Classical Physics
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Lecture No. # 36

I promise to do discuss Lawrence invariance and the associated four vector formalism let us systematically, so let me do that, we start off with that as follows. We start by saying that experiment has told us the special relativity postulates have been verified now for over a 100 years. And I wrote down the principle of relativity and the postulates of relativity, they followed from the fact that if you have two space time events.

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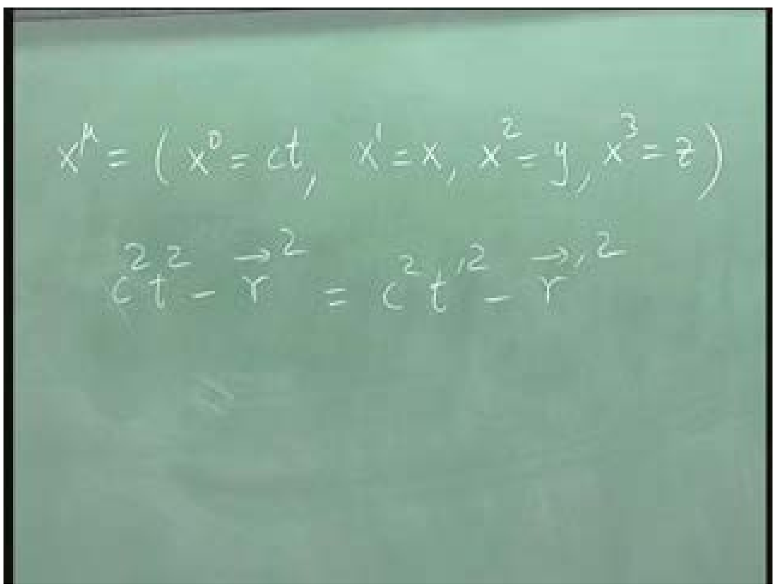

You have a space time event occurring at the space time point x mu, which is x naught equal to c t at some instant of time t in a frame of reference. And then x 1 equal to x, x 2 equal to y x 2 equal to z, I used this notation now the super script index notation. Then we know that, this quantity here x mu; we know that this quantity C square t square minus r square which is the square of x squared plus y squared plus z squared the sum of these squares.

This is the same in two frames connected to each other by a Lawrence transformation. So, we have the invariance of this quantity, this leads to the rest of what is going to follow. And it is convenient now just as we define the three dimensional vector as a set of three quantities, which transforms and rotations of the co ordinates system in the same way as the coordinates themselves did that was the definition of vector.

We now define a four vector as the quantity which transforms like x mu under Lawrence transformations, which would include rotations of the coordinate system as well as velocity transformations or boosh from one inertial frame to another. Now, a little bit of notation here is helpful, so we going to introduce this notation. And the most important thing that I want to would like to introduce is this quantity, so this thing here is a four vector. It is the primary instance of a four vector, so anything that transforms like this transforms like a four vector.

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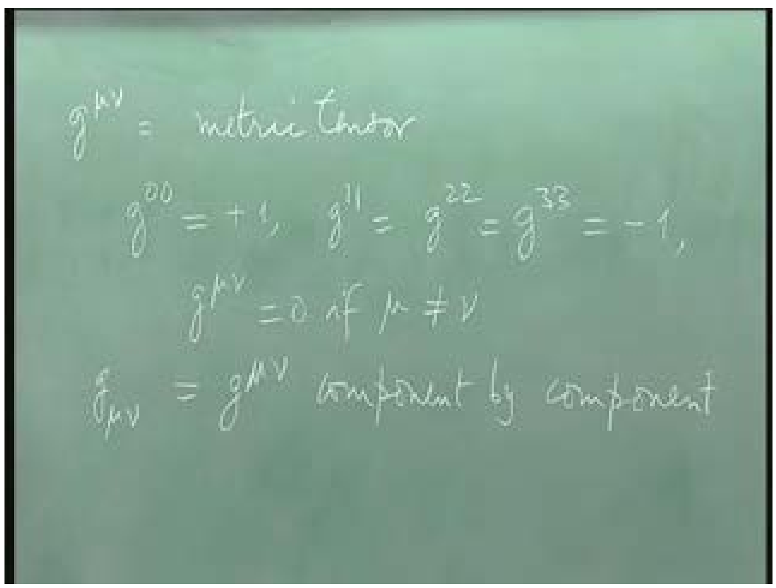

Now, this quantity g mew new is called the metric tensor and it is numerical value is the following g 0 0 is plus 1, g 1 1 equal to g 2 2 equal to g 3 3 equal to minus 1 and g mu nu equal to 0 if mu naught equal to mu. All my creak indices mu nu sigma row extra will run from 0 to c 0 is the time component, 1, 2, 3 are the space components. So, this metric tensor and it is analog the downstairs metric tensor, I will explain what is mean by putting indices downstairs, equal to g mu new component wise.

In the flat space time, we are talking about where special relativities valid it is useful to introduce this quantity g mu nu called the metric tensor the two versions of it. There is a version with super script indices and this version with sub script indices will become familiar with these this distinction in a short while. But, as far as the tensor g mu nu concerned these are the numerical values. So, g 0 0 is also plus 1 g 1 1 equal to minus 1 and so on. Now, the reason for writing this is because in general it turns out that when you have manifold, namely spaces with smooth smoothness properties different points.

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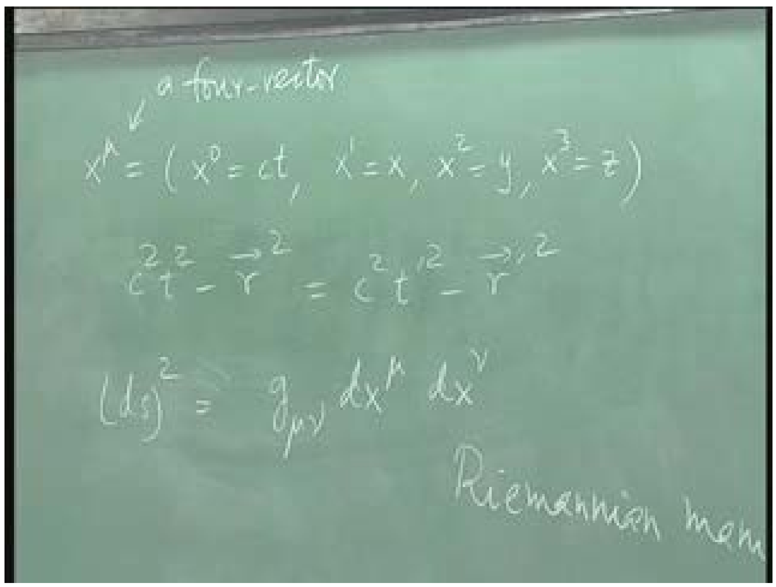

Then the square of the interval between two space time points in this manifold can in general we written down as g mu nu d x mu d x nu. This is the general statement here a whole class of spaces exists called Riemannian manifold for which the infinite decimal separation between two points in this manifold. The square of that separation can be written in this form, it is a scalar and can be written in this form here dx mu is the increment in x mu d x mu using increment in x mu. And mu and nu run over the dimensionality the indices the set of indices label in the dimension of the space in this case 0, 1, 2, 3.

Such a general statement is called Riemannian manifold, general space is the Riemannian space or a Riemannian manifold. In our case in our special case where the space time is flat, what we say technically a flat the coefficients of this of whatever multiplies d x mu d x nu this set here. The coefficients are constant independent of the coordinates themselves but, in general you can consider for situations where this g mu nu is the function of the space time point, after point labeling in the manifold itself.

Then of course, you do not have flat space time but, our special case has this, in general relativity when you include the curvature of space time this quantity will become this thing here will become a function of the coordinates themselves. In fact, I can give an example of that in a situation yeah just a minute in a situation, which are familiar with; let us just look at two dimensional space, the real ordinary two dimensional space Euclidian space, but now the surface of the sphere.

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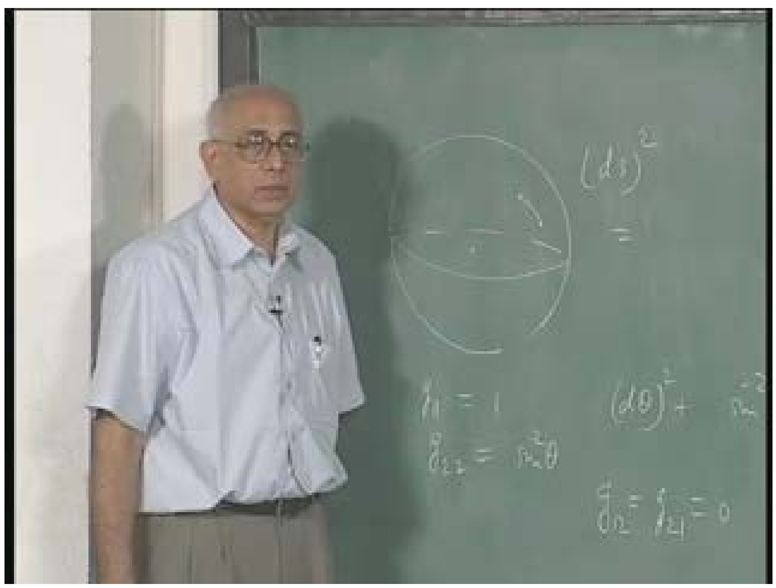

So, here is the unit sphere and on this sphere I have points and I would like to label these points here. Now, how do you write down the distance between two points, what do I mean by distance between two points all the surface of the sphere like the globe. Well one way to define the distance would be to say it is a shortest path between these two points, that is the distance between these two points.

