Classical Physics
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Lecture No. # 35

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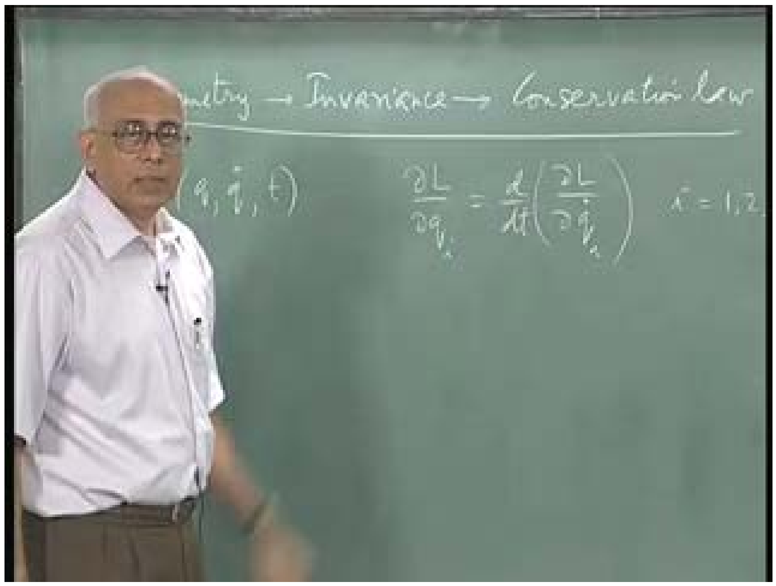

Let us start today with discussing the connection between symmetry invariance and conservation principles, which is what I mentioned the last time. Symmetry leading to invariance leading in turn, leading in turn to a conservation law; and this is captured in this theorem called Noether’s theorem.

The full power of this theorem is actually apparent only when you do field theory continuous number of degrees of freedom both classical as well as quantum. But we will talk about it in the context of plain mechanics Lagrangian mechanics; and the idea is the following we have kind of scattered around this whole thing back and forth without ever formularizing it.

I mention that in Hamiltonian dynamics, you have a set of constants of motion in involution with the Hamiltonian; and if you have a sufficient number of them, then you end up with an integrable system. Now, the point is that every time you have a constant of the motion, not only do you reduce the difficulty of solving the dynamical equations, but you actually get a little more information.

You get some information about the symmetry of the system; and this is the idea behind the constants of the motion. And the idea, the point that they are the significant of the fact that, they are in involution with each other helps you by via the live ornal theorem to show integrability and so on; and we said, we saw a little bit of what the geometrical implications were we should now look like what the algebraic implications are. The idea is that, if you have a constant of a motion in Hamiltonian dynamics that quantity serves as the generator, as a generator of a group of transformations called the symmetry transformations of the system under, which the systems equations of motion do not change at all.

So, generators of symmetry transformation come from constants of the motion. So, that is really what the algebraic significance of the constant of the motion is and we will see examples of this. Noether's theorem helps you to find these constants, so let me start by very knive terms by asking what is meant by a symmetry; by a symmetry, I mean by example for instance some set of transformations under which the problem does not change in a very general sense. If I look at the problem of particle in a central potential then both the kinetic energy as well as potential energy are invariant unchanged under a rotation of the coordinate axis. So, I would say this problem has spherical symmetry, because transformations, which take you from one orientation of coordinate axis to another orientation, do not seem to affect the Lagrangian or the Hamiltonian.

Therefore, you would expect that the set of solutions of the problem would remain unchanged not an individual solution that would of course, change. But an individual solution would go to some other solution look like some other solution under the transformation of coordinates and the set of solution would be unchanged; or the set the equations of motion would remain unchanged in form. This is what I would mean by a symmetry a dynamical symmetry of a system something, where the equations of motion do not change.

The moment you have this symmetry, it means that there is an invariance under a certain group of transformations or set of transformations, Noether's theorem then tells you how to find the corresponding constant of the motion. And this is how it goes, I start by using this Lagrangian mechanics you could translate this to Hamiltonian mechanics subsequently. So, I start with an L, which is a function of some q q dot and if it is non antonymous system there is also a t sitting there then of course, we write the Euler Lagrangian equations down.

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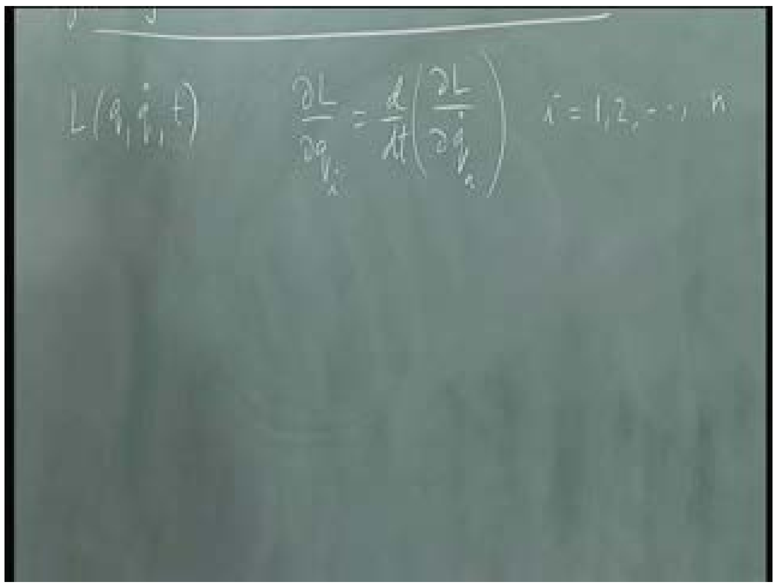

And the Euler Lagrangian equations are delta L over delta q is d over dt delta L delta q dot and this is true for each degree of freedom I running from 1 2 up to n, if you have an n freedom system. And this stands here, this short hand, this is short hand for all the q i's and this is for all the q i dot's. Now, on the solution, on a solution trajectory, this equation is satisfied of course, and of course, the solution becomes unique once you specify a sufficient number of initial conditions.

Now, the point is if I make some transformations such as changing the coordinate system or shifting the origin of the coordinate system or anything else at all I make some transformation of some kind on the variables when I ask, what is the change in Lagrangian as a consequence of this transformation?

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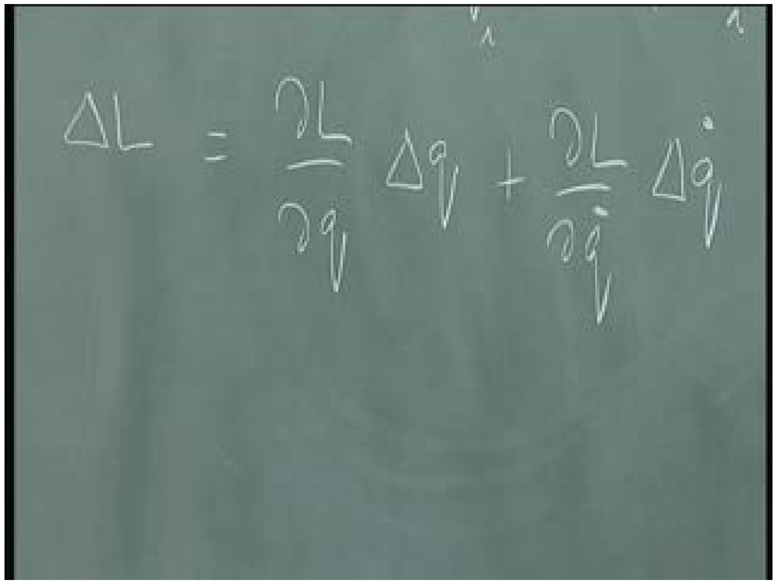

That change let me denote by capital delta L, I do not want to use little delta L, because I use that for variations, which led me to derive the Euler Lagrangian equations let us call it delta l and I will give the examples. This delta L of course, is equal to delta L over delta q delta q there is a summation over I implied here, so we are not going to write this explicitly. If delta q is the change induced in the coordinate q under this transformation plus delta L over delta q dot, delta q dot, if this change does not affect time.

Of course, they saw there are transformations, but you may even change the time you may scale the time or shift the time origin and so on for the moment let us not look at those cases then this is the change in the Lagrangian. L goes to L prime, which is given by this quantity here; but now we are interested in what happens on a solution trajectory always therefore, on such a trajectory delta L by delta q is satisfies this equation here.

