Classical Physics

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Lecture No. # 28

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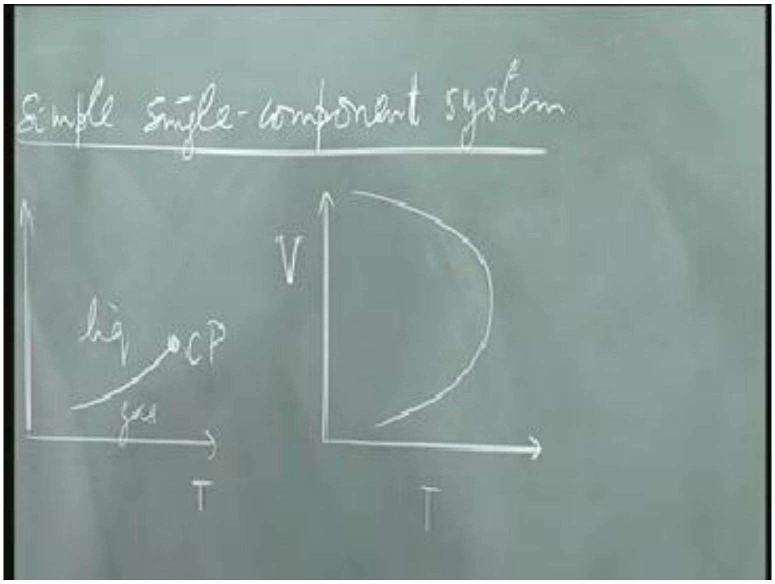

Let us resume our discussion on phase transitions and I quickly recap what we did last time? We took a simple single component system and for this system we do the P V diagram, the P T diagram, the T V diagram and our focus was to recall in the P T plane. Our focus was the critical point and this was the liquid phase and this is the gas phase. The focus was on this phase transition here from liquid to gas and corresponding to this. We found that in the V versus T diagram roughly speaking we had some kind of curve like this, which did not display too much symmetry, schematically something like this as T c and this as V c just as this is critical point T c and this is the critical pressure P c.

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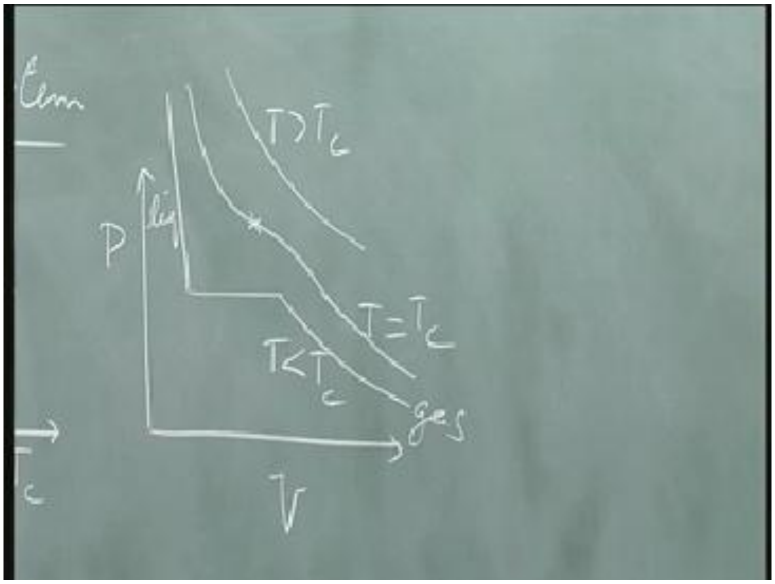

And finally in the P T plane, the V T plane and then the P V plane. We found that there were high temperature isotherms then the critical isotherm, which went off like this with an inflection point and then isotherms of this kind which corresponded on this side. In the gas phase and this side to the liquid phase and this was T greater than T c this is the critical isotherm T equal to T c and T less than T c. Now, what I would like to focus on is the fact that near this critical point. You have some kind of universal behavior in a very deep sense many systems behave independent of the actual interactions or details. They behave identically near the critical point and this was one of the fundamental observations, the theory of phase transitions and critical phenomenon. We will try to see why that is so and we will try to see what is universal about that point? So, T c this was P c.