In the shortest path between any two points, here to here would lie on a great circle, namely a circle which goes through here with this usual grid circle like longitude forms grid circle for example or the equator is the grid circle. So, between any two points you draw a grid circle and that distance is called the geodesic distance or the shortest distance between this two points. You can define geodesic distance is on other spaces as well even if you does different kinds of curvature.

Now, what does, what this thing here means what does d s square in the present case what would this be. If this sphere had a radius r, then you would say that the distance. In general between two points and spherical polar coordinates would be something like d r whole squared plus r squared d theta whole squared plus r squared sin squared theta d pi whole squared.

This is what you would say is the distance written in spherical polar coordinates between two points in three-dimensional Euclidean space. But now you are in the surface of the sphere where r is the constant. So, there is no question of d r these terms go away and you have this thing here if this sphere has unit radius and this r is the constant you will remove this just unit this is it. Now, you see the d x mu that I have x mu the mu in case should really run over just 2 values 1 and 2 one referent to theta, one referent to phi. And this case I would write this metric down in a very simple way I would say g 1 1 is equal to 1 g 2 2 equal to sin theta sin square theta and that is it.

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g 1 1 but I will use the fact that the super script and sub script are exactly the same thing numerically, so this is it and I would say g 1 2 equal to g 2 1 equal to 0. So, you see what is happened here is that the coefficients the g is components to the metric tensor depend on the coordinate they depend on the value of theta in this case here. Therefore this is not a flat space any place any time you have a situation of this kind where these g's are constants you have a flat space. If all the g's are plus 1 then you have a Euclidean space.

But some of them are plus 1 and some of them are minus 1 the number of plus 1 minus number of the difference between the number of plus 1's and minus 1's is called the signature of the space. And where did this minus comes from, it came because the interval is c square t square minus x squared minus y squared minus z squared. So, there is one time like direction which contributes the plus d t whole squared and there are three space like direction which contributes minus, minus, minus.

So, I would say it is pseudo Euclidian, it is not Euclidian pseudo Euclidean still constants, so it is flat. Unlike this case where this curvature in this space and therefore, this depends on the coordinates itself could dependent in much more complicated way. And of course, in this case as in that case you see that there are no cross terms there is no d theta d phi term at all. But, in general there is no reason why they should it be in general in an Riemannian manifold there are cross terms here, this need not vanish and this need not equal to mu.

But in this pseudo Euclidean space of special relativity this is not the case and this is, in fact, the metric as it is says. Now, the use of this g mu nu is that it enables you to write two different kinds of vectors. The moment you do not have a Euclidean space something special happens and you can have two different kinds of vectors I will explain the difference between these two kinds of vectors. Corresponding to this x mu you can also define an x mu downstairs with an index downstairs this is called a contra variant vector and this is called a co variant vector in the old literature. But I am going to use slightly different terms for it this can here.

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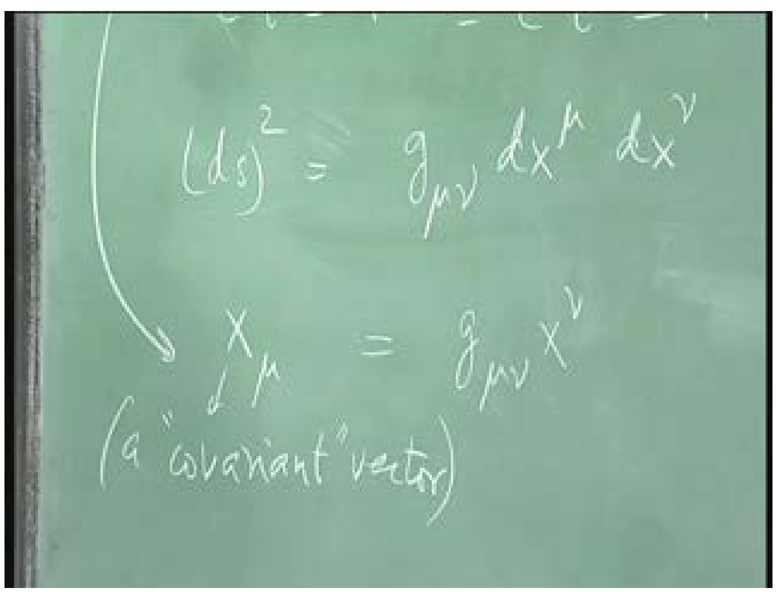

Contra variant this is architect terminology it is defined as follows. It is obtained from this contra variant vector by contracting this index nu in other words every the convention the Einstein summation convention is that every time I repeat an index it is some lower. Over the allowed values which is 0, 1, 2, 3. Just like in this case of Cartesian tensors.

So, what is happened here is that this is g g mu 0 x 0 plus g mu 1 x 1 etcetera. There is one free index mu on the right hand side and one on the left hand side, so that both of them are vectors with the down stairs index and I called those co variant vectors. Now, the real thing that is happening is that, when you have a vector space a linear vector space you have elements in this vector space. Then you can define the natural dual to this vector space which is also a linear vector space.

And you can have objects in it which are the duals of the objects you have the original vector space they called dual vectors. And what is really going on is that these are vectors and these are vectors in the dual space. In the language of differential forms these are called vectors and these are called one form. So, I do not want to get into the technicalities of calculus of manifolds it is really the right language to look at over this whole thing.

But we will use this somewhat old terminology, but I would not call it vector contra variant vector and a covariant vector I just use loosely called both of them vectors. Or if I want to really be careful and distinguish I call this a vector and this is a co-vector. So, that is the better terminology co-vector, whereas here I just call it a vector on that side. Now, what is this give you, this is equal to what well if x mu has these components then the question is what is this x mu have and this is equal to g mu 0 x 0 plus up to g mu 3 x 3. But, since g vanishes unless the two indices are equal they only thing that contributes here is the 0 component here.

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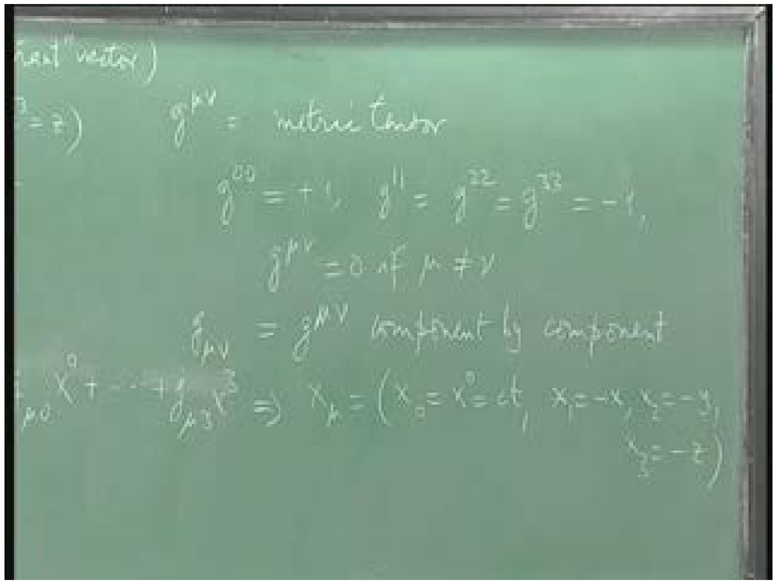

So, this immediately implies that x mu has components x 0, which is also equal to x 0 equal to ct component wise, because g 0 0 plus 1. And it has minus x 1 equal to minus x x 2 equal to minus y x 3 equal to minus z. So, the co-vector the space components of a co vector are the negatives of the space components to the corresponding vector.

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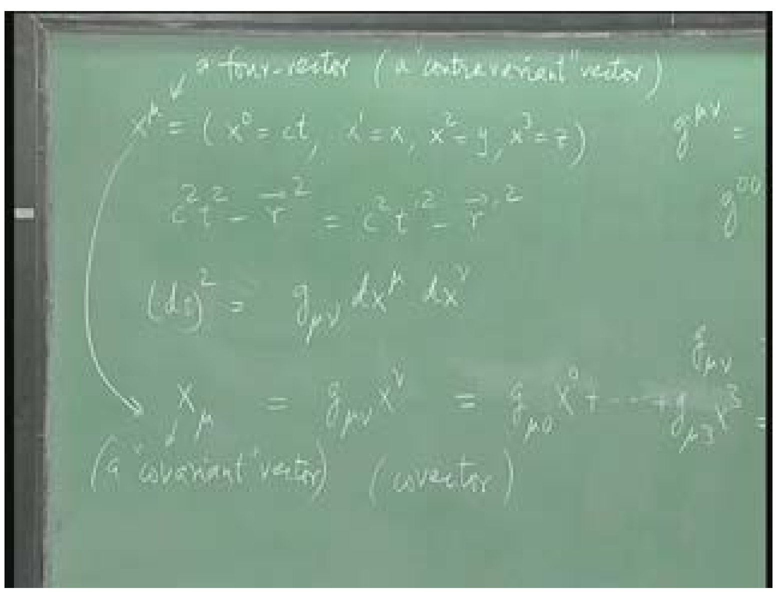

Now, what is the advantage of doing this, the advantages I can write dot products in a very simple way. As you know in a linear vector space, the only way you can found the dot product of two vectors the inner product of two vectors always is to take one object from the vector space. A dual from the dual vector space and you contract that two and you call it an inner product or scalar product.

Now, the reason you do not see this when you do ordinary vector algebra in three Euclidean dimensions is because Euclidean space itself do it. And therefore, it turns out that the co variant in the contra variant objects looks exactly the same. So, when you do the a dot b you imagine that a and b are both in the same space. No, the a is in the dual space is an element of the dual space and the b is an element of the original vector space.