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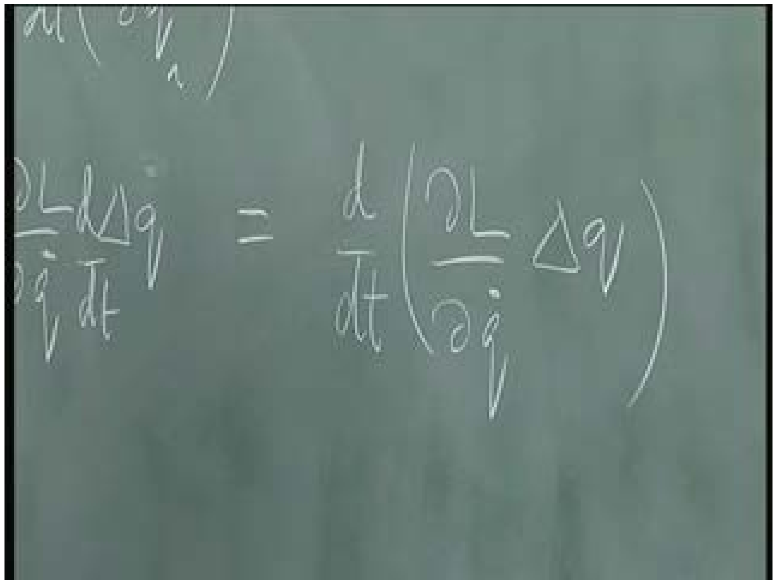

And you put that n and you immediately see this by the way is d over dt of delta q because this change has nothing to do with time evolution or anything like that. I have one set of coordinates I make some change of variables and I have another set of coordinates; and therefore, the d over dt operation here commutes with this delta operation. And therefore, I can write d delta q dot as d over dt of delta q and I put in this expression here and you can see immediately that this becomes a total derivative.

So, I have assume that I make an infinitesimal change of variables, transformation on the coordinates for instance and that is delta q and immediately there is a change in Lagrangian, which is d over dt of this guy here, where you recognize that this is just for the if this were q i dot this is just pi by definition the conjugate momentum.

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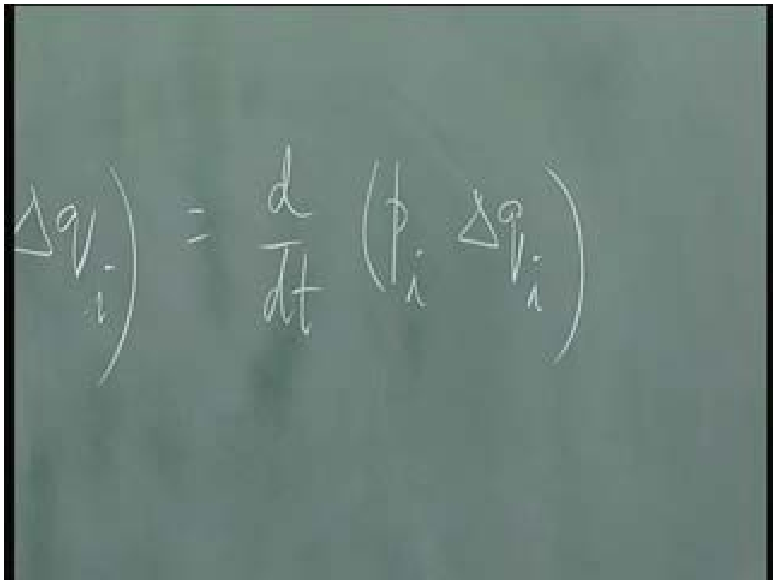

So, we could write this as d over dt, so let us put in all the i's d over dt of pi delta qi. So, using the Euler Lagrange equations I can write an arbitrary infinitesimal change in the coordinates in this form, in the form of a total derivative. On the other hand, I know that the action does not change, and the equations of motion do not change, if the change in the Lagrangian is a total derivative of some function of the coordinates in time.

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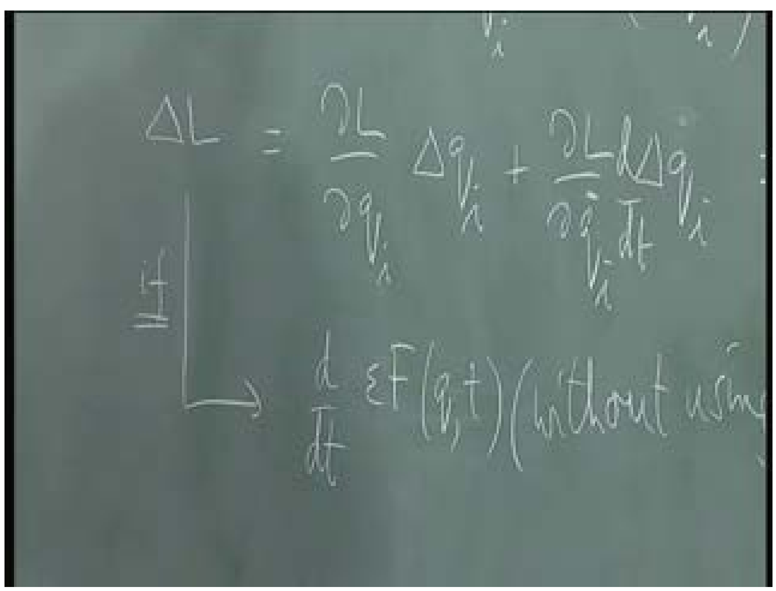

So, I also know that there is invariance provided the equation of motion do not change provided delta L is also equal to d over dt some function of the coordinates in time. And I am imagining infinitesimal transformation, so let me pull out that infinitesimal always and write this as some epsilon times f of q and t. So, this is just a scale factor to tell me this is the order infinitesimal the same order as delta q and delta q dot. So, if is a big if, if under that transformation it turns out that the delta L is of this form without using the Lagrangian, without using the Euler Lagrangian equations.

Because that is been used there without equations using this, then on the one hand using the equations of motion tells you that delta L must be of this form. Directly calculating depending on what the transformation was tells you that delta L is of this form if it is of this form.

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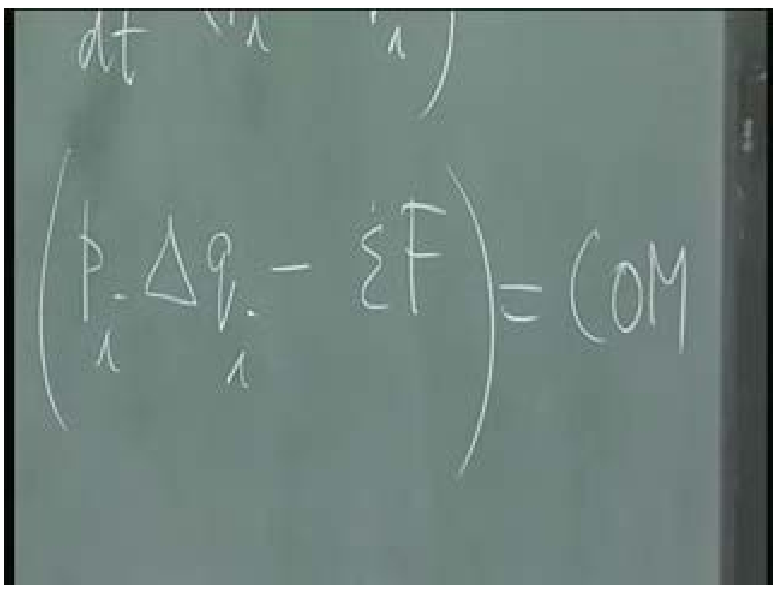

This immediately implies of course, that these two must be equal implies that pi delta qi minus epsilon f is a constant of a motion, because d over dt of that is 0 on the solution set. This is the content of Noether's theorem of course, I have said this in the simplest way possible the whole thing is made much more general it applies to field it applies to relativistic cases and so on and so forth quantum field theory etcetera. But, in ordinary Lagrangian mechanics this is the way the theorem goes on. So, please notice the ingredients that are borne in. On the one hand, if I take some arbitrary coordinate transformation or any transformation, what so ever, and it should turn out first or infinitesimal transformation.

And it should turn out that the Lagrangian change in the Lagrangian is a total derivative set that aside, but then using the equations of motion and fact that L is function of q dot and t the change in the Lagrangian is expressible in this form. Then, equating these two fellows you conclude that this combination must be a constant in time as the system evolves. So, this is how you derive the actual expression for a constant of the motion.

Now, let us try and apply this right away and let us apply this to a simple case of a particle moving in some potential.

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which derivative did I get which derivative.

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If if if it turns out that I have to tell you what the transformation is I put it into the Lagrangian and ask, what is the change in Lagrangian directory without using the equations of motion. And if it turns out to be this, then these two are guaranteed to be equal.

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Because, d over dt of it must be 0 d over dt of this is equal to that, so d over dt of that minus this is equal to 0 total derivative and therefore, it is a constant of motion.

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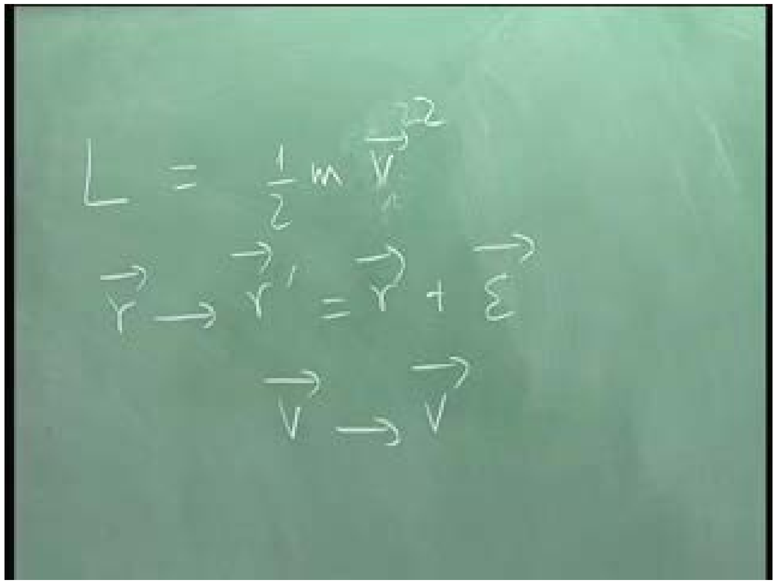

Now, let us apply this I know that if I take a particle in some potential say a central potential for example. So, let us see what happens then so I start with l for a single particle equal to the kinetic energy half m xi dot square x 1 dot square plus x 2 dot square plus x 3 dot square if you like or if you like half m v vector square minus some potential v and let us say it is a central potential.

