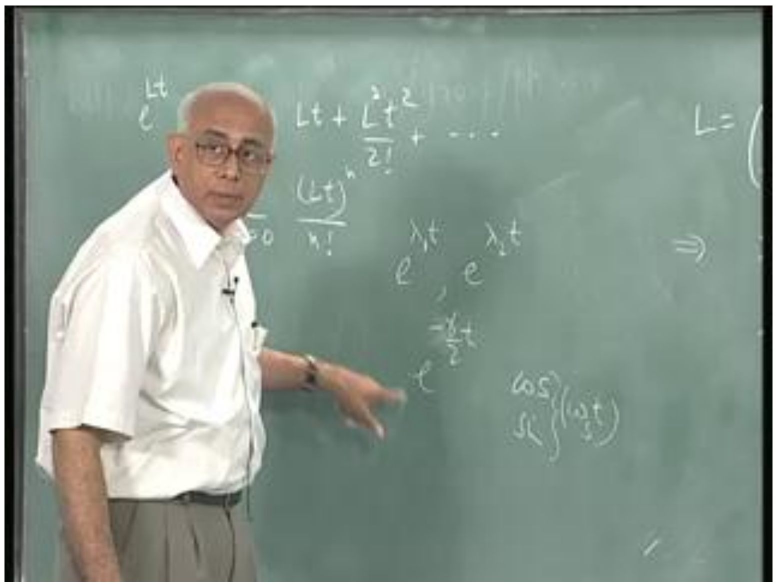
That is a secular equation and the roots or lambda 1, 2 equal to minus gamma plus or minus square root of gamma squared minus 4 omega naught squared over 2. And if I am interested under damped case and then this becomes a imaginary numbers. So, let me write this is minus gamma over 2 plus or minus I times omega s, where omega s is square root of omega naught squared minus gamma squared over 4 pull out the 2.

And I define a shifted frequency omega sub s, which is omega not squared minus 1 quarter gamma squared taken the square root, provided omega naught as greater than gamma over 2. So, this root correspond to under damped omega naught greater than gamma over 2 and this is over damped. In fact, this monotonic decreases, so it is just do this, over damped omega naught less than the critically damped cases omega naught exactly equal to gamma over 2.

And that is a case we looked at and talked already about, where this quantity vanishes and you have a lambda one and two each equal minus gamma over 2. I did that a carelessly it would it would part of the spiral, going in some passion depends on the initial conditions. It is not very interesting to me at the moment, because I know I can always go to the over damped case by going back to this writing by going back to this formula.

The point is that in both cases in the over damped as well as under damp case the system is actually reaching a state of equilibrium at the origin finally, because in the under damped case there is no doubt about it this point this part here.

(Refer Slide Time: 35:47)



So, e two lambda 1 t and e to the lambda 2 t they both co like e to lambda minus gamma over 2 t; and then the exponential of e to the plus i omega s t or minus i omega s t that does not the matter, because these things are essentially cosines or sine's of omega s t which are oscillatory functions. In the other case, but they are damped due to this, in the other case one of the roots minus gamma minus this square root defiantly damps out very fast. And the other root minus gamma plus this number minus gamma dominates over that, because it is bigger than this square root here and there for it still damps.

So, both the roots have damping they are not explosive roots both damped down to 0as we expect we have friction. So, whatever the oscillators does where you start eventually the oscillation die down and this thing goes to the equilibrium point at the origin; whether it does. So, by by oscillating across this point or does, so monotonically matter of detail it is not important at this stage. But this what phase trajectory looks like, what is the lesson from this everything depends on this Eigen value, on the Eigen values of this matrix L that controls everything.

So, once the real part of this Eigen value if it is complex set of Eigen values, once the real part of negative you would have damping automatically. So, this is a new set new kind of equilibrium point it is not a center and it is not a saddle point but, it is something else towards which system asymptotically tend the system tends asymptotically. So, we have to classify this but, before I do, so let me settle a small mathematical point here I mentioned that you start with this L.

