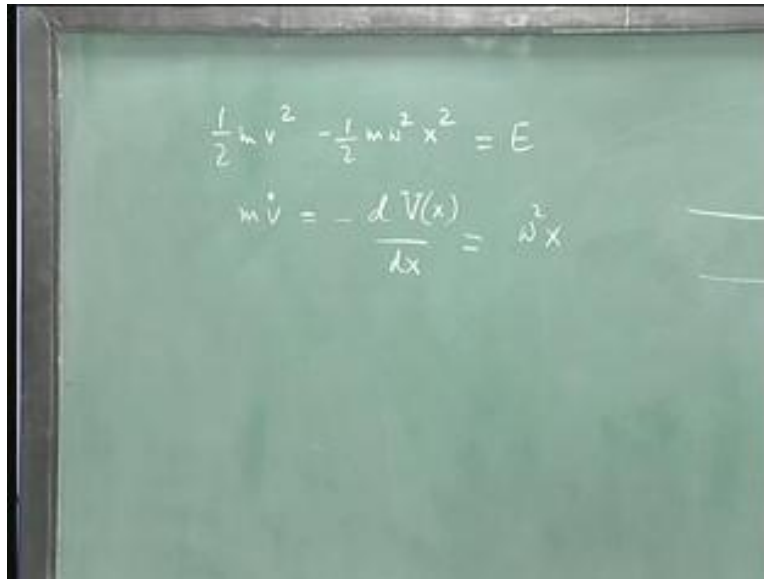


Classical Physics
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Lecture No. # 03

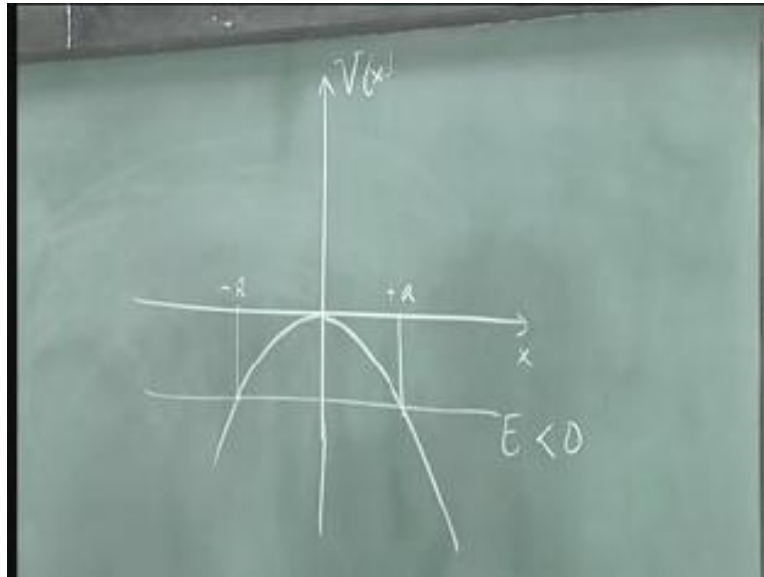
The last time we ended by talking about not just the simple harmonic oscillator but the inverted oscillator.

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$$\frac{1}{2} m v^2 - \frac{1}{2} m \omega^2 x^2 = E$$
$$m \dot{v} = - \frac{dV(x)}{dx} = - \omega^2 x$$

So, which I recall to you, the total energy was given by 1 half m v square minus 1 half m omega squared x squared; and this was the energy of the oscillator, function of the velocity and the position. And the question was what sort of phase trajectories do you have in this system?

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So, let us draw those phase trajectories here this was left as an exercise for you. If I plot x versus the potential v of x , then of course this is an inverted parabola of this kind. Because of the minus sign and the point x equal to 0 is the maximum of the potential it is an unstable point as you can see point of equilibrium but an unstable one. And the question was what kind of phase trajectories go with this kind of behavior. There are clearly no periodic motions involved here for any initial conditions.

Moreover unlike the case of plus 1 half omega squared x squared for which you have an upward looking parabola. In this potential the energy e the total energy E of this system of this particle could be any number between minus infinity and plus infinity. You could have negative energies as well, because if you have a total energy of this much for instance. So, here has as a typical energy E , which is less than 0, then it is evident that the particle can never find itself in the region between these two points. Because in between these two points the potential energy is greater than the total energy and therefore the kinetic energy is negative. And that is not allowed, because it is a square of a real quantity. So, this region is inaccessible to the particle classically quantum mechanically, it could be different story.

And then for positive values of energy the entire region from minus infinity to infinity is accessible to the particle. And our task now is to draw the phase trajectories in this system. We

do not want to sit down and solve the equations of motion, which are easy enough to write down. Because, that is obvious that Newton's equation would be $m \dot{v} = \text{mass times acceleration}$ is equal to the force on the particle; that is equal to minus derivative of the potential dV/dx and this is equal to $\omega^2 x$ itself.

Because, there is a minus sign here there is another minus there, they cancel and you end up with this equation. This is not a simple harmonic oscillator for which you would expect the minus sign. Therefore, the behavior is very different from what it is for the harmonic oscillator for which the phase trajectory is where family of concentric ellipses. This equilibrium is an unstable equilibrium point, what would it, then look like well; let me plot here for convenience.

So, here is x and here is the velocity v and the idea in a phase portrait as you know phase trajectory is that you eliminate time. And we directly plot v as a function of x and the system is specified by specifying v and x at any instant of time. And as time goes along this point this representative point moves in this plane, which is called the phase plane, and executes a trajectory called the phase trajectory.

And in this case since there is no periodic motion the phase trajectories are open curves they can't be closed curves which would be then representative of periodic motion. So, what would this look like, well it is quite evident, since we have had this we have already seen that this region is forbidden for $E < 0$; that the behavior of the trajectories are at least the qualitative shape of this trajectories could be very different for $E < 0$ as oppose to $E > 0$.

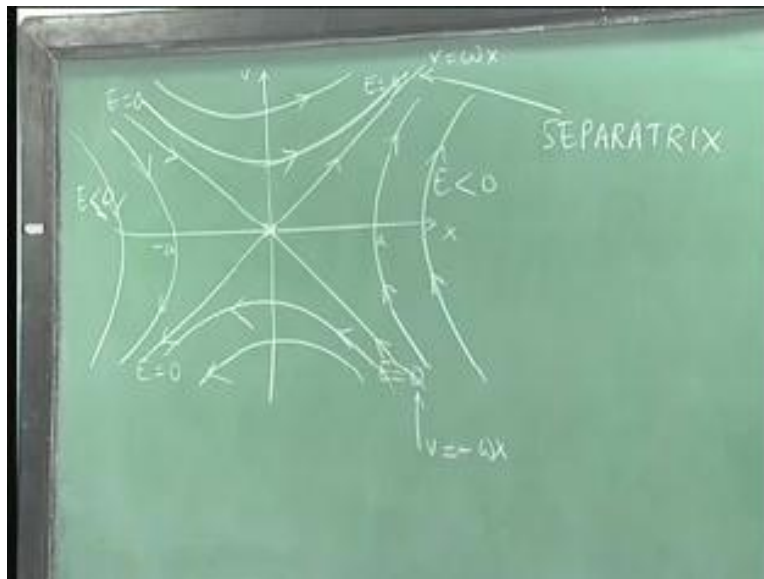
Let us look at $E < 0$ first and it separating these two is one very special set of trajectories corresponding to $E = 0$ on this line itself. So, what would it look like, well we can draw this in many ways but let us argue this out in words and then we will try to justify this by writing down the equations. Incidentally what sort of equation is this? It is a hyperbola, it is a family of hyperbolas and of course you have one set of hyperbolas for $E < 0$. And another set for $E > 0$ and that precisely, what the phase portrait is look like.

