Classical Physics
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Lecture No. # 34

Let us resume our discussion of rotations in three-dimensional spaces, and then go on to understand what the rotation groups in three dimensions looks like.

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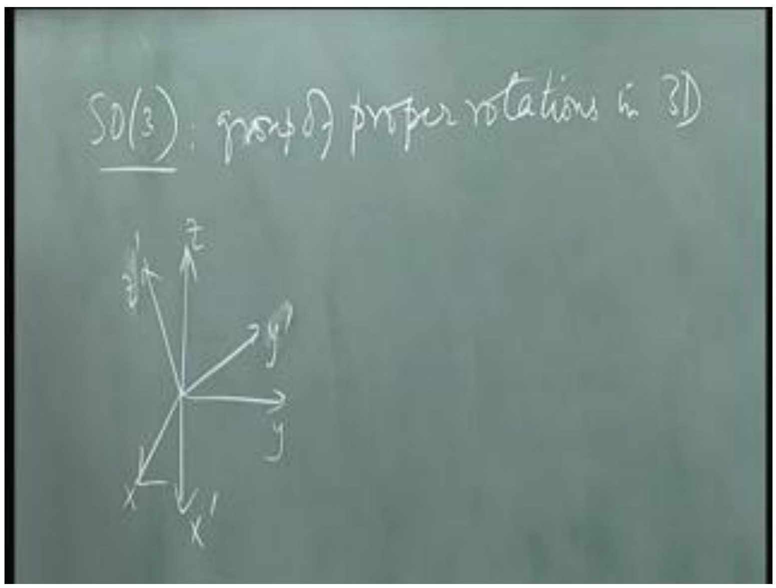

So, our target immediate target is to find out, the parameters space of the special orthogonal group, the group of proper rotations in three dimensions. So, ordinary Euclidian three dimensional space and we would like to understand what the properties of rotations in three dimensional space are. Now of course, I can represent these rotations by orthogonal matrices on the real numbers with real entries with determinant equal to plus 1. As you know if I include improper rotations namely, those which include parity for instance, or reflections about some plane then the determinant could be minus 1.

So, SO (3) is that portion of O 3, that portion of group of orthogonal matrices 3 by 3 matrices which has determinant plus 1. O 3 itself will comprise of matrices which have determinant either plus 1 or minus 1 nothing else, is possible for an orthogonal matrix and the portion we are interested in is that portion which can be continuously for found from the identity transformation. No, rotation at all means determinant plus 1 it is a unit determinant, unit operator. And we would like to find out what this group looks like, what is the lie algebra of this group, what is it structure and so on.

Now, the first thing we should do is find out, how do we parameterize this group now clearly since, SO (n) has n times n minus 1 over 2 independent real parameters SO (3) has real three real parameters. You could regard these three as the 3 Euler angles. So, if you want to go from any coordinate systems of this kind this is x y z. And you want to rotate this coordinate system, and go to a new set of coordinates X prime Y prime sorry x prime y prime z prime by a some arbitrary rotation. We can do so, in many, many ways you can reach the prime coordinate system from the unprimed one in any number of ways. Many prescriptions, not unique or anything like that, you could rotate first about the x axis then about the z axis then again about in new x axis; that is one possible set of Euler angles and you have many many such possible sets. What is unchanged is that you need three angles.

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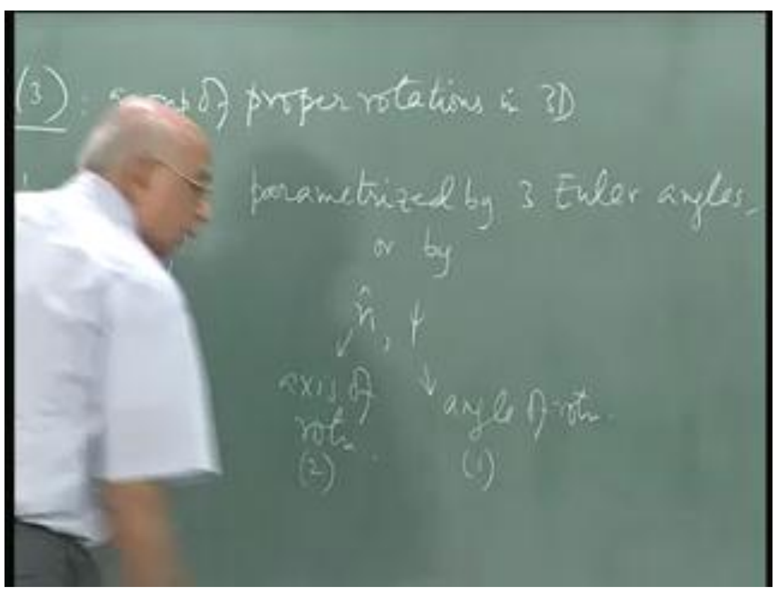

So, a rotation of this kind would be represented when acting on coordinates would be represented by a 3 by 3 orthogonal matrix parameterized by 3 Euler angles. For instance, but that is not the most convenient way of parameterizing a rotation for our present purposes, this is slightly a better way of doing this and that is the following.

Any rotation like this could always be regarded as a rotation of the original coordinate system about some axis, which is specified by the chalk to a certain angle. So, identify some axis in space and then I rotate through an angle psi for example, and that gives me the new coordinate system. So, I can parameterize these either by 3 Euler angles or by n and psi this is the axis of rotation, and this guy is the angle of rotation is this clear?

So, I could always choose if I want to go from here to that coordinate system, there is some axis in space pointing along like this, and about that axis I rotate in this fashion in the plane perpendicular to this axis, I rotate through an angle psi, and I reach the new orientation. So, again I need 3 parameter parameters, because this angle is 1 parameter the amount of rotation, but to specify an axis in space with respect to original coordinate system, I want to specify the direction of this arrow I need latitude and a longitude. It is a unit vector and therefore, need just a polar angle and an azimuth angle to tell me, where this point is, where the tip is.

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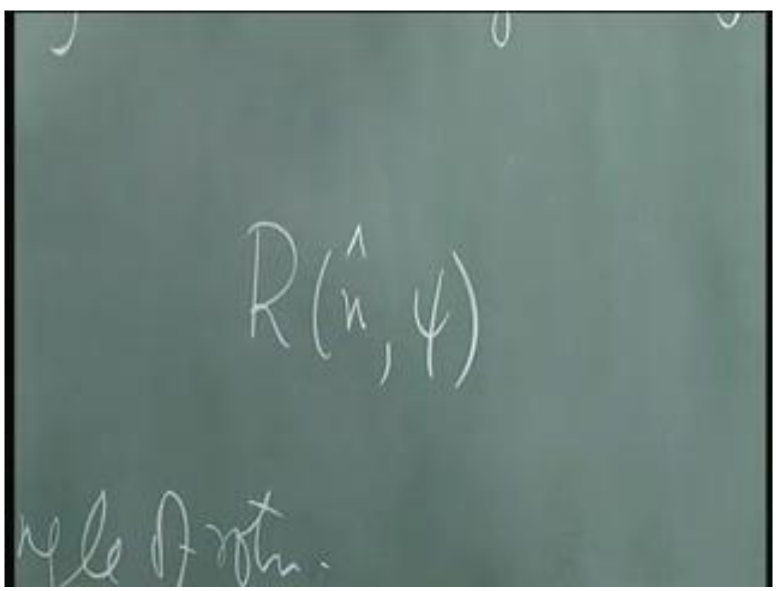

So, once again I have 2 parameters here, and I have 1 parameter here, that gives me the same set of 3 parameters. And then having done this let me represent the rotation by R of n, psi. This is now a 3 by 3 matrix, which will act on x y z and give you x prime y prime and z prime as linear combinations of x y z. And what we know about R is that it is orthogonal and it is got determinant plus 1. Now, I would like to find out what the parameters space is like, what is the space of these 3 angles? Like well clearly this unit vector n is specified by a polar angle theta and an azimuth angle phi theta runs from 0 to pi and phi runs from 0 to 2 pi and psi itself the rotation amount of rotation in this plane is 0 to 2 pi, so that is my range of n the parameters specifying n and psi. But let us see how to represent this in pictures, after all it is a three-dimensional space and I should be able to draw picture and tell you here, is the three dimensional spaces.

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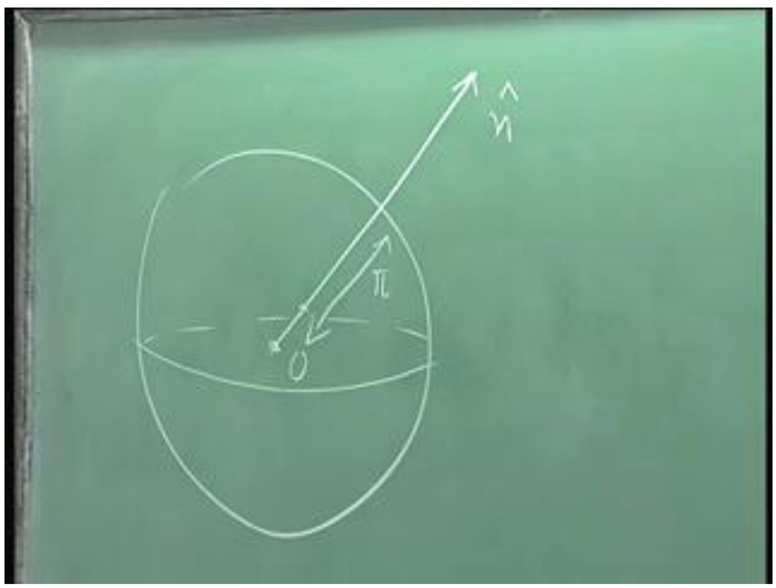

So, let us draw a sphere, and this is the space of rotations it is not the physical three dimensional space, it is the space of rotations namely the space in which the components of n and psi there that is three dimensional. So, I draw a solid sphere and now the convention is that from this origin any direction like this is the direction of this vector n. So, I move in any direction in this solid sphere and that tells me, the direction of the axis of rotation clearly all four pi's and solid angle is covered by this all possible axis are covered by this. But, in addition I also say that the distance from the origin to the point in this space is psi that is one way of parameterizing it. So, if I say I go from here up to this point this is the amount of angle by which I rotate. So, what should the radius of this sphere be, to cover all possible rotation? You would say 2 pi it should be 2 pi because, then I go right up to psi equal to 2 pi, but then a magical property of three dimension takes over for which I need a solid object to demonstrate it. So, if you pass me the phone I would try to demonstrate this for you.