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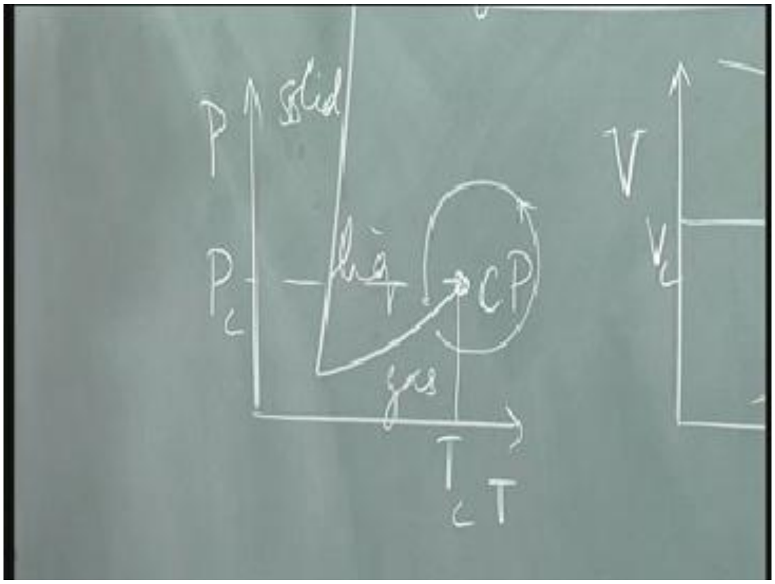

The first point to observe is that in a phase transition of this kind. When you have a critical point here and this curve of course, discontinuous transitions ends abruptly at this point. It is possible to start with some thermodynamic equilibrium state here and take it by a process which goes through process at equilibrium all the time. So, you could take a path of this kind and come to that point there. So, you converted a gas into a liquid without encountering the sharp transition at any point, this is possible because this curve is ended on the other hand. If you look at the liquid solid curve keeps going for ever in principle there is no way of doing this and the reason this curve does not end in a critical point. Because the solid has crystalline symmetry it is symmetric under the group of transformations corresponding to the phase group of the crystal a set of rotations and translations and so on.

On the other hand, the liquid is homogeneous and isotropic it has the same properties in all directions and therefore this has a much greater degree of symmetry than the solid. The solid has order crystalline order long range translational and rotational order. On the other hand the liquid does not have this order at all. It has at best some transient short range order so; order and symmetry are kind of opposites. The more ordered the phase the less symmetric. It is in the sense that the set of transformations under which it remain unchanged is smaller and smaller. Normally, intuitively we would associate order with symmetry it is just the other way above the most disordered state.

I think that is completely random it is in fact the most symmetric, because it looks exactly the same in every direction. There is no change at all. So, statistically both the liquid and gas phases are isotropic phases. They are homogeneous, if you have an infinite expansive liquid or solid or gas it looks exactly the same, but a crystal is not like that at all. You have a discrete symmetry so, what happens is that the Euclidian symmetry of a liquid. The factor is exactly the same on the average under all rotations, under all translations is broken down into a discrete symmetry called the space group of a crystal.

We come back and talk about the broken symmetry little later in the course, but that is the reason why? Either you have symmetry or you do not have symmetry, discrete symmetry. You cannot do this in a continuous way that is the reason why this curve does not end in a critical point. On the other hand between liquid and gas there is no symmetry breaking, it is exactly the same kind of isotropic behavior on both sides. Therefore since there is no symmetry breaking involved in this phase transition. It is perfectly acceptable to prefer it, but of course the other thing that happens is that the line of discontinuous transitions meets another line and that is quite a common occurrence.

So, many possibilities occur the most significant ones being that a line like this. If for example, for water it goes in other direction could actually end on the physical boundary at T equal to 0 or at T equal to zero, P equal to 0 either of them equal to infinity whatever. So, either of phase transition line ends on a physical boundary in this plane or it hits another phase transition, first order transition line or it ends in a critical point. There are other exotic possibilities can occur, but for the simple system we are looking at these are the only ones and of course there is also a triple point here. So, three phases could co-exist. Now what we would like to do is to show that this system behaves more or less like a magnet does and a fluid and magnet have close analogies.