So, what does it look like for $E < 0$, well let us take this energy, typical energy. Now imagine, that you start at this point here; and you let the particle move from rest at this point, in which case it is obviously going to roll down this potential well and disappear to $x = 0$

minus infinity. And all the while it is velocity is going to become more and more negative as it moves down, moving leftwards with greater and greater velocity.

So, what would it look like here well if I call this point, let us call this point a minus a and plus a for instance. Then it is clear if we going to start at some point minus a here this is plus a and then what would this trajectory look like, it would be part of the hyperbola. And it is moving with greater and greater velocity towards negative x and the velocities directed in the negative direction. So, it is evident that it is going to move down in this fashion. You could also start at minus infinity and give it just enough energy to crawl up this potential hill up to this point. So, it starts of very large positive velocity and by the time it reaches the point it is velocities has gone to 0, after which it will roll back.

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So, it is clear that the rest of this trajectory is look like this, in this fashion and this is the direction of the trajectory. Similarly on the other side we would have exactly the same picture and that would be the other branch of the hyperbola. So, we will have another hyperbola at this point, which looks like this in which direction should I plot this, which way it would go, upwards or downwards. It would be upwards really, I will put this if I had an even lower energy down here, some energy of this kind, then there would be another hyperbola this fashion.

And therefore you get a whole family of hyperbola, now what happens as the total energy is increased till it is zero. Then it is quite clear that a particle shot up from minus infinity would have just enough energy to reach the point 0 asymptotically. And as it crawls up here, since the force on this particle is minus the derivative the potential and the derivative is vanishing at this point. Vanishes the force becomes smaller and smaller; so the particle takes longer and longer to reach this point.

So, it is clear that there is a straight line and by the way at E equal to zero the trajectories are given by v equal to plus or minus ωx . Because I just set this equal to 0 and you get a pair of straight lines they are the asymptotes, these hyperbolas. So, you have asymptotic behavior of this kind and you have another asymptotic here, this is v equal to ωx , this is v this guy here is v equal to minus ωx . And in which direction to these trajectories point, incidentally they never reach the origin.

Because, the origin is a point of equilibrium by itself its an isolated point; so if I start with the particle at x equal to zero precisely with v equal to 0, it remains there forever and is that trajectory by itself. So, really speaking I should erase this little bit and say that there is a there is a equilibrium point at that stage and the rest of these trajectories actually asymptotically reach this point. Because, remember phase trajectories for an autonomous system can not intersect themselves they can not intersect each other either.

So, the point at the origin is a special trajectory by itself a single point is a phase trajectory. And these other trajectories asymptotically reached or asymptotically go away from it. So, by continuity this arrow has to be inwards, this points inwards; whereas these arrows points outwards, because they are asymptotic to these outgoing lines.

So, this set of trajectories corresponds to E equal to 0 and these trajectories correspond to E less than zero. So, for every E less than zero there are really two trajectories possible one on this side one on that side. At E equal to 0, there are four trajectories possible corresponding essentially to starting at minus infinity and moving up crawling up to this point. Or starting close to this point and rolling down this potential hill or the other way moving up here or rolling down those are the four straight lines.

For positive here has a typical E greater than zero it is easy to see what the trajectory is going to be like; because now if you shoot a particle from here with this total energy. It is going to go up here overcome this barrier and fall on to the other side, because when it comes here it still has this much kinetic energy, the potential is zero but still has kinetic energy and it therefore crosses the barrier overcomes it falls on to the other side.

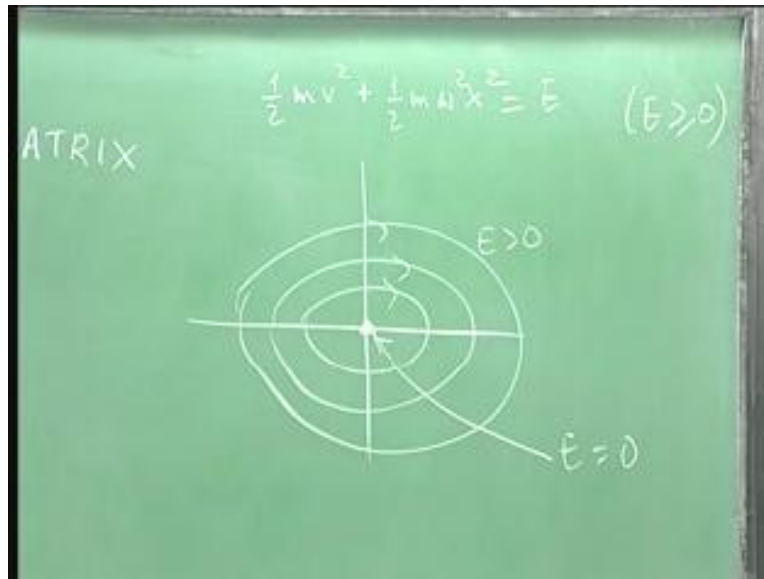
This would mean that the trajectory starts here and overcomes the barrier and goes off to plus infinity from minus infinity. And right at this point it crosses, you can see that it is kinetic energy is that a minimum the v is lowest possible value, because at this point if this is the potential this is v squared whereas at this point this is v squared, therefore it is a least at this point.

So, the trajectory looks this draw this little more symmetrically, this fashion and similarly here you have a family of trajectories, corresponding to positive E going to be other connection. And this is the full phase portrait for this particular system, which has an unstable point and unstable point of equilibrium, I will explain what the critical point is as we go along.

Now, it is evident that these two straight lines, they separate two different kinds of motion. One motion where the particle is either restricted to the left side of the x axis or the right hand side of the x axis but does not go from here to there. But second kind of motion, where its able to access the entire coordinate axis minus infinity to infinity.

In other words this set of hyperbolas, from this set of hyperbola and this set of trajectories separates these two kinds of behavior and they are called separatrices. This trajectory is a separatrix and these four trajectories are the four separatrices, they are special very special. At that critical value of energy at zero, you have these five possible trajectories, you have these four line straight line segments as well as the single point the equilibrium point itself. This equilibrium point is very different from the equilibrium point, when you had a plus sign, when you had a simple harmonic oscillator.