So, look at what happens here, now let us look at this object and here is some thing which conveniently looks like arrow, so let me draw little arrow here to show you where that let me draw an arrow here. So, I have drawn an arrow on this, and now I start like this this arrow pointing upwards looking at me and I am going to rotate about this vertical axis through pi, in the counter clock wise sense, in the positive sense. So, I rotate and there is your arrow, but then this was the original configuration and I rotate by pi about the vertical axis pointing upwards and I have reached this configuration. But, then I could also rotate about the opposite axis again through pi, so if I rotate in the positive sense about the opposite axis I rotate like this and I reach the same configuration. So, this tell us you that in 3 dimensional spaces rotation of pi about an axis in the positive sense is equivalent to rotation by pi in the opposite again in the positive sense about the opposite axis. This immediately says, you do not need a radius of 2 pi, a radius of pi is enough. Because, you can cover all possible rotations up to pi here, and all possible rotations up to pi here, and that has covered your full 2 pi.

So, this space really has the radius the size of this radius this length here, need be only pi not 2 pi because, you have covered everything this full circle either, by rotating about the axis pointing upwards or by rotating about the axis pointing downwards. And since, this n unit vector is over all solid angles it covers all possible directions, but more than that something more than that happens. So, you would say this space of rotations is very, very simple it is a solid sphere, it is a three-dimensional manifold so, it is solid sphere of radius pi with the understanding that from the origin from the center going in any direction to any particular point gives me the axis of rotation and the distance I travel gives me the amount of rotation.

But the unfortunate thing is or the fortunate whichever, way you look at it is you saw just now that a rotation by pi about a vertical axis is equal to rotation by pi about the opposite axis. Therefore, this space is not as simple as it looks, this point is mathematically the same as the antipodal point because, they both represent the same physical rotation. Because you took the solid object to a new orientation by two different ways and give the same final orientation. So, both of these points are exactly the same point, it is as if there is an invisible thread connecting these two they have to be identified, but that is true for every point on the surface it is true for this point. That point is mathematically identical to this point it is not true for a point which is infinitesimally close to the surface, but not on the surface. Now that corresponds to a rotation by an angle little less than pi and that is very different from this. This identity happens only if you go up to pi which you have to in order to cover all possibilities.

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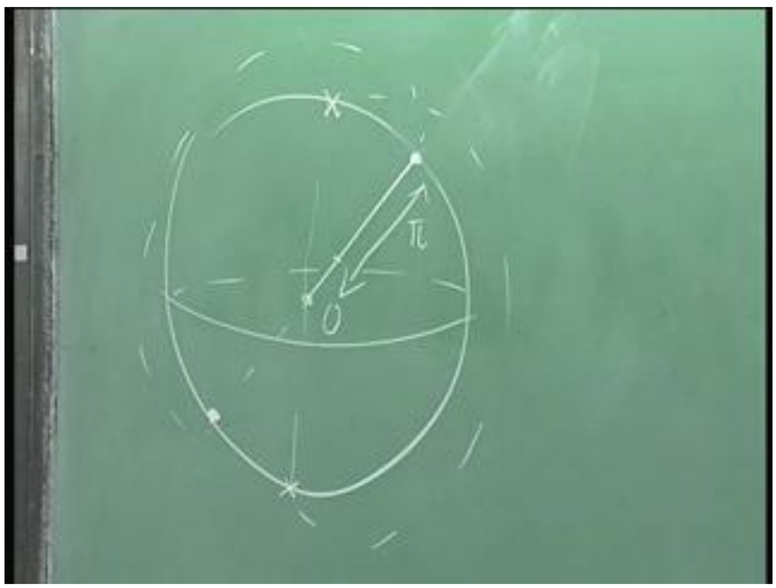

So, it says that there is some kind of invisible thread connecting these two points and kind of connecting these two points and so on. Every point on this surface is identified is mathematically the same as the antipodal point? Now, such a surface such an object cannot be written down in three dimensional Euclidian spaces that is why I am not able to draw it here. This space is complicated it cannot be you cannot embed it in three Euclidian dimensions, there are lots of such objects you need higher in dimensionality, we have to represent this object to see it and so on.

But, we have the benefit of only three physical dimensions have way but mathematically, we can identify these two points then the question arises, what sort of space is this, what is the connectivity of this space is it simply connected or not? Now, if those identifications on the surface did not exist this is just a solid sphere and therefore, every closed path in this solid sphere can be shrunk to a point without leaving the sphere and this space would be simply connected. But that is not true here, in this space is not simply connected it is doubly connected and let me now show you why it is doubly connected? Because there are lots of closed paths in it, which would do the following.

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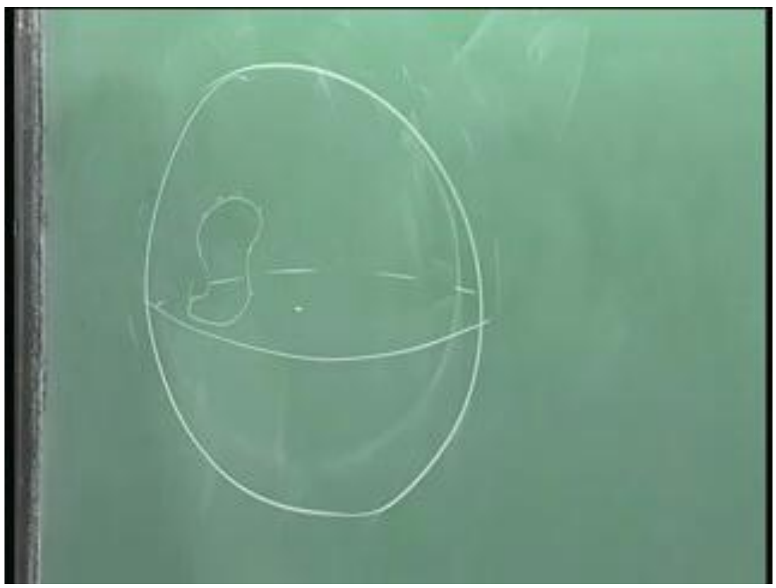

I could start at some point and then take a little path like this closed path which will say I look at a sequence of rotations of three-dimensional space of my axis specified by this point or this point or this point or this point etcetera and I come back. The sequence of rotations since, I come back finally can be reduced can be shrunk like a rubber band to a point, which means I do not do any rotations I just leave where I am where, I start with this point, and I end with that point and that is it.

So, there is a clash of closed paths which can be shrunk to a point, but there is another class which cannot be shrunk to a point and that class is the following. I start at the origin I do a sequence of rotations till I hit the surface, so this is my path in the parameter space. But this point is the same as that point and therefore, if I start here, and go back this is a closed path because, it starts at o goes to this point.

But that is the same as other point comes back to o and this is a closed path in this space, but you cannot shrink this to a point because to do so, I would start moving this point if I move it here, conversely enough this fellow starts moving there and there is no way I can close this. So, it is just not possible to close this physical path, but there is one way in which I can do it and that is I retrace the path once again. What does that mean, what is this physically meant? It says you go the full pi and when you come back and come since, so that is a full rotation, a complete rotation by 2 pi, because you really gone like this, and then you have gone like this.

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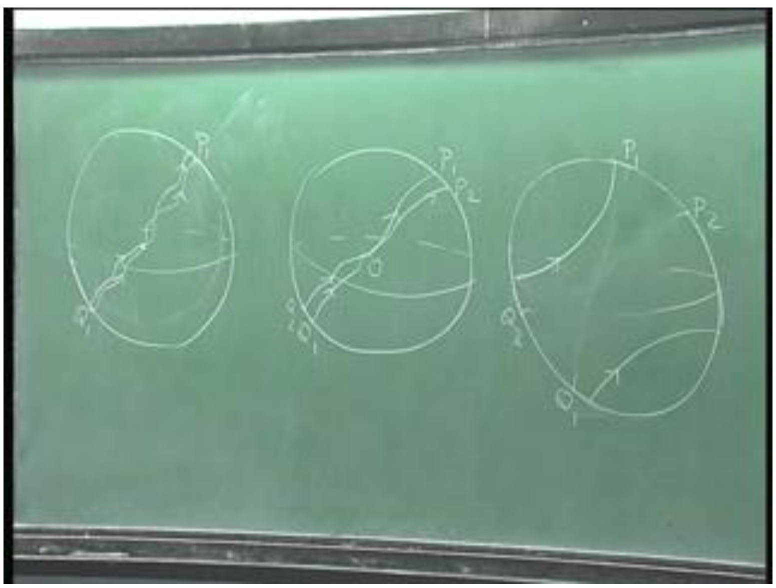

So, this is the 2 pi rotation, on the other hand I could do the following I do the 2 pi rotation once again. So, I start here, I start at this point go up to this point, let us call it P 1 that is the same as this guy here Q 1 come back here, and then I go once again here and do this. Now, look at this path I have traced this closed path twice, I start here and I can do this and come back, but what is this equivalent to? Let me draw this for clarity has to pass through this point in each case, now let me draw this for clarity by separating them slightly.