And it is finally, the Eigen values of L that make a difference but, to get to this point to justify it I said let us assume L is diagnosable. You have to understand that all matrices cannot be diagonlised by similarity transformations only some of them can but, you do not need the diagonalization. Independent of that this is what the solution looks like but, as an aside, when can you diagonalize a matrix, when can you diagonalize matrix.

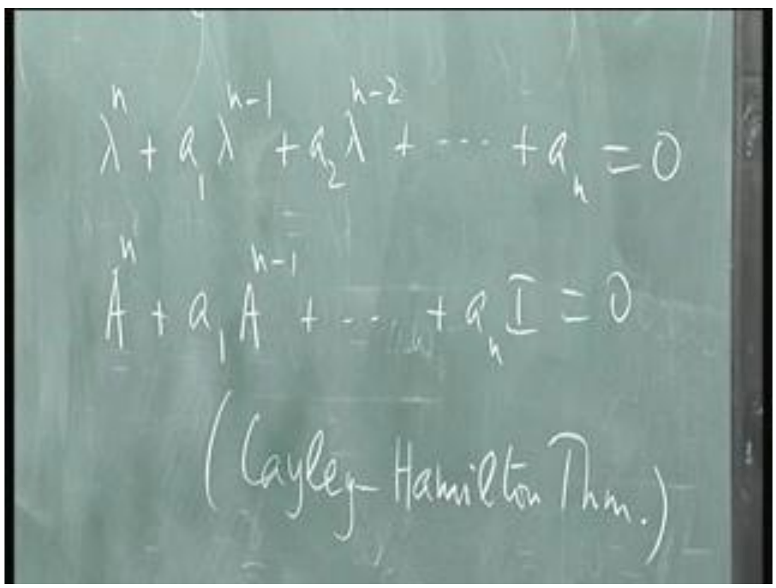
Matrix is real symmetric you guaranteed you can diagonalize it, in fact, you diagonalize by an orthogonal transformation but, what is the general case when can diagonalize n by n matrix. This is a problem in linear algebra it is nothing to do with this course but it is good to get. Yes indeed indeed yes. So, well here is a simple sufficiency condition it is not necessary but, a sufficiency condition.

(Refer Slide Time: 38:52)

If you have a matrix A and it is Hermitian conjugate A dagger by that I mean take its complex conjugate and transpose it is called Hermitian conjugate of the matrix. If A communities with A dagger, which mean that A A dagger equal to A dagger A by the way this stands for complex conjugate transpose.

The Hermitian conjugate of this matrix, if it commutes with it is Hermitian conjugate namely if A dagger is equal to A dagger A, does not matter in which order you multiply, this is called commutation. This is sufficient to ensure that you can diagonalise A by A similarity transformation, by that I mean you can find a matrix S such that, SAS inverse is equal to the diagonal diagonal matrix, it is sufficient it is not necessary. We will also like to have sufficient and necessary condition, well you know that a every matrix obeys a polynomial equation, its own characteristic equation this is called Cayley Hamilton theorem.

(Refer Slide Time: 40:12)



So, if you write down determinate of lambda I minus A equal to 0, you get a algebraic equation whose solutions form the set of Eigen values of matrix A, and this equation looks like in general, if this an n by n matrix, it is look like lambda to power n plus a 1 lambda to the minus 1 plus a 2 lambda to the n minus 2 plus a n equal to 0. That is the secular equation, in general this is what it looks like and the constant a 1, a 2, a 3, a determined from the elements from the matrix A.

