Classical Physics

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Lecture -33

We have been talking about groups, and I want to introduce large number of related concepts one by one, so by that purpose I have actually jotted down some of these things, so that I do not forget them. But in case I do will come back and try to rectify matters, the first of these is the following, we defined what a group was, and I did not mention what would happen, if we did the group composition law in two different orders.

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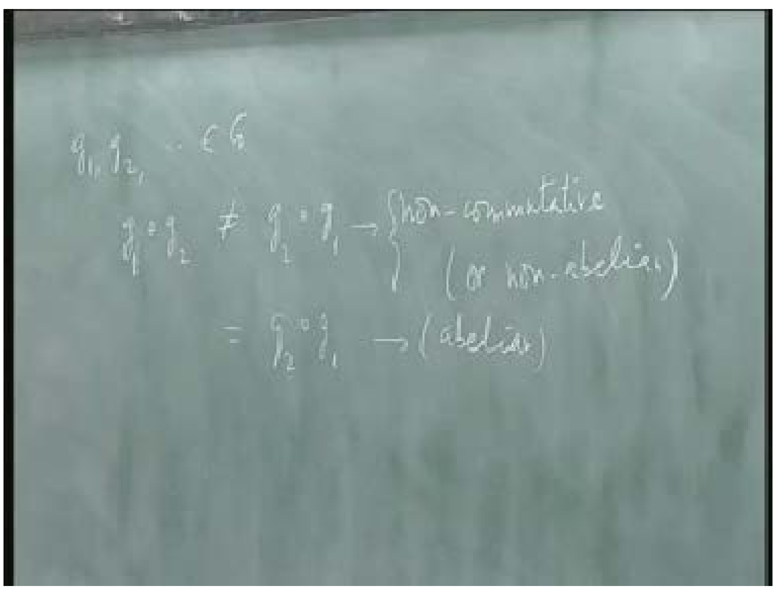

For example, if g 1, g 2 etcetera are some elements of group G, then you could ask what about the composition of g 1 with g 2, as supposed to the composition of g 2 with g 1 in the opposite order. And of course, if these are matrices, these are matrix groups then in general this is not true, g 1, g 2 is not equal to g 2, g 1; and then of course, you have a non commutative group, so if this is not equal to that, the group is non commutative or sometimes written as non abelian, and if its equal to g 1, g 2 the group is abelian.

Now, of course the G group that we looked at yesterday, the simplest of that namely r 1 with addition as the operation, that is very much abelian; you can add numbers in either order, you get exactly the same answer. What about the group SO2 is this abelian or non abelian, its abelian because, this contains rotations in a plane but, we represented it in many different ways; and one of the representations was to write this as, set of matrices of the form cos alpha sine alpha minus sine alpha cos alpha, and these matrices satisfied this relation.

This is an abelian group, it is very easy to verify, that R of alpha R of alpha prime is T of alpha plus alpha prime is equal to R of alpha prime R of alpha, in either order. And physically what it means, if I rotate in a plane by a certain angle alpha, and follow it in angle alpha prime I could have done this in the reverse order, I get exactly the same final state. SO3 and SO4 and so on, higher dimensions they are not abelian, because rotations in different planes do not commute with each other, abelian groups have a great deal of simplicity in them, as you can see.

But, in general groups are non abelian and therefore, you have to deal with that idea; the second concept that we talked about, that I mentioned was the idea of compactness, compact group by this I mean, by compact I mean you have a group whose elements are parameterized by a set of parameters, if the range of the set of the parameters, the values that these parameters can take is bounded. And if the set is closed in the sense that all limit points are also in this set, then its called compact, R is R compact is R 1 or R n this is of course non compact, because this goes all the way to infinity.

But what about SO2, this is rotations in a plane and the range is 0 to 2 pi for this angle and therefore, this is compact group, what about S 1 the circle group this is in fact, SO2 its an isomorphism because, we saw that you can put points on the unit circle in the complex plane. You can associate with it in arbitrary point here, is some e to the i theta and if the group operation is multiplication of this complex numbers of modulus 1, then clearly this is compact because, e to the 2 pi n is equal to unity once again. So, this is a compact group, there are many ways of writing this group S 1 is one of them SO2 is another groups are all isomorphic to each other.

You could also regard numbers of the form e to the i theta as one by one matrices, you can regard it as a matrices, but a one by one matrix it is just a complex number, and it is a unitary matrix. Because, the unitary matrix is U tag and U is equal to the identity, and you have e to the minus i theta e to the minus theta is equal to 1, and that group of matrices is called U 1, these are called different names. For exactly the same thing you write represent them in different ways, but they are isomorphic to each other, is S 1 compact very much so, all these are all compact rules, what about SO3, the set of rotations in three dimensional space, this is also compact.

Because, these are angles that you rotate about, and these angles are bounded from above and below and this is also compact SOn is also compact, these are all examples of compact things. What about the group of translations in a plane, non compact can go all the way to infinity that is like R what ever it is, group of translations like the parameter goes all the way to infinity, non compact. What about later I will show that the set of Lawrence transformation from one frame to another, is a group rotations and velocity transformations is a group.

Do you think that is compact or non compact?

We will prove this properly, we will prove this properly, it is a non compact group and we will prove, and say why this is so that group is non compact, but rotation is compact, this is a compact group. What about N-dimensional unitary matrices, U(n) compact or non compact, what do you think, the immediate guess would be yes it is a compact group, yes it is and if you impose a further condition, we are going to talk about this group a little bit more.