So, it looks like this there is one path which is P 1 and this is Q 1 So, it started here went up to that point meant you ended up here, and you came here and you went here. And I start once again and do this P 2, but that is the same as Q 2 and then I come back. I have shown them a little separate just for clarity. If is start distorting P 1 Q 1 will start moving in this direction always, but this is a path which has gone twice through this point and certainly I can rewrite it like this. So, certainly I can take this point and then separate it out, in this fashion so, this is suppose to be diametrically opposite this is P 1 P 2, this is Q 2 Q 1. So, notice that at this point I can attach this piece from Q 2 to this piece up to P 1 and this piece from Q 1 to the origin to o, I can attach to origin to P 2, once I do that I can separate the two, because I start moving P 1 Q 1 moves that is fine no problem, but I can move Q 2 up here P 2 moves that is no problem.

And after this distortion this path can be made to look like this, and then of course, each of these can be brought arbitrarily close to each other and knocked off to a point.

So, it actually closes this path is shrinkable to a point. So, it is clear that in this space there are closed paths such, that you have to do a rotation about 4 pi, before you reach the original configuration. Although, there are lots of paths, where you have to do a rotation about 2 pi and you are back to the starting point it is clear that there are objects as there will be representations I will see which would come back to their own original values only if, you rotate by 4 pi make two complete rotations instead of one.

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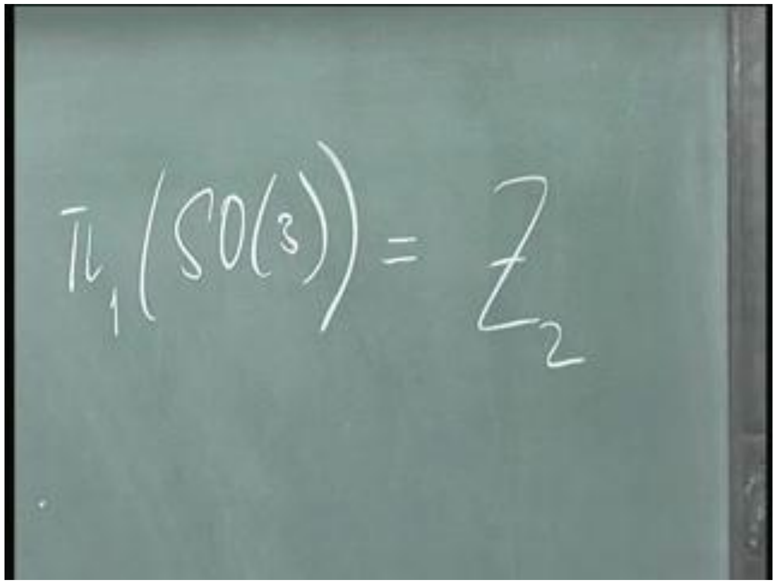

The technical way of saying this is that pi 1 of SO (3) is not equal to the trivial group with just one element, it is not simply connected nor is it z, nor is it the set of integers which is what S 1 was pi 1 of SO (3) has two elements, there are two equivalence classes of closed paths those which can be shrunk to a closed path to a single point immediately and those which can be shrunk to a point only after twice traversing the path. Therefore, there are two equivalence classes and this group is isomorphic to the group of integers modulo 2. The cyclic group of order 2 and we say that SO (3) is doubly connected. By that we mean it is a fundamental group, hemotropic group is z 2.

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But you cannot get away from the fact, that it is a property of three dimensional spaces that a rotation about pi about an axis is mathematically identical, to the rotation about pi about the opposite axis. This you can never get away from, no matter what parameterization you choose?

yes there is, there is because I labeled points in the sphere and three dimensions and there are opposite directions. This is the property of three dimensional spaces so, it is an intrinsic property, as it stands and you cannot get away with it from it. There are some similar properties in higher dimensions I should say right away that there SO (n) where, n is greater than equal to 3 is also doubly connected, but SO (2) is infinitely connected SO (2) is the fundamental group is z. So, in some sense two dimensions is more complicated than three and higher dimensions and this is finally, responsible as we will see for the existence of bosons and fermions in the universe this double connectivity.

We will see that there are wave functions which you transform like scalars, vectors, tensors and so on. And then, there is wave function which would transform like spinners which would correspond to half integer's space in quantum mechanics. But right now, we are just talking about groups the structure of this three dimensional space. Now, what does this imply, what is the covering group? Now, just as we found that even though S 1 was infinitely connected, SO (2) was infinitely connected there was a map from SO from S 1 to the real axis to the R 1 which was the mapping which went from one to many and the covering group for S 1was in fact, R 1itself.

The question is what is the covering group here? The universal covering group and there is a theorem which says that the universal covering group of a lie group is simply connected. And once things are simply connected we can do many many things on it, because we know that space is very simple to look at. And the answer is found as follows so, now, let me find out, let me show you how to find the covering group of SO (3). For this we need a slightly better way of writing down a point in space and of course, representing a rotation. So, let us look at that map.

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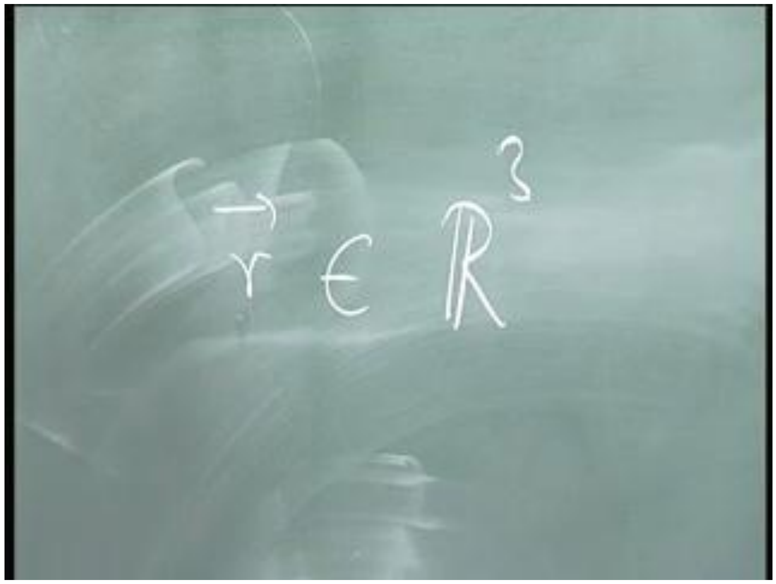

A point in three dimensional spaces is some r is an element of r this stands for x y z x y z components of any point, from this R 3 there is a map and the map is as follows.

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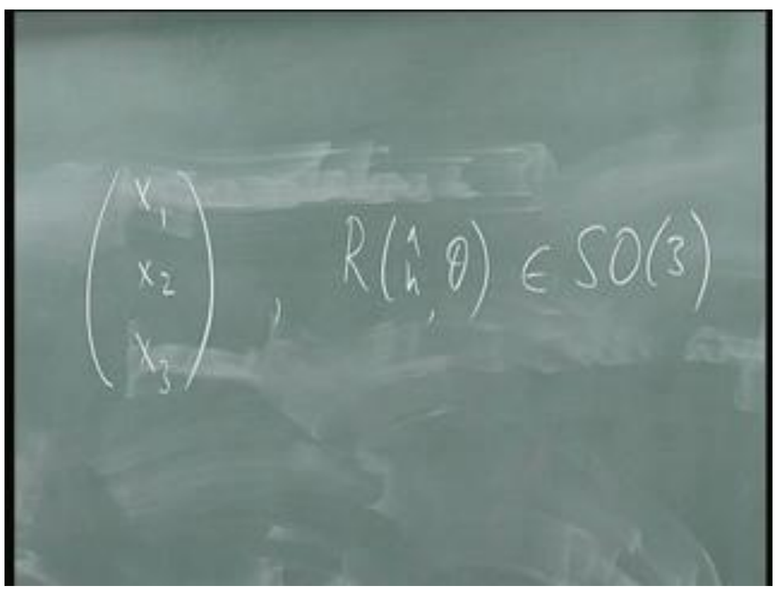

Normally I would write point r in space as either x comma y comma z or I would simply write it as a column vectors like x 1 x 2 x 3 this is the way I would represent it and then rotations would be represented by R of n n, theta element of SO (3). So, the point itself the points are represented by column vectors and the rotations are represented by 3 by 3 orthogonal matrices with determinant 1. Now, I come along and say, let us not do that let us make a map from R 3 to a different space a one to one map, so for every point in R 3 there exists an object in this new space.

