Classical Physics

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Lecture No. # 32

We will start our study of groups of symmetry is by looking at groups and groups of transformations and so on. Specifically class of objects called lie groups which are extremely useful in physics in many areas and in other parts of science as well. And I should say right in the beginning that the theory of lie groups is a very old one but it is still a current topic there is still a lot of research being done in various aspects of it and as we go ahead with physical science, we discover more and more interesting connections between abstract mathematics and theoretical physics and physic and the physical world. So this is an ongoing quest we are not going to give up we are not going to devote our time to whole course in group theory or anything like that.

So certainly we would not talk about abstract groups for too long. But I will talk about a few practical groups which are needed in the physical sciences. Such as the rotation group and subsequently the Lawrence group which is going to be useful in our study of relativity. So let me start by giving a few informal definitions these are not too formal they are kind of heuristic and there are lots of books which would give you exact definitions of the objects I am going to talk about.

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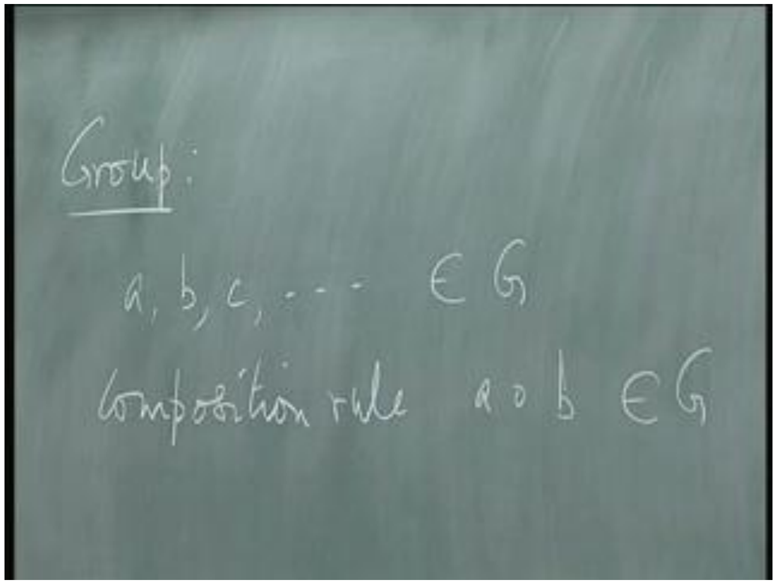

Now the first of these is what we mean by a group and very quickly, it is a set of elements in which a composition law is defined. So that you take two of these elements and you compose them in some fashion and you get another element of the same class so it is basically a set of elements in which some properties are specified. So the elements of the group could be a, b, c, etcetera and a composition law composition rule is defined such that, a composed with b is also these are all elements of some group g, this is also an element of which this composition could be in the simplest instances. We are going to talk about could be addition, could be multiplication of matrices and things of that kind. Some composition rule, such that you can compose two of these elements and get another element which also belongs to the same class.

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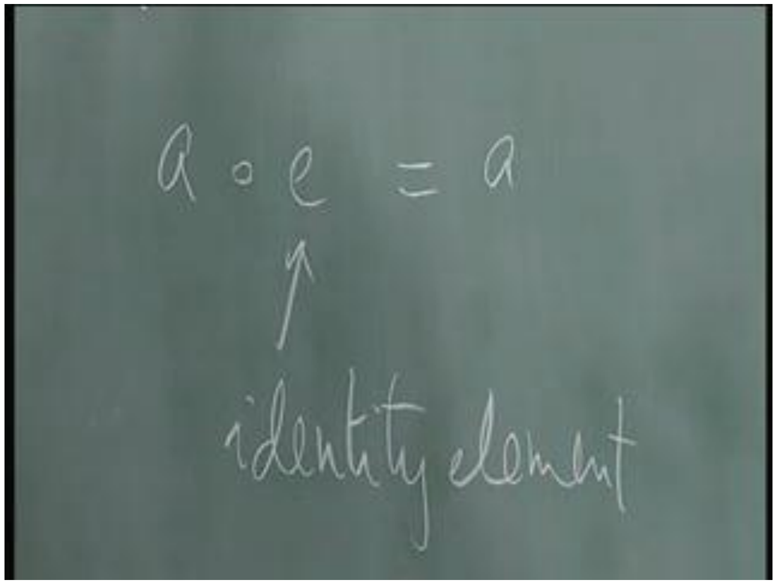

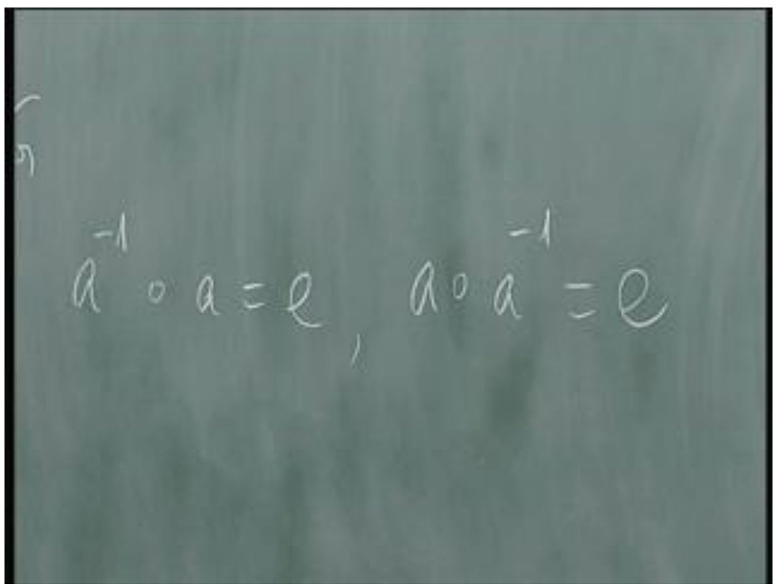

And most importantly we would like to ensure that there exist an identity element a composed with e is equal to a itself. This quantity, this object is called the identity the identity element so there is a unique identity element for this class of objects, such that you compose with the identity element and you get back to the original element itself and when inverse is defined for every element there is an inverse which I will denote by a inverse such that a inverse compose with a is the identity element.

So the left inverse and right inverse are the same and each of them gives the identity. So this suffices to define a group and abstract group and we will look at examples of groups there could be all kinds of objects these things the elements of this group could be all kinds of things, they could be numbers, they could be matrices, they could be operators, they could be functions, it could be any kind of mathematical object, they could be just some abstract objects in which some rules are defined. Now, of course immediately several kinds of groups you could have a finite number of elements or an infinite number of elements that is the first thing that happens you could also have an infinite number of elements parameterized by some quantity by some parameter which takes on continuous values. So we have a continuous infinity of elements not even countable.

For the moment we are going this is it, this is it and now we are going to put extra stuff on it and then we will say for what groups are rings are fields are and so on and so forth. We will talk about examples now some simple examples. Now I am not going to talk too much about discrete groups, groups where the set of elements is discrete and the operations are discrete but I am going to talk about continuous groups namely groups whose elements are specified by some parameter or parameters which can take a continuous set of values the moment you have continuity then other notions come into play like connectedness, compactness and so on come into play and those are what going to be interesting to us.