For example, it does not have to be let us say simple potential its clear that if I change the origin of coordinates nothing should have happened nothing should happen at all. So, how do I show that momentum is conserved as a consequence of this lets take an even simpler example free particle no potential at all. In this case the linear momentum is conserved the reason is that the system is invariant under a translation of the origin. So, let us go from r to r prime equal to r plus a vector epsilon an infinitesimal amount i shift the origin by an infinitesimal amount.

It is immediately clear that v goes to v itself because d over dt of this is 0, therefore this Lagrangian is invariant under a translation of the origin. So, let us plug that in into this business and what do you get here. It tells you that delta L is identically equal to 0 in this case, because there is no change at all r dot is the same as r prime dot. So, delta l happens to be 0, which implies if this portion goes away and you left with this and that is suppose to be constant, so what does that give you.

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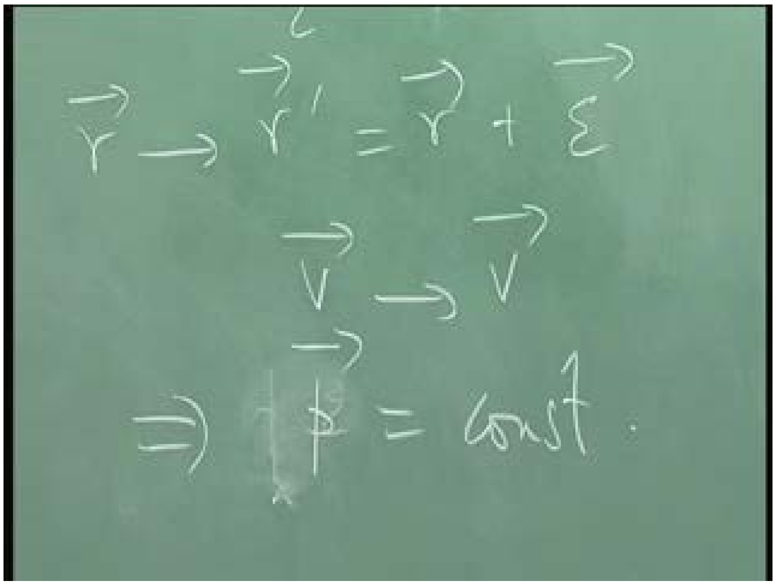

For each component what does it tell you, it says pi times delta qi that is the epsilon, each epsilon independently I take three different directions and I translate and each component p 1 epsilon 1 is constant, p 2 epsilon 2 is constant, p 3 epsilon 3 is constant, so it tells you this implies that p equal to constant. So, this says that, if your Lagrangian is invariant under a translation of the origin of coordinates then the linear momentum is a constant. This is the rather trivial example, we need to do a little better than this. So, now, let us put a potential and see what happens the moment I put a potential linear momentum is not constant anymore because there is a force on the system.

And the rate of change of force, rate of change of momentum is equal to the force, so the momentum is not a constant anymore.

So, let us put a central potential and see what happens minus v of r now this is no longer true. But I know that under rotations of the coordinate system, the Lagrangian does not change once again, that is obvious just by looking at it, because the rotations of the coordinate system keeps distances unchanged. And therefore, v of r is unchanged and definitely p squared is the scalar, so that does not change at all now, what does the rotation do we have to now look at a rotation matrix and ask, what an infinitesimal rotation does?

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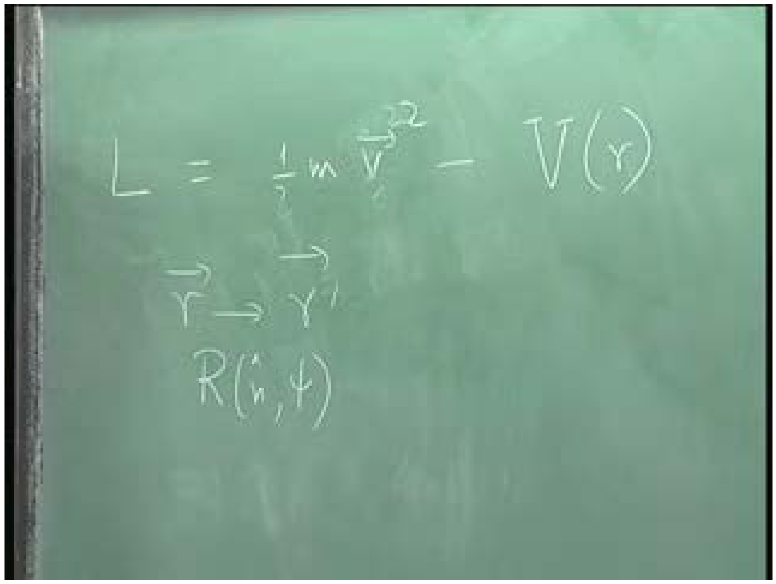

So, let us do that we have r goes to r prime under a rotation r of n psi, a rotation of the coordinate system through an axis about an axis n through an angel psi, we have a general formula for this. We should really need just an infinitesimal version of this, which is much simpler, but I have not derived that explicitly to write rotation matrixes and do it or we could take a special case, we could say the rotation appear occur in the xy plane, then I can write a very simple formula for the yz or zx.

But I want to do the general I want to show you that take an arbitrary direction and I rotate and nothing changes now let that direction be the unit vector n.

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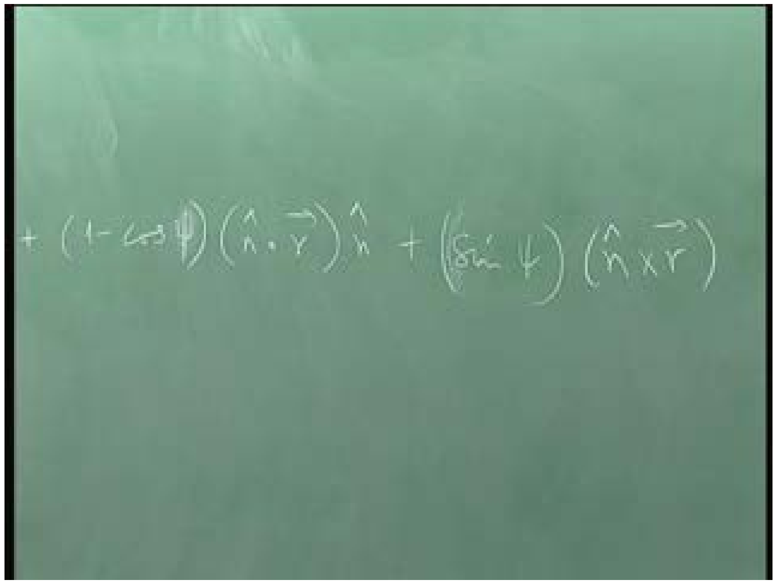

Now, I know that we already have a formula for the rotation of a finite by a finite amount of this vector and if you recall this quantity was equal to cos theta times r plus 1 minus cos theta cos psi 1 minus cos psi and then it was n dot r n plus sine psi n cross r. This is what a finite rotation did through a vector r, I take an arbitrary vector r i rotate the coordinates system about unit vector n through an psi and this is the new vector. I am interested in an infinitesimal transformation infinitesimal rotation.

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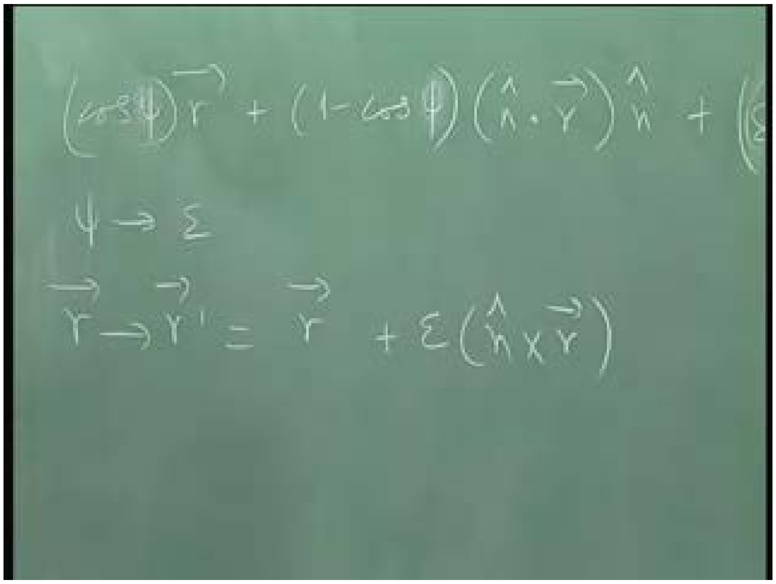

So instead of psi i replace it by an epsilon and work to first order in epsilon, then what does r do? Well, cos psi to first order in epsilon is just 1 itself, so this is r, this goes away this cancels out, but this remains and sine psi is psi itself, so plus epsilon n cross r. It is easy to verify that this is indeed what happens, if you change coordinates if you rotate by an infinitesimal amounts this is what any vector occurs, it is the portion which is proportional to n cross r.