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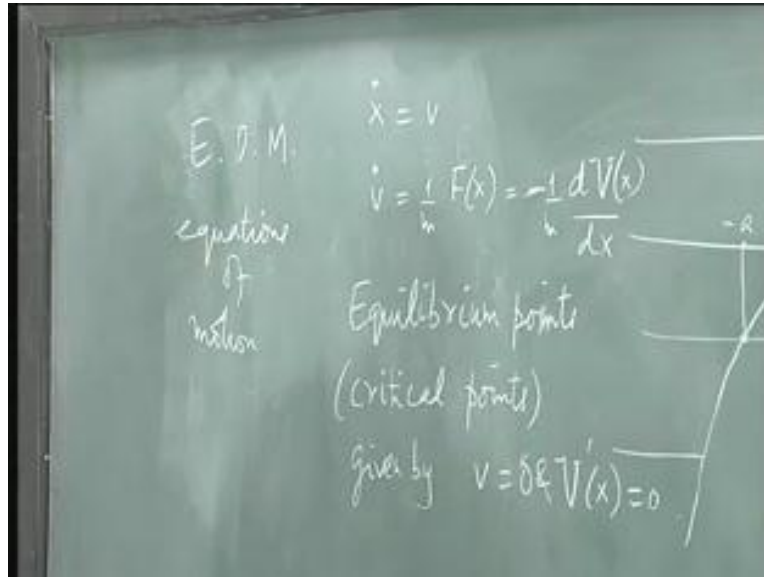
You recall just for comparison, recall what happens when you had half $m v$ squared plus 1 half $m \omega$ squared x squared equal to E . In this case E was restricted to non negative values and the set of phase trajectories. In this case was a set of concentric ellipses of this kind and this is for a typical positive energy and you had a point here, which corresponded to an equilibrium point for E , exactly E equal to 0.

So, we begin to see, that the behavior of this phase portraits is largely governed by these equilibrium points. Is they kind of decide what happens in the rest of the phase plane in these simple examples at least. This point here a stable equilibrium point of this kind has a special name, we will see its part of the general classification. What would you call this point a stable equilibrium point, what would you call this. Technically this point not a very imaginative name but a perfectly it is reasonable name is called a centre this trajectory is going round centre.

What would you call this name this this one, this point well its not centre of a hyperbola or anything like that it is the unstable equilibrium point. And around it around it you have hyperbolic behavior let us called a hyperbolic point very reasonable it is also called a saddle point. It is look a saddle as you can see, I will explain this little later this is called saddle point.

How do we find this equilibrium points well in the kind of problems, we are looking at where we have not started really writing down formally equations are anything like that. Recall that the equation of motion is always of the following kind, we looking at a particle moving on the x axis, under some given potential, which is time independent but coordinate dependent.

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So, the equations of motion this is called to be my standard abbreviation for equations of motion. They given by \dot{x} equal to v and \dot{v} equal to $\frac{1}{m}$ the force on the particle, which an all kind of problem is $-\frac{1}{m} \frac{dV(x)}{dx}$. Now, what would we say the equilibrium points, how would you define equilibrium points? I define it by nothing changes with time and something does not change with time, it is time derivative vanishes.

So, the equilibrium points, which I would like to call by a more general name, I would like to call them critical points. And I will explain why, why and why this is a mathematical term for equilibrium points, because it slightly generalizes the concept of equilibrium points. These are given by setting the right hand sides to 0, no time derivatives they are they all vanish, there obviously therefore given by v equal to 0 and v' of x equal to 0.

Remember these are critical points I will just call them critical points from now on, in phase space, not in the space of x alone not in position space alone but in phase space. And these points

I given by setting the right hand side to 0. So, since this is the trivial equation these always 0 and you have v prime of x equal to 0, when does v prime of x vanish, well in this example yes but in general I have a potential, so when does it vanish.

At the extreme of this potential could be a maximum, could be a minimum could be more complicated than that but whenever v prime of x vanishes the force vanishes that point. And if v prime of x is a reasonable potential then of course the set of solutions to v prime of x equal to 0 is a set of isolated points not a whole range, although it could happen, if the potential is completely crazy, you could have a whole range of points.

For instance suppose I had a potential, which looks like this like the bottom of the bucket, then of course, these are all if you like minimum but not in the usual sense of being able to differentiate this potential, because there are point where you cant differentiate potential. We are going to avoid cases of this kind or at least treat them separately but in general this would be a set of isolated points the extreme of the potential there correspondence; mind you the extremer do not have to be simple minimum at all.

For instance let us look at an example right away, what happens if I have a potential v of x equal to some constant k and x to the power 4. Let me put a 4 here, because I am going to differentiate it to find the force and I want to get rid of these unnecessary factors.

So, what happens when I have a potential of this kind and let us say k is greater than 0, if I plot this potential he has x versus v of x it still has a minimum at x equal to 0. In an isolated minimum but this is not a simple minimum it is a very flat kind of minimum and as you see the potential goes like this very flat here, goes of this fashion.

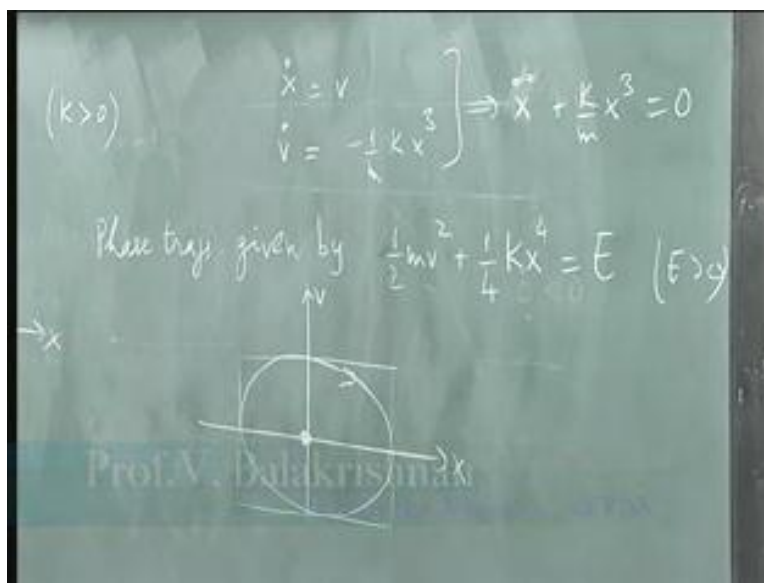
Now, tell me how do I know it is a minimum, what is the condition that we should have a minimum at any point, well in this problem, the first derivative vanishes but showed as a second, how. The first non zero derivative should be an even derivative and then we have an extremum, if the first non zero derivative is an odd derivative, you have a point of inflection. You have a point like what happens for x cube, if I plot y equal to x cube as a function of x , as we know it it has this and of course the first derivative vanishes, so it has the second that point.