In other words we are going to talk about topological groups. So you have a group whose elements are specified by some parameter and this parameter takes on values in some continuous set of values in some continuous region then we can talk about connectivity, we can talk about compactness metric and so on and so forth. So we will talk about topological groups without defining topological groups in too much detail but let me give an example of a group of a continuous group R 1 just the real line is a group but you have to specify what operation gives you this composition and it is addition.

So you take two real numbers between minus infinity and infinity you add them you get another real number and the identity element of course is zero in this case the ordinary zero. So R 1 is such a group the parameter space of R 1 is of course the line because the elements of this group are specified by specifying a point on this line and it is clearly continuous. It is also connected because you can go from any element to any other element by a continuous path which remains in the space.

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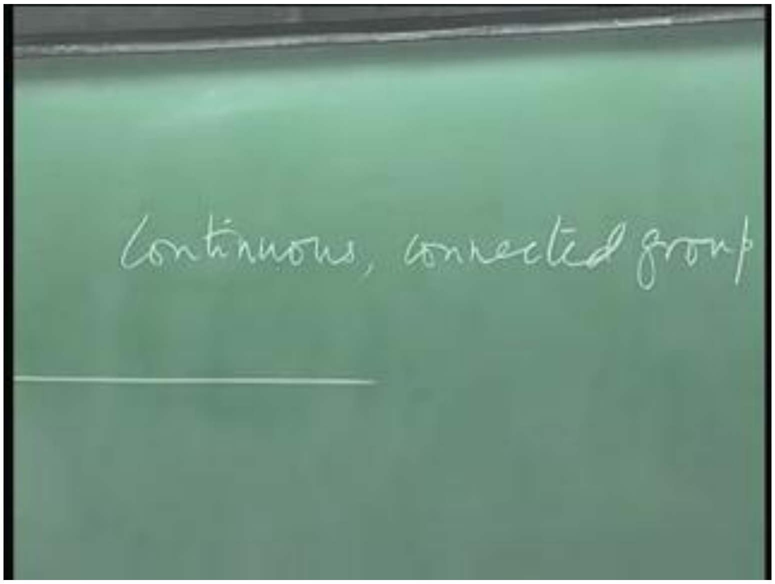

So this is an example of a continuous group not at all not at all does not have to be connected every group does not have to be connected. You could have a group of the two elements, there is no questions of connectedness or anything like that just two elements I have to define a composition law Z 2 group of integers modulo two cyclic group of order two this is just two elements which you could call zero and one and the rule is binary addition. For every group, we can always find the set of parameters that is why I am going to specify the group always there need not be parameter I could just specify the elements themselves like zero and one, why continuity and connectedness are completely mathematical term right connected means that you should be able to draw continuous paths from one point one element of set to another element of the set simply connected is another story means that every path should be shrinkable to a point and so on.

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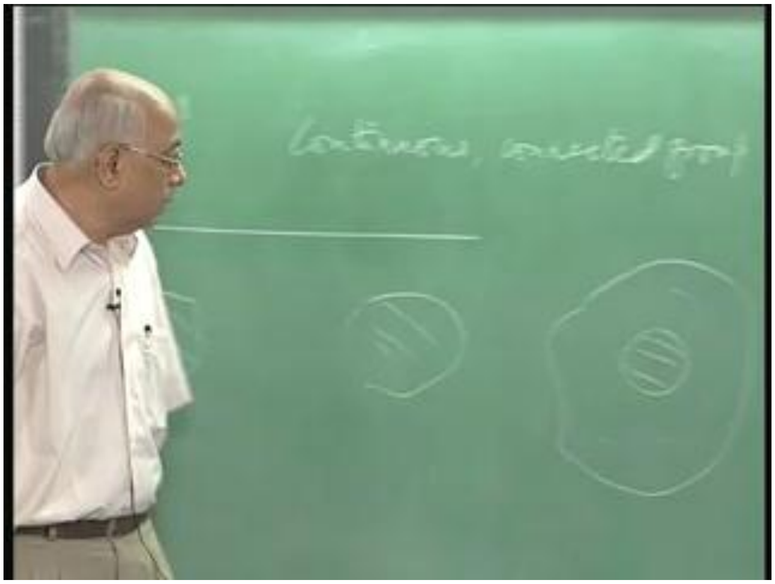

But certainly it is connected, but you can think of disconnected sets you could have a set of elements here and a set of elements here and this is something else this is outside. So this and this within this, this is connected that is connected but it is not simply connected this is completely disjoint sets. On the other hand if I had this kind of parameter space with forbidden region here and this is my region this set of points is connected from every point I can arc wise connect to every other point but it is not simply connected because there is a hole here. On the other hand this is not even connected. So at yes of course I should define open sets neighborhood so on and so forth. So all the epsilon deltas we will put in when necessary, otherwise it becomes very, very abstract and I would like to get some examples right away so as I said this is not a formal course in group theory with all due respect to the you know rigger and things like that we would like to use it like we use it.

And I think once we start using it, then we will learn all the definitions backwards and so on right now what happens is we give all the definitions you have a whole lot of definitions and then I make a statement you will have to check whether definitions whether it is really part of this definitions or not, whether it follows from it or not and by the time you forgotten what we are trying to prove at least I would.

so let us do this in a very heuristic way and waving way this is not the way to learn group theory you should learn group theory the right way to learn group theory is to take a text book on group theory and go from the beginning to the end and at the end of the day you have learnt all that there is in the book.

It is all given done that is the right way to learn it the useful way to learn it is to have a physical problem for which you won not apply group theory and you discover that is the way so you take the book out and then you sniff through the index you look at the index then you say, ah page sixty seven has it you go to page sixty seven then you discover there is a concept there for which you have to go page twenty four you go there then it says to really understand what is written on twenty four you have to go to two hundred and fifty two.

So you go there do this in between you sort of smell the book little bit you know and get used to it and so on. Then you apply it you make a few mistakes then you look at another book where it is given right or something like that and that way you learn the subject non linearly never read a book linearly.

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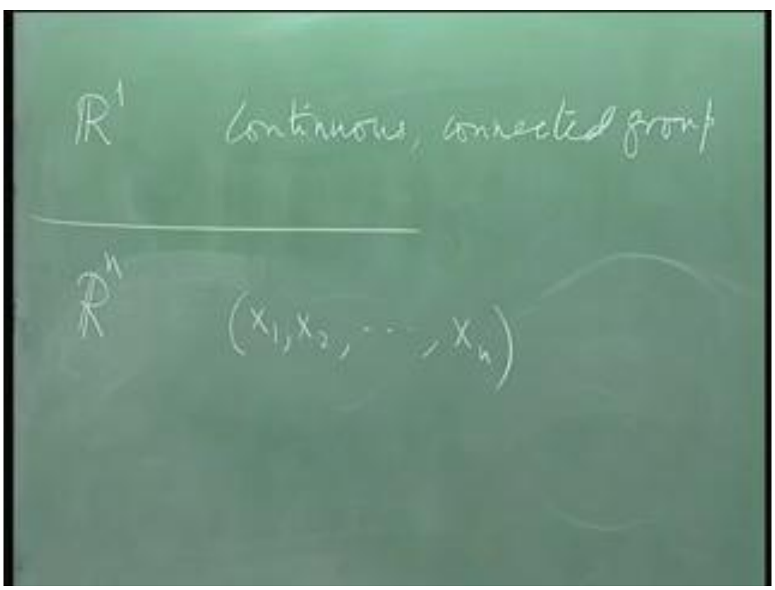

So this guy is a continuous connected group in fact we can go further R n is also a group. Because this is a set of numbers real numbers and n tuple of real numbers whose elements can be specified like specifying x 1, x 2 up to x n here

And the usual law of vector addition in n dimensions, this is a Euclidian space once I give a metric to it but right now we are just talking about R n this is also a group the set of n tuples of real numbers in which I add the addition the rule is again addition but component wise. S x 1, x 2 up to x n, y 1, y 2 up to y n the composed thing is x 1 plus y 1 x 2 plus y 2 and so on that too is a connected group continuous group. But there are other groups which are not so simple to look at lets pick an example for instance would be the following and I will come back to this s 1.