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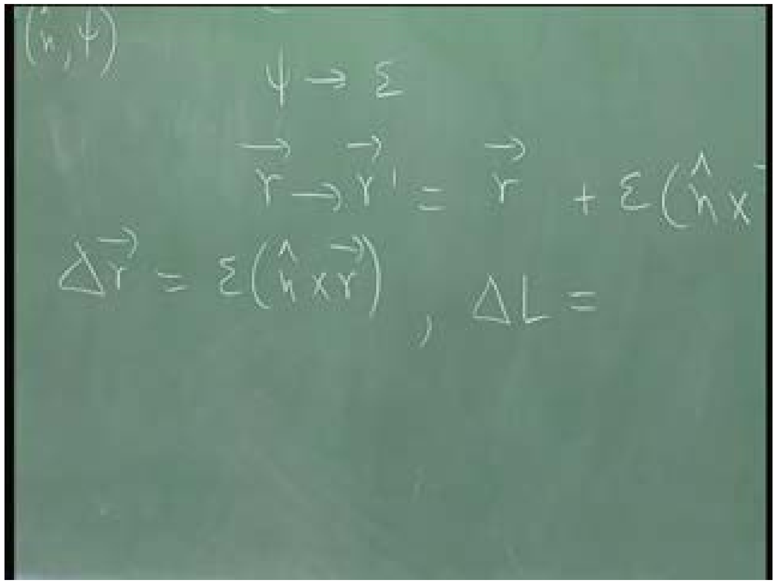

So, in this case, what we know, what the change in coordinates is, we know that delta r is equal to epsilon r n cross r. Therefore, what happens to L then what happens to L? Does this change, does this change at all? It is a scalar, so this does not change at all does that change it does not, because it depends on r prime is equal to mod r prime equal to mod r. So, that does not change. So, delta L in this case is also 0 trivially 0. So, what is the constant of motion here you have, you have only this and this is p dot delta r; and you do for epsilon its true for every epsilon.

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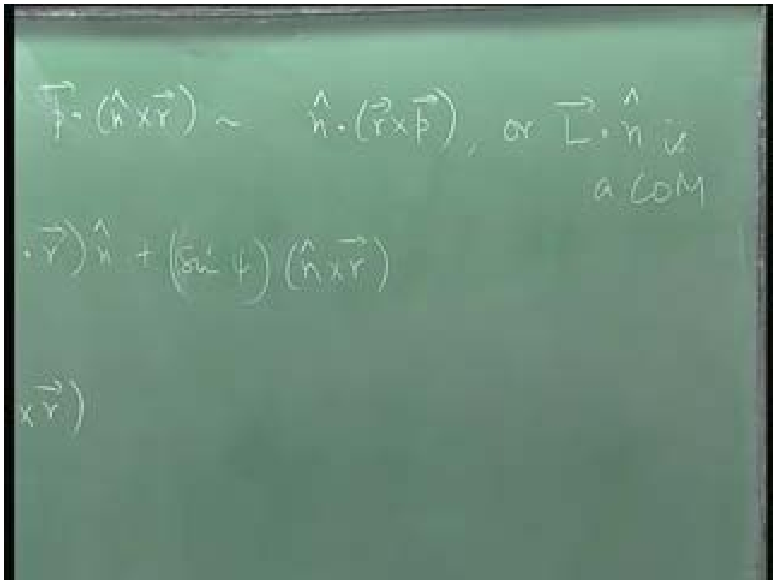

So, what is the constant of motion says p dot n cross r is also equal to n dot r cross p. You do cyclic permutation or L dot n the component of angular momentum about the direction of rotation is constant. But this is true for every n, I could have chosen any arbitrary n therefore, angular momentum is crossed. So, this tells you that angular momentum conservation is a consequence of rotational invariance. Just as linear momentum conservation was a consequence of translational invariance, if the invariance is lost you do not have this anymore.

These were cases were delta L was 0, this portion ended up being 0 let us look at a very simple case and even simpler case than all this where delta L is not 0 and you get a constant of a motion which may even be time dependent. So, let us look at that in fact, it can help you even solve the equations of motion in this case completely trivial.

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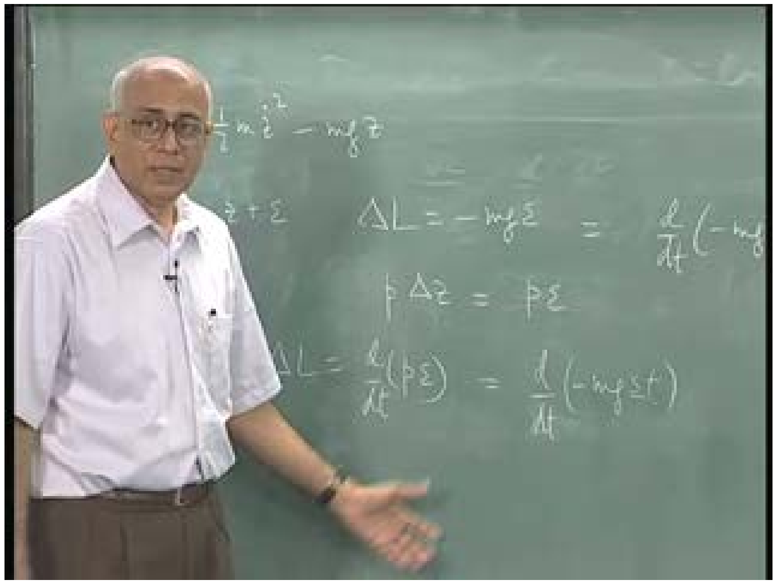

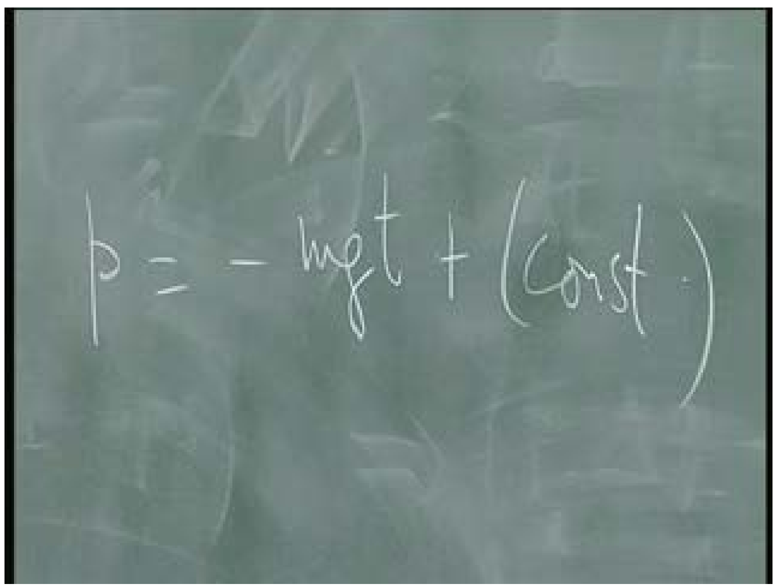

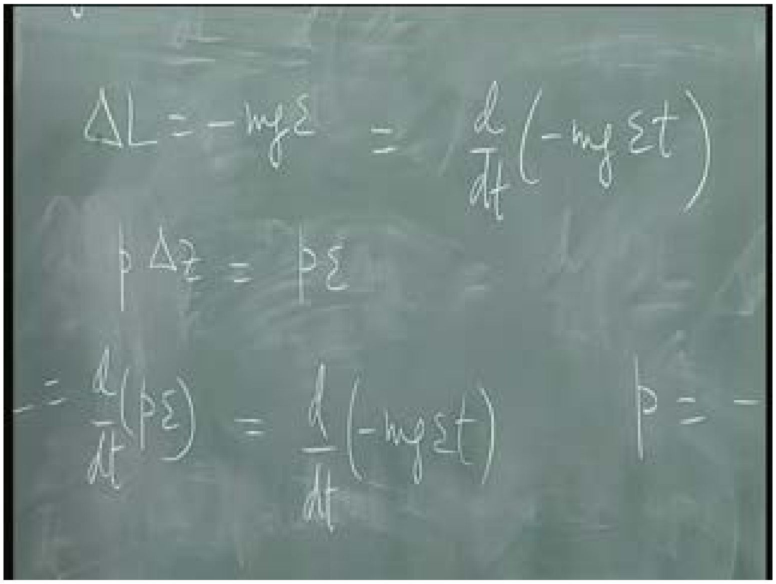

So, let us look at the problem of a particle moving vertically in the vertical direction under gravity. So, L is equal to half m z dot square minus m gz the problem dropping from vertical height is from a given height, it falls down this is the Lagrangian of the particle I am only interested in the z direction in the motion. Now, I ask what happens under z goes to z plus epsilon, I know that if I change the reference level the physics does not change add a constant to the potential energy it does not change.