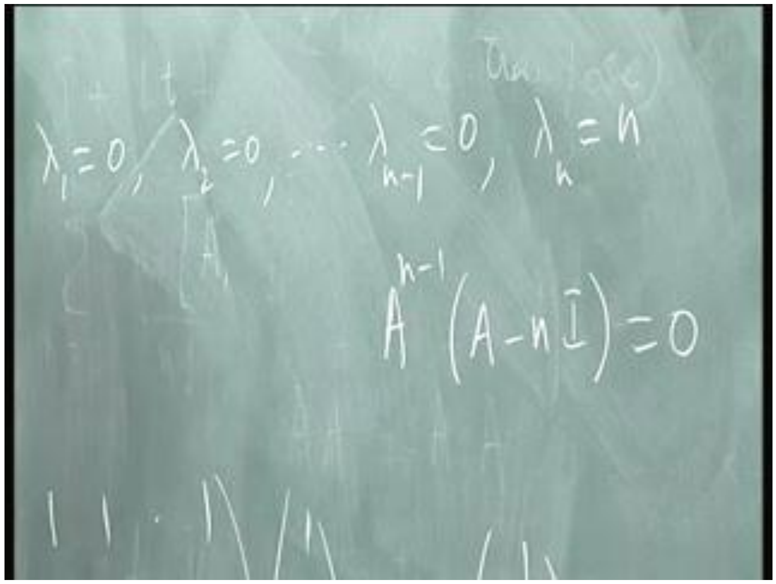
And Cayley Hamilton theorem says that, the matrix A itself satisfies this equation in other words, guaranty that A to n plus a 1 A to the n minus 1 plus a n I equal to 0; and this the Cayley Hamilton theorem the statement is every matrix satisfy its own characteristics equation. But, of course, it is nothing to stop the matrix from satisfying another algebraic equation, polynomial equation of a lower degree, this could well happen; that the matrix satisfies an equation of degree lower than n.

And the lowest such equation polynomial equation is called the minimal polynomial of this matrix, the lowest degree polynomial equation that polynomial equal to 0, that is called the minimal polynomial. And a sufficient necessary and sufficient condition, that A can be diagonalise by a similarity transformation is that the roots of minimal polynomial should be simple roots, there are many ways of saying this.

One of them has to do with the rank of matrix, which is mention in directly here, but this is necessary and sufficient condition, we will try to use it little later. But let me give you an example here suppose you have a matrix, n by n matrix all of whose elements are 1, 1, 1 everything is 1, and that is a n by n matrix what is the Eigen values of this matrix 0 certainly an Eigen values, because the determine of this matrix is obviously, 0. What happens if you do this, this is just playing with this, but what happens if you apply it column vector 1, 1, 1 what you get, you get n times the same column vector n times the same column vector.

So, would not you say n is an Eigen value n is an Eigen value and all the Eigen values are 0, every other Eigen vales is 0, because if you took this matrix it is determinate is 0, if you took the first minor that is also 0. So, you hid one of these columns and rows and rest of it also has a 0 determinate, and that is keeps going till you hit this elements itself, that is not 0. So, all the minors are 0 and therefore, this matrix has 0 Eigen values n minus 1 of them, the last one is n.

(Refer Slide Time: 43:56)