If you impose a further condition that the determinant of this matrix e equal to plus 1, then you put an S here, just I put an S here to show that its a special orthogonal group, determinant plus 1 similarly, S U(n) would also be unitary matrices, n by n matrices with determinant plus 1. What is the general statement you can make on an orthogonal matrix of this kind, an n by n orthogonal matrix, in which you have given that R transpose R is I that is it, what can you say about its determinant, its plus or minus 1, what can you say about the determinant of a n by n matrix which is unitary, the modulus of the determinant is 1. So, what is the determinant itself?

It can be anywhere on the unit circle, it could be some pure phase factor e to the i alpha, unlike this case where this has to be plus or minus 1, there you have a whole continuum of possibilities. And because of that, the moment you put S U(n) condition the number of independent parameters goes down by 1, and that is in happen here, and we will come to this and I will tell you how to count parameters, we need to be able to count the number of independent parameters, specifying each of these groups.

What about the group, that is the mother of all these groups, these continuous groups, the general group G, general linear league n, R, this stands for general linear group in N-dimensions on the real, in other words the set of all n by n matrices with real entries, and which are non singular, the determinant is not 0. So, the set of non singular n by n matrices, this is a non compact group, because you could have arbitrarily large entries in some sense therefore, this is non compact.

And it is the group which is huge one, and then n composes practically everything, and then after that there are subgroups, all these other groups you are writing down are actually sub groups of this, general group here, you could also complex if I things and make this on the complex numbers. So, the entries could become plus and they can form group also, as long as a non singular matrix, and that is denoted by g l n, c. A sub group of this is this group, a sub group is SL (n, R), which is a group of general n by n matrices non singular with determinant equal to plus 1.

And that is the special linear group, again non compact, again non compact simply because, you can have arbitrarily large entries

We have everything possible, of course yes, it could be scalar transformations, it could be deformations of all kinds, its just a linear transformation; a general arbitrary linear transformation, you have not put any condition like distance that should be constant or anything like that, completely general statement. So, we are going to sit down and count the number of parameters in all these groups very shortly, but for that first I also want to introduce the idea of connectedness, and we did that, and I pointed out that the group space is connected, if there is a path an arc wise path, from any point in the space to any other point without leaving the space.

Now, of course you have groups which are not connected, they have disjoint sets and so on, for example the group of orthogonal matrices over n, those matrices would contain two disjoint sets, those with determinant plus 1 and those with determinant minus 1, and you cant go from one to another by a continuous path. Because, the determinant does not change continuously, it changes discontinuously from plus 1 to minus 1 for one set, so that sort of group contains, so the group O (n) its space contains two pieces. And this contains determinant equal to minus 1 and this determinant plus 1, and this is what we call SO (n), and that is the full space of these matrices.

The number of disjoint components of a space is denoted by pi naught of that space, disconnected components or sets disconnected, and that is two in this case. Obviously, in a discrete group every element is disconnected and then of course, this is not a useful concept, but for a continuous group you have to appreciate the fact, that there are pieces which are continuous in one side and then other side which is continuous; and then there is no connectivity between these two lines.

But you can still form a complete group; this is now a subgroup of this group here, so the idea of a sub group is if you take the root g, you could have a set of elements in it, which would by themselves form a group and that of course, what you call a sub group. The identity element always form a subgroup of every group, it satisfies the group axioms and that is trivial, the full group itself is its own some subgroup in some sense, but what I mean by a sub group is a proper sub group, namely something that is not the identity element, and now its not in full state itself in between.

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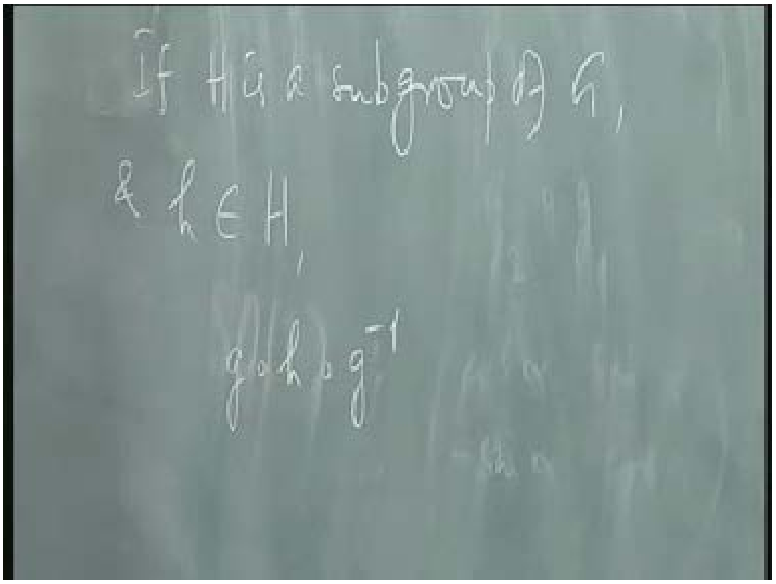

Very useful in the idea of subgroup is what is called the normal sub group and invariant sub group, let me explain what that is, so let us take a group G and g is the element of G arbitrary element. If this has a subgroup, if H has a subgroup of G and h is an element of H an arbitrary element, then you ask what happens with g composed with h composed to g inverse, so you do a similarity transformation in the sense of matrices for instance, on this element h with g, h, g inverse.