But as you can see the curvature undergoes with discontinuous change at the point and in this case you have a point of inflection its not a maximum not a minimum the slope of course vanishes at that point as well. Now of course, if you put a particle in a potential of this kind, it is clear that this is an unstable equilibrium point.

Because for it to be stable you should be able to displace the particle in any direction whatever and still have a restoring force which brings you back to the equilibrium point. In this case if you move it to the left, it just falls of therefore, it definitely an unstable equilibrium point.

For a stable equilibrium point you need an absolute minimum and as the number of dimensions increases as potential becomes the function of more and more coordinates this thing can get more and more interesting because the shapes would become more and more complicated now we will see that. By the way what would the phase trajectory correspond to in this problem, what would the equations of motion correspond to...

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Well once again you have \dot{x} equal to v . And then \dot{v} equal to 1 over m times derivative here; so this is equal to $\text{minus } 1$ over $m k x^3$, these are the equations of motion. So, it quite clear the equilibrium point is still v equal to 0 and x equal to 0 , what would the phase trajectories look like in a one dimensional problem, they are of course given by a single curve a single

equation in the phase plane. And we know these are conservative systems we have started with the assumption and therefore the total energy the statement that the kinetic plus potential energy is constant E the total energy, this suffices to specify the phase trajectories completely.

In more complicated system you need more conditions but, this is enough for one dimension. And therefore this is given by $\frac{1}{2} m v^2 + \frac{1}{4} k x^4 = E$. What are the possible physical values of E , non negative definitely non negative, because each of these two terms on the left hand side can not be negative. And therefore E is greater than equal to 0, E equal to 0 is the equilibrium point and what do these trajectories look like.

So, this is definitely a point of stable equilibrium, now I will use a little convention when I put a little cross that is a point of unstable equilibrium as in the other case, when I put a dot it is a point of stable equilibrium. We will define stability and instability very precisely but, this is immediately obvious as soon as we look at, what do the traject, what do these things look like are they ellipses, they are not ellipses but it is clear each of the terms is non negative and they are algebraic functions.

So, it is quite clear that in some kind of distorted ellipse and overlap some kind it is quite evident that when x is 0, v attains it is maximum magnitude, I mean v is 0, x attain it is maximum magnitude. And of course when v is 0, you have this equation there and remember that E is greater than 0 for physical trajectory and you have 4 roots, what would they correspond to, two are imaginary, two are imaginary and the remaining two roots would be the ends of the amplitude it is specify the amplitudes.

So, it is a evident from this problem that the motion would occur in some kind of rectangle of this kind and it is something which is not quite an ellipse but, an overlap some kind. So, I don not need to solve the problem, why is it so difficult for me to solve this problem. As oppose to the simple harmonic oscillator, I have these equations of motion, what makes it difficult to solve this equation.

Well the obvious thing to do is to say ok the way I solve this is the way I solve the harmonic oscillator immediately eliminate v and I write this as $x \ddot{\ddot{m}} x \ddot{\ddot{x}}$

plus k over m x cube equal to 0. What so, difficult about this it is a second order ordinary differential equation what so difficult about solving this equation.

What so difficult about it well in the case of the harmonic oscillator I had an ω^2 there and then an x . And then of course you know what the solutions are cosines and sines of ωt what, what happens here the motion is still periodic. It still periodic because I know the phase trajectory are closed phase directories have have an explicit equation for them.

But, why is it so hard to find x is a function of t , through out this course through out this course the mathematics and the physics will be very inextricably linked each other. Every mathematical fact will have a physical consequence and vice versa; I am not going to distinguish between the physics part of it and the maths part of it at all.

So, what is difficult about this equation it is non linear, it is not linear in x , so for one thing the principle of super position no longer is val is valid. And for another you can not use trial solutions its not so easy. So, the motion certainly the solutions whatever it is cannot be cosine and sine. Whereas it would take $\cos \omega t$ and you differentiate it twice you again get $\cos \omega t$, which is $y = x$ double dot could be balanced with minus $\omega^2 x$ that is not true here. The moment you have x^3 you immediately have $\cos^3 \omega t$ which involves $\cos 3 \omega t$.

So, harmonics get in into this problem and it quite obvious that trigonometric functions are not solutions to these equations right away. Of course we can solve it numerically you can always solve these equations, numerically to any desired accuracy. But, in principle you have to appreciate the fact that this problem has major difficulty in that it is non linear. This our first experience of non linear problems and we are going to look at a lot of non linear problems and how to handle this. But you see the power of the phase trajectory method and the phase space method, I do not need to solve the equation. I already know everything that I need to know practically everything I need to know about the nature of the motion, that is periodic and so on, so forth.

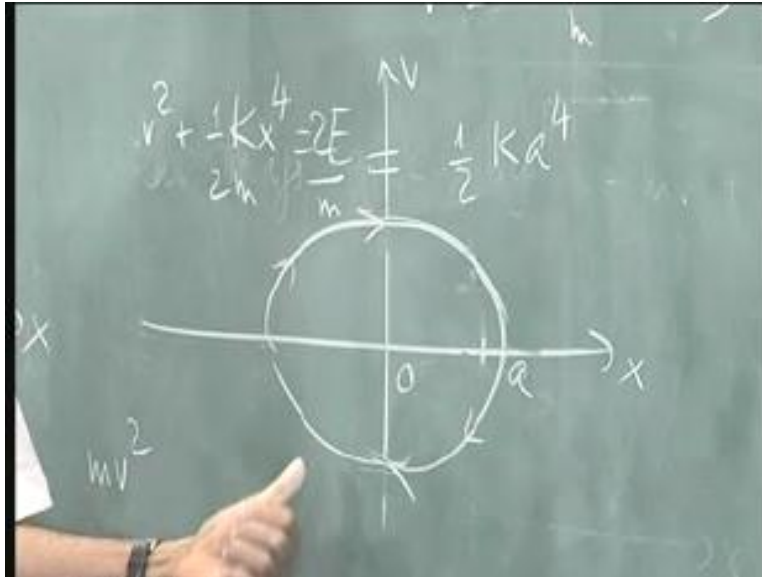
So, this is an non linear equation, no simple method exists to solve these equations this particular equation it turns out can be solved in terms of what are called elliptic functions. But,

there is an arbitrary for an arbitrary non linear equation there is no reason at all why the solutions are writable can be should be writable in terms of quote and quote known or tabulated special functions no reason.

Incidentally while we are on it this non linearity has some very, very interesting effects as you can see. You could ask would the time period of motion in this problem, would the time period be independent of the amplitude or do you think it depends on the energy; for the simple harmonic oscillator it was independent of the amplitude. No matter how much you stretch the spring the time taken for a complete oscillation was exactly the same. Independent of the energy of the oscillator would that still be true here, that is not only for the oscillator is this magic true, only for the simple harmonic oscillator.