So, I would like to build up this fluid magnet analogy to show you that near the critical point, which is called the Curie point in the case of Ferro magnet, Para magnet transition. You have identical behavior with the fluid and this was the start of the modern critical phenomenon. When this was recognized in a systematic way for that we need a model of magnetism. This model here is some kind of Vanderwaal’s what we used here and then we fixed this Vanderwaal’s and corrected it by adding the tie line and so on and so forth. But all sudden done a liquid is a very complicated system and then it depends on the interactions during a detailed theory of liquids is quite hard.

We are not interested in that, we are interested in finding out what a critical behavior is? So, let us do it by looking at the fluid magnet analogy, but before I build the analogy I have to tell you what a magnet is? And refresh your memory about what a Para magnet is? So, let us look at very simple model Para magnetism to start with then we go on to Ferro magnetism.

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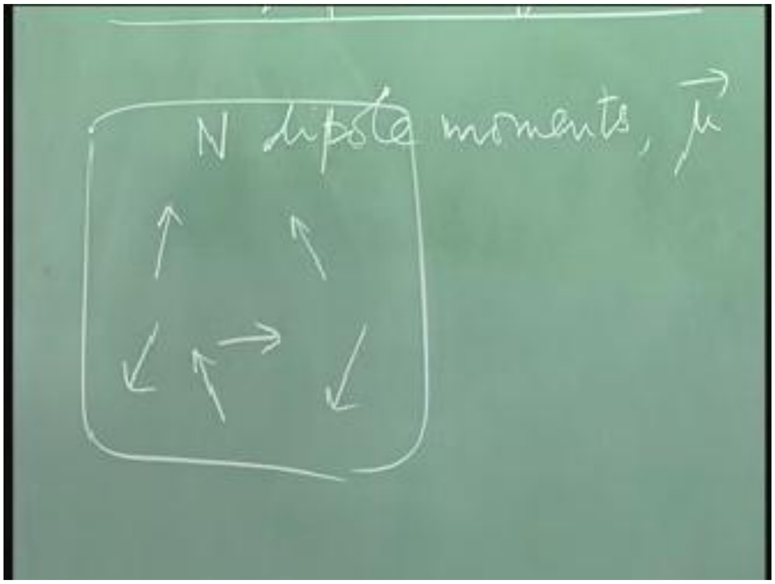

The naive ways of looking at things we are going to assume that the system, we are looking at consist of elementary magnetic dipole moments. We know that many atoms have permanent magnetic dipole moments or whatever it is the reasons are quantum mechanical. We will come back to that in the next course at the moment, we are only concerned with the fact that you may have a system, which has a lot of magnetic dipole moments oriented at random in the absence of external magnetic field. The moment you put a external field on these dipole moments tend to orient themselves along the direction of magnetic field, because that is what minimizes the potential energy and then you have some kind of possible order. So, what is our model? Our model consists of N dipole moments atomic dipole moments.

So, you have one of them like that one of them pointing like this and so on. They are all independent of each other and in the simplest approximation; we assume that these fellows are all sitting in a system in a thermal bar at some temperature T. For example, it could be in a solid and then the solid is maintained at a particular temperature and these dipole moments are oriented at random completely. That is the initial state and it is supposed to be in thermal equilibrium. The energy of a dipole moment once I apply an external field H this is what I call magnetic field, it is actually the auxiliary field.

I should talk in terms of the magnetic induction vector P, but you know that in linear materials these are related. So, let me call it just H modulo of some constants.

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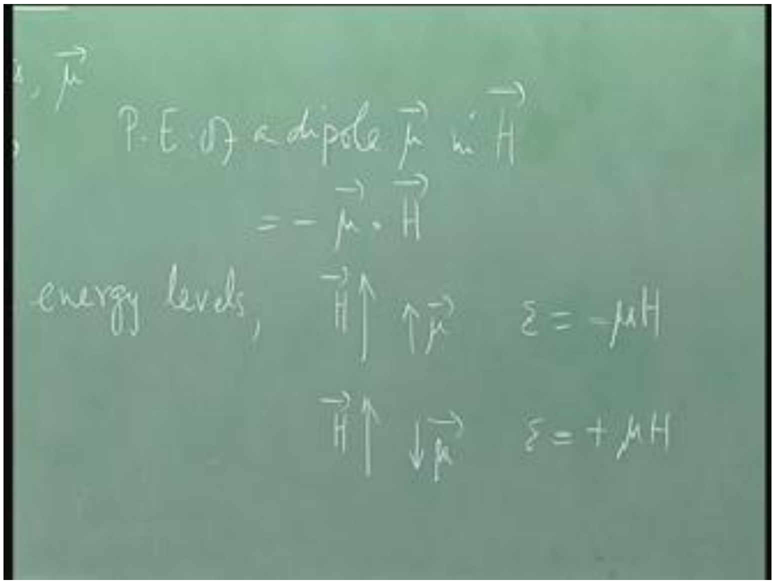