Of course, if you put h prime and h prime inverse, which belong to the sub group H this would go on to the sub group H itself, but you take an arbitrary element of G and you can struct this, if for every g and for every h, element of capital H and every g element of the full group. If this is an element of H, so it stays within that sub group after you do this conjugation, then then H is an invariant its an invariant or normal sub group; so a concept is involved here, you take the sub group and you take the sub group and elements in it, and you conjugate them with arbitrary elements from the full group, and if you still remain with in the sub group, then it is a invariant or normal sub group.

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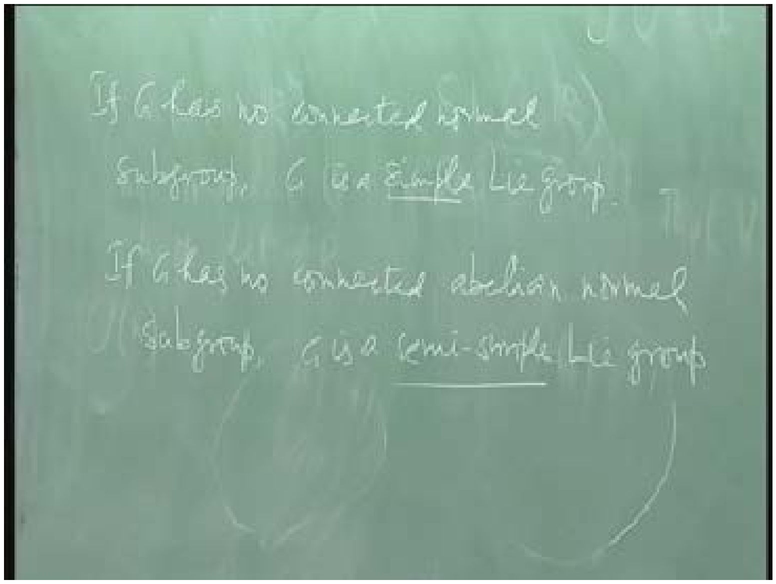

If there is no normal sub group at all, the connected league group does not have a connected normal sub group, G is a simple, I only talking about lie groups , this might be too restrictive very often, when I ask for a slightly less restrictive condition, if G has no connected abelian normal sub group, G is a semi simple. This has proved to be the right concept, so this is less restrictive and it is clear, that the simple group is also semi simple, but the converse is not true necessarily. So, all your asking is is there a connected abelian normal sub group, commutative invariant sub group, if the group is semi simple.

Now, most of the lie groups we are going to look at, like SO (n) like simplistic group SP2n the unitary groups SU (n) and so on, these guys are both simple and semi simple, and a semi simple lie group has very, very interesting properties, you can actually start cataloging representations, classifying the groups and so on and so forth. Make possible by the factor is semi simple.

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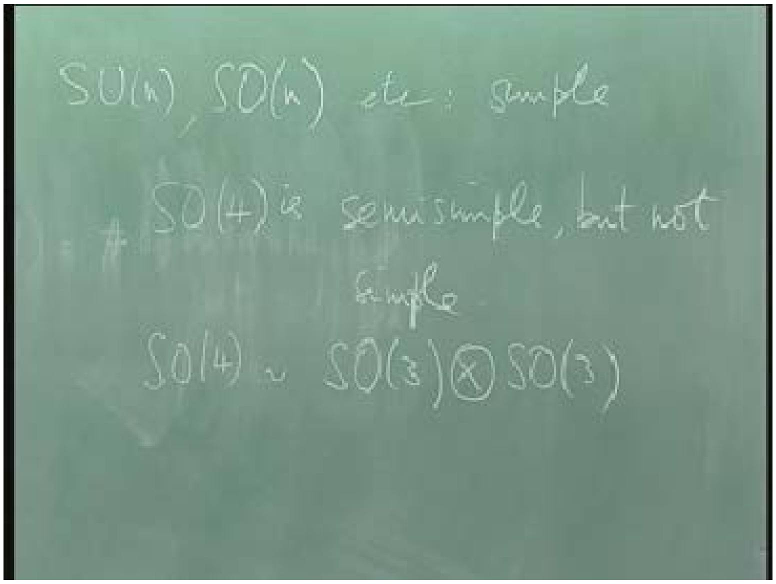

SO (n), let me straight away give an example SU (n), SO (n) etcetera these are simple and therefore, semi simple also, there is one exception among the rotation groups, SO (4) is semi simple, but not simple; just happens that SO (4) alone, is its not simple, its semi simple. It has no abelian invariant sub groups, but it does have non abelian invariant sub group, in fact it turns out that SO (4) is isomorphic to the direct product of two SO (3), SO (4) is the group of rotations in connected to the identity, in four Euclidian dimensions.

How many independent planes are there, how many generators it would have, 6 generators, 4 C 2 and in three dimensions you have three generators, so you actually have 3 plus 3 which is 6, and it so happens that in this case SO (4) can be written as direct part of these two, so each of these is a sub group, because each of them is a sub group, and is non abelian the group is it is not simple, but it is semi simple, these are not abelian, so this condition is not satisfied, but this condition, this condition has no semi simple no abelian, sub groups.