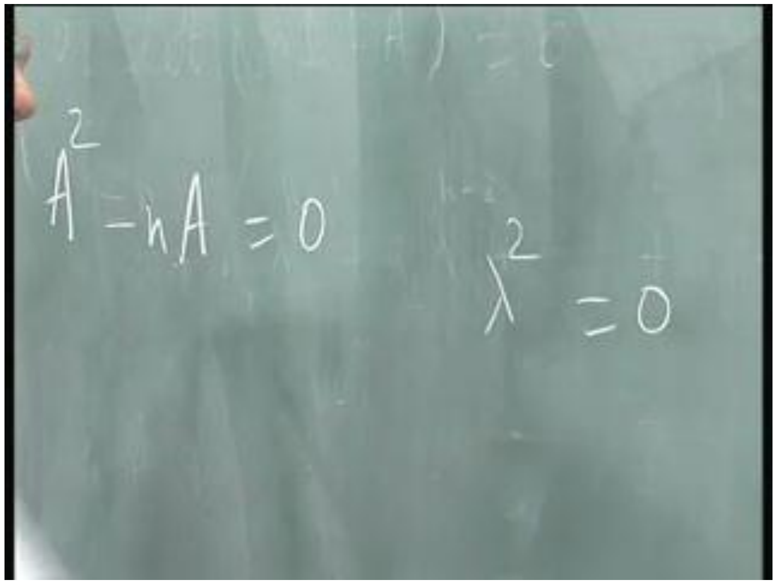


So, this matrix has lambda 1 equal to 0, lambda 2 equal to 0, lambda n minus n equal to 0 just one of them, so it is easy to write down the characteristic equation of this matrix, it is immediately obvious from here what the characteristics equation is. So, it says A to the power n minus 1 A minus n, that is the characteristics equation of this matrix, it is a nth order polynomial equation, and it guaranties that n minus 1 roots are 0, and 1 root is n instead of A you replace it by lambda and that immediately follows.

But this is not a minimal polynomial of this matrix, what is the minimal polynomial of this matrix, what is the least lowest order equation that this matrix satisfies, what you think is lowest order equation, but it cannot be a linear equation, because A is not equal to constant, A is not the identity matrix. So, it cannot be alpha a plus beta i equal to 0 cannot be an equation of this kind

The quadratic

(Refer Slide Time: 45:19)



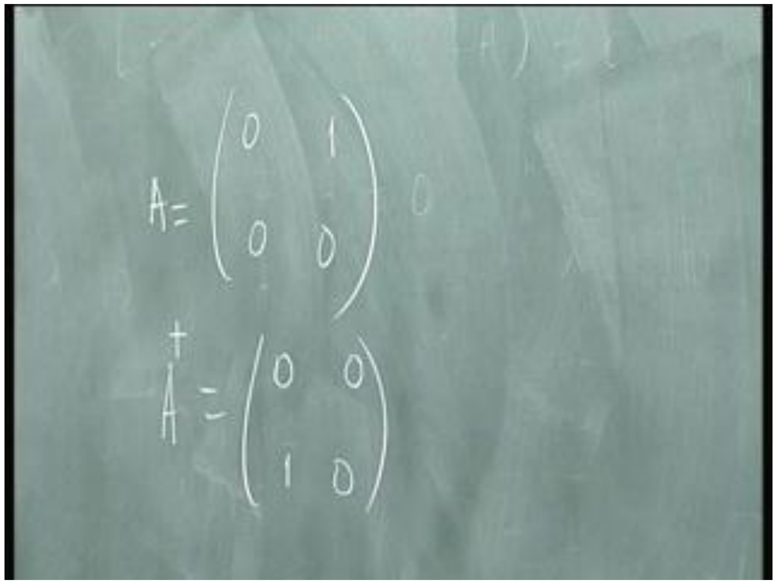
What happen, if you square this matrix A

n A, A square is n A, so we know that, so the minimal polynomial is in fact a quadratic in this case, what are the roots of the minimal polynomial?

0 and n 0 and n, are they simple roots yes, so you guaranteed this matrix can be diagonalise by similarity transformation, and absolutely guaranteed

What are the roots of this equation 0 and 0 that is a double root, say repeated root, when I say it is simple I mean the root is not repeated, it is multiplicity is 1.

(Refer Slide Time: 46:29)



What about this matrix 2 by 2 matrix, can you diagonalise this by a similarity transformation you think, well you can easily see A dagger is going to have 1 here, and they do not commute with each other. These two do not commute, you write A A dagger and you get one matrix you write A dagger a you get another matrix, they do not commute with each other. So, it does not satisfy this sufficiency condition, but off course it may not be necessary, a question is do you think is matrix can be diagonalized by similarity transcription at all.

What are the Eigen values of this matrix, 0 and 0 because, any triangular matrix in which everything below the principal diagonal is 0s the Eigen values of diagonal elements themselves, and that true for either upper or lower triangular matrices. So, the Eigen values of this matrices is 0 and 0 the characteristics function therefore, must be of the form a squared is equal to 0; so square of this matrix is null matrix, that is also the minimal polynomial because, the matrix itself is not 0 nor the identity matrix.