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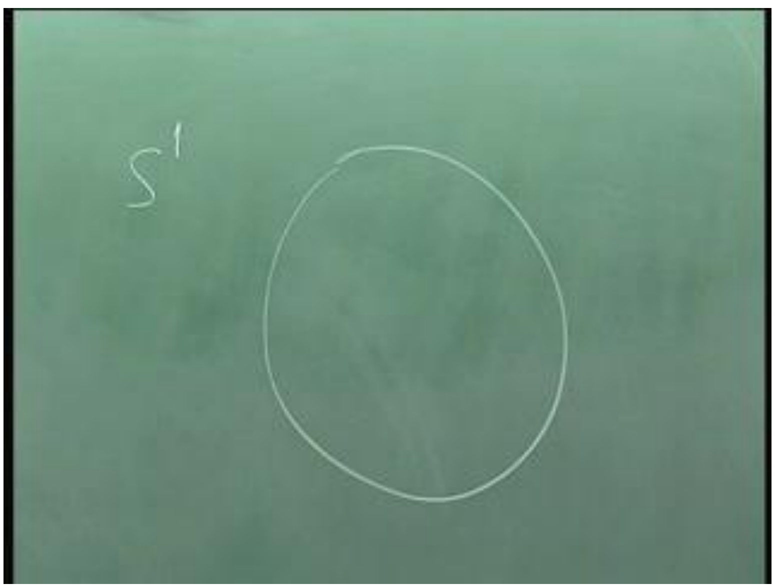

This is just a set of points on a circle but now the group law is multiplication so I take points on say the unit circle in the complex plane they are of the form e to the 2 pi i theta and I multiply e to the i theta and I multiply two numbers by simply saying e to the i theta e theta prime is e to the i theta plus theta prime and it remains on this circle so that too is a group but that is got much more complicated properties than R 1 has and we will see the relation between them. Now I am going to look at groups where the elements I parameterized by one or more parameters but the elements are analytic functions of these parameters smooth functions things, which can be differentiated at least twice differentiable ideally infinitely differentiable.

Any analytic function is infinitely differentiable. So I would like to look at groups where I have in addition to a group set of elements I have an underlying space of the parameters and this space is actually an analytic manifold well it is actually some space where I can write these elements as a function analytic functions of the set of parameters that is the kind of group I would look I would look at and these are called lie groups.

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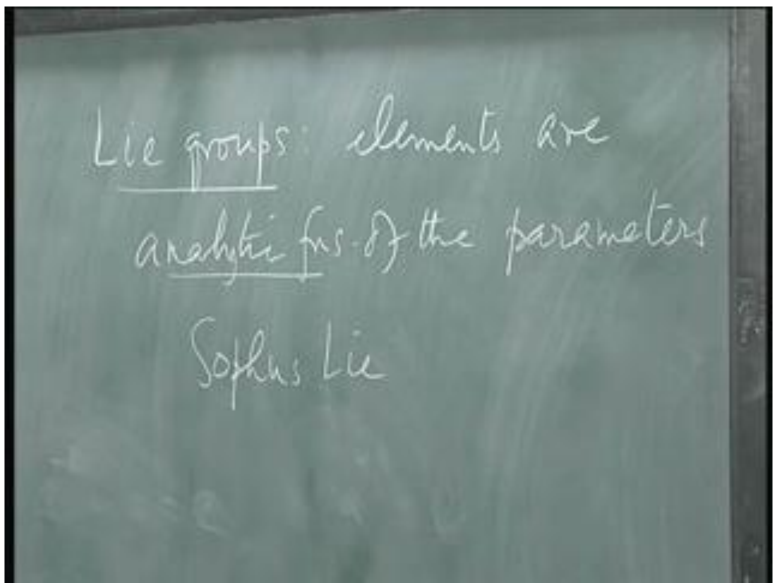

The elements are analytic functions of the parameters, this is important because this immediately means that in addition to continuity, in addition to connectedness and so on. I have imposed some further conditions have actually said these are analytic functions. So I can differentiate them I can ask for infinitesimal and so on. So calculus can be done using these set of elements. So this is the first step and this was taken by a very great mathematician Norwegian mathematician Sophus Lie and lived in nineteenth century eighteen twenty five or something like that up to eighteen ninety nine and in the process of his mathematical investigations he laid down the foundations of the modern theory of groups of continuous groups.

So it is got close links with topology is very close links with the differential geometry and so on. Which we will not get into in many of these subjects but we will talk mainly about lie groups and all the groups we are going to look at are essentially lie groups this kind. Having with that let me point out that you can represent a group in more than one way.

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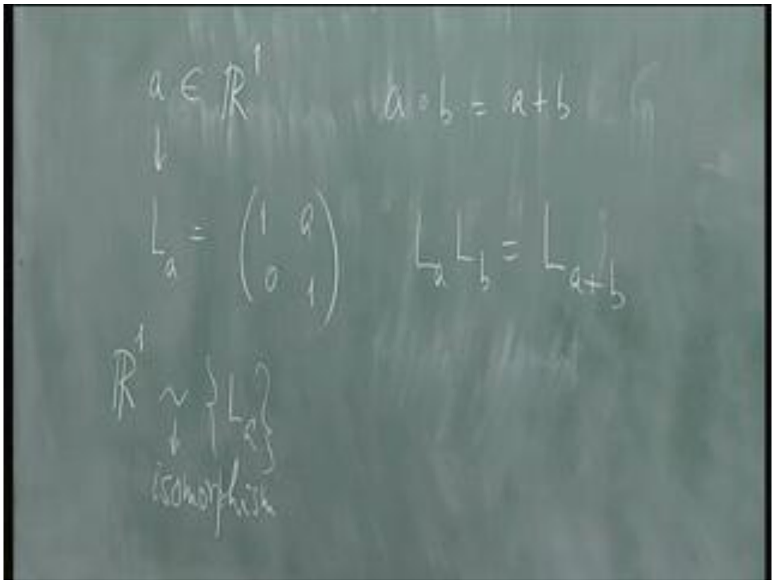

So, let me come to the idea very important idea of representation of a group and it is not to be confused with the group itself representation you could always represent the elements of a group by other objects and form a rule to go from an element of a group to that other object then that set of objects is a representation of a group.