But, it does have connected normal sub group and so on, so which is therefore simple but not semi simple, but not simple that is the right answer; I should have used the negation of these statement. Now, we need to count the number of generators, because we said that for SO (n) I have n times n minus 1 over 2 generators, I need to know what it is for SU (n), we need to know what it is for simplectic group and so on, we need to be able to count generators. For that concept, its better to work not with these groups themselves, and the group elements, but with what are called the infinitesimal generators. I pointed out that we are going to look at group elements, which are continuously obtainable from the identity element, in a continuous way and the parameters the group elements are analytic functions of these parameters, under very general conditions they are actually exponential functions of these parameters.

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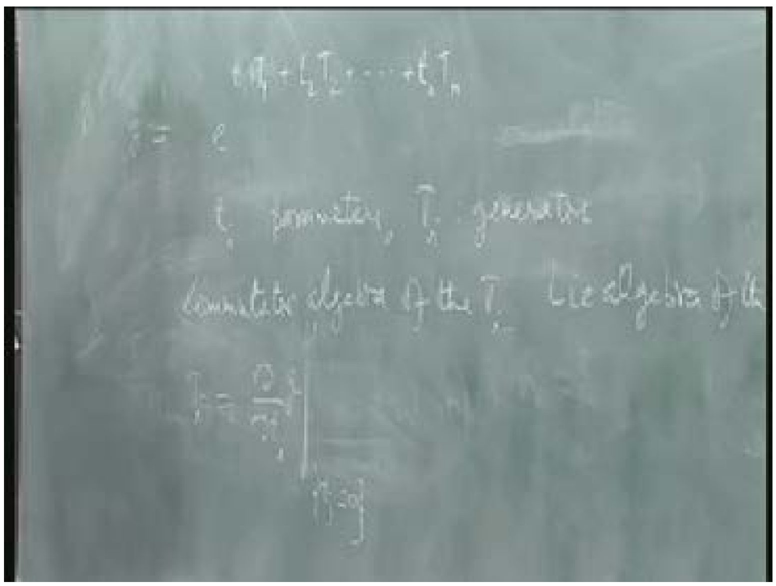

So, under very general conditions an element g of one of these matrix groups, could be written can be represented by e to the power something of the form t 1, let me use this symbol I want to use g, let us use T 1 plus t 2 T 2 plus etcetera, up to t n T n, where t i are the parameters take them to be here at the moment, and the T i are the generators. This is the matrix group, then this capital T's would be matrices, and you exponentiate this whole thing and you end up with general group element. Now, the point the whole point is that if there is more than one capital T, there is no reason why they should commute with each other they would form a non trivial algebra.

And that is the lie algebra of the group, so the commutator algebra, rather the algebra of the commutators of the T i is the, it is called the lie algebra of the group. So, the bilinear operation that I define among the generators is the commutator, T 1 commutator T 2 and so on, which means T 1 T 2 minus T 2 T 1, in the case of matrices its just the matrix commutator and they would be linear combinations of the other T's and that is the lie algebra of the group.

So, once you find the lie algebra, the whole idea is that the group elements can be generated by this procedure of exponentiation, the hard part is that of course, the various T's is going to commute with each other therefore, you cannot write this plus that as e to this plus that e multiplied by that is the whole point. Now, of course you have chosen representation in such a way that if all the T's are 0, you get the identity element, and then for infinitesimally small t you start generating the elements of the group.

The whole idea would be to ask using the lie algebra, what information can i get about the group elements, for example if I want to compute what T i is its equal to delta over delta t i g which is a function of all these guys, evaluated at the set t j equal to 0. If I differentiate with any one of these say T 2, I bring down that T 2 here capital T 2 and then I set all the T to 0 and I get this expression. So, this is the connection between the infinitesimal generators on one hand, and the group generators on the other hand, you could ask the exponential is not the only analytic function, other functions which would be non exponential functions of the parameters.

Then I form group elements, yes indeed the theory of groups does not necessarily require things that you must have things, which are exponential in this fashion, but by and under very general conditions all the cases, we are going to consider is certainly true, although there are some exceptions. So, we are going to exploit this in order to count things, and let us do that, let us look at for example the group of let us look at rotations itself.

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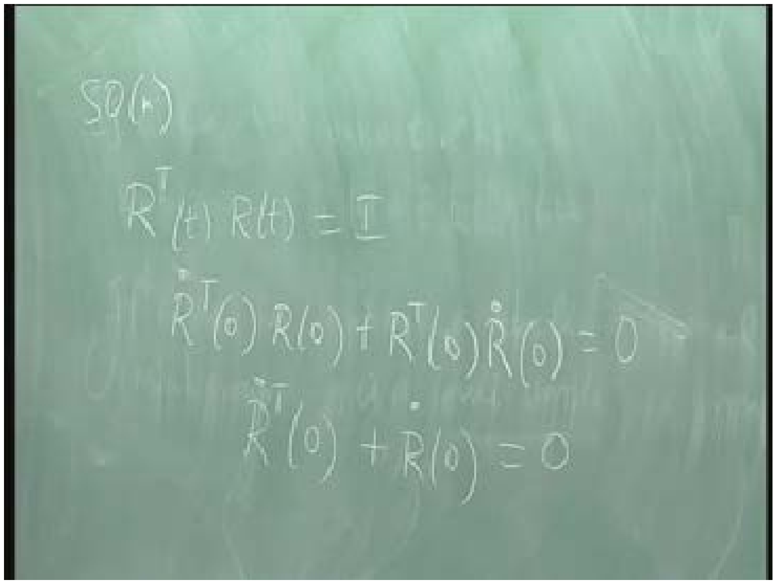

So, let us look at SO(n), this is represented by the matrices of the form R transpose R equal to I, if I write this in terms of infinitesimal generators, what would be the condition that I get. Write this as e to the power some generators, something of this kind and similarly for the other R, and then what would you think, well put a parameter here, so let us put R of parameter t equal to I, t is not the time its just the parameter of this kind, I differentiate this expression with respect to t, and set t equal to 0.