But, when you have more complicated objects or vector spaces this difference is definitely there. If you are familiar with quantum mechanics or better still, there is one way to look at it is to to keep its distinction is the vectors you imagine to be represented by column vectors, column matrices, and the dual objects you imagine to be represented by the corresponding rho vectors. Then of course, as you know to find the scalar you need a row in the left and a column on the right. So, that is all it is.

In the case of quantum mechanics, you know that these vectors are the really kate vectors in some (( )) space and the dual space gives you the. So, called brak vectors and what you have to do is to take dot product of the brak and the left and kate on the right gives you a number. It is exactly the same thing with co variant and contra variant vectors. So, you need to contract this object you need to find the dot product of two vectors you need to take co vector and a vector and dot the two and contract with it.

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That is right I am, no I am saying that this x mu, good question is this x mu a vector or is it a component is his question. So, when I give you three Euclidean dimensions and give you a vector I give you a set of three vectors, I am going to loosely call that x mu a vector actually it is a set of four quantities that is the vector, good point. So, I am going to use an notation for it. I will loosely call a component of vector a general component of vector, but I should not do that. I should really write x as a vector.

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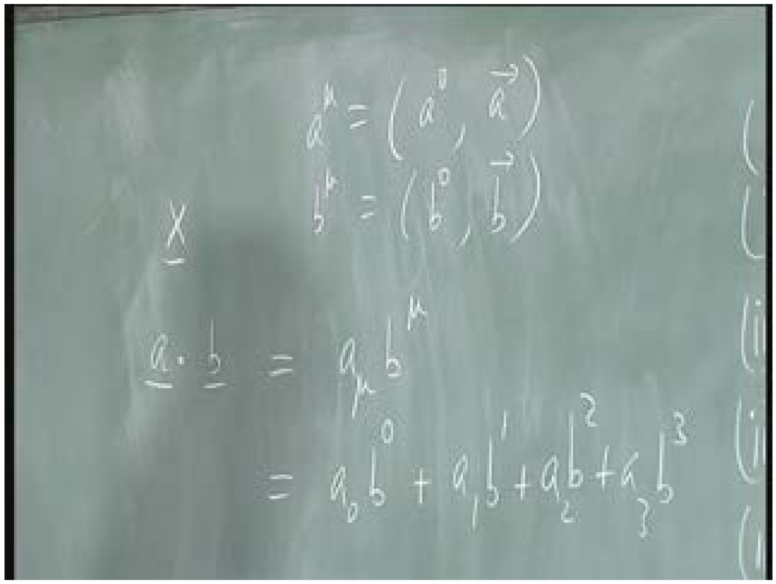

So, I will use x under bar for a vector when I ask what is a dot b two vectors. Really I know that one of these has to be a co vector and the other has to be a vector. So, this is short hand for a mu b mu and mu summed over just has an Cartesian in the Cartesian case I would write x i dot y i and call this is the x dot y. But, you see this is equal to a 0 b 0 plus a 1 b 1 plus a 2 b 2 plus a 3 b 3.

But, my vector a mu a is really a set of four quantities, which is a 0 and three space like components, which I will combined and call them ordinary three dimensional vector a b also is b zero b in this passion. But I am going to use the same notation, because there is no confusion that can arise, I am going to use the same notation for the dual vector also. So, that is the reason I preferred to write this as a mu and b mu. But, that is abuse of notation to some extent because you can see it is really a set of four quantities and not a single component.

But, if I write it in this passion, then you can also ask, what is a mw what does that comprise of what is that equal to this is a 0 downstairs. And then a 1 a 2 a 3 downstairs indices, but we know those are minuses of the upstairs indices. So, this stands for a 0 equal to a 0 numerically minus a minus. And similarly b mu stands for a set of four quantities b 0 equal to b 0 minus b and now what does a dot b stands for, what does this equal to.

This is a 0 b 0 alright the time component scale multiplied and then this is a 1 a 2 a 3, but those are minus of the components of a. So, it is minus a dot b in the ordinary three dimensional sets. So, you see this minus sign, which was bothering as, this minus sign has been taken care of by the notation automatically. So, that is the advantage to doing this whole thing. when you define a scalar product normally you would say its a one component of this times the one component of that plus plus plus etcetera, but in this metric you need a minus and that is automatically taken care of by realizing that you always contract a downstairs index with an upstairs index. So, that is the rule for contraction you cannot have two the same index appears twice upstairs or twice downstairs. Once downstairs once upstairs but it is not hard to see that this is also exactly the same as a mu b mu. So, which one is downstairs and which one is upstairs is relevant unlike the case of brazen kits, whether there is no this kind of symmetric does not exist in this particular space it does.

Because everything is real to started just as an assign you know that in quantum mechanics psi with phi is equal to phi with psi complex conjugate complex conjugate, but this is real vector spaces. So, that distinction is not there and therefore, a dot b is same as b dot a and this equal to this number here.

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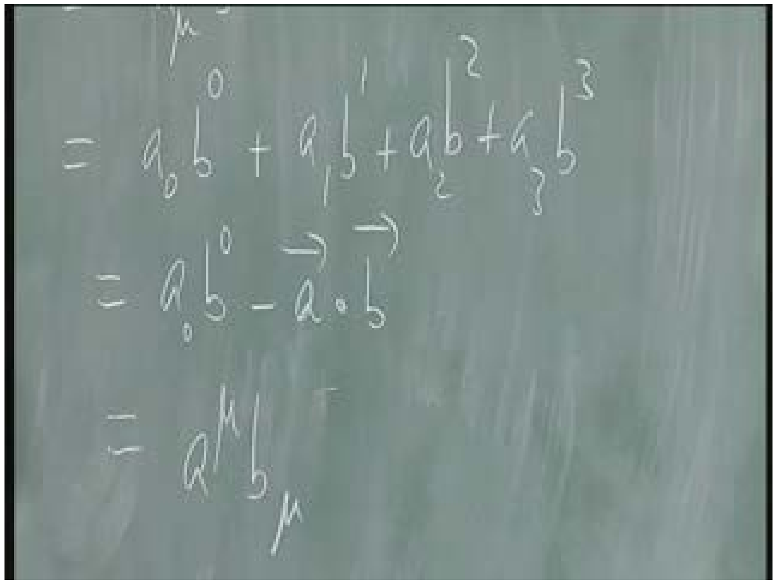

This is a dummy index, so I can use any thing I like call it sigma, rho whatever you like it just contracted over, there is no free index left here this is a scalar quantity. And it is invariant under Lawrence transformations. It is of course, invariant under rotation if time access is not affected at all, you just rotate the co ordinate access then this portion is not affected at all and a dot b is invariant under rotations.

So, you can see this automatically preserves distances and rotations, automatically like in case of usual three dimensional vectors. But over and above that it says this combination of 4 points of quantities is invariant under Lawrence transformations. So, this g plays the role of lower in the index. Similarly, you can rise the index if you give me the upstairs vector.

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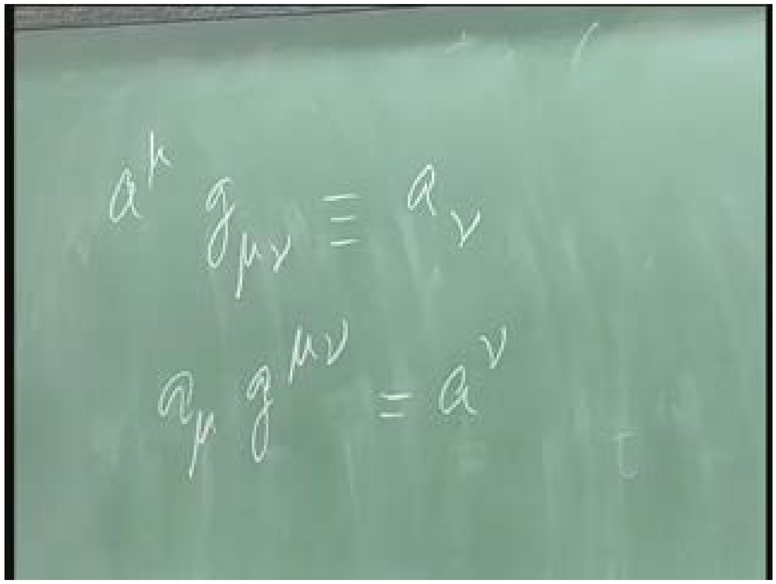

So, you give me any vector a mu and I do g mu nu this is equal to a mu automatically by definition this is true. Similarly, if you give me a mu and I do g mu nu this is equal to a mu. So, please notice that mu is summed over the repeated index appears one upstairs once downstairs is summed over what is left is an upstairs index therefore, this must be upstairs. So, the rules are very simple, if an index appears once it is a free index, if it appears downstairs once on the left, it must appears downstairs once on the right, similarly, for an upstairs index.

If an index appears downstairs and upstairs once in the left hand side of the equation is summed over it is contracted and it is gone. If it appears three times, then you made a mistake is as straight forward as that, now

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We are not going to talk about that at all, I am not going to ask I am not going I am not going to do that here; because I am not going to talk about general covariance, what you expect of this g what it is determinants is I am not discuss this at the moment. If time permits I will come back to this. Little bit of algebra now, let us little bit of manipulations, what is this equal to g mu nu g mu rho equal to what would this be.

Well now you are contracting g mu whatever is summed over is called an index that contracted over it is just contraction appears. So, now whatever appears on the right hand side must have an upstairs index rho and downstairs index nu. So, it is a mixed tensor it is neither fully covariant nor fully contra variant but, a mixed tensor in the whole tensor language.