It may be a faithful representation or in other words there may be a one to one mapping between the elements of this group and the set of objects we are talking about but there may be one to many as well if you have a one to one mapping it is called an isomorphism and if you have one to many it is called homomorphism and I will give examples let me take a representation of R 1 which looks nothing like R 1 when you start off R 1 is just a set of real numbers on a line but here is another representation of R 1 to corresponding to every real number a element of R 1 I associate corresponding to this a matrix let me call it l sub a and it is a two by two matrix which is 1, a , 0, 1 and now the addition law here I replace by the multiplication law for these matrices.

And it is easy to see that a composed with b is the same as a plus b that is what I mean by the composition law in R 1 but here l a l b is equal to l a plus b in the sense of matrix multiplication and that is trivial to check this is upper triangular matrix so multiply another one with a b and you are going to get a plus b. For every number here there is a unique matrix here and vice versa. Therefore this set of matrices forms a representation of the lie group R 1 buts it is not the same as

R 1 these are R different objects the rule also have been changed. So there is no problem I could instead of R 1 I could talk about this and it is a perfectly good representation. So this technical statement is this set of matrices forms a group and let me just call it l sub a this is actually a group this is isomorphic to R 1.

There is a one to one mapping and it is a two by two matrix representation of real numbers it is not the simplest representation but it is a representation and everything you can do with real numbers you could as well do with this may be not very usual very often but it is certainly a representation and by this symbol I mean an isomorphism right, that is a lie algebra that is a lie algebra and we will talk about lie algebra indeed it will turn out let me anticipate a little bit that once I say that the elements are analytic functions I would like to write these analytic functions down and very often they are exponential functions and then if they are close to if they are derivable from the identity element in a specific way which I will demonstrate then you can generate the entire set of elements by using a set of what are called infinitesimal generators.

And these generators will obey the lie algebra corresponding to the lie group itself. So in a sense you can replace the study of the group by the study of the lie algebra and this is the idea behind writing down things which are analytic functions. The parameters are continuous, if the parameters were discrete I have a discrete group and I am not going to talk about that at all lie groups are not discrete groups they are continuous groups more than continuous groups they are actually analytic in some sense. So it is little more than that. Now the familiar groups that you do in that you look at in crystallography like the group of symmetries of an equilateral triangle or the group of symmetries of a cube these are discrete groups, they are finite groups; they have a set of elements which obey a certain rule.

Now that group operations the group elements are actually transformations on this geometrical object but you can represent those transformations by other objects which are matrices in general. But there could be other objects as well. So we have to distinguish clearly between the representation of an abstract group and the abstract group itself very often I would not distinguish between them that is it once you have relation like this I would just as well work this instead of this everything that you can do here you could also do here. There are other representations of R 1 here is another suppose I map R 1 to S 1 so corresponding to every x.

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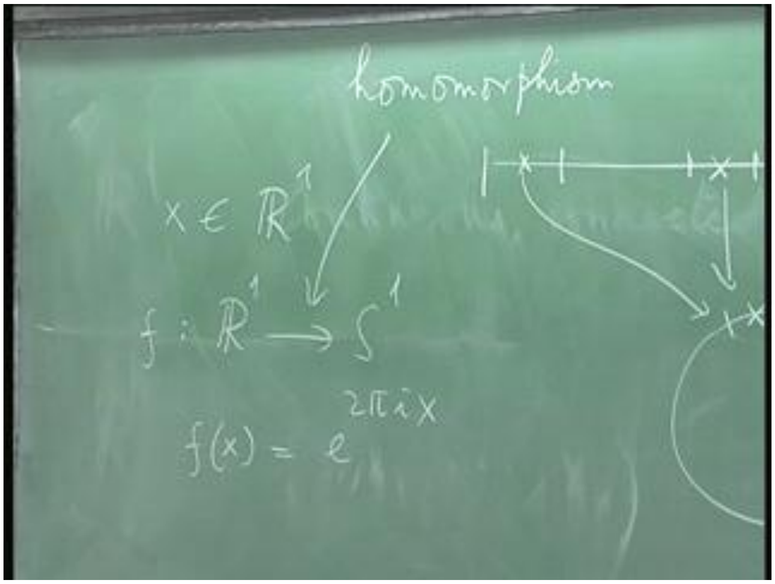

So, I start with some x element of R 1 some real number and I put a map f such that R 1 goes to S 1 this is a function of x which takes me from a real number to a complex number in S 1. And this map is f of x equal to the e to the 2 pi i x, for every x I replace x by e to the 2 pi i x e to the 2 pi x is the complex number which lies on a circle unit circle and now the group addition law is replaced by multiplication law. I certainly have x plus y is an element of R 1 here. Here it would be f of x multiplied by f of y is f of x plus y. So this is a representation but it is not a single valued representation.

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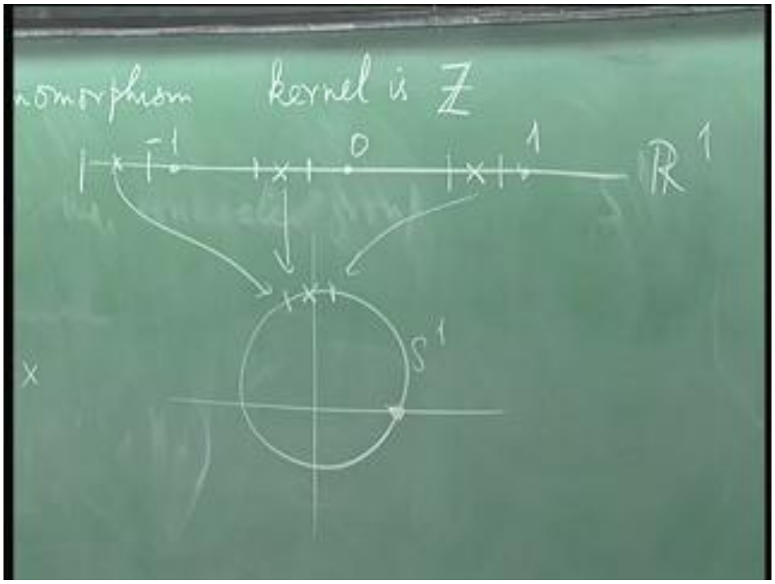

Because you have R 1 here and you have S 1 here this by the way is a mathematical notation for a circle it is one dimensional sphere the object is one-dimensional. S 2 would be the two dimensional sphere what we would normally call the surface of a balloon in three dimensions S 3 would be the three dimensional sphere and so on and we can people have studied all properties of S n in general they are not as trivial as you think and we will see there are some deep differences between for instance S 2 and S 3. S 1 is also a lie group, R 1 is a lie group but you see every time you increase by 2 pi you increase x by an integer you come back to the same point on S 1. So it is clear that this point if it is mapped here and the neighborhood of this point is mapped to a neighborhood here, there is a point here which is separated by an integer which is also mapped to this guy this is also mapped to this guy and so on.

So an infinite number of points separated by integers are mapped on to the same point in S 1 and that is true for every point honestly. This is a representation however it is not a one to one representation it is a homomorphism this guy here is homomorphism it is not an isomorphism but it is isomorphism is one to one that means you go back you can go back it is get back to original thing uniquely but that is not true here many elements are mapped on to one element here and then the question is which of those elements is mapped on to the identity element here this is e to the 2 pi i x. So let us put some coordinates on this, this is theta equal to 0 theta equal to pi over 2 and so on till 2 pi. This is the identity element here and identity elements come from the point 0, it comes from the point 1, it comes from the point minus 1 and so on.