So, I would like to see if this invariance of equations of motion leads to a constant of motion or not. The Lagrangian itself changes of course, z goes to z plus epsilon this changes, but then you say that is just an addition of a constant to the potential energy. So, if i differentiate and find the Euler Lagrange equations they would not change, so it is apparent that they do not clear, so let us work this formula and see what happens? In this case, delta l equal to minus mg mg epsilon.

This is the change in Lagrangian, but we also know that delta L over delta z dot this guy which is the momentum p times delta q, P times delta z equal to p times epsilon. So, we have on the one hand to change in the Lagrangian must be equal to d over dt of p epsilon, by using the equations of motion or the Euler Lagrange equation. But, this must also be equal to this quantity here which should be writable as a total time derivative and that of course, is completely trivial because I can write this as this as d over dt of minus mg epsilon t. So, you guaranteed that p equal to minus mg plus constant.

But that is of course, true it says the velocity depends on what your initial condition is that constant of the motion is the initial condition; and it helps you to solve this problem, because, if I now put p equal to m z dot it says its initial velocity minus half dt square initial position minus half dt square, so on and so forth. So, it solves the equation almost in this particular cases, but notice this trick here this was necessary for may be to be able to use this as a guaranteed constant of the motion and therefore, find the first integral and solve the path. So, here is a case where you need the delta L part it was vital. In fact, that is where all the physics was everything was sitting inside here.

One can now do this in more general cases etcetera you remember we long ago we looked at a charge particle in an electromagnetic field then I said under the gage transformation of the potentials the Lagrangian change by a total derivative of a function of coordinates and time. So, that was a case where you explicitly had delta l with a function f sitting it was in fact, the gage function; and now you should ask what quantity is conserved as a consequence of this. Well in the mechanics case, it becomes completely trivial unfortunately because now the initial portion of it the change in the coordinates, what kind of change do you have in the coordinates.

I make a change in the gage, I do not do anything to the coordinates at all, so in that case the initial the first part of this constant of motion the first portion vanishes and its only the second part that is important and you get charge conservation. So, we know that charge is already conserved, so in a particle picture in a field this is different this is actually give you a conserved current.

It tells you the continuity equation gives you the current the appropriate current. So, we want to find the in a field, which is interacting with the electromagnetic field for instance, what is the conserved current you have to go through this procedure. And Noether's theorem will give you this conserved current and recall from the conserved quantities things which satisfy equations like continuity equations you derive conservation laws. Let me do that recall this to you very briefly.

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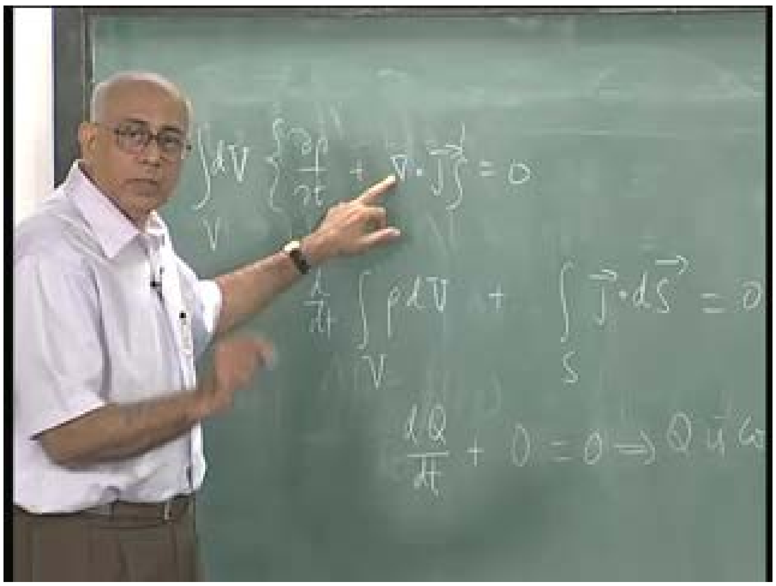

If I have any equation like this equal to 0, then the standard way of finding a conserved law conservation principle is to integrate over an enormous large sphere. And then assume that j vanishes out there, so what we are talking about is currents and charges moving about in some finite region of space and I integrate over a very large sphere where there are no currents at all.

Then these things says if I integrate it over dv of this whole thing, I still get a 0 and of course, if the volume is one large fixed volume which I am going to let go to infinity eventually this first term gives you d over dt of integral p over dv. It is now a total derivative because, the range of integration is independent of time and therefore, I can bring it out and it becomes a total time derivative and this plus integral j dot ds equal to 0.

This of course, is the total charge in the system, so or total mass or whatever mass provided that, this current vanishes on that surface provided all your devices and charges and currents are moving in some finite region of space and the current is actually 0 out there this equal to 0 implies q is conserved.

This term may not be 0 sometimes there could be something at infinity or flux at infinity in which case you have to include that also.

So, this is the standard way in which you go from a local equation of continuity to a global conservation law in this fashion and this is more fundamental. Because, this is really saying something at local point, so suppose this is global integral and all are equations in physics are all local statements.

We always tell you that if you move something here a little bit something near it it is affected and so on, rather than global statements any integral statements are actually derived from local statements. Like Maxwell's equation in the local form as oppose to faraday law of induction in the global form you have to start with the local statement and then you derive global statement the integral form.

We talked about translation of the coordinate axis, so r goes to r plus some epsilon; we talked about rotation r goes to r prime, which is a rotation matrix acting on r. We could also ask what happens when t goes to t plus epsilon; I could induce a change by changing the origin of time and asking do the equations of motion remain unchanged.

If the laws do not change with time then I have time translation invariance I am going to leave it to you as to work out and find out what the conserved quantity is in that case what would it be? It would be the generator of time translations and what is the generator of time translations the Hamiltonian, Hamiltonian generates time translations that evolution is given by the Hamiltonian itself.

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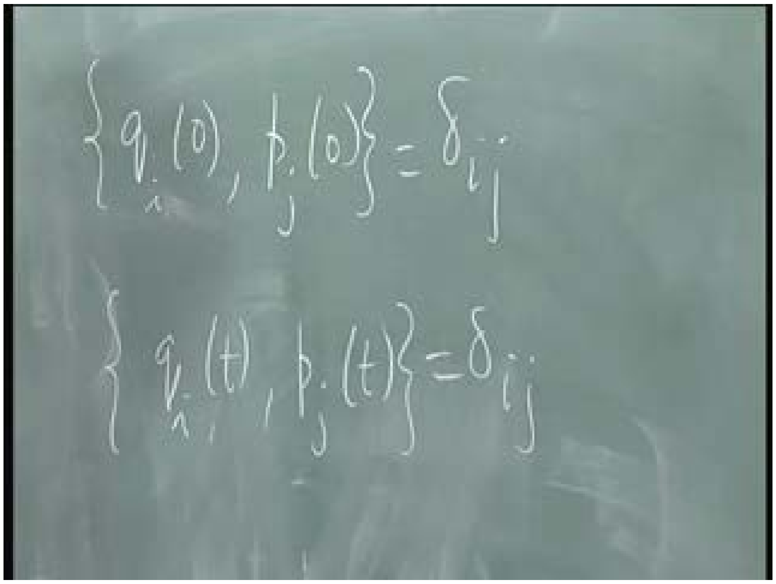

So, in this case, it is obvious that this is so in the Hamiltonian picture remember that evolution under time under a Hamiltonian was itself a canonical transformation, because we know that if q at time 0 qi is 0 pj 0 was equal to delta ij, then we know we are guaranteed under Hamiltonian evolution that this is also true.

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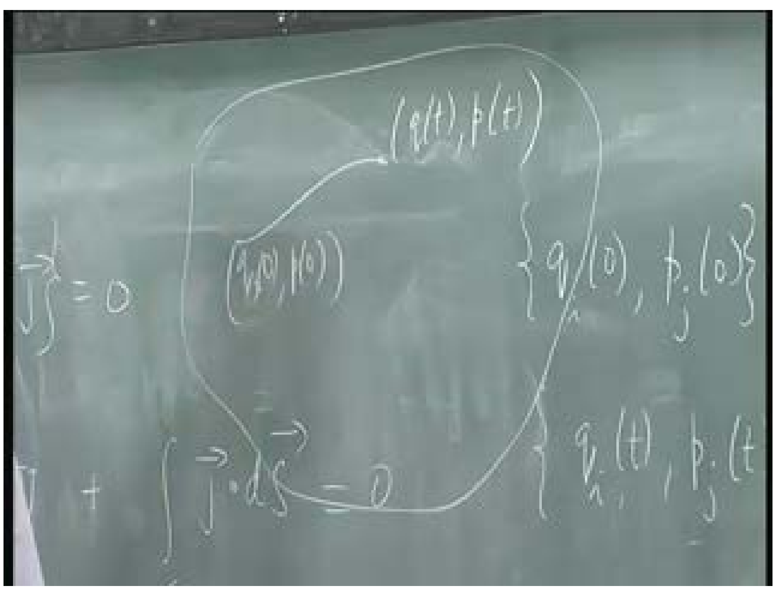

So, evolution under a Hamiltonian is itself a canonical transformation the coordinate at some, so if you start here and you go there and this is q at 0 p at 0 and this is q at any time t p at time t in some phase space. Then you can look upon this in two different ways one is you fix your coordinates, this is your initial state condition and as time evolves this point in phase space this state moves to that state, that is one way of looking at it. Another way to say is that this is just a transformed version of this in another coordinate system and that transformation is a canonical transformation.