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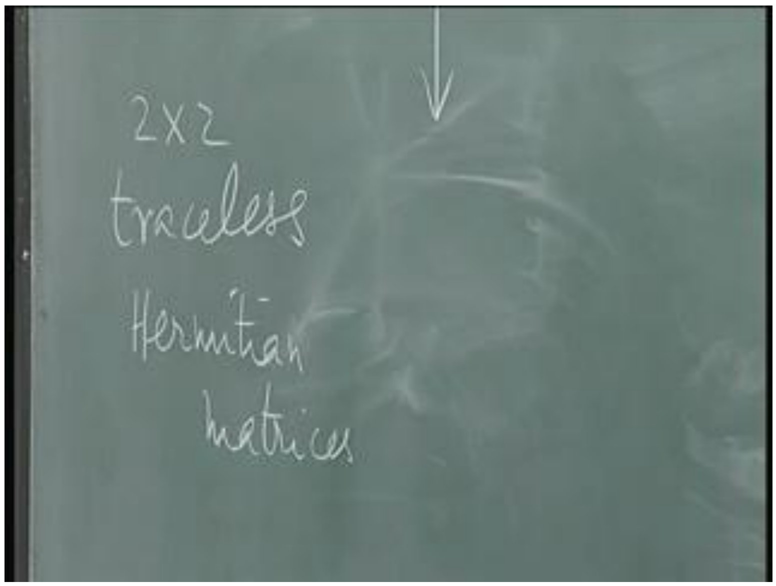

And this space is that of 2 by 2 traceless Hermitian matrices, you can always do this just as I showed you there were many representations of R 1 itself you could write R 1 in terms of functions space and could be various ways of writing R 1in exactly the same way. There is a map which takes you from R 3 to 2 by 2 matrices and that is as follows.

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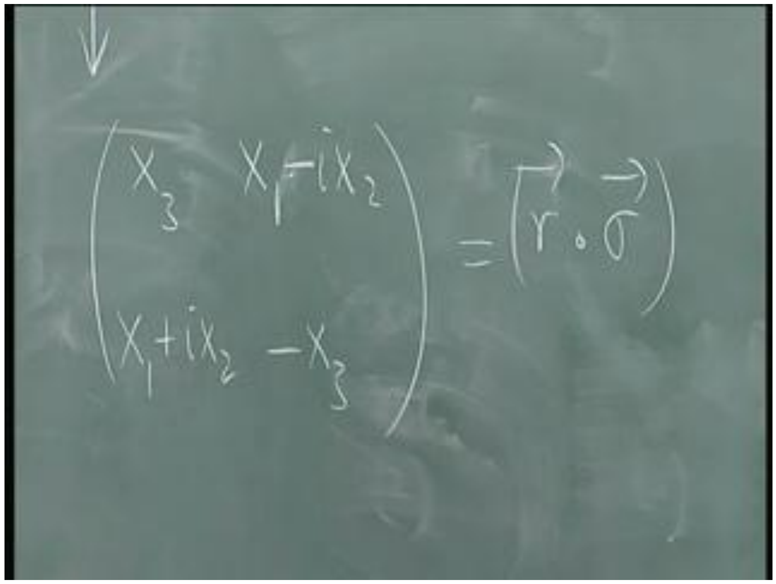

Let me simply write this instead of this let me write it as x 3 x 1 plus i x 2 minus i x 2 x 1 plus i x 2 minus x 3. It is clear that for every given x 1 x 2 x 3 there exist a matrix of this kind and vice versa. This matrix is traces and it is Hermitian because, complex conjugate transposes is equal to it is complex conjugate transposes x 1 x 2 x 3 are real numbers between minus infinity and plus infinity. Of course, you will immediately recognize that is this is also same as r dot sigma where, sigma is set of Pauli matrices which we talked about little earlier.

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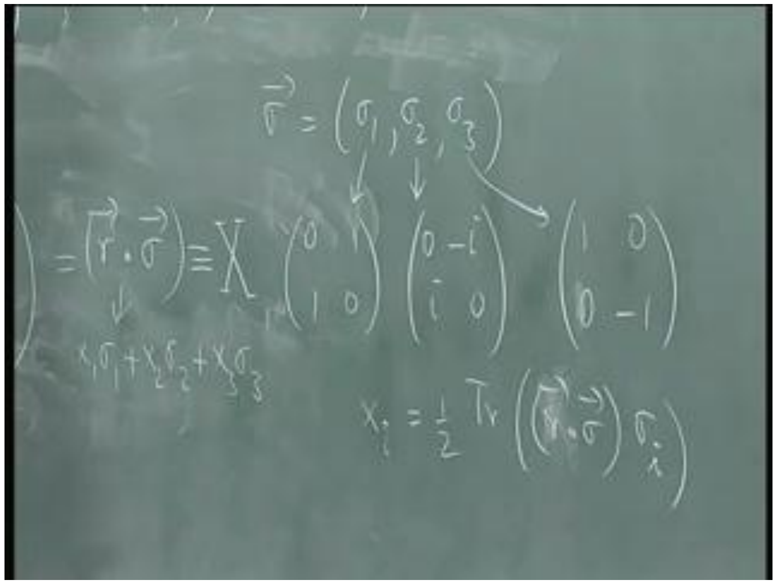

So, sigma stands for the set of 3 matrices and sigma 1 is 0 1 1 0 sigma 2 is 0 minus i i 0 and sigma 3 is 1 0 0 minus 1. And let me call this equal to for want of a better name let me just call it capital x capital x is a 2 by 2 matrix. So, instead of writing vector r in three dimensional spaces I write capital x is 2 by 2 matrix and I give you the same information. And symbolically, r dot sigma stands for x 1 sigma 1 plus x 2 sigma 2 plus x 3 sigma 3.

So, this here is short hand for x 1 sigma 1 plus x 2 sigma 2 plus x 3 sigma 3 it is a matrix. It also looks like a scalar product, a dot product and it is, so because sigma actually transforms like vector under rotations sigma itself transforms like a vector, but we will see that little later. But right now there is nothing to stop me from exchanging this column vector for this 2 by 2 Hermitian traceless matrix. Then the question is two questions, first can I invert this map namely if, you give me capital x can you give me this r is there a prescription to do that and is it one to one then it would be a one to one map, and the second question is what does a rotation do to x? I know what a rotation does to column vector x 1 x 2 x 3 I just write this 3 by 3 orthogonal matrix and I get a new linear combination what does the opposite what does it do in terms of x?

Well it turns out that the inverse is quite obvious this inversion is quite obvious. In fact, it is not hard to show that x i equal to half the trace of x or when we write specifically r dot sigma times sigma i. So, I leave you to check out that if you took this matrix and multiplied by sigma 1, sigma 2, sigma 3 in turn and took the trace half the trace you would get x 1 x 2 and x 3, so this is the inverse mapping. Therefore, you give me any r dot sigma any capital x any Hermitian traceless matrix, and I tell you what the corresponding x i.

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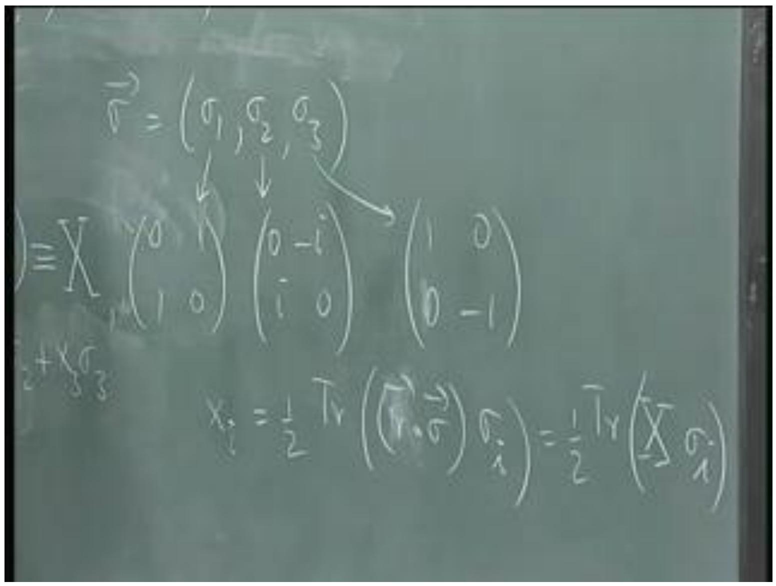

So, r dot sigma is a matrix it is this follow. So, this stands for equal to half trace x sigma i because I have an i and an i there. So, if I want x 2 I have to put sigma 2 there and so on. Then the question is how is this represented, how is this rotation represented as it acts on x capital x? The answer turns out to be that R of n and psi let us call it psi the angle by which I rotate because I would like to keep theta for the polar angle of axis n theta and pi so, let us call the angle by which you rotate through which you rotate as psi.