So, its clear R transpose dot 0 times R of 0 plus R transpose of 0 R dot equal to I equal to 0, the right hand side has no t dependence, this is true for every t, but what is R of 0, the identity element by definition, I have chosen my parameterization in such a way, that as shown as the I put 0 for all that it is, it just goes away. So, it says R dot transpose of 0 plus R of 0 R dot of 0.

Well, I do this Partial differentiation and then, its true for each of them I just put one therefore, algebraic simplicity but, imagine doing this for each parameter, so the moment I do it with respect to the I th parameter, this becomes the I th generator here; so it immediately says the generators must obey T transpose is equal to minus T, because this is just the generator itself, and its transpose here and therefore what is the condition that you get.

The generators must be anti symmetric matrices, and if the generator is anti symmetric and its an n by n matrix, how many its on the real numbers, how many real parameters do you need to specify the matrix

n times n minus n over 2, because you remove all the diagonal elements they are all 0, the half diagonal elements has split into half, because once below the diagonal or minus the ones above the diagonal. So, this implies n times n minus over 2 generators, so this is how you count the number of parameters, this is true for SO (n) on the reals, I could also look at orthogonal matrices n by n, as an abstract, as a matrix group on the complex numbers then of course, you double the number of parameters. But I have got only real entries, and this tells you that you have these many generators here.

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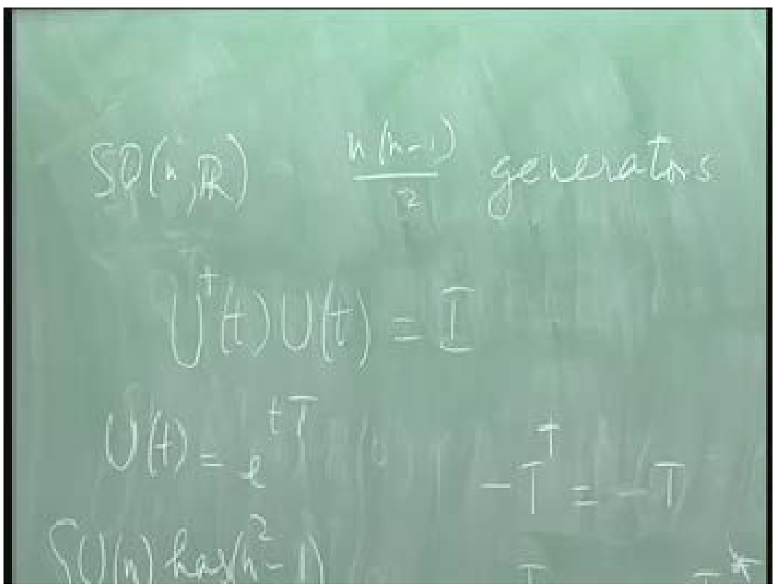

How many generators does U, n have, so I am going to look at U dagger of t U of t and these are n by n matrices, how many generators do you think it has, I do the same thing as before, if I write this general U of t as equal to e to the power t times some generator actually a sum here, I write it for simplicity in this fashion. And I differentiate with respect to that parameter, then what you get T dagger equal to minus T, exactly the same way as before this is going to be when I differentiate this, I am going to get U dagger dot its going to be T dagger and T dagger plus T is equal to 0, so T dagger is equal to minus T.

Therefore, this generators are antihermitian was Q hermitian, if T equal to T dagger it will be a hermitian matrix equal to minus T dagger therefore, how many independent parameters are there in U (n), U (n) is equal to n by n matrices, but unitary. Now, we are saying that these generators are antihermitian, but remember now you allow complex entries

So, how many independent parameters are there, we got to count a little bit carefully, if I take this T, and this T is written in the form of n by n matrix it is got diagonal elements. Let us put the condition on the diagonal elements first, and then see what happens, it says T 1 1 dagger T 1 1 star equal to minus T 1 1, diagonal element does not get transposed itself, and says this is equal to that. So, what does it mean, it says that the real part of this T 1 1 is 0, but there is a imaginary part pure imaginary part and that is arbitrary.

Therefore, along the diagonal you have n, you have n real numbers along the diagonal, now what about the off diagonal elements, so lets take for example it says, T let us put the minus here, it says T 1 2 is equal to minus T two one star complex conjugate, and transpose and this is the condition you have imposed on it. Now, what does that mean, it says that the imaginary parts are fixed, the real parts are fixed, once you give me T 1 2, you have given me 2 T 1 completely, it is real part is fixed its imaginary part is fixed, therefore all that you have to worry about at these fellows, and how many of those numbers are there?