Now, what would this be it is; obviously, a set of sixteen quantities, because mu goes 0, 1, 2, 3 rho goes 1, 0, 1, 2, 3 and new is summed over in between. But you can easily see that whatever be the values of mew and rho this guy is 0 unless mew equal to new and this fellow zero unless mew equal to rho.

So, this implies that the entire symbol is 0 unless mew equal to rho. And when mew equal to rho the answer is 1, because whether it is minus 1 times minus 1 or plus 1 times plus 1 you are going to get 1 in any other case. So, this index this symbol here is really the kronecker delta expect that you must write it in this passion.

Except for that (()) that you must write it in that passion it is the kronecker delta. Another way of saying it is if I take the matrix g, if I write it as a matrix right other g has matrix the square of this is the identity matrix, because kronecker delta is just a component wise way of writing the unit matrix 4 by 4 unit matrix.

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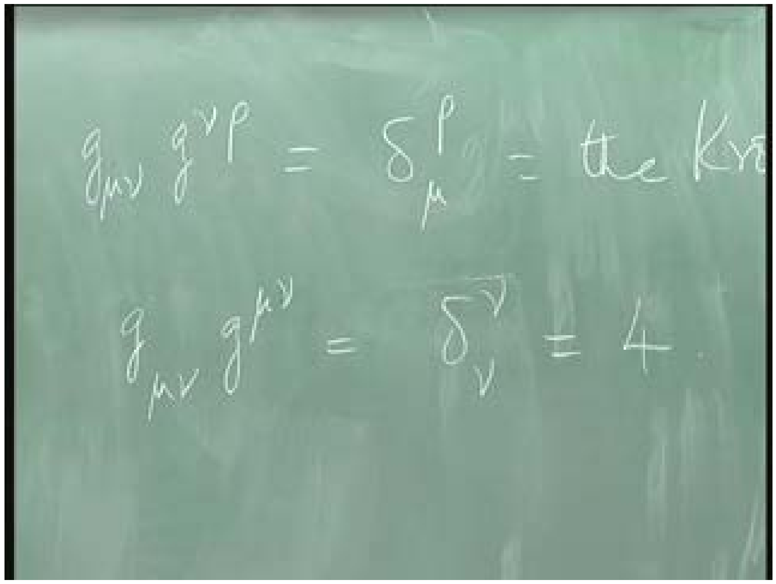

What does this mean this is equal to g by our rule the index by the way g mu nu is equal to g nu mu I assume this is symmetric. Then it is clear that this thing here if I finish of the mu it is says it is equal to delta nu mu naught that is equal to, but you have to sum over nu. So, you have to sum over it over as many dimension as there are, so what is that equal to it is equal to 4. So, the complete contraction of this g gives you 4.

Now, we begin to see how a large number of things that we write down in normal physics, in normal relativity physics once you include relativity the notation this notation makes things much more efficient to write. So, let me start giving you examples, but before I do that, we need to ask also we have the symbol for a co ordinates x mu what about the derivatives what about the gradient operator that is less trivial.

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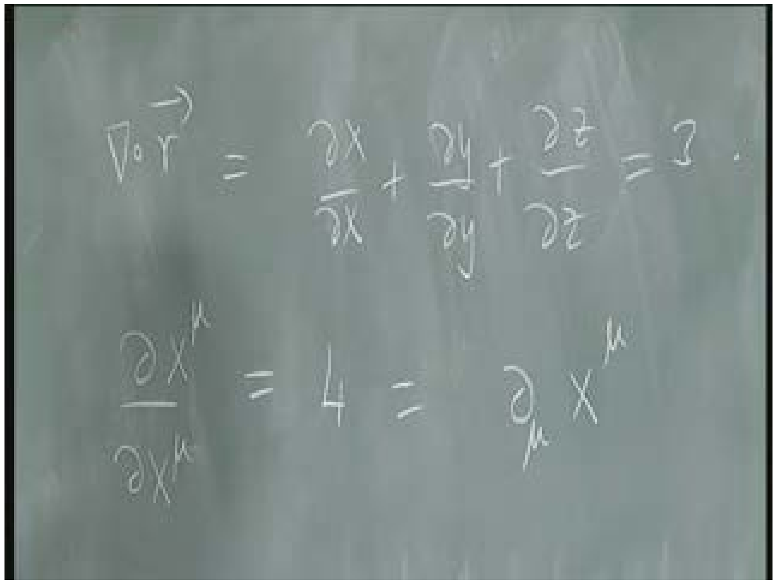

So, let us try to find that what I mean by the divergence of something please recall this, please recall that del dot r in ordinary three dimension stands for delta x over delta x plus delta y over delta y plus delta z over delta z equal to 3. This of course, an ordinary three dimensional vector calculus, you know this to take this del operator, whose components are delta over delta x delta y delta z etcetera.

Similarly, you can define a four dimensional del operator. What is delta over delta x mu, if I put mu equal to 1 2 3 etcetera, 0 1 2 3, then I would still expect delta x mu delta x mu this guy here should be equal to what 4, because it would be 0 1 2 3 I sum over mu therefore, this should be equal to 4, but this stands for x mu there and I want to use the symbol for the del operator.

So, let me use del here and a mu here in keeping with our convention that contraction is always done between co variant index and a contra variant index. This is consistent provided del mu stands for a set of four derivatives, which are delta over delta x naught delta over delta x 1, delta over delta x 2, delta over delta x 3 provided it is stands for that right which is the same as saying, this is 1 over c delta over delta t, because x naught is c t, when c is constant.

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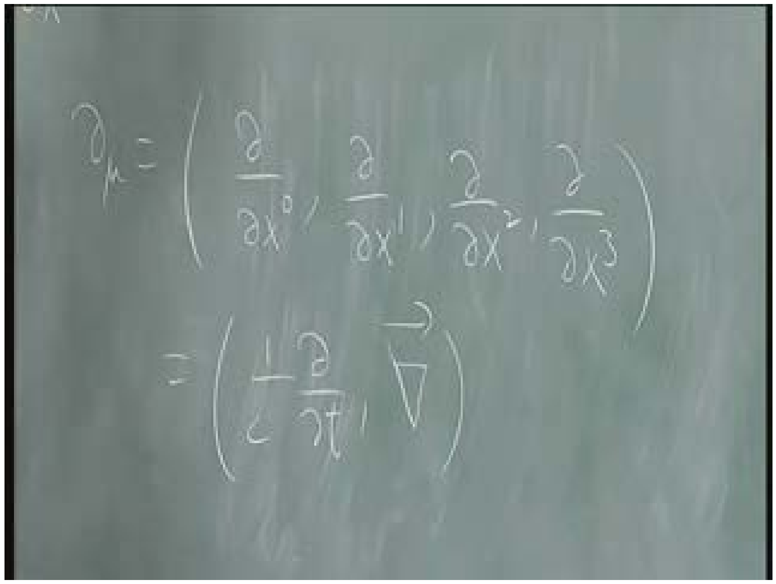

And the components with respect to x y and z you differentiate those three guys can be combined into a del. So, that is very straight forward, but what is del mew upstairs, how do I define this? This is my definition is g mu nu del mu move into raise the index are you have to put a g over del. And what is it is component become this is equal to 1 over c delta over delta t, this remains unchanged, but we know that the special component g 1 1 2 2 3 3 are minus 1's. Therefore this is minus you must remember this this is important.

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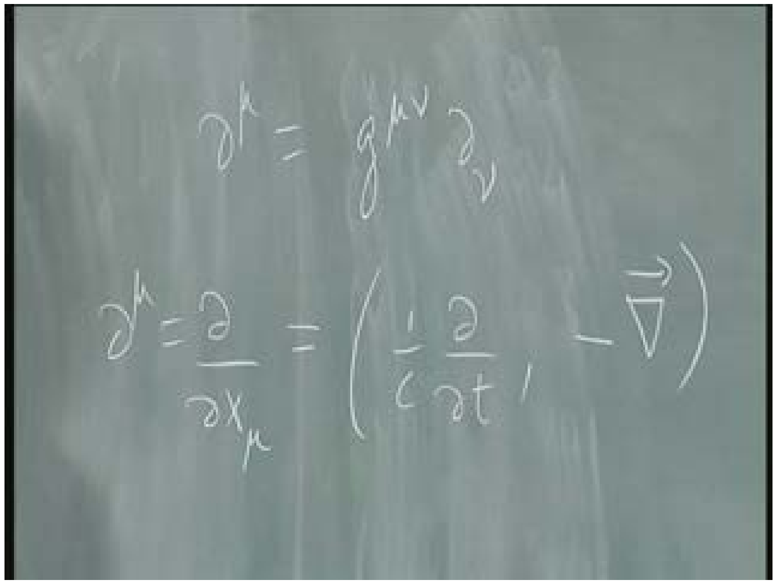

Because it says del mu equal to delta over delta x with mu downstairs here, it is the derivative with respect to the covariant vector here, components and that has a minus sign here contrast this with. So, let me write it just just to empathize this x mu equal to c t r del mu equal to 1 over c delta t delta over c delta t minus del, but x downstairs is c t minus r del downstairs is delta over c delta t plus.