So, it cannot satisfy linear equation, the next equation is it satisfies the quadratic equation, but that must be characteristics equation, because it is a 2 by 2 matrix, and that is a squared equal to 0, and the roots are not simple. So, you guaranteed this matrix cannot be diagonalized by a similarity transformation. But, it does stop you from finding the Eigen values here and here, so the best you can do to any arbitrary matrix is not necessarily diagonalise it, but put it in what is it called Jordan normal form.

So, what you can do is to take this matrix this huge matrix and you can put it in blocks of different dimensionalities etcetera. So, you can have something here, you can have something here, you can have something here and so on. In diagonal from and in each block corresponding to a given Eigen value of some multiplicity you can bring it to the form lambda 1, lambda 1, lambda 1 or 1 here, 1 here, 0 lambda 1, 1 etcetera and then zeros everywhere.

For instance if this Eigen values lambda 1 has multiplicity three it is possible finally, to bring the matrix to a form, where you lambda ones everywhere in the diagonal and upper triangular matrix with ones on this side and similarly for these other. If it is simple Eigen value then you just have 1 by 1 matrix lambda 1 in that block this is called Jordan normal form. And that is the best you can do for a arbitratory matrix you do not need you may not be able to diagonalize it at all.

So, so, much of linear algebra we would not need all this, but it is useful to know all we needs there is the Eigen values. Now, that we have this example let us go immediately and generalize more general arbitrary case of a dynamical system and I would like to now define a dynamical system in the following way. Many definitions and there is a little bit of digration in to more general mechanics, then what were really planning to do but, I like to do this because it puts thing in a proper frame work.

Now, what we have learnt looked at different kinds of equations for very simple systems particles moving in potential and so on. We discover we can write these equations of motion Newton’s equations in first order form is a set of first order deferential equation. And we are interested in the qualitative behavior of these solutions not the quantity of behavior specific initial conditions and so on.

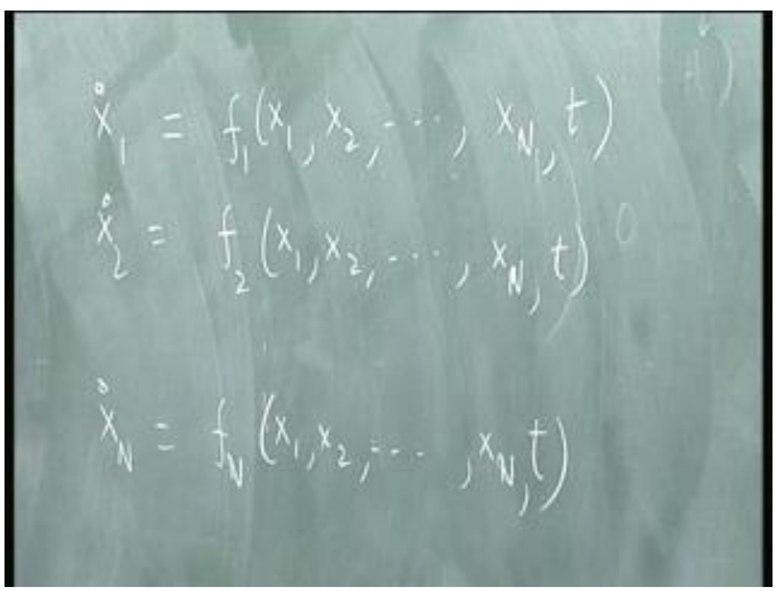
But qualitative behavior of the solutions as a whole and we had the simplest examples we found that, you have stable equilibrium point unstable equilibrium points. Sometimes equilibrium points around which the trajectories move and other cases where they tend to be these equilibrium points and yet other cases where they repelled outwards. So, I would like to generalize this and I start by saying independent of mechanics, let us assume that all our dynamical systems are defined by a set of variables, varying continuously in time. And that time is parameter arena in which these variables move.

And you have set of evaluation equations that is, it I do not make any further assumption. So, I start by saying my system is defined by a set of variables x 1 x 2 up to x n; n of them. I do not care by weather they are positions or velocities or momenta, I would not even care if they are actual mechanical objects they could be very complicated objects.