They are all mapped on to this single point here. So an infinite number of points all the integers are mapped on to the identity element in S 1 and that set of points which is mapped on to the identity element in a homomorphism is called the kernel of this map. The kernel is set of integers, representation is basically a mapping but it could be one to one, it could be many to one and so on. So, this mapping obviously if it is many to one it does not have any inverse you give me a point on S 1 I cannot tell you which point on R 1 it came from I can do that modulo a set of integers.

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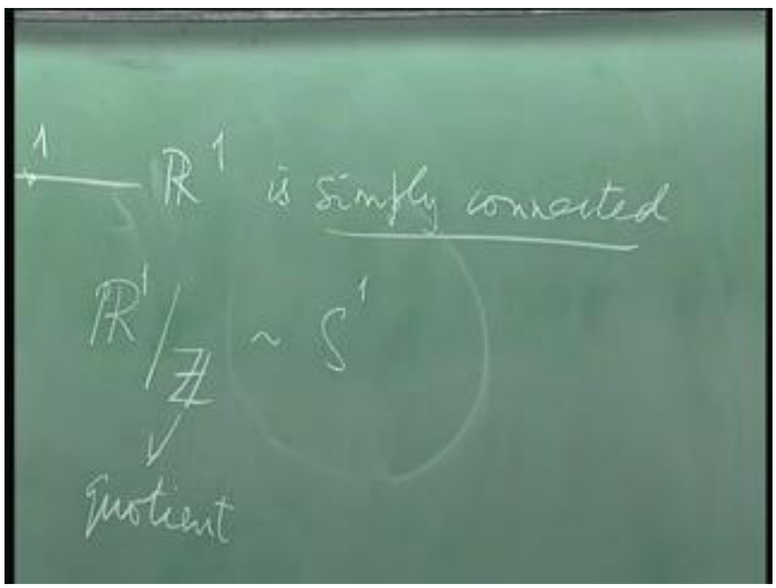

Therefore what one says is that R 1 quotient it with Z is S 1 this is quotient so the symbol is R 1 slash Z we will quotient it. Therefore modulo in an integer you can identify which point it came from. You take a point on S 1 you can say where it came from in a R1 modulo an integer up to an integer could always add an arbitrary integer to it right and the kernel is Z therefore when you quotient this space with this Z is S. Now we come to this question of connectivity locally, locally this thing looks like that in a sense if you go very close to this, this fellow looks like a straight line segment I am saying this very loosely there is a tangent here and that looks like that so the properties of this and properties of that are more or less the same locally but globally that is not true it is quite clear because then the curve which are takes over and so on.

And then this is not true. So although locally there is in a homomorphism you have this mapping which looks one to one globally it is not one to one and there is a non trivial kernel there we will say something more about this kernels and so on. As we go to other groups but this provides a cover of this for this in the sense that this thing here R 1 is simply connected by connectivity I mean if you took two elements of the set you should be able to go by a continuous path from one element to the other element without leaving the set of elements that is connectivity simple connectivity means I should be able to shrink all close paths to a point without leaving a space.

So, I could start on this line go somewhere and I stay on the line come back and just like a rubber band could be shrunk to zero without leaving a space at all then it is simply connected that is why I said if you take a plane and you puncture it you remove a hole from it, you remove a point from it then any path which encircles this hole cannot be shrunk to zero to a point without crossing without leaving this space. S 1 is not simply connected it is infinitely connected because it is clear that if I took a rubber band it is a stretchable thing so now what is a close path on S 1 I start here I go there, and I go back that is a close path now that can be shrunk to a point without leaving the space.

But the moment I go around once like this and complete the path this path cannot be shrunken to a point it is like putting a rubber tube round the rim of a cycle you cannot shrink it now without tarring or trucking or anything. You cannot smoothly deform it to a point in fact you could take the rubber band and put it twice around the circle this rim and that cannot be converted to either zero winding or one winding once you could also wind it in the other direction twice and that cannot be shrunk to anything else. So the question of shrinking path become very crucial in all these continuous spaces and this is part of a subject called homotopy and we will talk about the elements of this homotopy, what we are doing is kind of very, very crude version of homotopy here.

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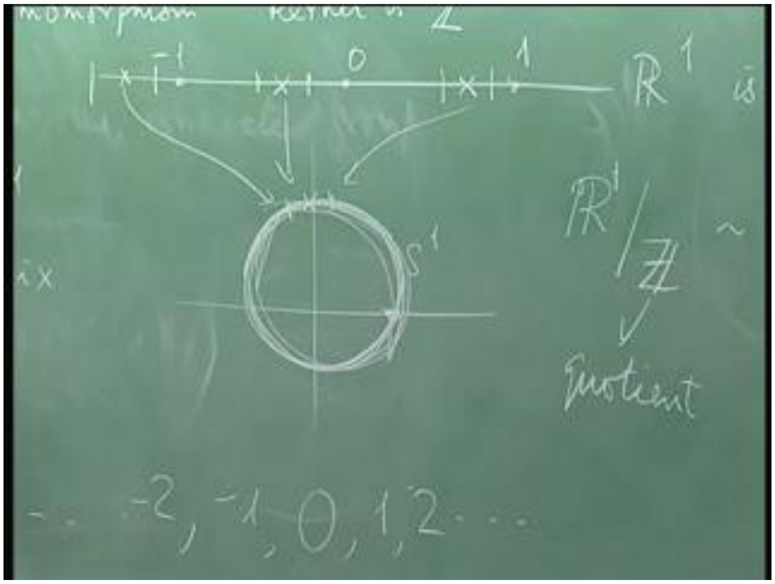

The set of closed paths on this S 1 can be put in one to one correspondence with the set of integers. This path or this path or this path or even something which goes all the way back and comes back without crossing can all be shrunk to a point and can be put into one to one correspondence with…

They form an equivalent class which can be represented by 0 the number of times you go around completely then the set of all paths which do this, once may be it also does this a little bit does not matter that can be put into one to one correspondence with the number 1. Because that is the winding number of this map similarly this set of paths which can be put into correspondence with minus 1 winding number 2 and so on and on this side minus 2 this side. So the set of all closed paths in a continuous space, which are deformable into each other without leaving the space form an equivalence class and there are many equivalence classes for spaces which are not simply connect. For something which is simply connected like R 1 there is just one equivalence class.

Every path on this line no matter what we do how many times we go zig zag back and forth you can continuously shrink it to a point.