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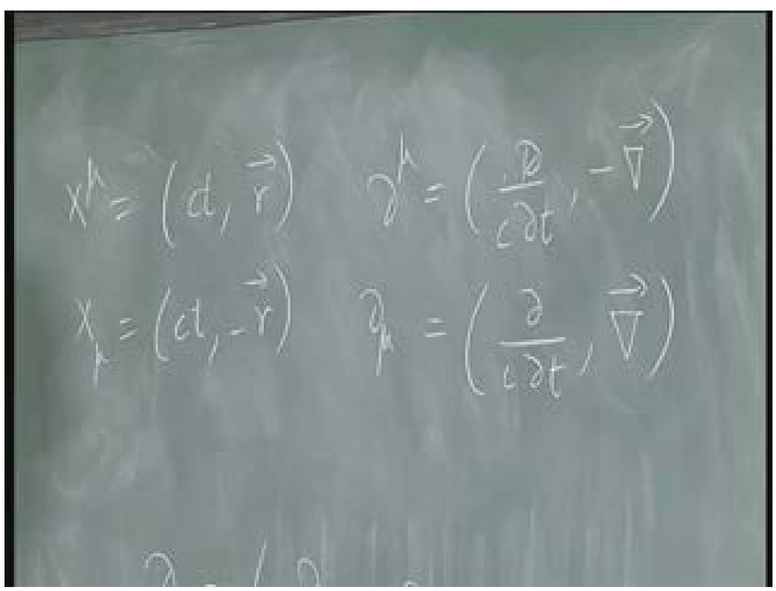

So, this you have to remember that del mu if its put downstairs as the plus del where as x mu if it put downstairs as a minus r. And it is inverted there and this is obvious, because of what I showed you namely delta x over delta x plus 1 by definition. So, with this chariot we can now start writing down all the rules that we know all the laws that we know in relativistic form, provided we identify the right quantities always.

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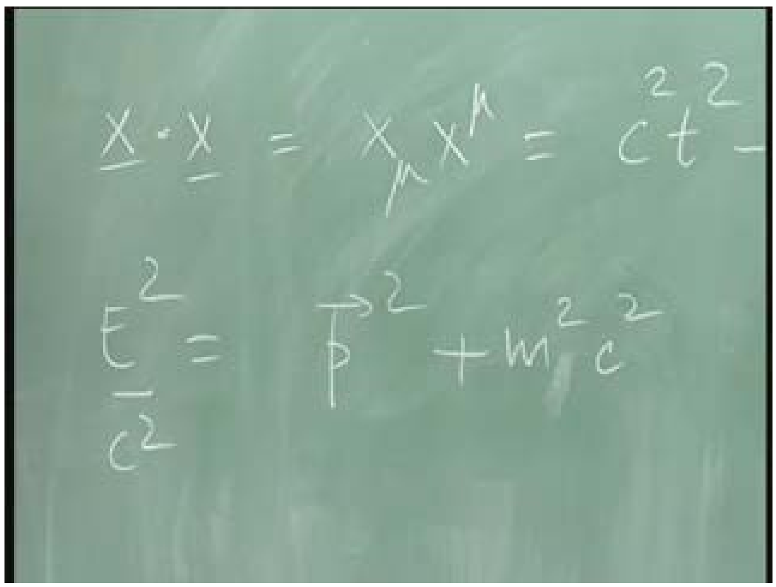

So, first what is x dot x equal to this, is this stands for x mu x mu and as you know this is c squared t squared minus r squared in a given frame of reference. You are familiar with the fact that the energy and momentum of relativistic particle are related to each other by a relation free particle which is different from the one which you have in non relativistic mechanics. And you, in fact, know that the relation is e squared c squared p vector squared plus m squared c four just call m the reference this relation.

And dimensionally you would like to have quantities of dimension momentum everywhere just as I put a c here to make sure that the time component of the vector has exactly the same physical dimension as the length as the space like components here. And exactly the same way you divide by a c square and this relation is this is this. Of course this immediately suggest that this equal to that is an expression of relativistic in variance.

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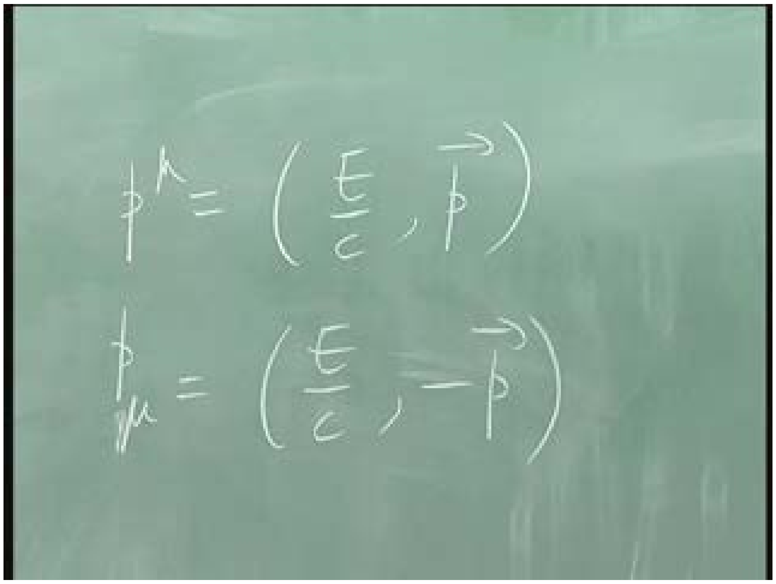

And indeed it is the definition of the four momentum p mu is equal to E over c p, the co vector is therefore, E over c minus p and that help you to identify what this quantity is this thing simply says that p mu p mu equal to m squared c squared. That is the correct energy momentum relationship for a free particle return in manifestly covariant form p mu p mu is a Lawrence scalar. So, it is exactly the same in all kinds of reference.

And it is numerical value is equal to the square of the mass multiplied by c square. So, this is the covariant way of writing this namely the way that shows explicit Lawrence invariance, because this guy here is the scale Lawrence scale. Similarly, look at some more rules here, we know that the charge density and the current form of 4 vector current j mu.

Together these are the natural quantities that combine to form a four vector current, then what is this, what is del mu j mu equal to it is the divergence the four divergence of the four vector current. And this stands for delta over 1 over c delta over delta t of c rho, plus that is important del mu downstairs is a del operator that dot j and this is equal to delta rho over delta t plus del dot j. And the equation of continuity says that if there are no sources and signs in a given region, then this quantity is equal to 0.

But, that just an expression of the fact that the four dimensional current the fourth current is divergence less but, since that is the Lawrence invariance statement the equation of continuity is valid in all frames of reference. And as you know leads to charge conservation that should not depend on a frame you are in. So, this explicitly shows that the concept like charge conservation is Lawrence invariance it is the same for all frames. So, if charge is conserved on one frame it is conserved in all other frames as well. And this is the very simple way of writing the electro the continuity equation.

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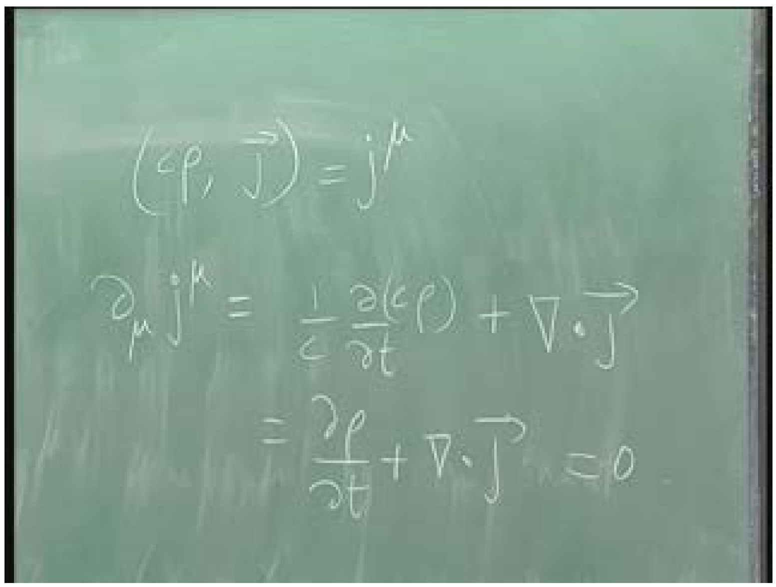

In the same way it turns out that you can define a four vector potential and that is equal to phi over c the three vector potential three dimensional vector potential. So, these two combines to turns out the scalar potential, and the vector potential in electro magnetism in to the components of a four vector potential; the 1 over c phi sees to it that have a same dimensions physical dimensions, then notice that we ask talked about the Lawrence gauge.

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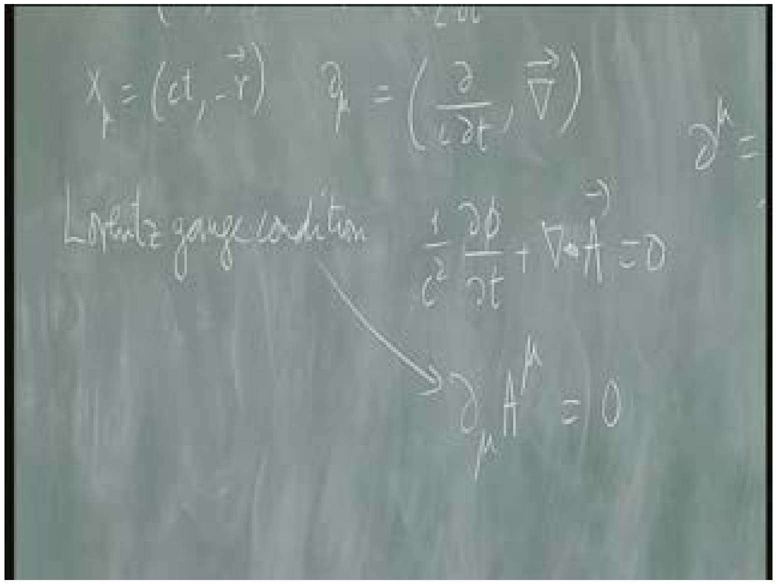

Under Lawrence gauge condition was 1 over c squared delta phi over delta t plus del dot a equal to 0 might have struck you that is a very strange combination, because I said once you put this condition n then the quantity a obey the vector potential a obeys the wave equation very simple wave equation, but might have struck you that this combination is very strange, but that is not a strange combination, because this thing here is really just del mu a mu.