Well, there are n into n minus 1 over 2 numbers here that many elements there, but then each of them is a complex number, so twice into 2 plus this n, so we have diagonals and then you have these n times n 1 over 2, non diagonal elements but they are complex numbers, and once that is fixed, these are fixed therefore, this is equal to U (n) has n square parameters, its got n square generators. What about SU (n)?

Now, you are fine because, we know that this matrix will have a determinant which is a pure phase factor, and now I put the extra condition that the determinant must actually be plus 1 therefore, it has SU (n) as n square minus 1 generators or but we are saying it is that dimensionality of the lie algebra corresponding to SU (n) that is written in this form, su (n), small su (n) that is n square minus 1, because that many generators, they form the lie algebra, they create the lie algebra.

What about has this many generators, what about O n?

No, no no yes no no I should have written SO (n, R) complex and trying to make it more complex is straight forward, but I am talking now about rotations in N dimensional space. How many generators does O n have, it has the same number because, the determinant is plus 1 or minus 1 its not continuously deducible from the identity, and in fact what example from O 3 is going to contain all those matrices, which orthogonal 3 by 3 real entries determinant plus 1, and then minus those matrices. So, it is got exactly the same number of independent parameters, you reduce the number by 1, only if you have a condition that is obtainable continuously from the identity.

So, SU (n) has this many generators, SO (n) has this many generators, so in some Naive sense you see its much bigger group for a given n, what about the simplectic group, what about the simplectic group that we talk about, what bout that? Notice that these conditions are becoming they are starting to look alike, on the one hand you have R transpose R equal to identity, then you have U dagger U equal to identity, what was our condition for simplectic group?

We introduced the matrix J 2, N dimensional matrix 0 I minus I, these are all n by n and this is a 2 n by 2 n matrix, anti symmetric matrix and then, we said that a matrix transpose J M equal to J itself. We want to now count the number of parameters, that this group has and this set of matrices formed Sp (2n, R), the group of canonical transformation of the group formed by canonical transformations of n degrees of freedom Hamiltonian system, their Jacobeans are simplectic matrices, and those matrices form a group Sp (2n, R), and we would like to know what is the generators of the group.

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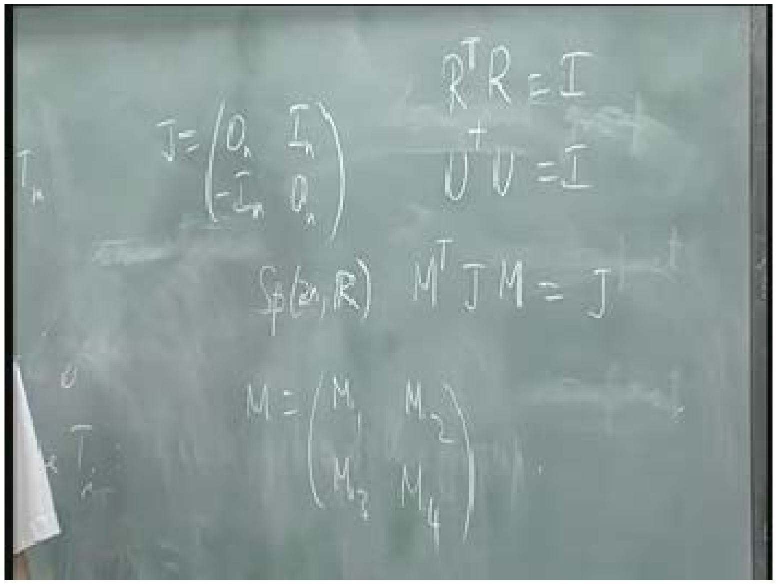

Well, we could put into this, and see what happens? So, if I write M as M 1, M 2, M 3, M 4, where these are n by n matrices, so you get a two big N by N matrix, and they are all real entries. Of course, you have 4 n square elements to start with real numbers, but we have to put this condition on it, unfortunately this condition is not linear, so what will happen if you impose this condition is that, it will say M 1, M 2 transpose is symmetric. M 1, M 3 transpose is symmetric similarly, M 3, M 1 transpose is symmetric, and then the diagonal part will say M 1, M 4 transpose minus M 1 transpose M 4 minus M 3 transpose M 2 equal to the identity matrix and so on, so that is going to help very much.

On the other hand, if I now take the same thing, and say that this is M transpose t J M transpose M of t and do this differentiation trick, then it says G transpose due to this guy, there is J here, and this is M of 0 and that is identity, and then I differentiate with respect to this plus J G equal to 0, because this is a constant matrix. So, this is the condition I get, and its linear in the capital G, I should use G, I should use T for this generator, now we can start counting, so suppose T is of the form, what shall we call it, let us say x y w z T is of this form.

Where capital X, capital Y are N by N matrices, and then let us impose this condition this times that, what do we get (x y w z) and we want x transpose, z transpose, w transpose, y transpose 0 I minus I 0 on (x y w z), this must be equal to, what did I do it was minus 0 minus I 0, time T which is (x y w z). So, let us work this out it says, x transpose and that part is 0, minus w transpose x transpose minus z transpose y transpose equal to minus on this side, let us get rid of this minus sign by putting a minus sign here, and a plus here this is equal to 0 minus w minus z and then I x y, so this is the condition that I get.