In that sense time evolution is just the gradual unfolding of a sequence of infinitesimal canonical transformations. Each q and p and little delta t later gives you another q and p, which is the canonical transformation of the original and the infinitesimal one and therefore, a symplectic transformation and so on and so forth. As it goes along the analog of this in quantum mechanics is that evolution under a time independent under evolution under a hermitian Hamiltonian is a unitary evolution.

Probabilities are conserved, what is conserved here in Hamiltonian flow, in general what is conserved? Volumes are conserved the density obeys theorem and therefore, volumes are preserved here. Corresponding thing in quantum mechanics is a probability is concerned as we will see. We talked a little bit about generators and the algebra of generators not too much of it. But what I would like to do now is to move on to relativity introduce special relativity talk about the Lawrence group and then try to link some of those ideas with the ideas we had earlier about generators.

We did a little bit about symplectic transformations now we are going to generalize this and do a little bit more about Lawrence transformations. And of course, one can do this in many ways I am assuming that you already have some familiarity with special relativity; but I will state the facts here and then write down the equations and then we will try to look at this from the point of view of sym symmetry transformation.

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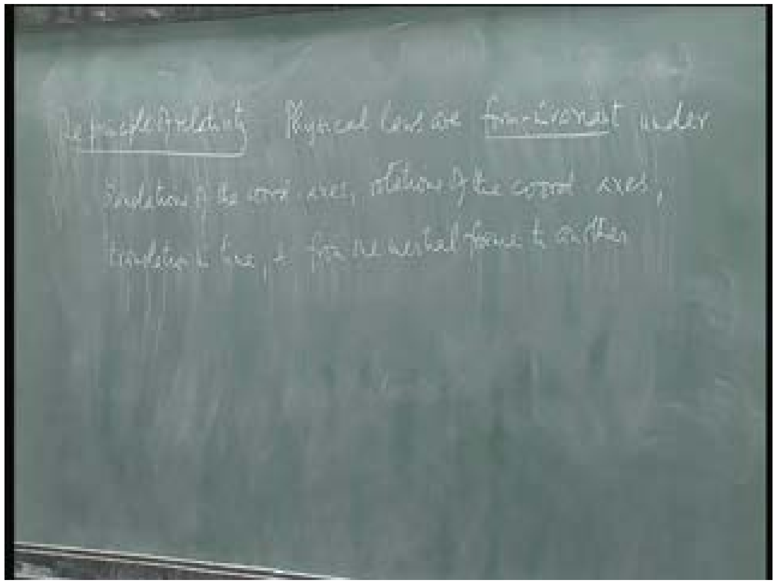

This is the way we would like to approach this subject right. The idea is that in Galileo in Newtonian mechanics it is been found that the laws themselves do not change if you make a translation of the coordinate axis. If you make a rotation of the coordinate axis is a purely spatial transformations time itself is suppose to flow in exorably from the past to the future and is not a dynamical variable in Newtonian mechanics.

And these are the two invariance’s, but you also have invariance under of all physical laws, when you move from one inertial frame to another inertial frame and this was Newton’s and Galileo’s big insight, namely the laws do not change in form if they are written down in two different frames of references, which are related to each other and fact that one of them is move that each of them is moving at a uniform speed with respect to velocity with respect to the other.

So, anything which is just a pure velocity transformation does not change the form of the equations of the motion this is no longer true if you have accelerations, if you have an accelerating frame of reference the form of the laws in that accelerating frame may be very different from the form of the laws in an inertial frame. Of course Newton’s big input is turned out to be not quite right was that he chose a preferred set of coordinate axis frames he chose a set of inertial frames he said these are inertial.

All frames which are fixed with respect to the fixed stars is reported are inertial. So, what has happened there is that, what happen was that the set of frames got chosen it is a preferred set of frame and there is no such preferred set this is what we know now. So, in a sense it was almost there principle of relativity was already there the postulate of relativity was missing and this is what we are going to write down. So, let me write this down and say, the principle physical laws necessarily it is little vague here, but we make it precise form invariant that is you do not change in form in appearance form invariant.

Translation of the coordinate axis, rotation of the coordinate axis, translation in time in other words shift of the origin of time is what I mean by a translation in time, and from one inertial frame to another. By that I mean two frames of reference connected by a velocity transformation. The other frame moves at uniform velocity with respect to the given original frame.

Of course as I said he defined a set of inertial frames as those frames which are fixed with respect to the fixed stars something, now you could ask what do you do on a cloudy day when you cant see the fixed stars how do you know you are in an inertial frame. Well, the correct way of defining this is to say an inertial frame is one where Newton’s first law of motion is valid. If you discover if you roll a ball on a floor and it turns out and it has no friction and so on and it keeps going forever at uniform speed, then you know you are in an inertial frame.

Now, this becomes a matter of measuring an precision and so on. So, we are not going to get into that on the other hand there is an unambiguous way of defining what an inertial frame is in Einstein’s relativity and that is we are working in flat space time no curvature of space, no gravitational fields at all, when you can define an inertial frame and other frames inertial with respect to it. And there is no such thing as a frame fixed, with respect to the fixed stars or anything like that this is not needed; so, you simply examine if your region of space time is flat in a specific sense; then all frames moving with respect to you at uniform velocity are inertial frames. Now, we already know in Newtonian mechanics that the moment you go to a non inertial frame Newton’s law is not valid Newton’s law of motion is not valid, because we know that there exist pseudo forces, so if you insist on writing mass times acceleration is equal to the force, on the right hand side you have to put the compensating terms to take into account the fact that your frame is in acceleration. Those are called pseudo forces they really should be called pseudo inertial or non inertial forces or pseudo accelerations and you multiply it by a mass and then you get pseudo force.

So, I am assuming that you are already familiar with the idea that you have a centrifugal force and coriolis force or an Euler's force and so on. So, I would not go into that, but let us now focus on this principle here and ask what does this imply. Translations of the coordinate axis have three generators the three directions. So, there are three generators here let us count parameters we have three here, you have three here three Euler angles or the n and the psi we talked about so you have three more here. Translation in time one here one inertial frame to another how many components do you have well you could go in any direction this velocity could be in any direction with respect to your fixed coordinate system.

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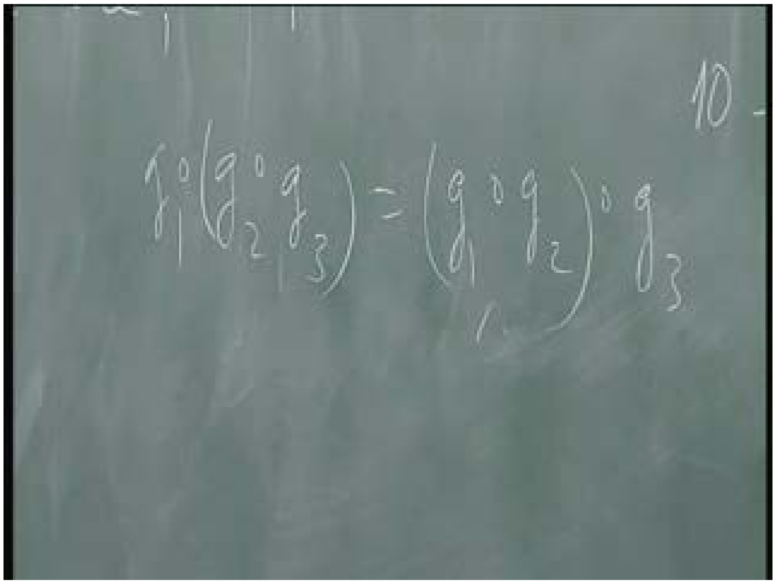

And therefore, you have three velocity parameters and you have 3 plus 3 6 plus 1 seven plus 3, so this is a 10 parameter group. And these transformations can be shown to form a group, satisfying the axioms required of a group namely.

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There is a group composition law, there is an inverse, there is an identity and the group composition law is associative. Namely, I compose two elements and then compose it to the third; it is the same as doing it in either order g 1 g 2 g 3; that is the group associated one and that is also needed for a group as he has pointed out.

And this group here is called as the Euclidian group sorry, the Galilean group. It is a 10 parameter group it is a fairly intricate group and its got complicated portions, you can see that the translations form an Abelian sub group, where these fellows can be done in either order, but the rotations do not and then the question is do these fellows form a group or not etcetera, so there are deep technical questions involved. Now, what Einstein's postulate did what is Einstein's insight did was to add to this the all important postulate of relativity.