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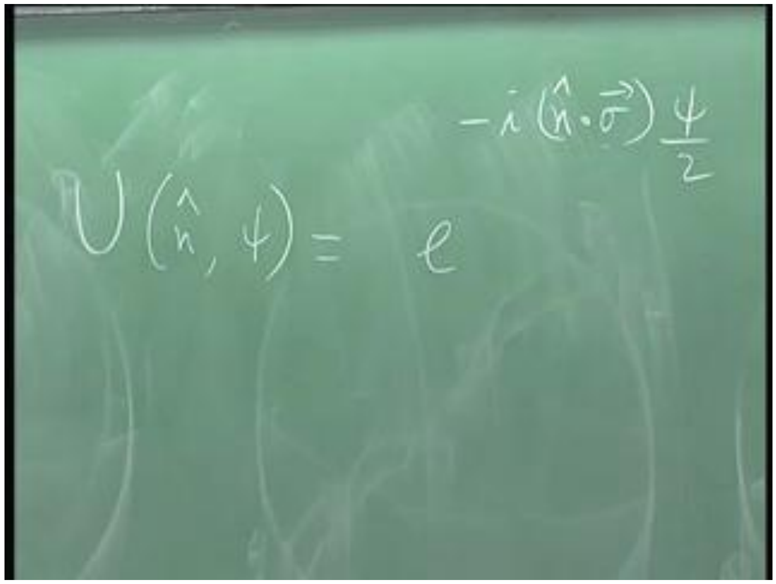

Instead of this, you represent it by a unitary matrix U of again parameterized by n and psi which happens to be e to the minus i n dot sigma psi over 2. Now, what sort of object is this U? Remember sigma is a 3 by 3 is a 2 by 2 matrix is a Pauli matrix. So, you exponentiate it to still get a 2 by 2 matrices. So, this quantity U this object U is again a 2 by 2 matrix finally, it should be because that is what can act on this this is, what can act on this guy, what sort of matrix is it? Remember n is direction cosines for some unit vectors and there are real components psi is real and sigma's are all Hermitian matrices and you have e to the power i times real parameter times the Hermitian matrix. What sort of object is that?

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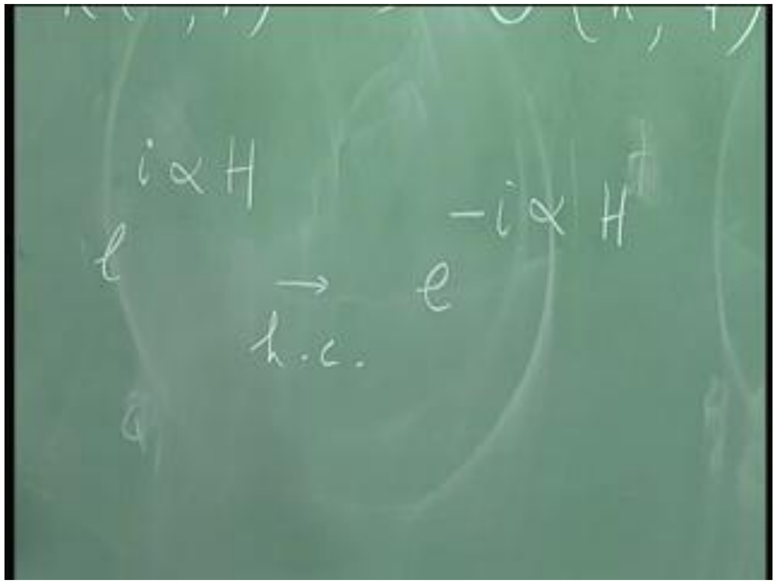

If you have e to the power i a real parameter times the Hermitian matrix here, Hermitian operator here, if I take it is Hermitian conjugate it is going to give me Hermitian conjugate is e to the minus i alpha is real h dagger the Hermitian conjugate, but this is the same as H. So, taking the Hermitian conjugate of this guy gives you, e to the minus this guy which is the inverse. So, it means that U is a matrix such that u dagger equal to U inverse.

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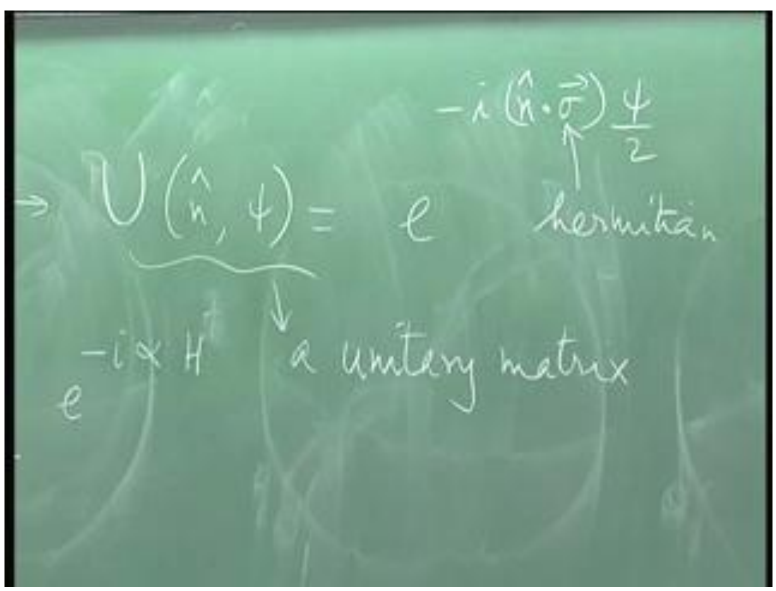

And what you call a matrix, u dagger equal to U, U inverse or U, U dagger equal to identity unitary matrix that is the reason I used U. So, this guy is unitary this guy is Hermitian.

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How did I arrive at u I worked backwards, there is systematic way to arrive at U I do not want to get into that right now. This is way you do if you look at that some books on Classical Dynamics we would treat what are called Kelly Klein parameters and this is essentially what it is.

So, right now I am just presenting their result and telling you what exactly is you could ask can I have other representation and so on y 2 by 2 and so on and so forth. There is a reason, technical reason for it this is in some sense the easiest representation, there is infinite number of representations possible, this is the simplest of all and you will see why?

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How do I get u this is.

The prescription you mean if you are given R it self, let me I have not written down R itself I have not done that. It is a 3 by 3 orthogonal matrix it is fairly messy it is in fact, much messier than U, you will see that U is fairly straight forward towards the end I will go back and show you what r does, how it acts and it does, so in a very interesting way. So, this is now my claim is that this represents the rotation as it acts on this x then the question asked is the following if you go from r to r prime.

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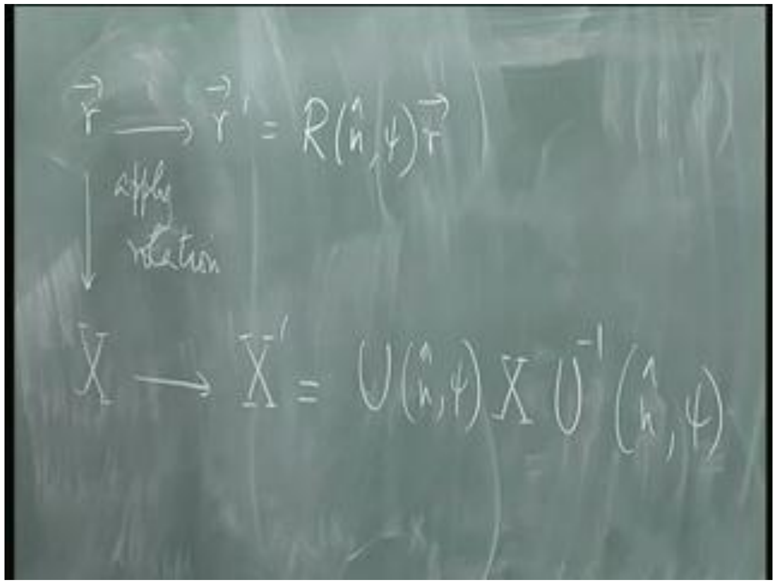

So, I start with the position a point r, and then I apply rotation I go to r prime which is equal to this R of n psi acting on r where, by this I mean the column vector and by this I mean the 3 by 3 matrix. So, it goes to some point r prime which is x prime y prime z prime, then the question is what does x do? So, this guy here is mapped to x is replaced by x and; obviously under rotation it goes to some x prime. And how is x prime given in terms of x? And the answer is that this is given by U of n psi acting on x U inverse on n psi.

So, this is the action of this U of this representation of a rotation group this is the action of it. So, you give me an x corresponding to a point in space and under the rotation represented by U it goes to a new matrix x prime where, x prime is given in terms of x by this object. And this is invertible, because U has an inverse really and it is clearly invertible you could write; obviously, U inverse x prime U is equal to this goes back in the other dimension. Therefore, rotations are represented either by 3 by 3 orthogonal matrices or they could be represented by 2 by 2 unitary matrices with determinant plus 1. It is easy to see that determinant of something like this is plus 1 because, if you set psi equal to 0 you get the identity. So, it is the connected part of this group and the determinant is plus 1. Now, the question is is this the one to one map or not?

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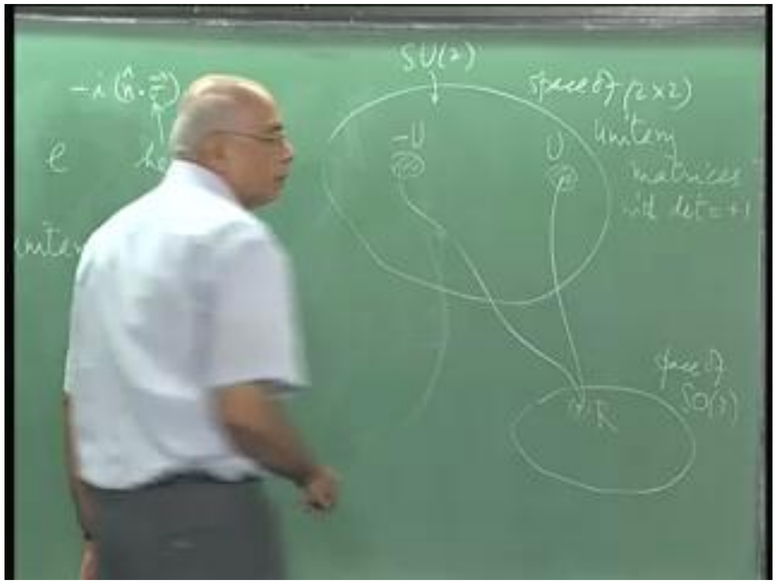

And the answer is it is one to two map because, in the space of the U's, here is the space for this U's space of 2 by 2 unitary matrices with determinant plus 1. And this is the space so, 3 of all rotation matrices, then corresponding to a rotation here and some small neighborhood of it there exist a U here, so this was an R. But, there also exist a minus U, the minus U will give you exactly the same rotation, so in fact, it is a 2 to 1 cover every point, every physical rotation represented by an element of SO (3) is mapped to two different U's here, which differ by a sign.