This immediately implies, that w equal to w transpose y equal to y transpose, that is these two conditions, and then x transpose is minus z and x is minus z transpose, so its z equal to minus x transpose. So, now how many parameters are there, it says this matrix is symmetric and since, its symmetric how many elements is it have, it is on the real.

n times n plus 1 over 2, it has n times n over 1 off diagonals plus 1 that is n times n plus 1 over 2 similarly, y also has n times n plus 1 over 2 element, and its clear that z is completely determined, once you determine, once the give me x.

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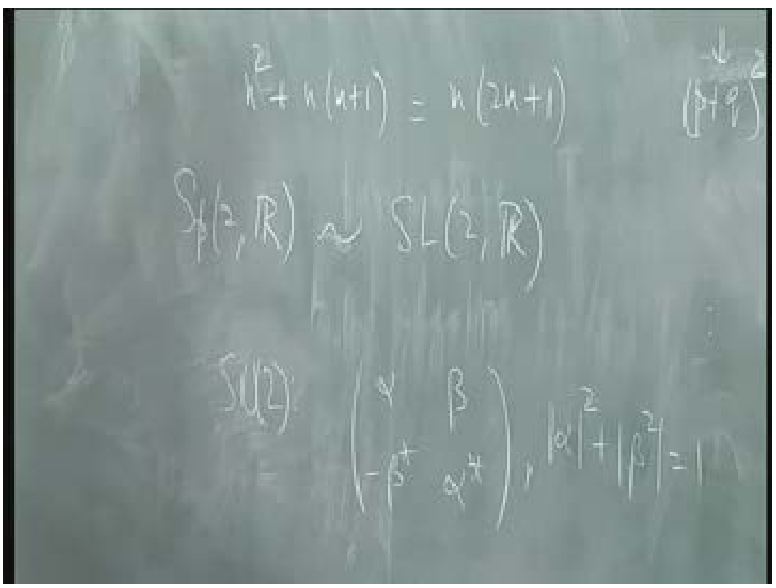

And how many elements are there in x n square, so the total number of parameters is n square plus n times n plus 1 equal to n times 2 n plus 1, so it says Sp (2n), this has n times 2 n plus 1, its bigger smaller, than the linear group, but its certainly bigger than the SO (n) and so on. I should mention here, Sp 2, R this guy here, is 2 by 2 matrices, singular degree of freedom 2 by 2 matrices with unit determinant, so it contains all matrices 2 by 2 determinant plus 1 etcetera. This guy has how many generators, 3 generators and how many generators does SL (2, R), write it is SL (2, R), how many generators does that have?

It has four real numbers a, b, c, d and a d minus b c is equal to 1, independent parameters these two coincide and turns out that these are actually isomorphic, so the conclusion is that, if you have 1 degree of freedom the Hamiltonian system, then the simplistic group Sp (2, R) is the same as the is the special linear group, and what it means is, for 2 by 2 real matrices imposing, the condition that the determinant be equal to plus 1 is enough to make it simplistic.

That is not true in higher dimensions, where you need more conditions, so this has this group has magical properties, such as very many important properties to talk about, but now to come back here, this condition M transpose J M equal to J, all these guys look like you putting some matrix on the system, and in fact you can generalize this and ask for SU (p, q) a set of matrices this kind p, q, where you require all those matrices, and let me again call it M such that, M dagger instead of M dagger M equal to I the identity report M dagger, and then there is metric g M equal to g and this matrix g is the identity matrix three dimensional, minus the identity matrix q by q matrix, and then 0 0, these are not square matrices.

So, this is the p by p matrices, and this is the q by q identical matrix, those matrices which satisfies this, satisfies these relation square n by n matrices satisfying this, where n is p plus q also belonging to a group and its called SU (p, q). Once again the number of generators of SU (p, q), this is p plus q whole square minus, and it could ask what does SU (1, 1) look like. Well, we know that SU 2 contains all those matrices, which are of the form alpha beta minus beta alpha star, where alpha and beta are complex numbers such that, mod alpha square plus mod beta square equal to 1.

All 2 by 2 matrices with complex entries, with two complex, arbitrary complex numbers alpha and beta satisfying this relation here, written in this form they form the group SU 2, what does SU (1, 1) look like? This is a set of all matrices of the form alpha beta beta star alpha star) such that, mod alpha square minus mod beta square equal to 1, and turns out SU (1, 1) is isomorphic in this case. So, you have deep deep relations, by the way is this compact, is this group compact?

Yes.

Yes, you see, because this alpha is in beta must lie on some hyper sphere, alpha 1 square plus alpha 2 square plus beta 1 square plus beta 2 square must be equal to 1, so magnitude of none of these numbers can become bigger than 1. On the other hand, there is no such restriction here, this sort of hyperbolic, this is not compact I must now introduce you to the idea of the universal cover of a group and connectedness. We already talked about, what happens?

No, no no absolutely no, this happens only for this case, even the number of generators will change, so not true.

Well, two groups may have the same number of generators, they must be having same number of generators otherwise they cannot be isomorphic, but if they have the same number of generators it does not imply that they are isomorphic.

SO 2 and O 2, isomorphism is very rare thing, it is very, very rare thing, it has to do with algebra, let me let me tell you

No, no which compact group, which one

These are not isomorphic, not at all, they are very different groups that is, why two different symbols here, these are very different groups, I mentioned isomorphism between SU (1, 1) and SL (2, R), so otherwise it is not true. So, let me talk about the idea of a cover of a group, you see the simplest of these groups S 1 or U 1 what ever it is was the circle group S 1, but I pointed out that there is a homomorphism between R 1 and S 1.