, $\vec{\mu}$

P.E. of a dipole $\vec{\mu}$ in $\vec{H}$

$= -\vec{\mu}\cdot\vec{H}$

energy levels, $\vec{H}\uparrow \quad \uparrow\vec{\mu} \quad \varepsilon = -\mu H$

$\vec{H}\uparrow \quad \downarrow\vec{\mu} \quad \varepsilon = +\mu H$

Then the potential energy of a dipole mu in a field H is equal to minus mu dot H. So, it is clear that if it is along the field, you have the least amount of energy minus mu H and if it is opposite erected in this fashion energy is maximum plus. Now, the system has put in thermal equilibrium with a heat path and you ask, what is the actual magnetization along the direction of the field, the average value of the magnetization? We can do this using the statistical mechanics we do it in number of ways but let’s look at the simplest case possible.

We make the assumption that this dipole moment can either be along the field or perpendicular or anti parallel to it just two possibilities. Simplest assumption in other words mu dot H is mu H cos theta and I assume that cos theta is a plus one or minus one. The reason I do this is because later we will see that some of these in many cases is magnetic dipole moments would arise from for instance. The magnetic dipole moment of an electron, which is related to the spin of the electron? Then, if you measure the spin of the electron along any given direction, it has only two possible Eigen values plus or minus half H cross and the reason for this is quantum mechanical.

So, I am going to anticipate and let us look at a very simple model such that mu dot H is either plus mu H or minus mu H in that case so, two energy levels. So, if the field is like this and the dipole moment is like this corresponds to energy minus mu H. On the other hand if it is in this form this epsilon plus, we will relapse this assumption and assume all possible angles and so on later on. But to start with just to get our bearing state this would be a very simple model, what happens next? This immediately implies that the probability well we can compute the magnetization directly.

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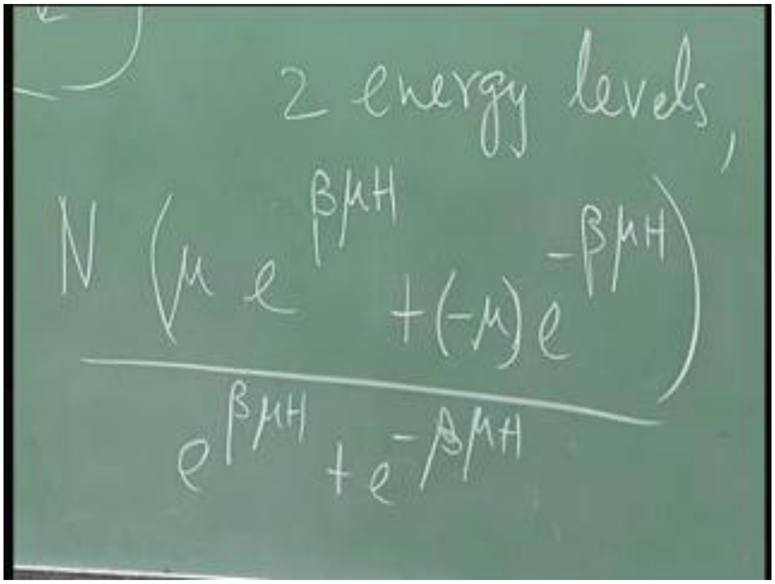

So, let us just do that the magnetization M this is the average value of the magnetization thermodynamic quantity M. Since, I am only interested in the magnetization along the direction of the field always then we call it a scalar number. This is equal to since these are all independent of each other the net magnetization just adds up there is no interaction between these dipole moments. I have not switched on any dipole interaction. This is equal to n times the average magnetization of each of these dipole moments well, if the dipole is in this fashion up there. When it contributes an amount mu, but the probability with which it does show is proportional to e to the power minus beta times. The corresponding energy that was our rule in canonical, the energy was minus mu H.