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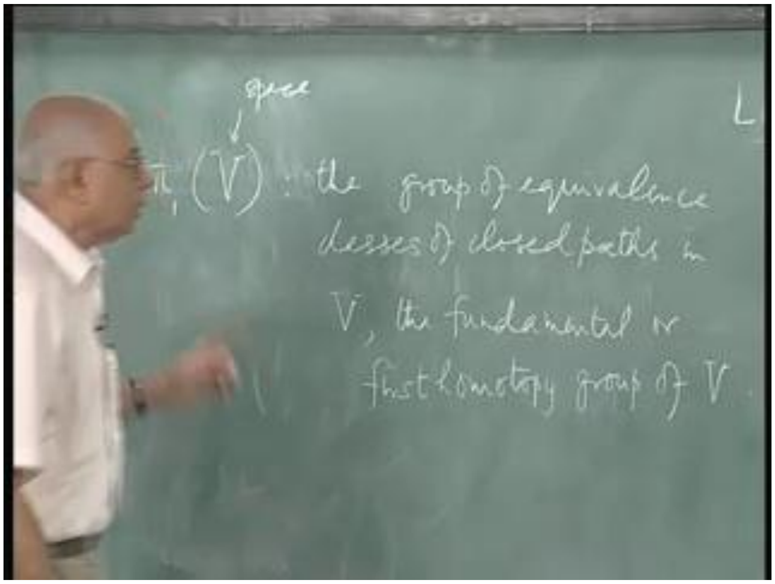

So there is only one trivial element one equivalence class but on S 1 you have as many equivalence classes of paths as there are integers the set of equivalence classes of a space we need to call the space V, V is the continuous space the set of equivalence classes forms a group and its called pi 1 a group of equivalence classes of closed paths is V and it forms a group.

In fact you can compose paths you need a rule for composing paths and its precisely the naive rule that you think about for instance if in a space, you start at a point, and then you have a closed path like this that is a closed path.

If I have this guy here and this space is completely simply connected, you could just as well convert it to this continuous thing could just become this and then it could be shrunk to a point completely. So you can compose paths you can add paths to it and so on. So under all these operations finally it reduces to just a set of equivalent classes of paths and that set forms a group and this group is called the fundamental group or first homotopy group of the space. I will look even I mean, how the paths on this are Z well I pretend there is a rubber band which I want to put it on a rigid cycle wheel then I ask as the infinitely stretchable but it is topologically deformable the only thing is I cannot break it.

Then if I start with a band here and go around like this once this path cannot shrunk to a point without cutting it somewhere. If I put it round twice in this fashion that cannot be converted to something which goes around only once or which goes around zero times without cutting it and so on and then I have a sense to this every path must have an every closed path I must have an orientation I must also have a sense in which this path is described and if I do it in the clockwise direction or counter clock wise direction I end with plus or minus as the winding moment.

So the paths are all classified according to the number of times they wind around and that is the set of integers yes then it is a yes is not it the identity element is connected to this one provided it is connected to the identity all elements need not be connected to identity continuously i will come to an example this is only true if entire thing is simply connected otherwise it is so there is it is possible there are elements which cannot be found continuously from the identity element then this is not true.

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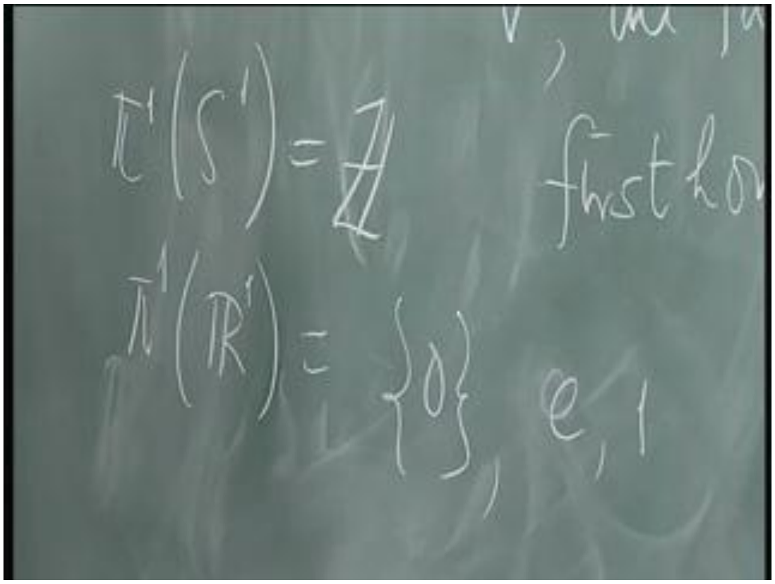

So, we so the idea of pi of V is clear so we can write this down pi 1 of S 1 is the set of integers what is pi 1 of R 1 what is this equal to it is a group, but there is only one element and one element group is called a trivial group. It is just the identity element you can represent it as you like it is just got one element to it you could call it 0 if you like sometimes people write it in this fashion some time people write it as just e that is not good notation because this suggest that this base of natural logarithm or something like that or sometimes they write it as 1 or whatever it is a trivial group it is a group which has only one element and anything which has got pi 1 or whatever is just one element is simply connected that space is simply connected.

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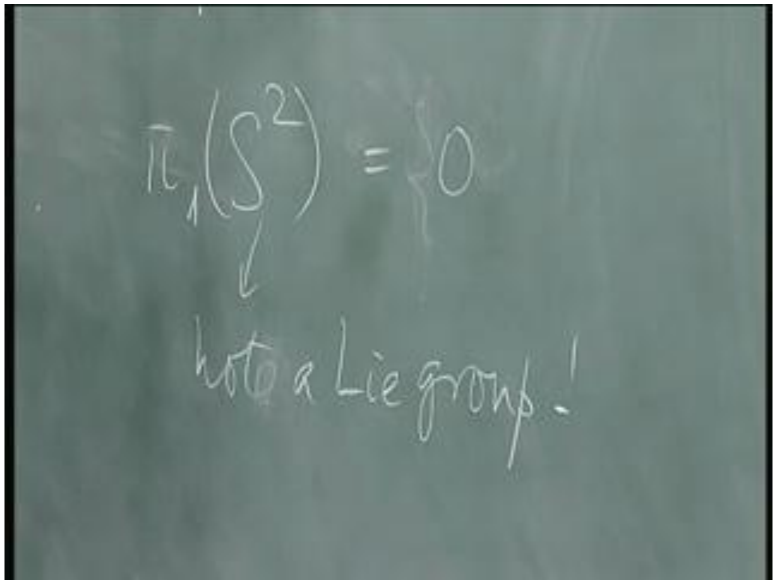

What is pi 1 of S 2 what is pi 1 of S 2. Some subtly is involved here S 2 is the surface of a sphere it is parameterized by two angles the latitudes and the longitude it need not be embedded in three Euclidian dimensions. You need at least three Euclidian dimensions to embed it but it need not be there I mean it is just the abstract space just the two dimensional surface described by surface of usual kind of sphere now what is pi 1 of that you put a rubber band on it. It is clear that no matter what you do to it you can always slip it off. You can always slip it off you can always shrink it to a point you cannot lasso a basket ball that is the way it is said in colloquialism and therefore this is equal to the trivial group. I will write it as just 0 I mean just a trivial group so already you begin to see between S1 and S 2 there is a deep difference is S 2 a lie group I have to define an operation on it in order to compose right this is a point on a surface of a sphere in the conventional sense in three Euclidian dimensions therefore I can specify it by a latitude and a longitude or I could specify it by a unit vector whose tip lies on the surface of the sphere.

if I add two such unit vectors do I still get a unit vector no so S 2 is not a lie group it is a space, it is a connected space, it is a simply connected space but it is not a lie group not a lie group where you think of any operation in which you may get a lie group this not a lie group no its just a space all spaces are not groups there are groups which have topological groups which have spaces with specific properties.