They could even been populations of species competing with each other, they could be all sorts of variables temperature, pressure, whatever I do not care. I have a set of variables which changes with time each of them is function of time. And I prescribe the way these variables change and I argue by saying, that I have a set of first order differential equations first order, because I assume that these are the only variables needed to specify the state of the system completely.

And once have that as an article of faith I start by saying let us assume that they obey a set of first order deferential equations. So, that specifying the initial conditions would tell me the future completely, what is the most general set of the equation write down write.

(Refer Slide Time: 52:44)



X 1 dot equal to something on the other right hand side it, could sum function arbitrary function of x 1, x 2 up to x n it could in fact, change with time. So, I could perhaps as t as well in general then x 2 dot equal to sum other function f. So, let me call that an f 1 and make this f 2 x 1, x 2, x n possibly t as well and i go down all the way. And I have x one dot equal to f n x 1, x 2, x n, t to start with yes I could think of more horrible competitions. I could think of a situation where this set of variables x is itself continuous, I certainly think of that I could think of x 2 be say the pressure field in this room and if assume changes a point to point and for time to time. Then of course, you have continuous set of equations and what would then have, what would you, then have we will come that very interesting question we will come to that in a minute. But for the moment, I assume it is descript set of variables the other assumption I have made is that I have assumed that they obey a set of the first of order differential equations. There could be problems were intrinsically you cannot find this set of equations we want ignore that for that moment; I have also assumed the time is continuous.

If I did not assume that if I said I monitor the system every year or every month or every hour, then time itself could be desecrate variable; and then have difference equations rather than differential equations. So, I have not been that general but I have been fairly general in this sense. Now I of course, I have said it could depend on time itself, this means this system is not autonomous the rules are changing as function of time.

But, then I would like to say let us look at simpler case where the rules do not change with time. I remove this explicitly dependence here by the by, even if you have the t dependence I can subsume it in an autonomous system as follows I play trick and I say. Let me define x n plus 1 to be equal to t define add a dimension define it in this passion. There of course I have x n plus 1 dot equal to 1 that is a nice function it is just 1 on the right hand side and all these are replaced by x n plus 1.

So, what I done is taken n dimension n plus 1 dimensional non autonomous system and replaced n dimensional autonomous system and replaced it with n plus 1 dimensional non autonomous system get rid of the t in this form. The physics could be very different of course, we would not get into that for the moment I will given example as we go along, what

I would like to point at the moment is, because of this possibility let us look at only autonomous systems; so I simplify have this.

And I play the usually trick I call this a vector a column vector with elements x 1 to x n and f 1 to f n another column vector. And then this set of equations can be written very compactly as x dot equal to some vector valued function of x. With this f underscore stands for a column vector with elements f 1, f 2, f 3 etcetera. And each element is a function of all the axis.

And this is the n dimensional autonomous system, now task is to try to analysis this in generals in formidable task. But life can be become very simple if you make certain assumptions on f which are not always valid and we would like to go beyond those the assumption but what would be the simplest assumption I could make.

Well if I, if this is linear if these are all linear functions if this is a linear function of all its arguments. Then the problem becomes x dot equal to sum L acting on x and write the solution down by saying its e to the L t acting on x. So, linear system is very simple very easy to do but, before I do even do that I like to ask. Do you thing as hope of solving such in a equation at all this is possible, the answer is no, the answer is no, because what would you do to solve without using a matrix method.

If this is non-linear then even matrix method would not be simple to do what you do naively is to say, whatever we did in the earlier case. If I have two equation for x dot and v dot I eliminate v and write it the second order differential equation in x, what would you do here. If I start with eliminating in principle you would say, let us get rid of x n, x n minus 1 etcetera.