And it turns out that, that space is a right one that is the universal cover, we have to show it is simply connected, we also show we have to show see what it is parameters space is? But we can identify that space, what you call the space of unitary matrices in n by n matrices on the real it is SU (n) and of course, if you put an s for determinant then it is the unitary uni modular matrices. And therefore, this is SU (2) this space is the space of SU (2). So, that is what provides the cover we have to show that it is simply connected, that snot hard to show because, what is the most general form of a 2 by 2 matrix which is unitary and which has unit determinant.

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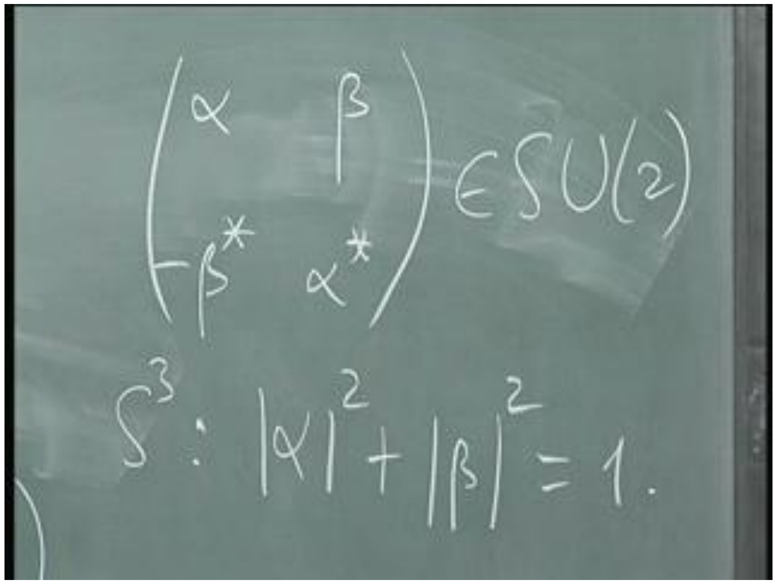

Well you could have as arbitrary complex number alpha here it is unitary and it must have determinant plus 1 these conditions are sufficient to show that the most general such matrix that you can have must be of the form some alpha complex number, some beta complex number, but what must appear here is beta star and alpha star and mod alpha square plus mod beta square equal to 1. This is the most general 2 by 2 unitary matrix with determinant plus 1.

It is parameterized by two complex numbers alpha and beta satisfying mod alpha square plus mod beta square equal to 1. But, what sort of object is this? It is alpha 1 square plus alpha 2 square plus alpha beta 1 square plus beta 2 square equal to 1 that in the space of 4 real parameters alpha 1 alpha 2 beta 1 beta 2 is the surface of the sphere of unit radius. So, this guy is S 3. So, this says that the rotation group in three dimensions SO (3) has as a universal cover SU (2), but SU (2) is simply connected. SO (3) itself was doubly connected it has a complicated parameter space, but S 2 S 3 is simply connected.

And I said S 3 is also a lie group it is in fact, SU (2) S 3 has the structure of a lie group which is SU itself. So, this is the origin and this is the starting point of saying that there are objects in three dimensional spaces which would have transformation properties under rotation group such that the rotation of 2 pi brings you back to the same object. And there are other objects were a rotation of 2 pi changes the sign of this object and rotation of 4 pi brings you back to the same object and the former are called tensor representations and the latter are called the spinner representations, that is the origin of spin half three half and so on. When you quantize and when you do when you add when you put in quantum mechanics on to this structure. But this is the basic point, this is doubly connected.

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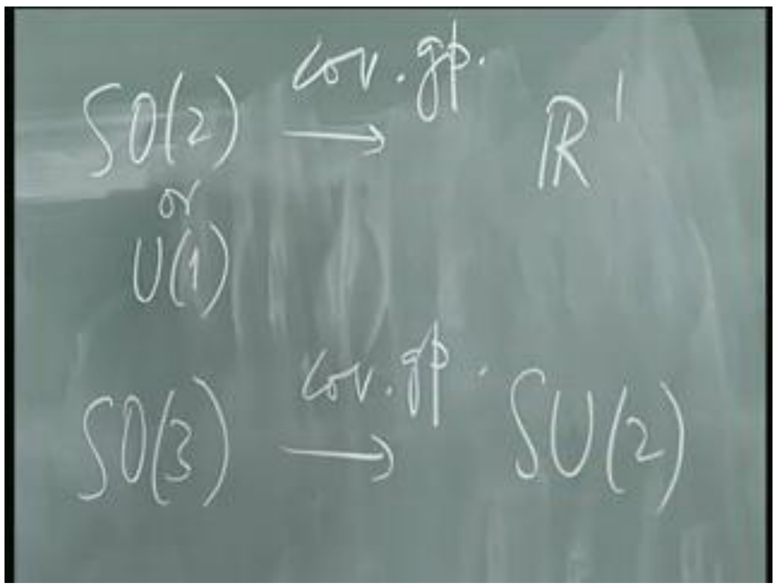

You could ask, what about the covering group for so, let us write the covering group you have SO (2) and the covering group, universal covering group here for SO (2) plus R 1 itself this is SO (2) or U (1) points on the unit circle or S 1 when you have SO (3) and the covering group is SU (2).

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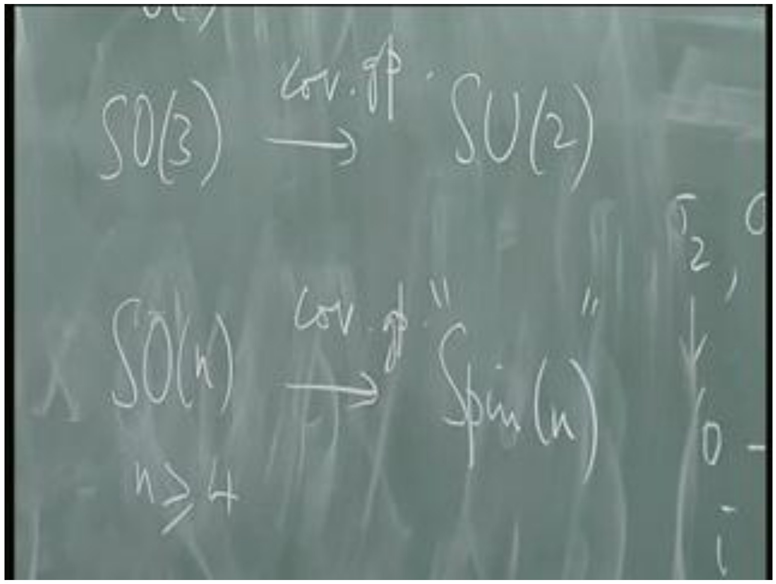

Now you could ask what about SO (n) n greater than equal to 3 n greater than equal to 4 well these are also doubly connected. And the covering groups here, are not SU (n) anything like that or SU (n) minus 1 or anything, the covering groups are called the spin groups. So, the covering groups here are Spin (n) that is just a name and the moment later we put some more structure to it, but all these spaces are also doubly connected.

Now, to go back and answer the earlier question what does r do, what does r of n comma theta look like? In general it will be a terrible mess of course, if I rotate about the x axis or the y axis or the z axis things are simple, but if I rotate about an arbitrary axis it is quite complicated, the 3 by 3 matrices are given in textbooks it is quite complicated. But let me write down the answer as to what finally, happens to a given vector and that is not hard to see, the there is a systematic way to do this and that is the following represent the point r by this matrix x apply this rotation.

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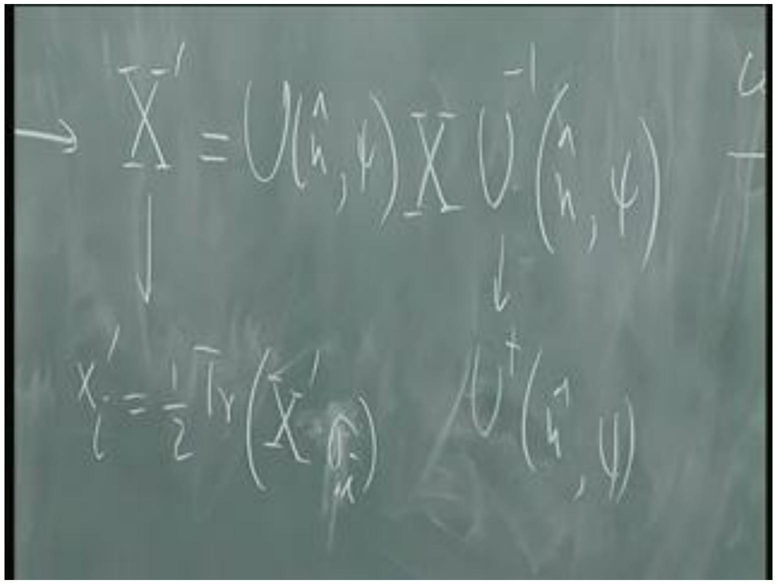

So, apply U of n psi on XU inverse of n psi where, U is equal to that so, apply this rotation here, find U inverse which is the same as U dagger this guy is the same as U dagger and then you have to multiply 3 2 by 2 matrices that gives you x prime, and from x prime go back to r prime because I know, that x i prime equal to half trace x prime x prime sigma I, that will give you the new components good exercise except you must know how to do this. You must know how to write the exponential of this thing here, but that is where the properties of the sigma matrices comes in the sigma matrices are very very interesting properties among which is the following fact.