Because, the 1 over c delta over delta t is del 0 and phi over c is a 0. So, it is del 0 a 0 plus del 1 a 1 plus del 2 a 2 plus del 3 that is del mu a mu. So, this in this is the Lawrence gauge condition del mu a mu equal to 0 very compact way of writing it. And since it is a scalar under Lawrence transformation it is says the Lawrence gauge condition is Lawrence invariant.

Even if you change to another frame you are still in the Lawrence gauge, that is no longer true if you are in the Coolum gauge, where del dot a alone is 0. Then this does not transform like anything simple under Lawrence transformations. This gets mixed up with the time component and it is no longer co variant. So, the equation of continuity turned out to have a very simple form, the Lawrence gauge condition turn out to have a simple form.

The question to ask is, what does the maximal equation, itself look like does not become simple or not. Well before we do that lets ask what Laplacian. So, we could ask what is this quantity, del mu del mu that is a four dimensional analog of the Laplacian operator this stands for second order differential partial differential operator.

So, this is equal to del 0 del 0 which is equal to 1 over c square delta 2 over delta t 2 minus del dot del minus. Because, the upstairs guy is minus radiant and this is plus radiant operator here. But, this is equal to 1 over c square delta 2 over delta t 2 minus del square that is the wave operator.

And it emerges naturally this is got a name since it appears all the time its written as box and its called the d'Alembertian is written in this simple form. Now, of course, we know that if something obeys the wave equation it is a scalar quantity this minus del square equal to 0 then it says box on that equal to zero that is the simple thing. But we know that the Maxwell equations, the Maxwell equations themselves the two that are left after you get rid of the after you eliminate things in favor of the scalar and vector potentials. And you go to the Lawrence gauge the maximal equation gave you 1 for 5 for the scalar potential and other equation the wave equation for the vector potential.

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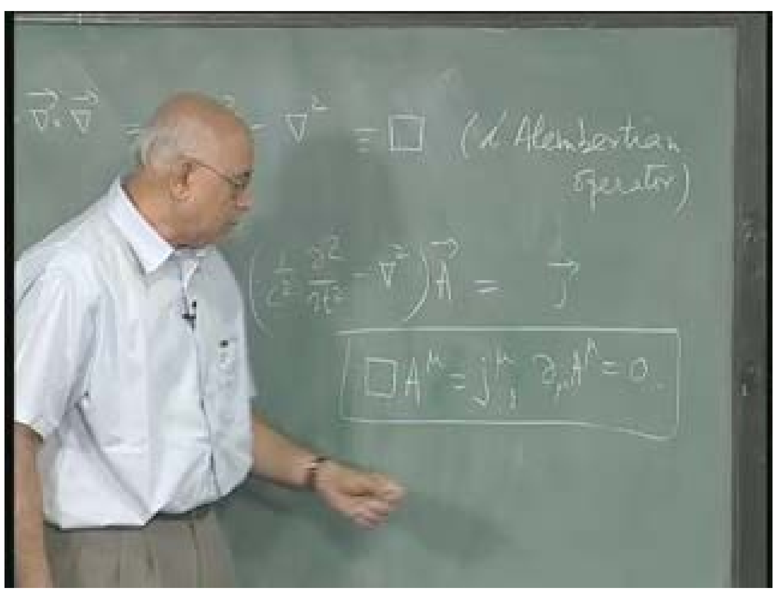

For the vector potential you ended up with 1 over c square d 2 over d t 2 minus del squared on the vector potential A equal to j in some units I do not recall the units now, but may be mu not mu not there j in this passion. And similarly there was an equation phi now verify that both those equations can be combined. And the equation is just box A mu equal to j mu that is it. That is the set of Maxwell equations all of them.

So, once you go to the Lawrence scale by imposing this condition then the Maxiwell equations simply saying free space simply say the other passion on A mew equal to j mu. This is the source and that is the field that is it. Of course you got to go back and look at to what happens to e and b. And e and b are three dimensional vectors they do not look at anything like this at all.

So, we still have to answer the question of what to do about e and b if you talk about a minute. But, I want you to appreciate the fact that this is the manifestly Lawrence invariant way of writing Maxwell equations. Because, every quantity that appears here has known transformation properties under Lawrence transformations. This is the scalar operator that is a four vector that is a four vector there and this is the scalar quantity. So, it is called manifestly covariant form of maximal equations.

In fact, we can go little further we would look at this equation here, this guy here and put in a little quantum mechanics and look at what is going to happen, we jumping a little bit. We will put in a little quantum mechanics. As you know in quantum mechanics you cannot describe particle vectors trajectories you have to use wave functions for that this is certain fussiness associated with things. But these physical quantities like energy momentum and so on. Become operator which act on a wave function.

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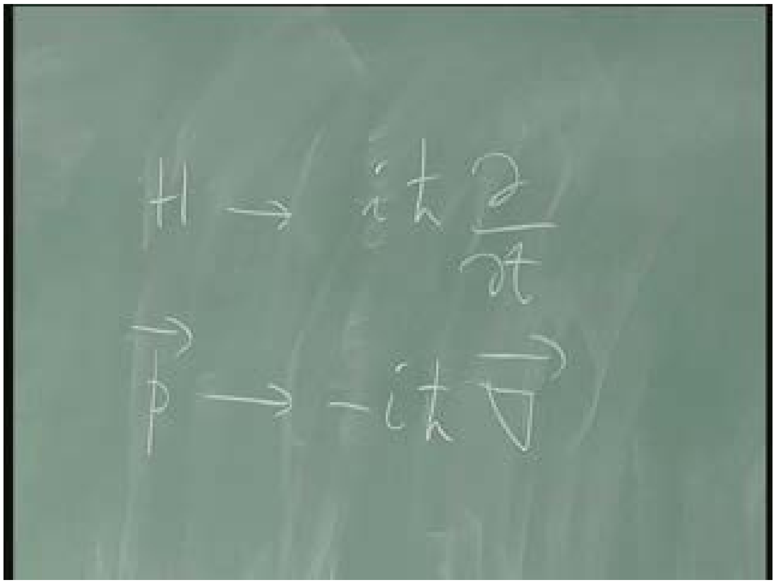

So, let us see where this gets us, remember that the classical quantum correspondence says that the energy are better say the Hamiltonian is replaced by i h cross delta over delta t. So, this physical operator called the Hamiltonian, when it acts on the wave function it is like acting on the wave function by i h cross delta over delta t.

On the other hand the three momentum is minus i h cross the gradient. These are the standard correspondence is which you learn from the Einstein (()) for example. Once you do this and you act on a wave function you get the wave equation extraordinary equation for instance. Now, we are trying to do this in the relativistic case, so we would like to do it here.

This E stands for Hamiltonian it is the energy here. So, what does this tell you together these two guys since this is E over c c times e these two guys corresponds to saying that, you have the correspondence p mu minus p mu to i h cross del that is the correspondence. But please notice that the change in sign is automatically included.

Is a plus here and a minus here, but it is already here, because it says the space component gives you i h times the space component of this del, but there is a minus del. So, this is already being included that. Then this relation here when I go to quantum mechanics should really be written minus I p mu del mu that is it.

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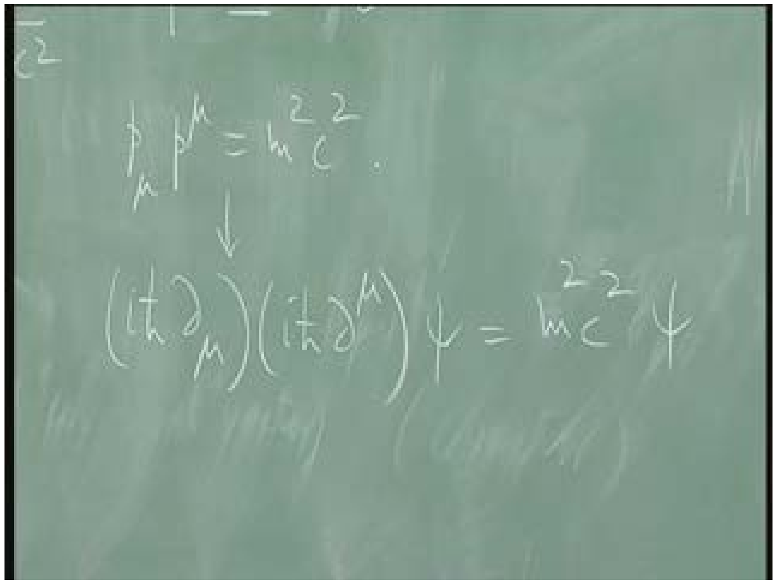

I just lower it putting the g and gives mu and i h cross del mu. So, this says i h cross del mu i h cross del mu on a way of function. Let me call psi equal to m squared c squared on a wave function psi, which is now a function of space and time coordinates this would be the relativistic wave equation for a free particle free relativistic particle. Where does where does that get you this gives you a minus sign.

So, lets bring it over to this side and it says del mu that is box, box on this side plus m squared c squared over h cross square psi. So, that is the free Schrodinger like equation this equation has a name for a relativistic particle its called the Klein Gorden equation. What is the physical meaning of m c over h cross or h cross over m c.