As soon as you have non linearity's the problem becomes immediately much more difficult. And in general the time period of periodic motion is not independent of the energy of the total energy or the total amplitude is the synonymous. I would like to ask can we solve this problem and find it the nature of the dependence on the time period without solving the differential equation. Of course if you solve the differential equation the job is done immediately. But now I would like to pretend that I do not have information about elliptic functions do not have any table of special functions. I would like to know how does the amplitude depend, how does a time period depend on the amplitude of motion is there a way I could solve this problem.

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Well as a hint since we moved into this non linear problem, here is a typical phase trajectory not an ellipse badly drawn, but it is some kind of symmetric curve it goes up and down. And I like to know what the time period of oscillations or at least how it depends on the amplitude of motion in this potential.

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$$T = \int_0^T dt = 4 \int_0^a \frac{dx}{v}$$

$$\propto \int_0^a \frac{dx}{\sqrt{a^4 - x^4}} \quad \text{Put } x = au$$

$$\propto \frac{1}{a} \int_0^1 \frac{du}{\sqrt{1 - u^4}}$$

So, let us do it in the following way its actually doable by very elementary means I start by saying that the time period is in fact 0 to T dt by definition, capital t is a time period for a given

energy. But I could also write this as equal to well by symmet in this problem; since the phase trajectories are given by a very symmetric curve $\frac{1}{2} m v^2 + \frac{1}{4} k x^4 = E$ this curve is symmetric in v and symmetric in x .

So, whatever happens here is reflected here, whatever happens here is reflected here. If it starts here it starts at the end of it is amplitude this energy starts at this point it moves back and goes back and forth. And we want to find the time period of oscillation. So, it starts here moves down comes to the origin but it is going backwards at high velocity and then it reaches this point stops turns and moves to the right. Again passes the origin once again positive velocity comes back to its end of the amplitude. It is obvious by symmetry that the time it takes to go from here to here is the same as this time, is the same as this time, is the same as this time.

So, this immediately says that is this is equal to 4 times the time it takes to go from this point to this point. And if this is the amplitude of oscillation a then, this is equal to 4 times the time it takes so $a \frac{dx}{v}$ or $v \frac{dx}{dt}$ is v , v is $\frac{dx}{dt}$. So, that t if you like dt cancels the dx cancels out and you essentially start with dt and you change it to dx .

But, then this is also equal to 4 times $\int_0^a \frac{dx}{v}$ but, what is v now you can solve for v here and it immediately tells you, that $m v^2 = 2(E - \frac{1}{4} k x^4)$. Well let us divide by m multiply by 2, so this becomes 2 and then we have divided by m , so 2 over m in this fashion. But I am not happy with a way I write this, this is equal to $\frac{1}{2} k a^4$, because when you are here the entire energy is potential energy, therefore E is also $\frac{1}{4} k a^4$.

So, once I write it in that fashion then apart from some factors this whole thing is square root of $a^4 - x^4$. There is no other a dependence anywhere else and then I these factors k over 2 etcetera I take out right. So, apart from some constants t is proportional to this and keeping track of the a dependence, there is a dependence here and there is a dependence here. There is no a dependence anywhere else that is the integral I have to do.

By the way I wrote v is equal to $\sqrt{a^4 - x^4}$ but, I should tell you whether its plus or minus, when I take the square root, what should I do; is it plus or minus? It is plus, because when I go from here to there it is plus v is positive I have taken this root, then I multiplied by 4. So, that is the reason I chose \int_0^a had I chosen this, this way the other direction, I had put the

minus sign there careful with that and it is a plus root here. And it is this integral we want to do, then of course this is not a trivial integral, because if we put x equal to a $\cos \theta$ it won't work, it works only if you had a squared minus x squared but not with any other higher power.

First we have to ask do you think this integral exists, because after all when x equal to a it goes up, the integral goes up. So, do you think the integral exists, it actually exists there is no problem, because you have to ask what part of it a singular. And that is interesting because a $4 - x^4$ is a squared plus x squared times a squared minus x squared. If you put x equal to a this part is harmless but, it is this part this vanishes well here too, you could put a plus x times a minus x . And if you put x equal to a this part is harmless and it is only this that matters and the whole thing hinges on integral dx over square root of $a - x$. Whether this exists at x equal to a or not but then this is like a power half in the denominator and then you integrate it you get a power half in the numerator.

So, dx over square root of x , when you integrate you get $2\sqrt{x}$ on top and that is integral put x equal to 0 everything is fine. So, it is an integrable singularity and physically of course we know that this thing had better have a finite time period it is a physical quantity. And I know its finite because, the restoring force never goes to infinity anywhere does not do anything crazy. So, definitely there is finite time period and we will see that indeed you can extract the information; so the integral exists but I would like to know what its dependence on a is, what should I do next.

Yes we can do that, we can differentiate we can use the formula for differentiation and the integral sign and so on. But, you see my my ambition here is limited I am not interested in the numerical factor which I know is finite I am interested in the dependence on a so what should I do; a depends does a dependents on the integral and there is a dependents in the limit of integration as well.

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I should take out a how should I do that.

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why

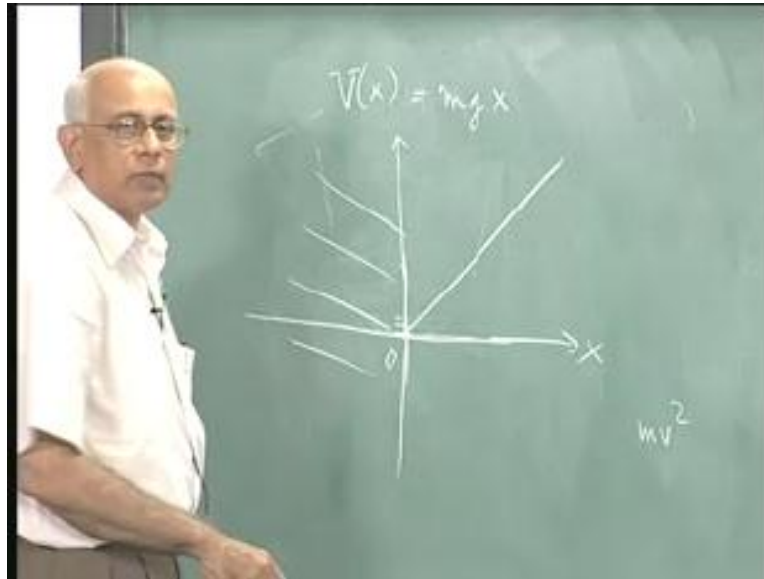
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Non dimensionalization absolutely right, I scale out with respect to a , I scale out I realize that in this problem there is a natural line scale in the problem a , a specified line scale for a given energy. So, what should I do is to put x equal to a times u , that is all that is all that you need because this immediately becomes proportional to an integral zero to 1. And dx gives me an $a du$ divided by an a to the 4 which comes out and you get a square $1 - u^2$ to the 4. And of course this comes out at ones; so it is equal to $1/a$ times this.

And the integral now is a numerical integral it is an absolute number and we had guaranteed its finite. It is some crazy gamma function something or the other quarter or something we do not care some finite number of the order of unity but, here is the difference. So, this tells you that in this fourth order or quartic potential the time period of oscillation, for this oscillation is less than the time period of oscillation for this and that is less than this.