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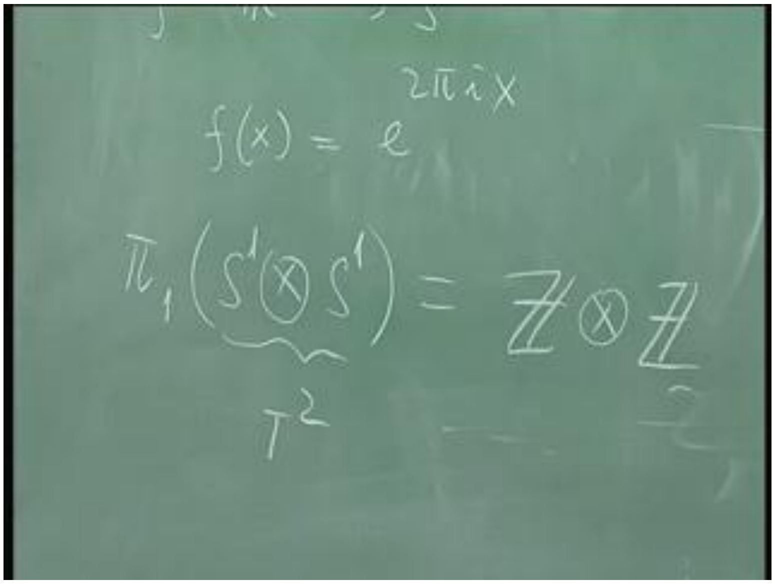

So, S 2 is not a lie group at all but pi 1 of S 2 is 0 that is for sure S 3 happens to be a lie group once again what is S 1 cross S 1 is this a group it is a torus, it is a torus, is it a group yes, yes, whatever is it for I should really write direct product.

That is for every point here you are associate the full space there and so on. So it is a Cartesian or direct product and this guy here is parameterized by one real number here on one quantity e to the i theta 1 to the e to the i theta 2 and this is a group is it simply connected no its not simply connected, what is it is fundamental group, what is pi 1 of S 1 cross S 1, Z cross Z that is it, because you can take a rubber band on the torus and wind it around m times this direction and n times in that direction and m and n are integers and these paths are homotopically not equivalent. So that guy is S 1 Z cross Z fine. But you can do the same thing that you did for this fellow namely you can unroll this imagine this guy here rolling on that and then of course every point here gets mapped there many times that is why it is a homomorphism you can do the same thing with this torus S 1 cross S 1 it is called the two tours.

It is T 2 and this T 2 is parallelizable in the sense that you can also put it into correspondence with what is the analog of R 1 that you put it in correspondence with R 2. But on R 2 you would now identify all points which in the x direction differ by an integer all points, which in the y direction differ by an integer and what you call that I call it as square lattice I call it a square lattice.

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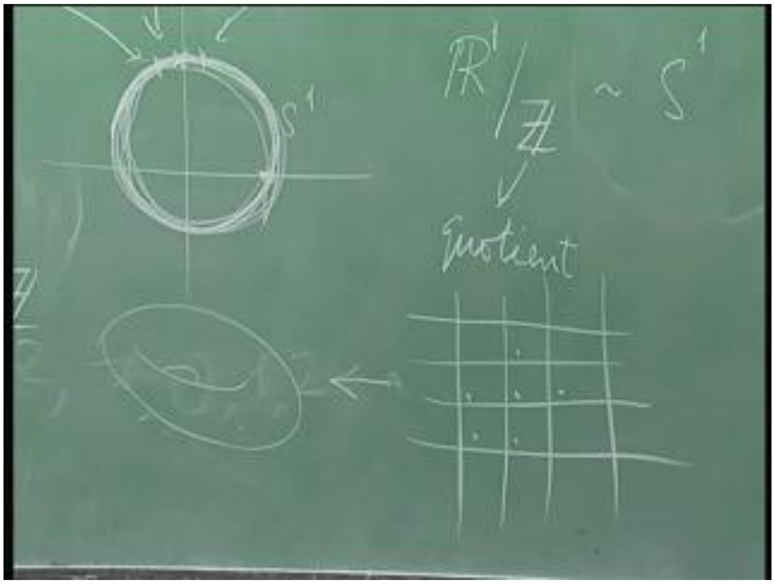

So, this guy here once you represent it in that fashion this torus is ma this plane in which you put a grid such that this point, this point, this point, this point, etcetera are all mapped onto a particular point here. This is the representation of the lattice the plane representation of T 2 and it is also not a simply connected group it is infinitely connected what is the kernel, what is the kernel of this group, what is the kernel of this mapping m comma m the set of pairs of integers all those numbers go on to a single point the identity element in T 2 and so on. You can generalize that S 1 cross S 1 cross S 1 arbitrary number of times and it will get give you a representation of rn not a single valued thing it is the homomorphism. So we have seen here is one more representation of R 1, the representation in terms of functions. So I could do the following I could say just as if a is element of R, let me look at the space of all infinitely differentiable functions of a real variable.

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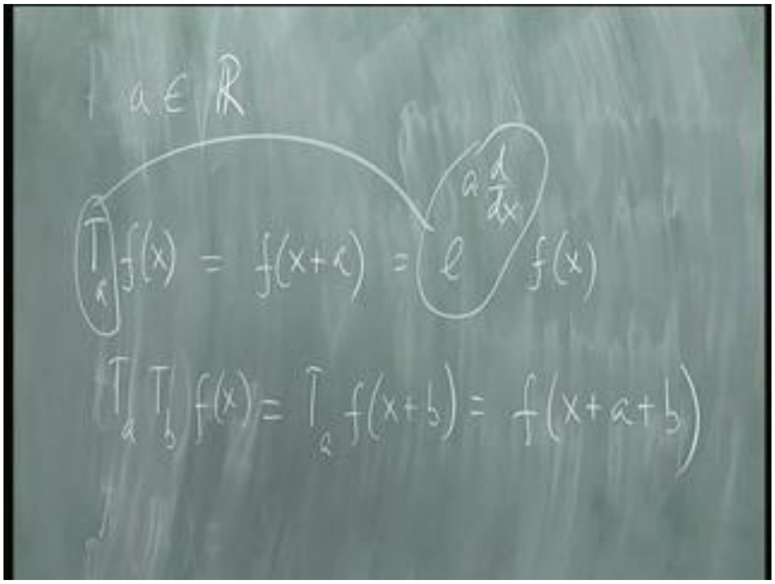

So every function f of x where x is a real variable taking values from minus infinity to infinity and I assume that this f of x is a function which has an infinite number of derivatives like e to the x or sin x or cos x it is up to you and then I put a rule saying that this quantity a is represented by an operator T sub a, which now representation by an operator such that this is equal to f of x plus a so it shifts the argument x by x plus a. Then what is the group composition law it says T a T b f of x is T a acting on f of x plus b equal to f of x plus a plus b. So just as a and b give you a plus b on R 1 this is the operation this guy, this is a representation, this T sub a now of course, you would say what does it look like what does this representation look like I do not care, I just tell you that T a is an operator such that it acts on the function f of x and shifts it is argument by an amount a.