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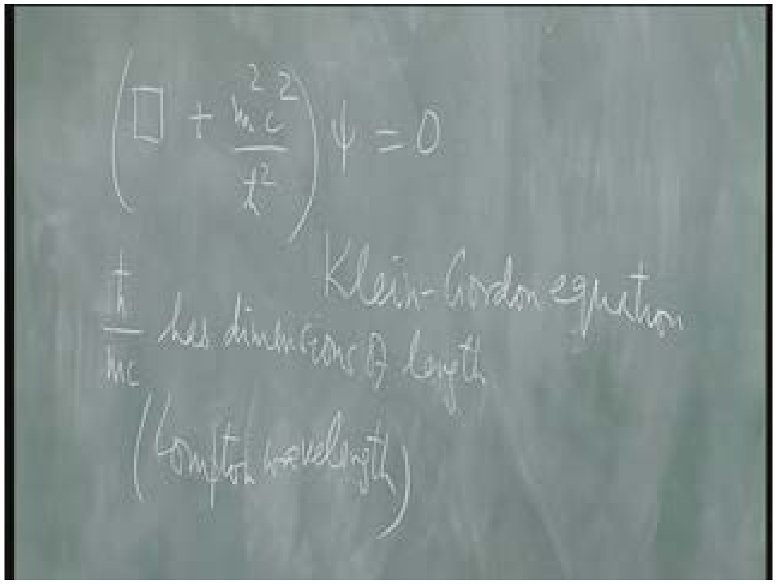

This is dimension 1 over length squared, because it has a del squared in it. So, h cross over m c has dimensions of length; m c is what would you called the Compton momentum of a particle. And h cross over m c by the the relation is the Compton wave length, so this is the Compton wave length. To the extent that you can associate a size with the quantum mechanical particle this is the natural size. And this is the Compton wave length what happens if you put m equal to 0.

You get box and phi equal to 0, but that exactly the equation here in free space in the absence of sources. You get box on a mu equal to 0, but then of course, unlike a scalar function phi you have a vector function A here. So, you have 4 equation here those are the free electromagnetic equations they describe a electromagnetic radiation.

The reason you need a vector index there, because the electro magnetism is described by two vector fields E and p and not by a single Lawrence scalar field, which has only four components. So, therefore, this thing here is only an auxiliary potential but you need more than one component it turns out you need for technical reasons you need four components here, then other ways of doing this.

But, this a mew describes the electromagnetic field from which you can derive the electric and magnetic fields. And it satisfies box on a mu equal to 0 to describe radiation. So, when m goes to 0 you end up with an equation just a box on this quantity equal to 0. So, Maxwell equations which would describe which in the case of the source region would be pure radiation in the absence of sources, would describe mass less particles; particles of zero resonance.

So, it is consistent to the facts that the photon has zero resonance, once you remove this it is gone and then you have this three equations, because you would ask what are E and b this is not a very trivial question. It is not immediately apparent how to combine things, but once we know what delta it is not so hard.

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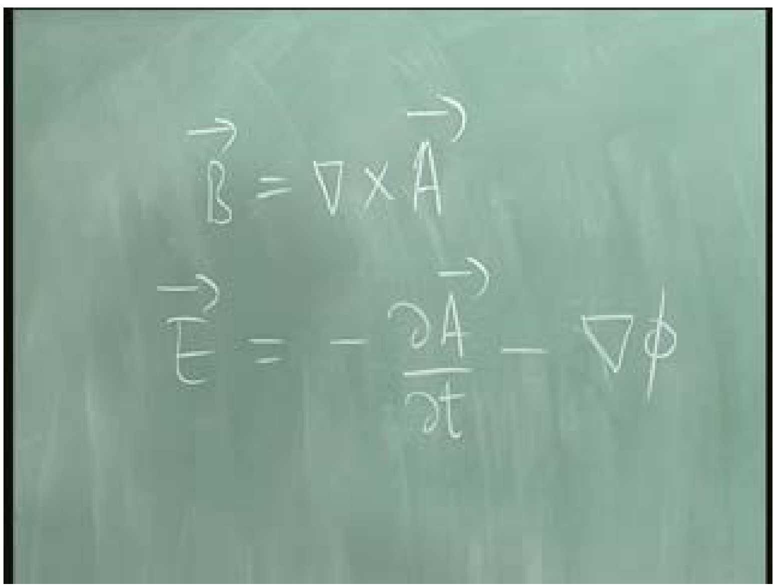

Remember that b that equal to del cross A and you have E equal to minus delta A over delta t minus del pi it is something like this, and these two equations do not look anything like each other very, very different things. Of course we got it in a straight forward way from Maxwell equations. So, you should ask why should Maxwell equations have the form that they had there.

But, you already know that once you do things relativistically Maxwell equations start looking very, very symmetric. Therefore, it is natural to expect that this condition also is some very simple kind of condition here. This is a curl condition here, but if I write this out write this out in terms of vectors and things like that, it says B any component of it.

Let us call it B i then now, put super script of this guys equal to it is a cross product here. So, we should really be careful here this equal to del del I del, we write this out explicitly. So, B x equal to delta over delta y A z minus delta over delta z A y and cyclic permutations. And similarly E x has space component of a differentiated with respect to time component of x mu and then the time component of x mu differentiated with the space component of the del operator.

So, it is suggest this is also like a curl it is mixing things up and indeed what we can do is is to define a tensor f mew new definition as del nu A mu minus del mu A nu. Define in electromagnetic tensor of this quantity it is a rank two tensor and by definition it is anti symmetric. Similarly, of course, you can also define f mu nu equal to del mu A mu minus del mu that just contract this guy over. And then, I want you to do this given the definition of del given the definition of the four vector potential a mu as phi over c A and the del operator as 1 over c delta over delta c and minus del.

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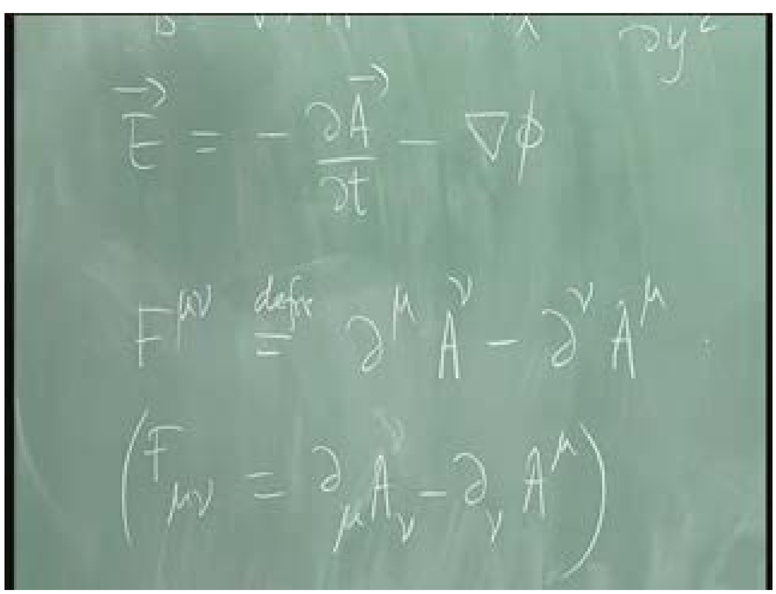

I want you to show that this f mu nu, which is rank 2 4 by 4 tense, four dimensional tensor has how many independent components. Well, since mu and nu to start with run 0 1 2 3 there are sixteen component here. But many of them are 0 the diagonal once are all 0. So, how many independent components does it have? There are sixteen component four on the diagonal are gone that leaves twelve, but it is anti symmetric.

So, the six below are return in terms of minuses of six above therefore, it has six independent components. And indeed six is precisely the number of components you have for three electric field components and three magnetic field components. So, this a is going to give you fully the electromagnetic field.

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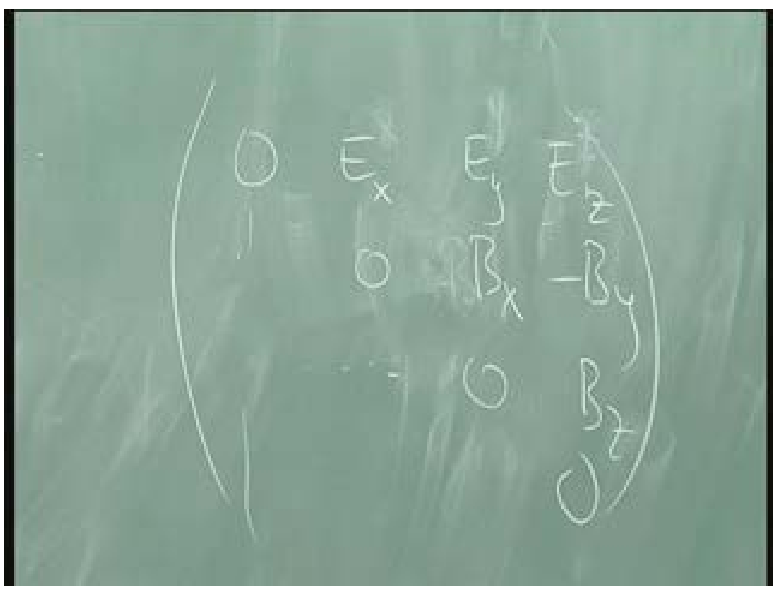

So, apart from science and so, on; I want it to find show that its zero’s here and then there be E x E y E z here with minuses here and then B I do not remember whether this is 2 3. So, this is going to be B x and this is going to be minus d y and d z in the standard way of writing it give what x some signs and so on so forth. This is what would happen to the f mu. So, I leave it to you as an exercise to complete this to write out this tensor an explicitly and write down f mu nu also completely.