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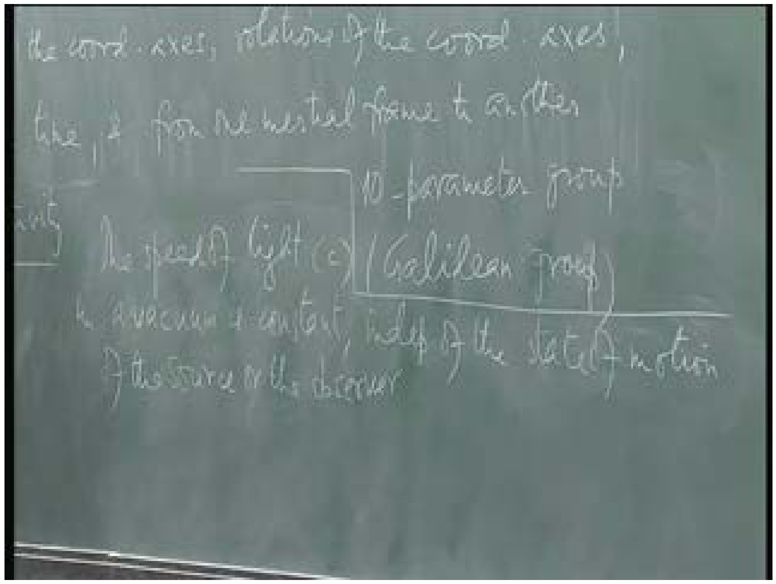

And since he did this in the context of light of Maxwell equation you specifically interested in the invariance of Maxwell's equations, the postulate was made in turn with reference to the speed of light in vacuum. But, actually the statement is independent of that the statement is that there exists a fundamental velocity which is same in all frames of reference.

And so, happens that light in vacuum provides a physical manifestation of this fundamental velocity it is an entity which travels at this fundamental velocity, but let us write it in the conventional way its constant independent of the state of motion of the source or the observer or the receiver is denoted by c. With that single input, this is physical input, now please notice that the rest of it had to do with general forms of laws and was not specific to any particular phenomenon on the other hand this is specific it is a physical input, it says something about a physical parameters.

It was lucky that we have an entity which travels at fundamental velocity; otherwise it should have been much more difficult to discover this invariance here. Now, what it implies immediately are the Lawrence transformation equations and I would like to focus on that this implies the following and here is the derivation of these equations. Just as you derive what rotations do by requiring that a certain form is kept unchanged namely the distance of a point from the origin is unchanged under a rotation of the coordinate axis.

This told you immediately that rotations are orthogonal matrices are represented by orthogonal matrices in exactly the same way this is going to tell us what sort of transformation can you possibly have, the idea is that if you have two frames of reference in which the origins coincide at t equal to 0 and say one frame is moving at uniform speed with respect to the other along the x axis for instance.

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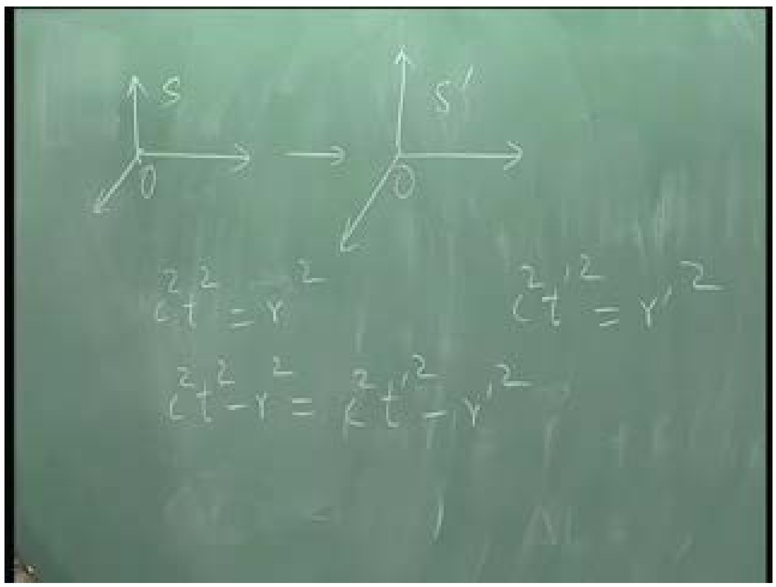

And at t equal to 0 in both frames the pulse of light is emitted this pulse travels as a spherical wave then the first person in the frame s and you have s prime moving along this in this direction in the frame s c square t square equal to r square. And that is the shape of this particle it is a spherical pulse at time t it has reached a distance ct from the origin. In the other frame, the c is exactly the same thing c square t prime square equal to r prime square, and c is the same in both frames and you would impose this is the content of this statement.

The moment you do that you have this possibility you have c square t square minus r square must be equal to c square t prime square minus r prime square then in Newtonian mechanics c is essentially infinite.

This is only way of sustaining this equation without changing t, if you say t equal to t prime which is what Newtonian mechanics does then the only way of sustaining this equation is to make c infinite. But in relativistic mechanics and this was Einstein's insight this is some kind of hyperbola, so that is some kind of hyperbola here right it is the invariance of this form that is required and this form can be kept invariant in more than one way and the second way is that t changes and r changes t becomes t prime r becomes r prime such that this quantity is kept unchanged remember for ordinary rotations you insisted that r prime square equal to r square.

Now, you saying this quantity is equal to that and then you ask, what are the possible transformations under which this can happen. And this is how the Lawrence transformation is derived and its very well known to you that consequences immediately the following in this situation I have pictured, where this guy is moving with speed v along the x axis.

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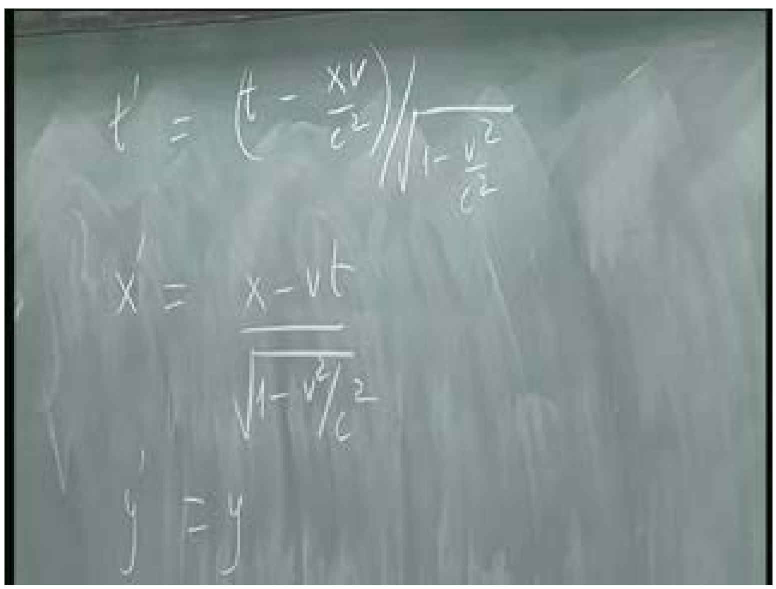

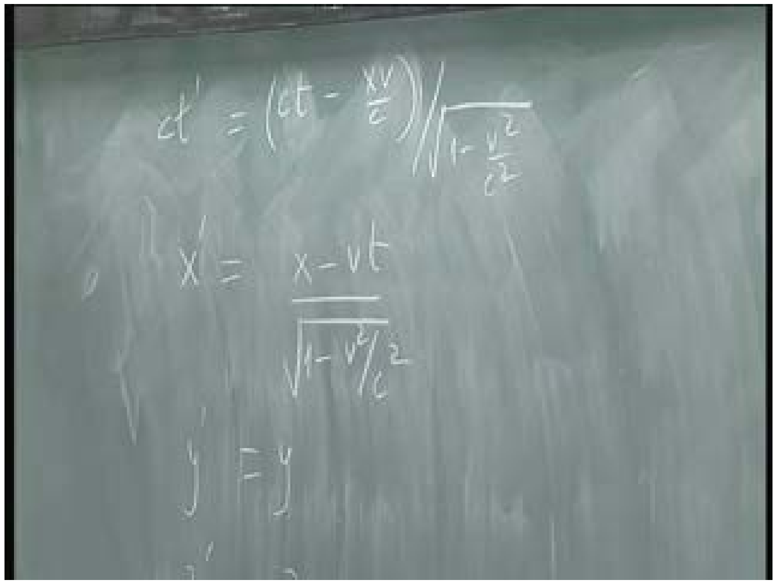

And we know that this implies that x prime is equal to x minus vt over square root of one minus v square by c square, Y prime is y z prime is z and t prime is write t prime in the beginning. So, those are the solutions to the equation which tells you this hyperbolic form should be kept unchanged and of course, now we want a better notation I would like to combine x as well as t x y z and t.

So, I would like to make this have the same dimensions as length and let us multiply it by c. So, you have ct minus xv over c and the same is this correct.