So, this becomes e to the beta mu H divided by a normalization, which we are going to do a partition function. We will put that in plus, if the magnetization is in the opposite direction it contributes an amount minus mu and the probability of it is e to the minus beta mu H divided by the total probability. There are only two possibilities this and that therefore this is just the sum of these two terms e to the power bet m u H.

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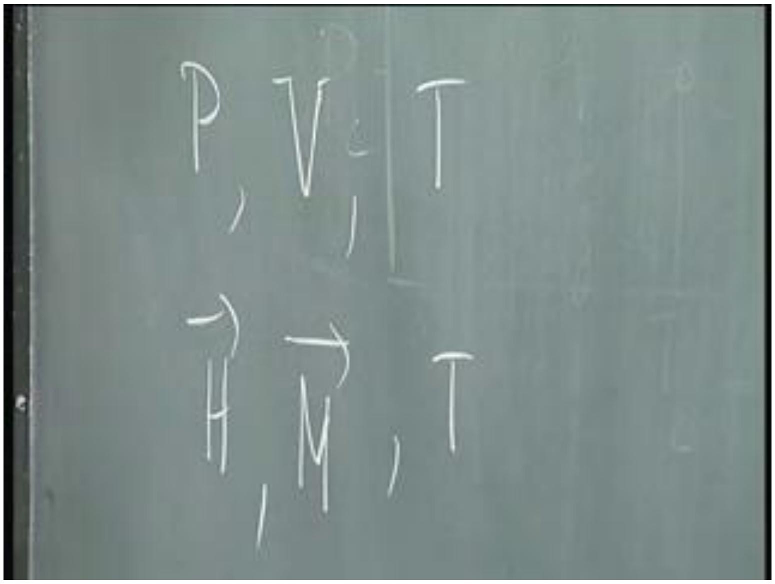

So, coming back here we will get back to these curves little later, it says that the average magnetization of the sample of N elementary dipole moments M equal to N mu tan hyperbolic beta mu H, because it is this minus that over this plus that and that is equal to sine hyperbolic over cos hyperbolic and that is the tan hyperbolic. This gives us a magnetic equation of state, because as you can see this is equal to N mu tan hyperbolic over k Boltzmann. I call the magnetic equation of state, because the fluid case. You have a pressure a volume and a temperature and in the magnet case you have the temperature of course instead of the pressure. You have the magnetic field H and the response of the system as you change the magnetic field is a magnetization. So, this is the analogy this is the way you go from one to the other and it is also clear. Just as P dV is the amount of work done is the contribution to d U distance that appears in the first law of thermodynamics.

Similarly, you have a contribution here instead of the form H T M. If you do a Lagrange transform it becomes M D H, but it is like V D P and P D T. So, this is the analogy we are going to use the pressure is replaced by the magnetic field. The volume by the magnetization and the temperature remains as it is the relation between P V and P is in equation of state. Here is a relation between M H and T and you need to know how much of the material is there and that is done by the number N of magnitude.

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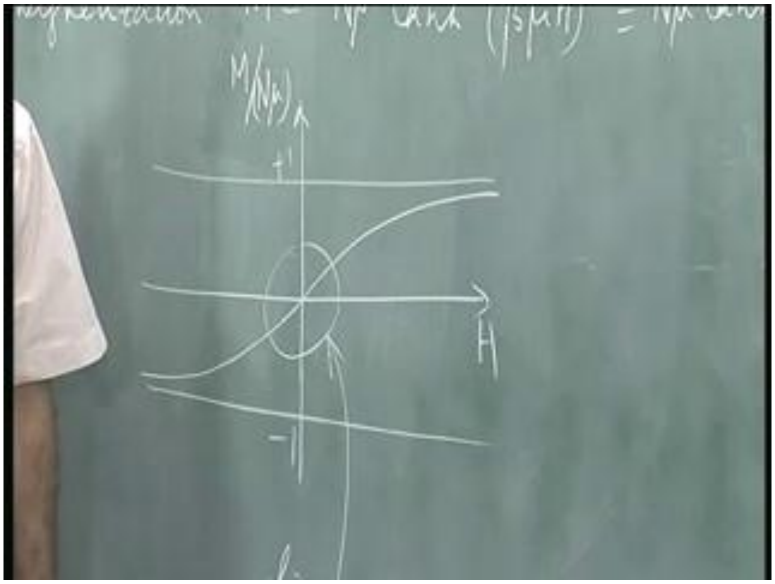