The downstairs f mu nu also this would interchange these fellow will become minuses and these would become pluses and there will be some changes here. Time space components of the electromagnetic field tensor this F mu nu is called the electromagnetic field tensor give you the electric field. And the phase space components give you the magnetic field completely.

Once you have that you can ask, what quantity what combination of E's and B's remains unchanged under transformations under Lawrence transformations. What quantity E and B do not remain unchanged and we going to write down very shortly the transformation properties of electric and magnetic fields. This is immediately obvious I have a charge here and I am stress with respect to it there is no magnetic field, I see an electric field and electrostatic field.

But, I am start moving with respect to it to me as if this charge is moving and producing a current and therefore, I see a magnetic field. So, you can immediately see that electricity and magnetism are closely linked and under Lawrence transformations you can expect electric fields and magnetic fields to get mixed up in going to each other. But, there are some quantities, which would remain unchanged and you can guess what that quantity is given, this as F mu nu you can constructs several invariants and check which of them are independent.

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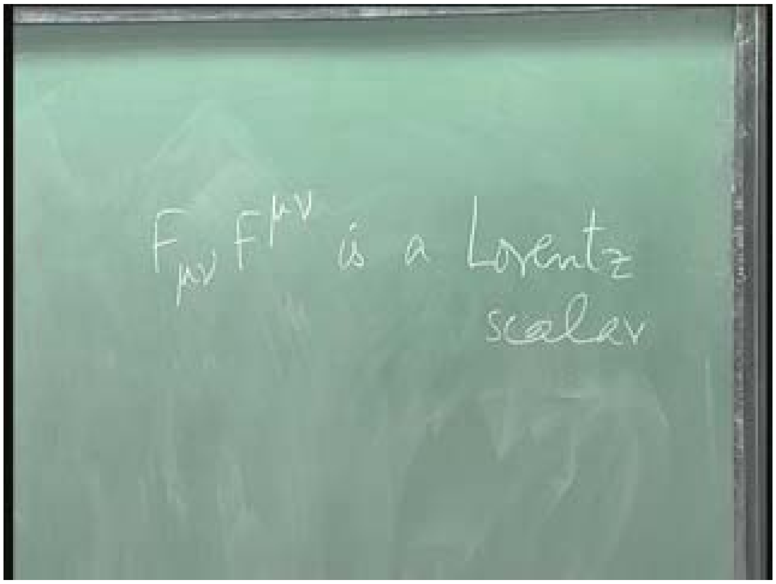

It is immediately clear that F mu nu F mu nu is the Lawrence scalar. So, I want you to write this matrix out the two metrics out multiply them and check out what combination is invariant. Incidentally you should be a little more careful I used the units that almost natural here in terms of c, but if you go back and write the Maxwell equations in terms of mu naught and epsilon naught and so, on. Then you have to be little careful in writing these components. So, there be various mu naught and so on, sitting here.

But, they are little uncomfortable, but whatever it is once you fix those units, I want you to check out what that quantities F mew f mew. Do you think its energy density of the electromagnetic field. Do you think it is, what is the energy density of an electromagnetic field electric field E magnetic field B.

Well in the units that you people use the standard international units in free space it is equal to half epsilon naught E vector square plus half mu naught b vector square right. This is what you call the energy this is the energy density of the electromagnetic field, would this be the same in all frames reference, do you think, what you think well I would say no because based on the experience with particles.

Where I know the energy of the particle is not Lawrence invariant right that is immediately true even in non relativistic mechanics the kinetic energy of a particle is not Lawrence invariant it is a free particle it only has kinetic energy. But, if I start moving along with it there is no kinetic energy in that frame if it is moving a constant speed. So, it is clear the energy is not anything in variant its the four the time component of a four vector.

In the case of the electromagnetic field, the energy density is the 0 0 component of a rank two tensor. And therefore, it is not invariant and yet some quadratic function of E's and B's is invariant. As you can see it is got to be a quadratic function because this fellow has E's and B's this guy also has E's and B's and when I multiply these two matrices I am going to get some combination of E's and B's everywhere.

So, I leave you to work out that Lawrence scalar and we will discuss that tomorrow what the scalar is it will turn out to be not E square plus B squared B squared minus B squared and that is invariant end up with a minus sign here. And it has the significance it is a Lagrangian density of an electromagnetic field. So, that is the density from which you can derive Maxwell equations by using the Euler Lagrangious prescription and that is the same in all frames of reference. So, the Lagrangian density of a field is in variant Lawrence invariant but, the Hamiltonian density is not invariant. There is another constant here which is exactly the same and we will discuss that as well.

You see what looks like light looks like light in another frame also, it should because we started off by making the postulates of relativity. So, if I have a electromagnetic radiation in one frame and inertial another inertial frame it should also be an electromagnetic radiation. Now, what is characteristic of the electric and magnetic fields of electromagnetic radiation, what sort of wave is it? Transverse waves, what is transverse waves mean.

There is a direction of propagation and what about E and B they perpendicular to each other; so, the fact that they perpendicular to each other must remain unchanged. Now, how do you state the fact that E and B are perpendicular to each other in terms of vectors. The pointing vector gives you the direction of propagation E dot B is 0 E dot B is 0.

So, E dot B equal to 0 must be a Lawrence invariance statement even though even though E and B themselves change. This is only a three dimensional scalar but, the fact that is zero must somehow be a Lawrence invariance statement. And we turn out that the other invariant that you can write down from the Maxwell equations is E dot B whole squared that quantity is invariant and if it is zero in one frame it is zero in all frames.

So, E dot B whole squared is the other quadratic invariant you can form from Lawrence invariant scalar that you can form from the electromagnetic field. And we will discuss tomorrow the transformation properties of electric and magnetic fields. That tell us how these fields change, when you go from one frame from to another and how this two constant how this two Lawrence scalar appear, how this guy appear and how the Hamiltonian density lagrangian density appears you will look at that next time.

So, the physical interpretation of these are important, because it tells you that radiation looks like radiation in any other field. You could also ask other invariance is there anything else and we will prove that they are not any more. Just these two fellows two quadratic combinations of E's and B's which are Lawrence invariant with specific transformation properties for the E's and B's themselves.

Finally looks like there are two different ways of writing describing in the electromagnetic field. One is in terms of a vector potential, which has four components and the other is in terms of a rank two tensor which is anti symmetric which have six component. This is exactly like saying once you give me A and phi I give you E and B. Because these guys have specific curls of relation and so, on.

So, there two you have two different ways of describing it and you could ask, which is more fundamental or the equivalent and so on. These questions assume a great deal of relevance in quantum mechanics and perhaps weight around in the quantum course are describe, how this is connected to the question of spin and representation of the Lawrence group. How these two equivalent ways of describing this spin one particle each has it is own physical significance we will talk about that in this stage.

Just one last point and that is when you write the interaction between an electromagnetic field and a charge particle. That too should be writable in an invariant form and exactly the same in all frames.

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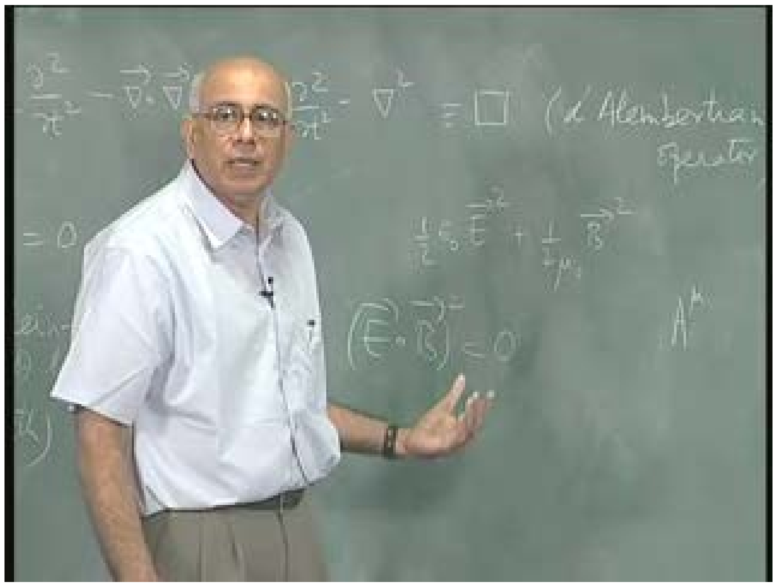

And it turns out that thing that leads to we wrote down the lagrangian of the charge particle you would ask where did this lagrangian come from I tailored it to get the current current Lawrence force equation. But, really I was go back to fundamental principles and say I must find something that is invariant Lawrence invariant. So, the equation looks same in all frames of reference and then the interaction.

Lagrangiam would should have something, which depends on the field and should have something that depends on the particle. The thing that characterizes the field is the four vector potential. And I need something to contract it to get a constant to get a Lawrence invariant quantity a scalar and that is j mu. It is linear in this its linear in that and this is the simplest thing that to couple the two and its called minimal coupling.

You are familiar with it us Amperes theorem. This is Amperes theorem just minimal coupling theorem. So, this is the coupling that used at now of course, the proof of the putting is in the heating and put the sentence uses the rules of relativistic quantum field theory to see if you get correct prediction. And the answer turns out to be yes this is the right Lagrangian in the scales not any more complicated thing in terms of f mew new etcetera.

So, let me stop here and tomorrow we will start off by writing down the equations the first the electric and magnetic fields in terms of the field tensor F mu nu. After which we will look at the f mew new and see what happens to be contracted and so on. And then we would write down the Lawrence transformation properties.