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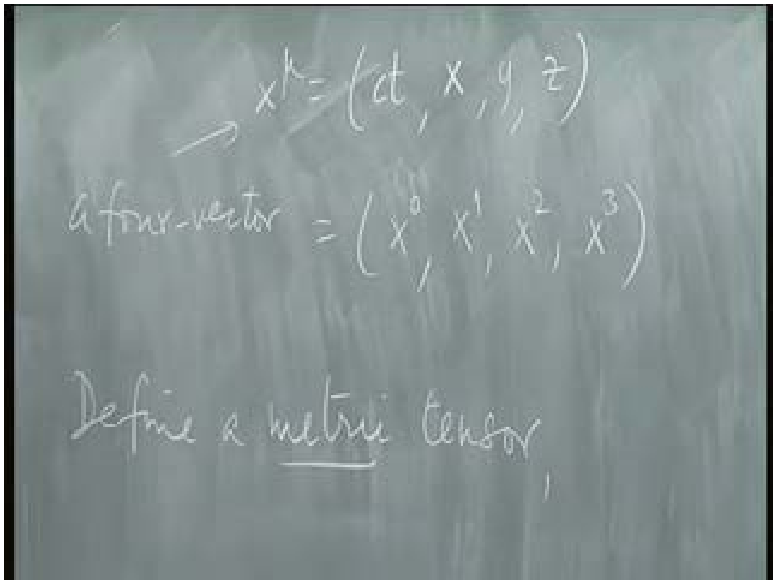

And the standard way of writing this is now to introduce a four-dimensional vector. So, I will introduce x mu equal to ct x y z and I have call this x not call this x 1 x 2 x 3, I put a superscript rather than a subscript become clear, why? This quantity which satisfies these equations, these transformation equations is called a four vector and the group of transformations under which this that quantity that this form is invariant is the Lawrence group just as you had the rotation group earlier. This is the Lawrence group, and it will turn out as very shortly that its represent able by 4 by 4 matrices which act on these coordinates, because these are like column vectors am going to write one below the other and these four by four matrices would not keep or not orthogonal these matrices are not orthogonal, but rather they keep this form invariant. And therefore, they are pseudo orthogonal and it is written as so 3 1, and this is called the Lawrence group special Lawrence group three and one because I am going to introduce a metric.

And there are three spaces like directions and one time like direction taking into account the fact that in this quadratic form there is a minus sign here. So, there is a c square t square minus x square minus y square minus z square I want to take that into account. So, this quantity I call a four vector sometimes called a upstairs is called a contra variant vector in old tensor analysis and it is a superscript of this kind is called a contra variant.

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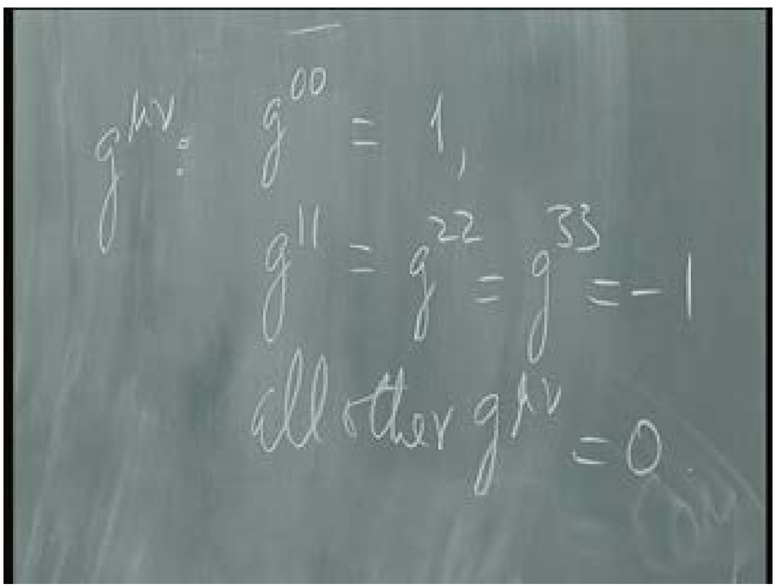

Then I define metric tensor g over mu v which is given by such that g is 0 0 equal to 1, g 1 1 equal to 2, 2 I am doing this very badly I should do this a little more systematically.

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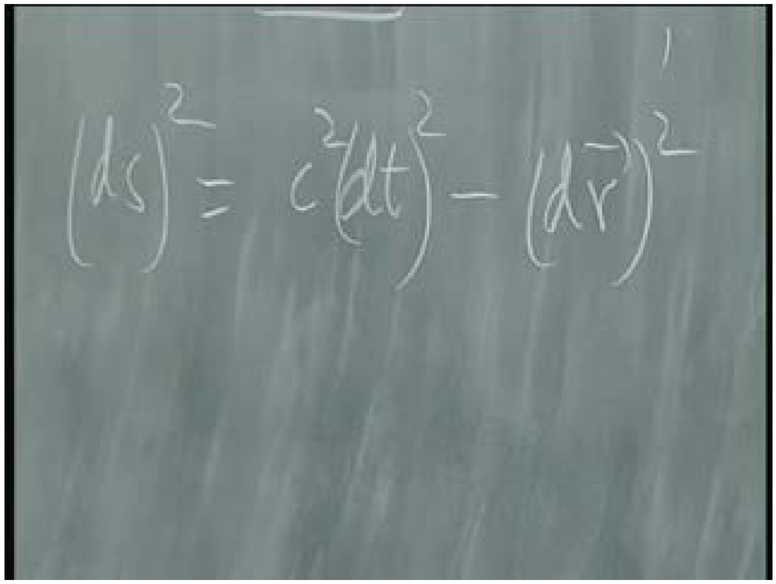

The square if you have a space time point x mu and a space time point x mu plus d x mu then the square ds whole square equal to c square dt square minus dr square is called the square of the interval between these two points. Under, under Lawrence transformation this interval is unchanged it is a scalar and that was a whole point now this quantity here can be written in compact notation. I should now define contraction and so on, let me do this let me do this next time slowly and carefully I should define for you the lowering index how to lower it, what is meant by a covariant vector and so on, I wanted to do this properly. I do not want to just define it as something which is done with genuine I want to show you that you really have a vector and you have what is called a co vector in dual space since I want to do that properly let me do this next time because we have run out of time and I should do this properly.

Let me ask you how much of how much of relativity are you already familiar with yes you went out of transformation in this form and then what Lawrence contraction and time variant length contraction and time violation, those are trivial consequences put together. The fact that these things form a group is that thing emphasizes the question this is the point this is what we must look at let me do that I will do that properly, so let me. You see you have to understand that Lawrence transformations involve not only velocity transformations like this, but also the rotations of the coordinate axis.

So, and you do not have to go along the x axis you could go in any arbitrary direction and rotate and then you can shift the origin. So, all these are possible and they form a ten parameter group once again. So, the set of rotations the set of translations in space the set of the translations in time as well as velocity transformations they again form a ten parameter group, but the element the group trans the transformations leave a form like this invariant.

And this is makes all difference now and this group is not a Galilean group it is called the inhomogeneous Lawrence group or a (( )) group of which the special Lawrence group this is a sub set is a sub group here. And this is the group of homogenous transformations which do not change which do not shift the origin of space or time. So, we need to understand that and we also need to understand that the structure of this whole thing

The structure of this group is such that it is not orthogonal not represented by an orthogonal matrices, but by kind of strange pseudo orthogonal matrices we need to recognize that. And above all it will become clear that the rotations actually form a sub group, but the velocity transformations do not and this is important, if I move to another frame moving along from s to s prime.

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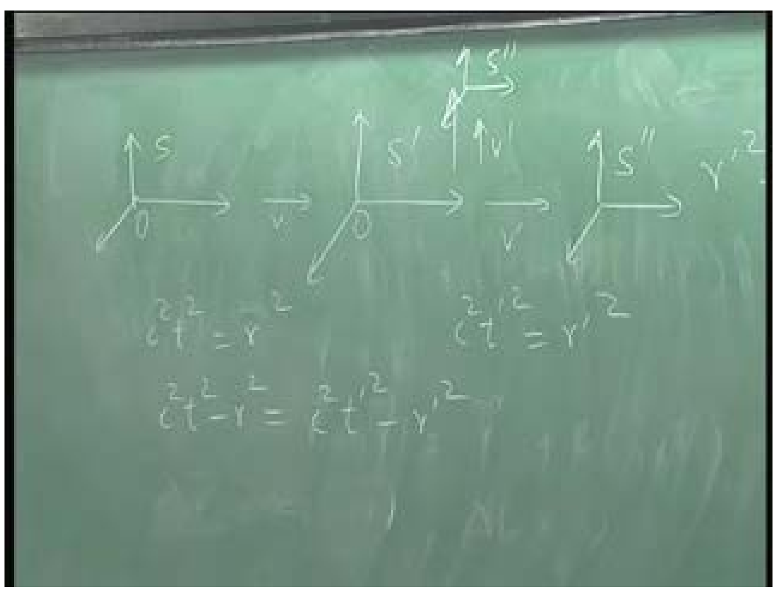

And then from here I move to s double prime which is moving with the speed v double prime in the same origin then this is s double prime is related to s by an Lawrence transformation.

But if this fellow was moving in this other direction this is s double prime with a v prime in the vertical direction then these transformations do not form a group among themselves these are called boost, because you give a boost to a frame of reference and get it moved here. So, the boost do not form a sub group now that is what complicates matters the rotations form a sub group and the boost do not form a sub group.

But the boost and the rotations together form a big group. So, the complicated things can happen. So, you move like this and then move like that it could be as if there is a rotation also effectively. So, I my my aim is to make you understand that that this is, this is, this is what I would like to focus on, and I would like to use an efficient notation.

This four vector notation is extremely efficient because it will tell you what is a scalar what is a vector what is a tensor and. So, on just like we use scalars and vectors in ordinary mechanics and it is much more convenient. I would like to use this notation. So, that is the, I have to lay the groundwork a little better I will do that.