So, the smaller the amplitude, the longer the time period sounds counter intuitive after all you have to traverse much greater distance, when you have a high large amplitude. But, yet the time period is smaller how do you explain that what you think is happening at at larger amplitude the slope becomes higher the restoring force is much greater, so it zips through the origin very fast at very low amplitude, this is an extremely flat minimum. It is a third order minimum very flat and therefore if this is your amplitude this is extremely sluggish, the restoring force is practically 0. Therefore it takes a long time to crawl through this zero go to the other side and come back.

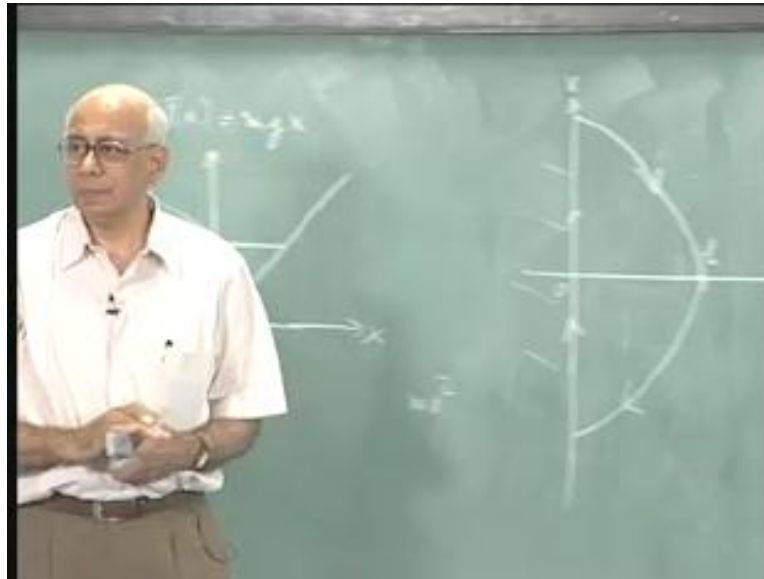
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Therefore this is energy to the minus 1 by 4 it is a non trivial statement that the time period of this quartic oscillator, non linear oscillator is proportional to the minus 1 quarter power of it is energy unlike the harmonic oscillator, where it cancelled out. Because remember that argument we went through if you had a square root here and squares here, then this a and that a would have cancelled out, and it becomes independent of the amplitude, whereas in this problem it goes in the other direction altogether. By the way imagine taking a rubber ball with perfect restitution no air resistance of any kind, I bump bounce it up and down on the floor, is the time period independent of the energy or the amplitude, what you think it does, what power what power of the amplitude the amplitude is just the height from which I drop it what power of the amplitude does it go high, now lets work that out.

So, here is a ball bouncing up and down and I would like to know how does the time period depend on the height, it should go like the height to the power 1/4 it should go like a to the power 1/4 half it should go like a to the power half and once check that out and see if its right or wrong. What is a potential look like in that problem v of x is equal to lets assume x is the height above ground what is v of x looks like mgx it is just mgx.

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And therefore that problem v of x is mgx that is it and of course we have an impenetrable floor and therefore this portion is not available to us the phase space does not include this part of axis that is only the positive side. And then you have an infinite potential value here it is an infinite potential barrier can't go through the floor. And a bouncing ball up and down it just executes oscillations there of some energy e .

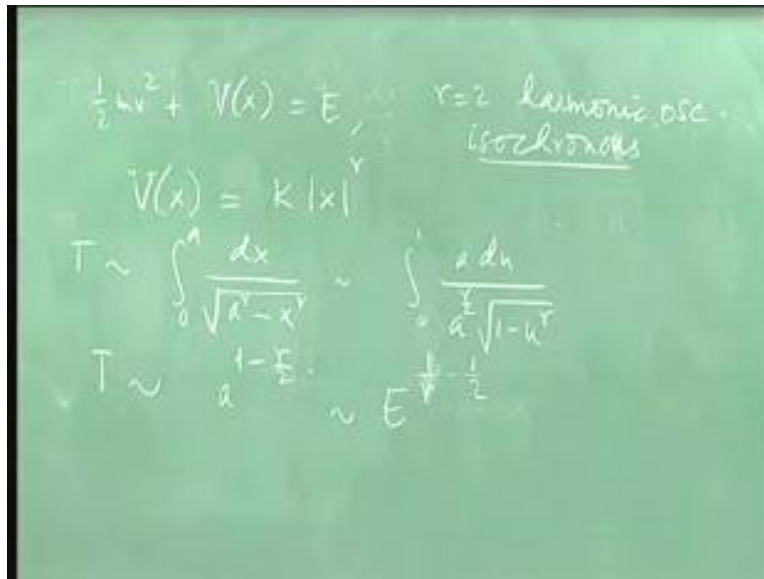
What do the phase trajectory look like in this problem what do the phase trajectory look like? Well once again here is x and here is v and you have to tell me now, what the phase trajectories look like well I have $\frac{1}{2} m v^2 + mgx = E$ and x is positive by agreement. Therefore e is only non-negative values positive values what is the phase trajectories look like, well remember $x < 0$ is forbidden completely gone.

So, I drop the particle from a height h and the particle's velocity goes in the negative direction. And what sort of curve do I have here, I have a parabola because I have a parabola. Because, the velocity v increases linearly with time in magnitude, while the position goes like $\frac{1}{2} g t^2$ whatever goes like the square of the time. Well as the velocity goes like linearly with time, which tells you v^2 and x match as you can see from the phase trajectory. It was therefore part of a parabola goes down in this fashion, the moment it hits the floor what it does, where it becomes discontinuous, the phase trajectory becomes discontinuous, in this problem.

Because, we have assumed this perfect restitutions; so it starts here at this point and then it completes the rest of the parabola and it jumps up in this discontinuous fashion.

Now, we would like to know what power of we know already we like to corroborate this that the time period is proportional to the square root of a height I have to prove this explicitly. Well let us do this in generality let us take a potential v of x .

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So you have half $m v$ squared plus v of x equal to E and let us take a potential v of x which is equal to and I would like to take a whole family of potentials, in with the harmonic oscillator is included. The fourth power is included the linear power is included and so on. And just to make life simple, let me assume that you have a potential on this side too but it is completely symmetric.

If I block it of then of course this time period is easily calculable and so is this time period calculable they are related to each other. I just take a potential which looks like that; what kind of function of x is that, mod x this is the mod x function. So, let us take v of x equal to some constant times modulus x to a power, some power whatever power, what shall we call it, let us call it r . If I put r equal to one I get the problem of particle, a particle falling under gravity except it goes inside and comes out it does not matter can you think about physical situation, where you

actually have a particle subject to a potential of this kind a mod potential. You see the force is constant, because if I differentiate x I get a constant but, then if the moment I put mod x , when x is negative it becomes minus x and the force, therefore is again towards x equal to 0.