And write everything as function of x 1, what order of differential equation would that be an n th order differential equation. Then you have a problem of solving very complicated non-linear equation matters are made verse by the fact, that there is no grantee you can do this. There is no grantee that in general you can eliminate all these variables and do this. If I started with n th order differential equation in one variable I can convert it to set of n coupled first order differential equations by defining the higher derivates to be new variables but, I cannot go backwards.

So, the real problem is this and you cannot go backwards you really stuck with this and the analysis of this is incredible complicated all sorts of possibilities exist. And in fact, beyond the small values of n really don not know, what is possible very, very complicated things can happen, extremely intricate things can happen.

And big discovery made actually long ago but, codified may be 25 or 30 years ago is that beyond n equal to 2 you already have unbelievable complications. You have phenomena of kiosk after n equal two till two nothing happens. But once you have n equal 3, 4, 5 etcetera; things can get really nasty they can get extremely nasty. So, nasty that it is not computable any more at all.

So, that is the big discovery and that is why we now say that kiosk in classical dynamics is generic, is the rule rather than the exception. And all things you study in normally classical mechanics are exceptions the real system is really very, very complicated much more, more systems are . But there is method in the madness we found lots of ways of handling this difficultly.

And this is sum of what I would like to communicate to you but, for I do that even I would like to point of that there is no difficulty what is so, ever. Once you give me nice functions f 1, f 2 etcetera; to put in this on a computer and solving the equations numerically and always do that.

Because, you give me an initial point in this n dimensional phase space, I use this set of equations you given me x at sum initial instant of time t equal to 0 at an initial infinitasble delta t. I used this differential equations to tell me what x is and then form that point is next delta t so, on. Therefore, once you give it me here I find out what it is here, what it is here, what it is here I, join all these guys and I have my phase trajectory.

So, it is really sort in that sense you have to be very careful you may need a very fine time step. If the function is very non-linear you may need a very fine time step but, in principle you can do this always. So, solvability is not the issue you can always solve it for some initial conditions for specific initial condition. Do you have much more ambitious program do you want to find out what is solution for long times arbitrary long times.

So, we would like know if I start here what happens after million years, this is of interest to me, because I could be talking about the solar system. It is deep interest to know what happens long times and maybe it is not possible to integrate, you may not have enough computing power or time to integrate. In fact, what happens is errors multiply exponentially and you cannot compute. So, this is really the difficulty in dynamical system but, in principal solvability per say is not issue at all, you can always solve these equation locally at any point. And there the hand you cannot write the solution down explicitly.

So, that is integrability, you cannot integrate this and that is the harder problem but, I want to emphasizes again solvability does not imply integrability. By integrable I mean, I should be able to write these solutions down as explicit formulas as functions of time.

So, I can put t equal to 20 million years and get what happens 20 million years from now it is not clear whether you can do this. And it is not just hypothetical in terms of millions of years because it is a matter of scale for the solar system, it could be millions of year or billions of year. But for elementary particles like particles inside a accelerator which are moving at speed comparable to the speed light, it could be once second and that is equivalent to millions of years of solar system.

So, within 1 second things become un predictable that is very bad news and therefore, you would like to know how to handle such systems. And there is no question of I would like to know what happens of long times, that is going to be the phrase I use I like to see if I can handle this kind of thing at long times.

Let me, start at this point tomorrow and show you, what this solvability means and why it is not integrable in general. After that we will go back to the general set of equations and be little less ambitious and start with 2 by 2 systems analyze that completely and then see what we can say for n by n cases.

And In fact, once one understand that frame work the rest of mechanics the rest of classical dynamics is actually special cases various interesting special cases. So, I would rather do that and introduces Langrage and the Hamilton and so on. Special kinds of dynamical systems which are part of this more general frame work here, is use full to know this kind of thing.

Especially the question of stability I would like to especially understand this because I have in mind situation whereas, engineers you would face not just mechanical systems not just electrical system. You could have electromechanical chemicals systems some of your variables could be machine parameters, some of the variable could be positions of rigid objects orientations and so on. Yet others could be concentrations of chemical species we would like to be able to have frame work, where you can handle all these situations in one shot. So, let me stop here.