This is our simple equation of state magnetic equation of state for a Para magnet substance whatever consequences well you can plot one against another. So, let us do that of course we have to be careful to realize that unlike P and V, which have to be non negative M and H could be negative or positive, that is along the given direction will be opposite to that plus or minus. So, if I plot let us look at the M versus H diagram so, I plot H here and M here and what this curve looks like well. I can normalize it by saying that let me look at the magnetization per atom or per dipole moment and the unit is mu. So, I write this for convenience as M divided by N mu and then it is just the tan hyperbolic function as a function of H. Now we know that the tan hyperbolic function cannot exceed one in magnitude.

So, this is minus one that is plus one will not exceed that and for small values of x tan hyperbolic x is proportional to x. So, this thing here is proportional to mu H over k T therefore, there is small linear region. But then it goes and saturates that side and then on this side and that saturates to minus one and this region is in linear region M increases proportionally only in that region. So, that it becomes non linear and it saturates what is the definition of susceptibility. How do you define the magnetic susceptibility call it high. So, what is the definition of the magnetic susceptibility so, we used this definition right M is equal to M divided by H M divided by H M equal to the H proportional etcetera.

This is the definition of susceptibility that you learn in high school, but that is not quite right, because it is clear here. That this is a non linear graph so, dividing M by H does not make much sense as H increases it simply goes to zero. Because M remains fine and h becomes infinity. So, that is not the right definition of the susceptibility this is a linear relation it assumes that M is proportional to h, but that is only true near the origin. So, the correct way of saying it is it is equal to M divided by H in the linear region in this region, but then where does this region stop. How do I decide where it stops pardon me to what accuracy do I want it?

I want it to arbitrary accuracy I want it till arbitrary accuracy. So, what should I do? What take the limit what is the meaning of taking the limit? What is the operation which takes the limit takes a little M and you take a little H and then you take the limit. You differentiate it to find the slope, where slope at the origin.

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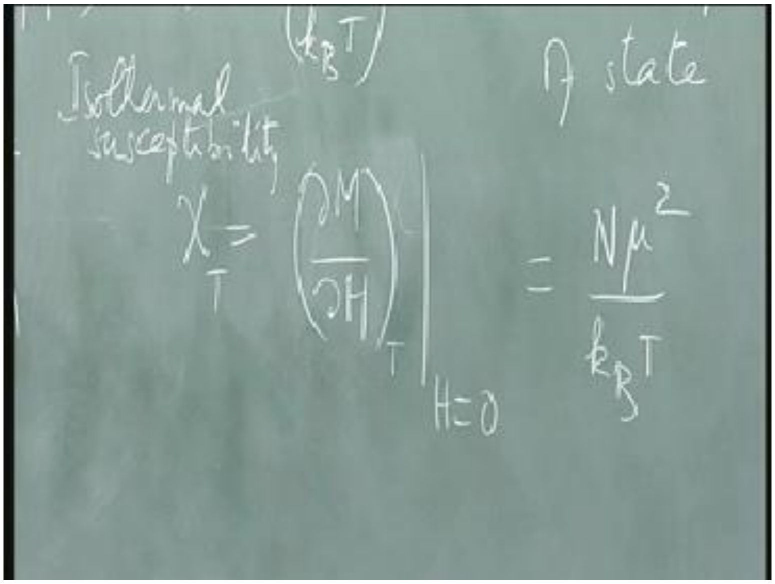

So, the right definition of the susceptibility this equal to delta M over delta H partial whatever, we are supposed to keep constant the temperature constant at a given temperature. So, we keep the temperature constant and of course we keep the number of particles constant also just like that. As we defined the compressibility as delta V over delta P so on. Keeping temperature constant, but we call that the isothermal compressibility we have kept the temperature constant.