So, so what kind of physical problem can (ϕ) , where you have a potential a force that is constant on the particle. So, when it is here or here or here, it still the same and on this side it still the same and magnitude but pointing towards x equal to 0. A sheet of charge an infinite sheet of charge laminar sheet of charge and you put a charge particle here, it sees the constant electric field and goes down and the other side it sees other way.

So, that problem is just a generalization of the gravity as you can see, so that is how that is how general problem. Now, when r equal to 1, you have this when r equal to 2, you have a parabola, when r equal to 4 you have an even steeper curve of this kind. And they all pass through 1 and our question is for a general r what is the time period do, when r equal to 2, you discover the time period is independent of the amplitude.

Now we would like to do this in general, we go through the same thing as before and the T is proportional to an integral from 0 to a dx over square root of a to the power r minus x to the power r that is it, apart from some factors this is true and I do the same trick as before and therefore this goes like zero to 1 $a dx$ divided by a to the power r by 2 times a numerical integral.

So, the conclusion I have is the T is proportional a to the power 1 minus r by 2 . So, when r is 1 I have the gravity problem and indeed it is proportional to square root of a , when r is 2 it is independent of a and the system is said to be isochronous. Because independent of the amplitude it is a same time and therefore it is isochronous and that is very special to harmonic oscillation. Otherwise it goes like this, when r is 4 it goes like a to the minus 1 , which is what we described. And so, you see by this scaling argument we are actually able to tell what time period is independent of the details of the potential in this simple instance.

By the way you could also write down what the dependencies on the energy that should go like energy to the power, what should I write? Remember the energy is proportional to a to the power r , therefore a is e to the power 1 over r and therefore this is 1 over r minus half that will become

relevant as we go along. So, that is a neat result the whole family of potential is taken care of in one shot the case r equal to 2 the harmonic potential is a very yes please yes.

Sir for example x by 4, when we put v equal to 0, we got some complex solutions, what does it physically mean, yes just says these are not real roots and therefore they are not actual physical points, because it is a fourth order equation x to the power 4 equal to E or whatever it is

Can we think of it, that in reality such potential would not exist

No, no, no the potential exist it is just that goes root do not exist right I solve an equation for the physical quantity which has to be positive for instance. And I end up with a negative root what does that mean I get a quadratic equation which happens all the time and one of the roots turns out to be negative what is it mean its not physically acceptable solution.

So, it does not have any particular significance in that sense which has done extra root in the problem, whereas we always have to impose physical condition to choose the roots that you need that is a good leads on to another point, when you solve these equations, when you solve differential equations depending on what you demand of the equations of the solutions. You get many, many solutions only under particular conditions do you get uniqueness and so on. If the solution exists it should be physically acceptable for instance I solve a problem in probability very often you end up with a root which is negative that is not physically acceptable.

When you solve the problem for a probability density it has to be non negative, it has to be normalizable its integral must be equal to 1 and so on. You could get other solutions which are not physical, there is no significance in general, we come across more such examples as we go. So, we have some idea now of this and there is a reason why I do this is because we need to know this, because when you quantize the problems, when you put make this particle quantum mechanical. Then you would like to know what the energy levels look like as a function of your quantum number. And this is going to provide us with a handle to get into this using the Bohr, Sommerfeld quantization, so this is something which it is worth bearing in mind for later on.

While I am on the stack let me also point out, that this argument about scaling is very powerful but, it work because the potential had this very simple structure. It was just a power of arc had it

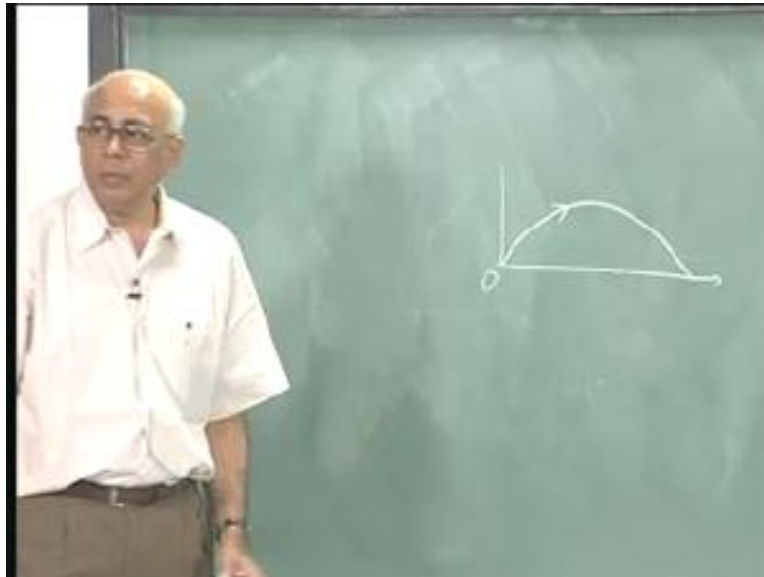
been a more complicated polynomial with line scales in it and so on; itself then you couldn't have done this.

But because it was simple monomial we were able to do this and let me illustrate that thing by pointing out to you a slightly more complicated problem where too you can similar powerful result without doing any work. And this has to do with the problem of planetary motion which all of you are familiar with Keplers laws of planetary motion. Do you recall this laws what is the first law say, does the planets move with a sun, with a sun at the center of the ellipse at one of the four side.

That is crucial its absolutely crucial that we will see as we go along, where every one of these laws every clause is written down for a particular purpose not for the sake of riming or anything like that but, because its needed if you remove the clause its gone. I take a balloon and I blow it up you are familiar with Boyle's law and I do this very slowly. Assume that I blow in ideal gas classical ideal gas and I do it very slowly, so the temperature is constant, the pressure times the volume, should be a constant right; pressure increases the volume decreases and vice versa. I blow this balloon up and I know the pressure is increasing I know the volume is increasing, what happen you are adding gas.

So, this word for a given mass of gas is very, very crucial leave that out, because the line is too long, then you finished. So $p v$ equal to $n r t$ there is nothing to stop you from keeping t fixed and changing n , which is what you are doing in this one. So every clause matters ok Keplers first laws says, that the planet move in ellipses the sun at one of the four side. And what is the second law say, what is it really saying says the angular momentum is constant. Essentially, it saying the angular momentum is constant of the planet about the sun it is constant you have to be careful you have to specify angular momentum about what specify that too.

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I take this object and I throw it in a parabolic trajectory is the angular momentum about the point of projection constant do you think it is constant do you really think so, here is the part of the particle. Here is the point of projection and that is the plane in which the particle moves is the angular momentum about this point of projection constant. No starts with r equal to zero, and then of course in between it is not zero about, what point is the angular momentum constant?