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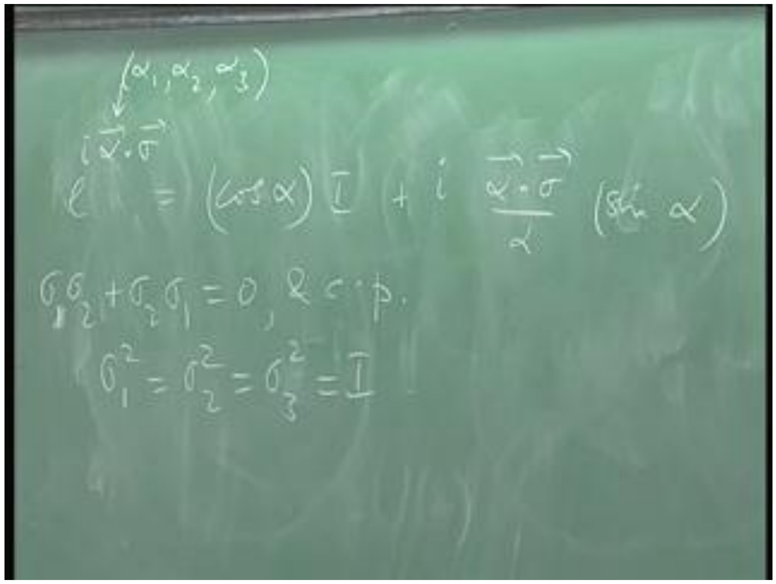

You see we have to do something like e to the i alpha dot sigma where, alpha stands for 3 real numbers in that case alpha was just psi over two times n with a minus sign, but in general if alpha 1 alpha 2 alpha 3 are 3 real numbers and I want to do e to the i times alpha 1 sigma 1 plus alpha 2 sigma 2 plus alpha 3 sigma 3. I use the fact that sigma i sigma 1 sigma 2 plus sigma 2 sigma 1 equal to 0 and cyclic permutations these matrices anti commute with each other. Moreover, sigma 1 square equal to sigma 2 square equal to sigma 3 square equal to the unit matrix. And then exponentiation is very easy you have a formula similar to the Euler formula.

The Euler formula says, e to the i theta is cos theta plus i sin theta for 2 by 2 matrices there is a similar formula, not for 3 by 3, 4 by 4 or anything, but for 2 by 2 there is a simple formula, and that follows from these properties of the sigma matrices. That is not hard to show that this is equal to cos alpha times i plus i sin alpha over alpha i alpha dot sigma over alpha sine alpha where, alpha stands for square root of alpha 1 square plus alpha 2 square plus alpha 3 square you got to be careful about the i's and so on. If alphas are complex, but if they all are real numbers it is straight forward this formula is not very obvious, but it is logical. Because you know that any two by two matrix can be written as a linear combination of the unit matrix and the three Pauli matrices e to i alpha dot sigma is also 2 by 2 matrix. Therefore, it should also be written writable as the linear combination of the unit matrix and the three sigma matrices and these are the coefficients and it looks exactly, like Euler's formula.

Now, given that all you have to so, is to substitute alpha is equal to psi over 2 time's unit vector n. And that will tell you, what U is and this is just the Hermitian conjugate on this side and put those matrices multiply through and take this space and you go back to what r prime is that is the fastest way of finding out what r prime is under a finite rotation. Mind you have not written down the matrix r itself that is quite a mess, but why do I need that matrix, I need it to find out only what r prime is have been telling you a formula for r prime. There is an even simpler geometric way of doing this heuristically and that is the following and that is as follows.

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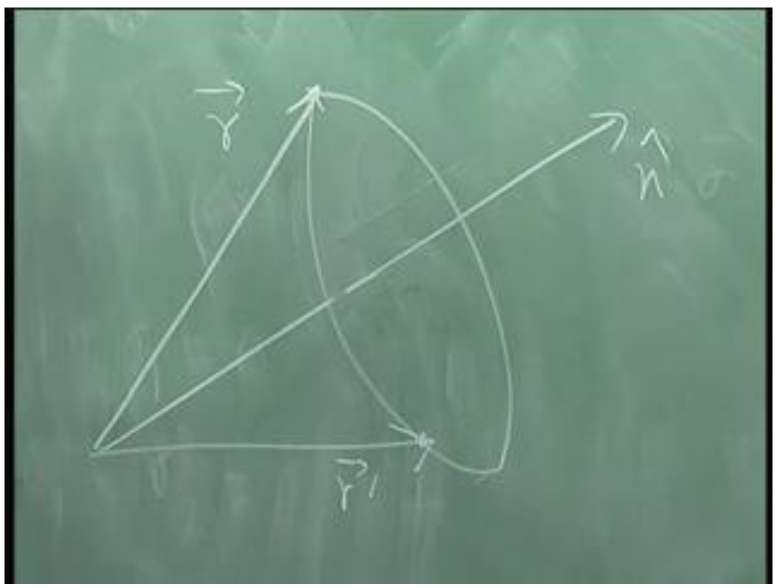

So, let us suppose this is the axis, n the original frame of reference this is the axis of n and this is my point r this point here and I would like to find out what does r do under a rotation through an angle psi about an axis n it is quite clear from this figure this is n. It is quite clear that r is going to move on the surface of a cone of angle psi of half angle the angle half angle which is equal to the angle between r and n. And it is going to move on this depending how much it moves depending on depends on what psi is, so let us suppose it ends up at some point like this r prime. So, we have this axis, and we have that vector and you rotate and go cyclic.

Now, the question is what can r prime depend on? Well there is the original r, and there is the n and it is a linear transformation therefore, every component of r prime must be a linear combination of the components of r. Look at r and n they form two vectors and they form a plane. Therefore, any vector in three dimensional space should be expandable along r along n and along r cross n.

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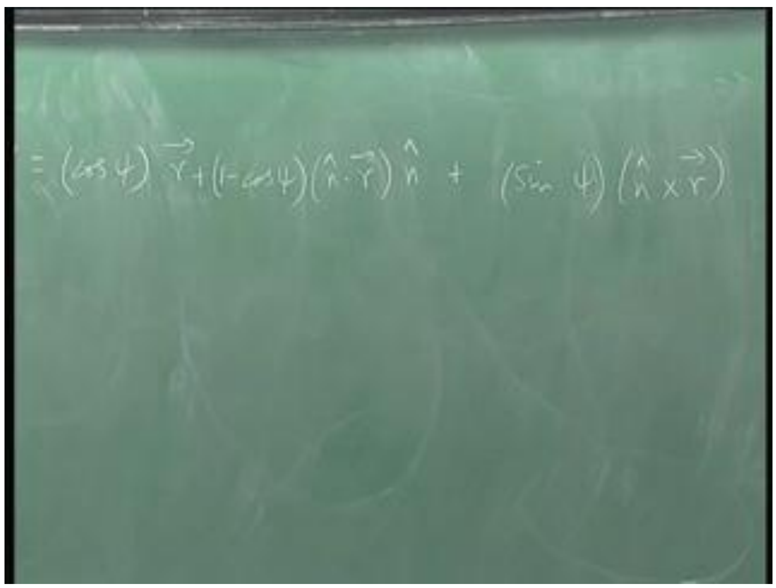

So, this whole thing finally, r prime must be equal to something times r itself plus something in the direction of n plus something in the direction of r cross n that is all it can be and; these something must be scalars they must depend on psi. Now, what can happen here is that this is not this term here must also depend on r linearly. So, it must be the component of r along n and therefore, you can in fact, further reduce it and write this as n dot r on n times something or the other. Now, every term is linear in r, and then if you go through this rigmarole, this long formula you discover there is very simple formula, which is cos psi r plus 1 minus cos psi n dot r n and then of course, no medals for guessing what is the last term is going to be.

What should it be?

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Sine psi sine psi that is it n cross r n cross r. So, that is the effect of a finite rotation about an axis n through an angle psi on an arbitrary vector r. I have to do this entire rigmarole this is the answer you get. So, I urge you to check this out to really get this concept. So, it is a very simple structure as you can see the portion along r portion along n with component n dot r and then n cross r. You could also combine this guy this cos part here, along with this and write it as n cross n cross r do that, but there are equivalent ways of writing that I find this easier to remember. Now, you can put in special cases you can put n along r and see that it does not change at all, to put n along r that last term vanishes and then this guy cancels r part of it and then you end up with cos cancels and then of course, you end with just r itself. So, from this you can work back and write what the 3 by 3 matrixes, so I urge you to do that. So, take n to have components theta and phi latitude and polar angles and azimuth angle and then find out what the 3 by 3 matrixes not the most convenient way of representing matrix rotations, but such as it is it is a simple orthogonal matrix.

Now, what I would like to do, is ask you a couple of questions for which will be useful later on and then, I would like to do the role of symmetry I would like to shift the symmetry we have been talking about rotations I would like to go to invariance and symmetry. Remember that I pointed out that you have matrices which are unitary look like this U dagger U equal to I elements n by n matrices which are unitary belong to a group of matrices satisfying this U dagger U equal to I.