So, what should I call this the isothermal susceptibility is delta M over delta H evaluated at H equal to zero. That is important only then does it make sense. It is the slope of this graph that is all it is, but what is that equal to present equation of state? So, what is that equal to I will plug it in so, what is that equal to what is tan hyperbolic x? Look like as x goes to 0 goes to x. So, what is tan hyperbolic ax look like, where a is constant so, what is the isothermal susceptibility? It is equal to N mu square over k Boltzmann. If I define it per unit, per atom or per magnetic moment dipole then I divide by M whatever it is. That is not the relevant part the relevant issue is that it is inversely proportional to the temperature.

What do you call that law the fact that the Para magnetic susceptibility is inversely proportional to the temperature? It is called Curie’s equation. This is Curie’s so; you have correctly got Curie’s equation, which was this Curie spear, who for the first time recognized first person to recognize the very important thing, which became very important later on. The role of symmetry in deciding the properties versus materials role of symmetry in deciding, how condense matter behaves? Now, that of course later on became a very big subject and today, of course it is combined with many other things. We talked about broken symmetry and the role of symmetry a little later.

But, this was the person who recognized Curies law he met a tragic end early in his life. Stepped out in cold Paris in afternoon and his scarf got caught under the wheels of the horse carriage, which was passing by at this time later on. He never recovered from this for long time, so many other things you did, but the fact is that responsible for understanding the role of symmetry first.

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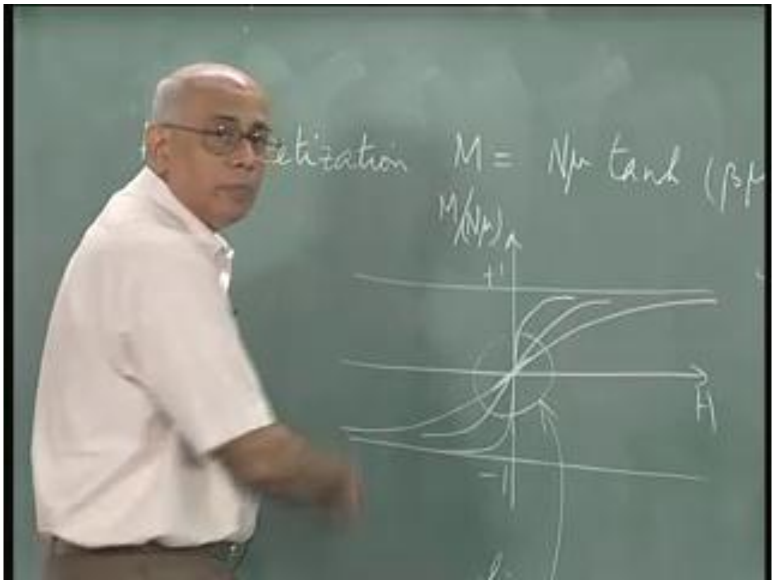

So, we have Curie’s law, which goes by one over T. Now, what is that imply as you lower the temperature? What would happen? What would happen to this curve? It would still saturate, but the slope increases. Therefore, it would start doing this and theoretically, what would happen, when you go to absolute zero? At absolute zero, this curve would start doing this it would simply become an extremely sharp curve like this.

Does that happen with real materials most often know, it does not do that, because many other things start happening and what is the one thing that you think on physical terms is going to happen that is what you neglected something there? We neglected the interaction between the dipoles completely. Now the reason we have got away with it is, because at sufficiently high temperatures imagine this guy is inside a lattice and it is being kept at temperature T. So, it is being kept around due to the vibration of the lattice that is the heat path. In this case then of course if that kicking k t is much larger than the typical interaction energy it is ok, because these fellows do not see each other's interaction very much, but as you lower the temperature and k t becomes very small. Then the interaction between these dipoles becomes more and more favorable and all phase.