About the center of the earth, because it is after all a central force and its part of an ellipse really you do not see the reason that it is an ellipse really part of an ellipse because that is, what it should be, what happen to the other focus its gone of to infinity. You assumed a constant gravitational field instead of a 1 over r squared force and essentially, we have assumed that the center of attraction is that infinity. It is gone of to infinity, what do you call an ellipse for which one of the 4 side has gone of with the infinity a parabola that is why its parabola is really an ellipse. So, which one of the four side gone to infinity nothing more than that.

So, let us write Keplers third law, now what is keplers third law, the square of something is proportional to the cube of something else, right T squared proportional to d cubed or something like that. Now, let us see if we can derive this law without solving the problem without solving it is not going to be a proof but, here is what happens here is the reason why you should you should

worry about why is the square getting proportional to the cube what is the significance of these squares and cubes and so on.

And the reason is the following, the equation of motion is this equal to the force of the particle the force on the particle is inverse square law force. So, this is equal to minus some constant, which involves gravitational constant and so on, divided by r to the power n e sub r , r squared that is the equation of motion.

Now, as always I would like to burry this problem or imbed this problem, in a family of problems and solve that in general after which I put n equal to minus 2. So, let us solve it in a situation, where the general force law is r to the power n , e sub r and n equal to minus two corresponds to the case of the planet to that of the Kepler problem inverse square law of those.

Now, I come along and say look this equation of motion which describes this particle may have bounce states, may have force trajectories, may not we do not care, we do not know. But, one thing I do know is that the physical trajectories, the physics of the problem should not depend on my choice of units, this is why in the old age, you could choose units. Where the length was the distance between some kings nose and his ear and so on and so forth and get away with it.

So, it is independent of choice of units so I come and say well you given me your equation in these units I choose different units. I like to write r equal to λ times r prime, I choose r prime, I choose the different unit of length and therefore what you call r is my r prime and they 2 are related by this multiplicative factor.

What happens to this equation here it immediately says this becomes $m \lambda d^2 r$ prime over $d t^2$ equal to minus k and then λ to the n r prime to the power n , e r is always got unit magnitude, whether I call it r or r prime does not matter, because r and r prime are parallel to each other. You measure things in centimeters I measure them in meters that's the only difference the λ is just a factor of hundred or may be one over.

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$$\left(\frac{R_2}{r_1}\right)^3 = \left(\frac{T_2}{T_1}\right)^2 \frac{d^2 \vec{r}}{dt^2} = F(\vec{r}) = -\frac{k}{r^2}$$

$$\lambda = \frac{r_2}{r_1} \quad \mu = \frac{t_2}{t_1}$$

Therefore this becomes lambda to the power n minus 1 and I get rid of this kind. Now since you have done that with length units there is no reason why I should let you get away with measuring time and seconds I would like to measure it in days, ugas whatever. So, I come and say well let us change units and put t equal to mu t prime what happens to this equation here, so there is a mu squared, which comes out of here. So, let us bring let us bring the lambda to this side and write this more transparently, lambda to the 1 minus n over mu squared. In other words, we discovered that the equation is exactly the same in form as before but, the only difference is that there is an overall constant multiplication.

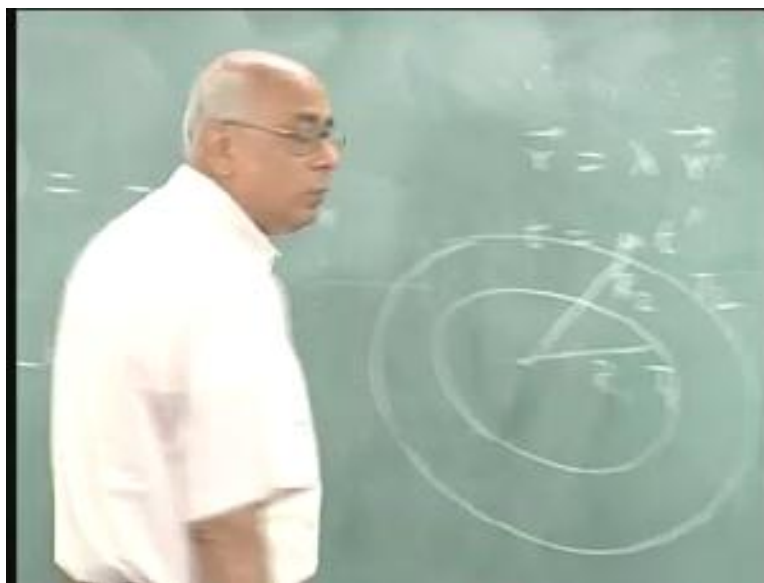
Now I argue by saying well this was for change of dimensions, change of units if you like lambda mu are pure numbers conversion factors. But, I could also look at in the following way, I could say suppose you told me, that there existed the closed orbit with some time period t 1 and mean radius r 1. Then I put it to you that there can exist any other orbit with time period t 2 and mean radius r 2, such that the ratio of line scales to the time scales is related in precisely this fashion.

If lambda is r 2 over r 1 then mu is t 2 over t 1 and both of us are describing the same reality if this balances that. So, lambda is the line scale param ratio and mu is a time scale ratio, in this in

units for two different possible orbits. And when you put n equal to minus 2 you discover λ cubed and t squared match, this is the reason why you have t squared proportional to d cubed.

So, what it saying is that it is not proving that there exist, such an orbit but if it says says if you have an orbit with mean radius r_1 and time period t_1 . There could exist the same potential can also support another orbit with mean radius r_2 and time period t_2 ; such that r_2 over r_1 cube equal to t_2 over t_1 squared that is Keplers third law.

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And we got this by scaling argument it does not prove that there are ellipses does not prove that these are close periods. And says if you have such a motion then there is another motion possible and that is all the content is its also true for hyperbolic orbits where the distance would be the distance of closest approach it could still scale in exactly the same and a final remark under what conditions of this potential do you think there is no dependence on the line scale of the time period.

If we put n equal to 1 it goes away if we put n equal to 1 it is this and that is equal to minus k/r what sort of force is that simple harmonic oscillator harmonic oscillations force proportional to the displacement. In three dimensions in such a case you realize therefore the harmonic potential

is very, very special you realize time periods cannot depend on the line scale and the problem, in other words on the amplitude of the problem of the oscillation.

So that also is a special case of this more general argument as you can see of course it works only, because the potential was a pure power but as good enough when you can approximate it by that you have very powerful result this kind of a argument physicist call it scaling. Very, very crucial the mathematicians have a slightly different name for it they call it homothety we will call it scaling. Because, it says cube of the line scales squares scales like the square of the time scales that is good way to remember it.

What was the crucial input here apart from the fact that the potential had that point this was a second derivative. It was a second derivative sitting there and this was linear, that plays role that immediately plays a role in the matter. So, at all stages you watch for what the crucial input is it leads to this result and the moment that goes the result changes qualitatively.

So, you notice it is completely independent I do not care what sort of potential you give me what kind of details you have and so on what this constant case and so on. But, the result in general a very powerful one, so when it works scaling can be powerful, this is just generalized dimensional analysis and we use this as we go along. Let me stop here this time and we will get back to looking at one dimensional potentials little more detail next time.