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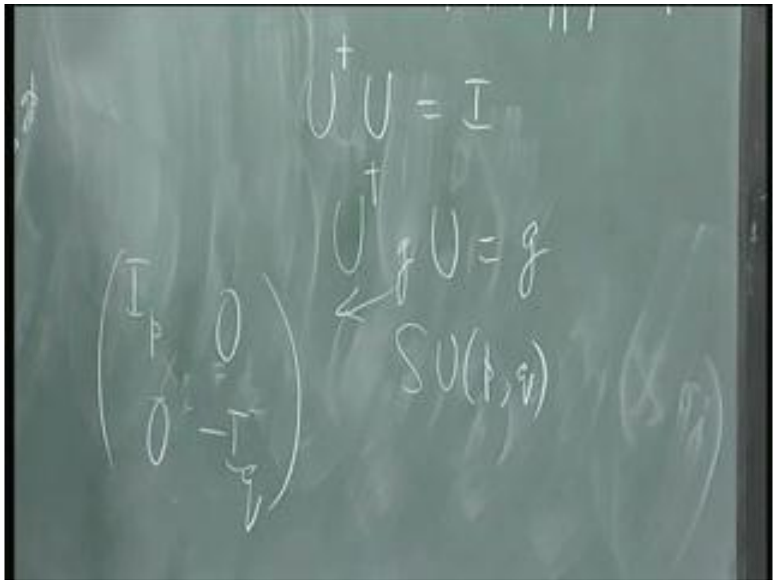

Now, the determinant of such a matrix is a pure phase, it is e to the i theta and if you insist that the determinant should be plus 1 you get the SU (n) group, and that has n square minus 1 generators. The space of this SU (2) for example, the space is a compact space, SU (3) is a compact space close bounded space, I also pointed out that if you had matrices which satisfy U dagger, some other quantity here which was not the unit matrix, but something slightly different I do not remember the notation I use for it something like g U equal to g and this g had a p by p matrix and a q by q matrix here in this fashion. This is the unit matrix and this is the minus unit matrix then I call the group of matrices satisfying this as belonging to SU (p comma q) with this matrix. So, you could ask we just talked about SU (2) these are 2 by 2 matrices we could also talk about SU 1 comma 1 those would be the matrices where this g would be a 1 0 0 minus 1 out here.

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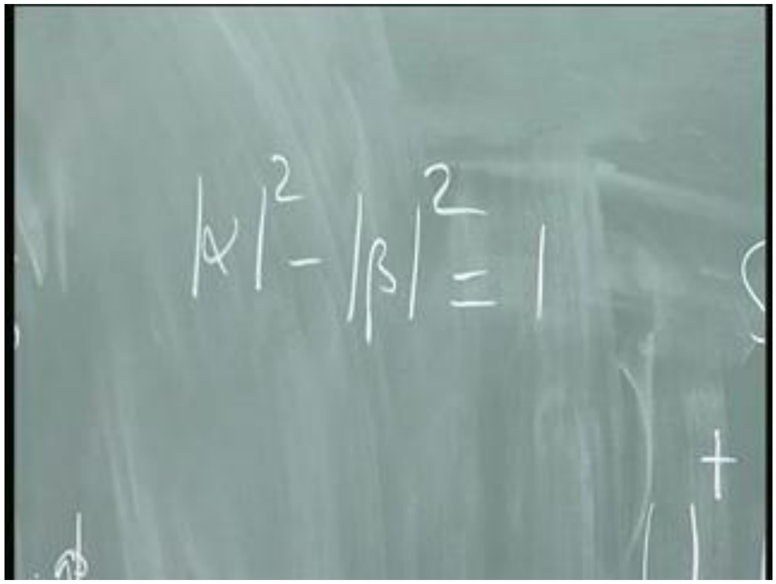

They would look exactly like this except that the condition would be alpha square minus mod beta square equal to 1, there would be a sign change here what sort of object is this? It is actually a hyperboloid, it is not a closed sphere it is a hyperboloid because you have a minus therefore, each of these quantities can be arbitrarily large and cancel each other to give you the difference equal to 1 it is like a difference between an ellipse and a hyperbola. Hyperbola is an open curve, but ellipse is a closed curve a bounded curve. So, a group like this is non compact, the parameter space goes all the way to infinity it is not bounded. Therefore, this change of metric is highly non trivial, the change goes straight away from a compact group to a non compact group. And we will see that the Laurence group the Laurence group the group of special relativity is a group in four dimensions space and time, which is the group of pseudo orthogonal matrices 3 1.

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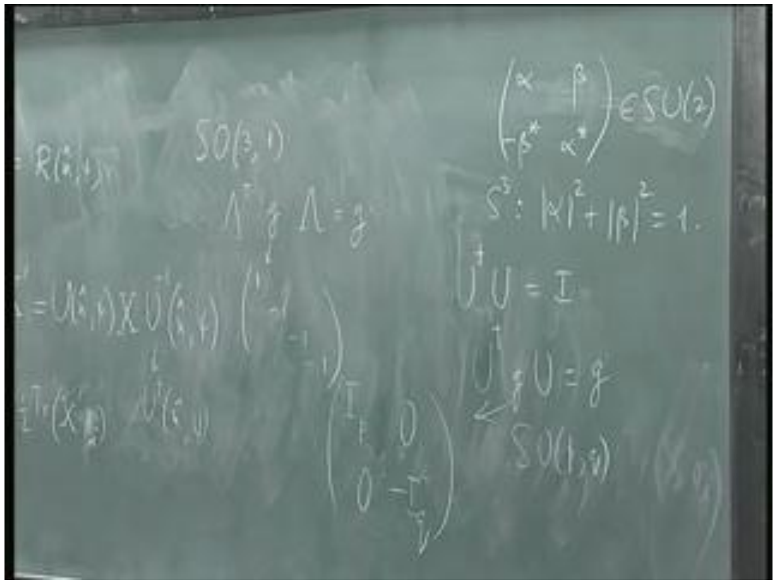

Three space like directions spaced like coordinates and one time like coordinate which would leave a particular kind of quadratic form unchanged and this is a non compact group this guy here right. And this would be group with the group we will focus it satisfies a relation very similar to this. So, if lambda is an element of this group then it says lambda transpose g lambda equal to g where this g would be 1 minus 1 minus 1 minus 1. So, there would be 1 time like direction and three spaces like direction sorry in this notation it will be q comma p the opposite way. So, we will do that next when we talk about relativity this is what we are going to do. But before I do that I want to come back a little bit too Classical Mechanics and show you how invariance and symmetries are closely related to each other and they are related by a theorem called Noether's Theorem see I have run over of time.

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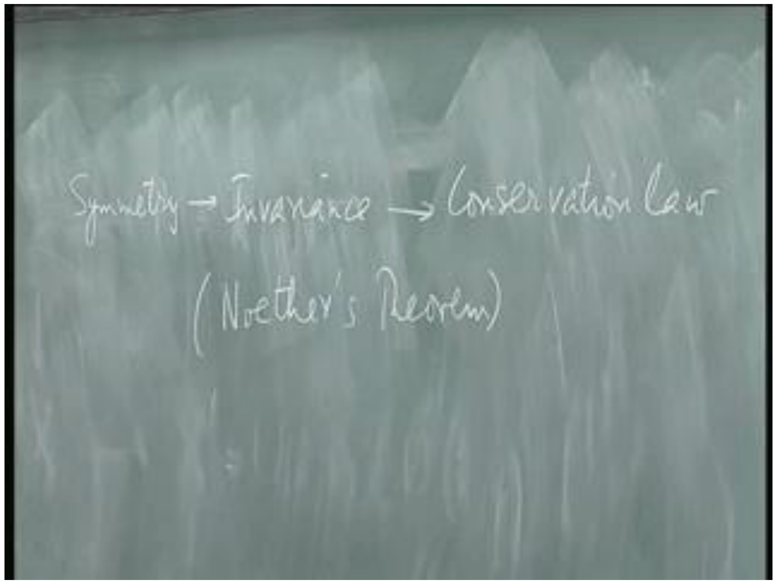

So, the connection in fact, is you identify a symmetry that leads to an invariance that leads to a conservation principle. It is a crucial thing in all of mechanics all of field theory and all of physics itself and we will explore this connection and it is summarized in something called Noether's theorem.

Next time I will talk about this theorem and show you how in Lagrangian mechanics this is a simple formulation which gives you this conserved quantity directly. So, the moment you identify for instance, I know that the planets move around the sun in a central force. And I know that a central force implies that angular momentum is conserved; and the way I prove it is to say that the rate of change of angular momentum is torque and there is no torque in a central force, but it is deeper than that we would like to understand where does this angular momentum conservation come from? It comes from the spherical symmetry of the system the fact that all the directions are equivalent when you have a central force. So, that symmetry leads to the fact that equation of motion do not change in form under rotations and that in turn leads to the conservation of angular momentum.

So, this is the path we need to trace to understand how the symmetry leads to invariance and how does that in turn lead to a conservation law. How do I calculate the quantity that is conserved and that is what is given by Noether's theorem? In general this conservation principle is in the form of a conserved current, the quantity satisfying some continuity equation and that in turn leads to the conservation of a macroscopic quantity like a charge or something like that conservation of mass follows from the equation of continuity. So, the you get a local statement and then you get a global statement, finally and this is done by a Noether's theorem. So, this is what we will talk about next to do this we start with simple example and go on like that.