Why should why do I need that it should be differentiable an infinite number of times why do I need that because I also have in the back of my aim the fact that I can write this f of x plus a in terms of f of x in a Taylor series provided all the derivatives exist. So you know this can also be written as e to the a d over dx on f of x, where this is 1 plus a d over dx plus a square over 2 factorial and so on. So I need an infinite number of derivatives to act on that and then I have an explicit representation of this operator so this operator is represented now by a differential operator but an infinite order differential operator. So I started with a humble real number a but now I am representing it by an infinite order differential operator so in the space of functions there is a representation as well and sometimes this might be very useful.

This is called this when you want to move an object, what you do you give it momentum right. So what I have done? If this were a wave function in quantum mechanics, I have translated it by applying momentum to it. So this will be this is the reason why the momentum operator would be represented by minus ih cross d over dx in quantum mechanics this is the underlying fields we are going to put some more conditions on it like the real eigen values and so on. But ultimately it is just this, this is the reason so here is a representation T a then we have a representation terms of l sub a and so on. So many ways of representing this is there that is the moral of the story and you must not confuse representation with the group itself that is an abstract set of objects.

Now the kind of things we are going to do is to look at groups of transformation of some geometrical objects such as a plane or three-dimensional space or any linear vector space or something and the whole of quantum mechanics in fact could be regarded as transformation theory in fact when Dirac first with all quantum mechanics in it is modern form he called it transformation theory that is it for want of a better word, and it was a very good word very good phrase. So we are going to look at groups of transformations of geometric objects like plane for example let us look at the simplest of these let us look at rotations on a plane this will bring us to gifts with something.

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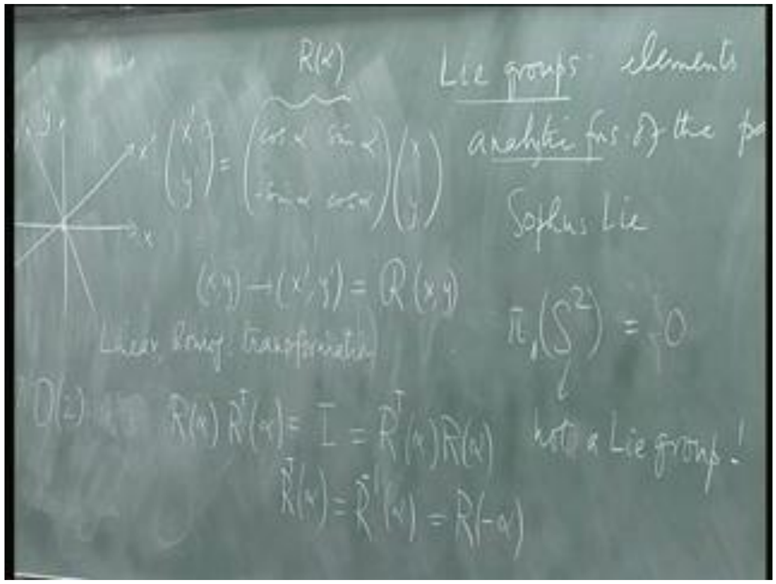

So I have a plane the xy plane and I am going to look at the set of rotations of the coordinate axis. So from x and y I am going to go to new coordinate axis x prime y prime by the usual rule which is x prime equal to so write it in this form equal to cos alpha sine alpha minus sine alpha cos alpha x y. So what I have done is to go from x y to x prime y prime through a rotation this is equal to some rotation acting on x y. So this is my group element the abstract act of rotating the coordinate axis the abstract rotation of the coordinate axis the abstract rotation of the coordinate axis this is a representation of that rotation, because I have chosen to represent a point in a plane by a column vector of this kind but I did not have done that I could have chosen some other representation of R 2 then I would have another representation of the set of rotations.

So is this clear that there is an abstract rotation, which takes a point on this plane to another point on the plane leaving the origin unchanged. But if I write this point on the plane as column vectors in this fashion this provides a representation of this R; it is a two by two matrix representation I can give you three by three, I can give you n by n, I can even give you a one by one matrix representation I say that all rotations are represented by the unit element one. So trivial representation it is not going to do very much for you and it is not certainly a isomorphism, but here this set of rotations is most conveniently represented by two by two matrices. Now, what you have to tell me is, is this unique for every rotation do I have just one matrix, I could have 2 pi plus alpha suppose I say alpha runs zero to two pi, then this is unique so I have to put some conditions on alpha.

If I say this is zero to two pi then that is it, it is unique thing. Now let us see what are what this actually implies this set of rotations means that the distance any rotation is a linear transformation of this plane, which takes you to some other axis x prime y prime such that distances are kept unchanged and this point is unchanged, it is homogenous in the sense that the origin remains the origin so no addition of constants otherwise, it is both the rotation as well as a translation that is a different story altogether that also is a group, but it is a different group altogether. This is a linear homogenous transformation is it invertible yes indeed this rotation can be inverted certainly it can be inverted there is an R inverse for every R and distances are kept unchanged and what is the condition that ensures that distances are kept unchanged.

That the transformation should be orthogonal that is sufficient on the the real that is sufficient so I know that R R so if I write it as let me call this R of alpha this is my abstract rotation this is the two by two matrix representing the rotation then R R transpose so just write it little better R of alpha R transpose of alpha is equal to the identity matrix which corresponds to the identity element of the rotations which means no rotation at all. This is also by the way equal to R transpose alpha and that says R transpose of alpha equal to R inverse of alpha and it is a trivial matter to verify that this is R of minus alpha. After all an inverse rotations means, you go in the opposite direction you change alpha to minus alpha, sine alpha changes sine and cos alpha remains.

So this is the group composition law written in terms of two by two matrices in this function this set of matrices forms a group that group is the orthogonal group, group of orthogonal matrices in two dimensions O 2 but now I ask what is the determinant of this matrix from this guy it is clear that the determinant the moment I say matrix is orthogonal it means the determinant of this matrix it is square is equal to one so I can only say that determinant is plus or minus one. So the group of two by two orthogonal matrices is not quite giving you these rotations, because the group of two by two matrices has matrices, which have determinant minus one and they cannot be formed continuously from the rotation here they would correspond to something where the x axis is flipped.

They go to left handed coordinate system so this contains this is a group no doubt but it contains more than the representation more than the rotations it contains rotations as well as reflections. If however I write this which means that this is a special orthogonal group in two dimensions those matrices, which have determinant plus one then this forms a group which is a sub group of O 2 and this set of elements is in one to one correspondence with the set of rotations and you can form this set of elements continuously from the identity. So you already see that O 2 is a bigger group than SO 2 and it is SO 2, which talk about which have proper rotations otherwise O 2 contains some improper rotations as well.

They contain they could represent the group of not only rotations but also reflections about either of the axis. O 2 is not a continuous rotation there are portion in it which are discrete different set altogether now what the so we will have to ask what kind of matrices are those guys the remaining matrices the ones which are in O 2 but not in so two that’s not very hard to find we are going to do that next time and then we will take this up to SO 3 and see what happens and SO 3 you will see will become considerable more complicated for the following technical reason.

This group is abelian in the sense that if I rotate by alpha and rotate by alpha prime, it is the same as rotating by alpha plus alpha prime and I could have done this is in either order, but the moment I have rotations in three or more dimensions, then the rotations are in different planes they do not commute with each other. So all the complications arise because in higher dimensions rotations about different planes do not commute the order in which you did this matters, and they form what are called non-abelian groups and this will bring us the idea of generators and the lie algebra also, so we will do that next.