In the sense that a point here, a set of points here, separated by integers all got mapped on to the same point here, by this map which took you from R 1 to S 1, I call that map f, and I wrote f of x was e to the 2 pi i x, in explicit form. But locally in the neighborhood of any of these points, this set of points will get mapped to this set, so locally this and that look alike, that is like saying you draw a tangent to a circle, then on the circle very sufficiently close to this point of tangent it looks almost same. But, the global properties of these two objects are very different, this space is infinitely connected and in fact, pi 1 of S 1 was in fact z whereas, pi 1 of R 1 was 0, was trivial, but there is a homomorphism.

There is a connection between infinite number of points in R and map into one point in on S 1 since, there is a homomorphism of this kind and since local neighborhoods look alike, the identity the neighborhood of the identity is going to look alike in both cases; the lie algebras are the same. So, when ever you have this sort of relation, the lie algebra is the same, but this guy here will have a global structure different from that, and this R 1 is called the universal covering group of S 1, and it is simply connected. So, every connected lie group has a universal covering group which is simply connected, and with which the original group is related by a homomorphism.

So, in technical terms R 1 is the universal covering group S of or U (1) or SO (2) they are all isomorphism, now it might happen that you have many covers, but there is one universal covering group and that is always simply connected, so that is the theorem, which I want to prove, but its very important to look at that covering group. Because, it says topologically the covering group is easier to handle, because its simply connected, this thing here, always simply connected, whereas the original group need not be simply connected, our intention in doing this is to ask, what happens to SO (3), the group of rotations in three dimensions very intimately connected with that, interested in that group.

It turns out and I will establish this, next time SO (3) is doubly connected another way of saying it is, pi 1 parameter space of SO (3) is equal to z 2, the fundamental group of the parameters space in three dimensions, is not z, pi 1 of SO (2) is z it is infinitely connected. But, pi 1 of SO (3) is doubly connected, that means there are only two elements in this group, and these two elements can be put in one to one correspondence with the cyclic group of order 2, z (2), which you can think of as the group of two elements, which could be taken to be 0 and 1.

And a binary addition or the set of even integers and a set of even odd integers, and you do addition modular 2, so that is denoted by z, because of this you should now ask, what is the universal covering group of SO (3), what group covers this SO (3) and the answer we prove turns out to be SU (2). But SU (2) is a set of all these matrices, it is a compact group, so is SO (3) it is a compact group and the cover here happens to be compact also.

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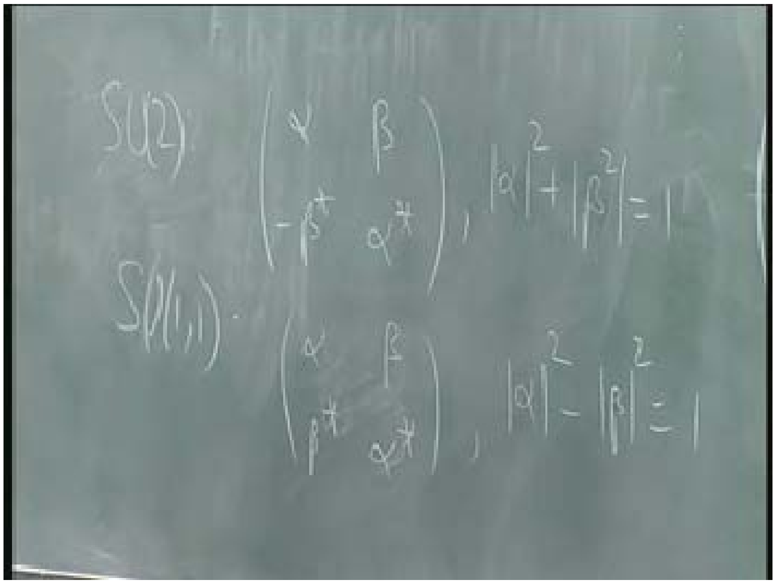

But this is simply connected, why is that, that is obvious here, because its the set of all numbers alpha 1 alpha 2 beta 1 beta 2 such that the square is equal to 1, the sum of the squares is 1, but that is the surface of a sphere in four dimension embedded in four Euclidian dimension that is compact object and it is simply connected because, I mentioned S n for all n greater than equal to 2 is simply connected.

So, what is this equal to you could also say that, this is equal to this is also a league group because, space here it is just S 3 that is what s three is the set of all these numbers so that is why I said that S 3 is itself is a league group S 1 and S 3 happen to be league groups among the spheres S 1 happens to be the same as U 1 or S 1 or SO 2 isomorphic under isomorphism and S 3 happens to be the same as SO 2. This will turn out to be the covering group of rotation group and therefore we have to look at the representations of this group they are all single values.

But as far as SO 3 is concerned you have the possibility of double valued representations one set of them will correspond to vectors tensors scalars and so on; another set will correspond to what are called spinners and this is the origin of spin this is the origin and goes on when you loose the rules of quantum mechanics.

So, it is it comes from this properties of rotations in three dimensions how this leads to that property when you do quantum mechanics we would study this group in great detail so it is worth knowing properties of all these in straight forward form so it automatically emerges even without quantum mechanics, it automatically it emerges just from geometry just from the property of three dimensional spaces and we will see this first, I have to show you that this space is doubly connected and after that how that cover, so we will do that next time.