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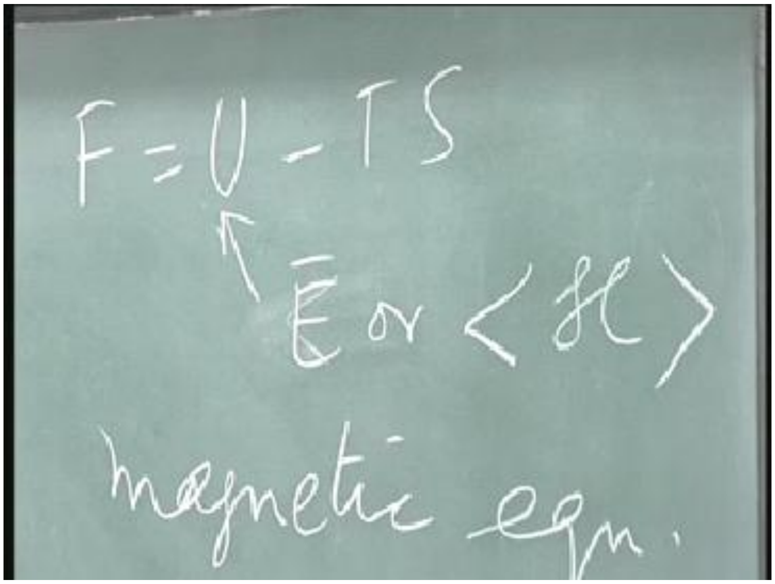

Transitions of this kind are dependent on the fact that some kind of free energy is minimized and if you look at the free energy f which is U minus T S U. As we have seen in statistical mechanics is just the expectation value the average energy is E bar or the expectation value of the Hamiltonian. I will use the script H for the Hamiltonian, because I have used H for the magnetic field. So, there is some interaction between the two constituents of the system. That is sitting in H the script H the Hamiltonian and this here tells you what is going on in real life? It so, happens that at sufficiently low temperature the ordering tendency of the internal energy tends to minimize potential energy overcomes the effect.

So, F is dominated by this quantity as you go to lower and lower temperature the T becomes smaller and smaller and S also becomes smaller, because S is k log omega and omega is number of accessible microstates and that is decreasing as the temperature decreases. So, in the low temperature phase you would expect the internal energy to dominate and decide the minimum of three energies. In general that could be an ordered phase at high temperatures. On the other hand the effect of this is overcome by the effect of the very large accessible microstates and a increasing temperature. So, that the entropy term governs the free energy behavior of the free energy and that is generally a disordered state.

This is the reason why underline reason why in general you find that the high temperature state is a disordered state. The low temperature state is ordered state it is just the competition between U and T S the Hamiltonian does not change the interaction between the independent constituents does not change. The interaction between two water molecules remains exactly the same. Whether it is a steam phase or a water phase or it is a solid ice phase does not change. What does change? However is the number of accessible microscopes these changes and the average energy change internal and external?

So, this is important for the member that is not the interaction is changing the fundamental interaction does not change at all. It is simply the effect of statistics as the temperature goes and of course many complex possibilities can occur. We will see some of them, but this is something we have to keep at the back of our minds. So, it is clear that in this case that the interaction is switched on. Then this cannot be the picture right down the absolute zero. There would come a state, when the effect of these interactions between these moments will start dominating and then what could you expect?

If it is dipole what would you expect? I have a dipole moment here. I have a dipole moment here and their interaction must be taken into account. What is the minimum energy configuration for this type both are like this? It is clear that parallel, anti parallel is lower potential energy is like this. This is certainly true for a dipole interaction; it is more complicated than that, because this interaction depends on the directions of the two dipoles at a given distance. This has one energy, that has another energy, and this has yet another energy and even if these two are parallel, this energy is different from this energy.

So, it is clear? That this interaction is not isotropic, it is going to depend on the interaction of the directions of both these guys and so on. But in a naive way, if you say this and this in two magnets bar magnets. It is clear? That the dipole interaction goes one over r cube kind of things mu one dot mu two over r cube to start with that would favor this another one. After that would go up, and then down, and then you end up with a configuration, which you will have an up and down and an up and down on an average the magnetization will be 0. So, if you relied on the dipole interaction inside the bar magnet, you will not have a bar magnet at all. These atomic dipole moments were anti parallel would order such that it is parallel and is finished.

So, what do you think is happening? But we do see permanent magnets, what do you think is responsible is a clear indication that the phenomenon of magnetism is more subtle than one imagines permanent magnetization. It is not connected to the dipole interaction. It is connected to a quantum mechanical effect called the exchange interaction, which actually favors this over that and in fact overcomes this interaction the dipole classical dipole. It is a very strong dipole interaction so, nearest neighbors would tend under favorable conditions to align parallel to each other and this interaction switches off exponentially fast.

So, magnetic dipole moment here another one here, the interaction would die down exponentially with a distance whereas, the classical dipole interaction would favor anti parallel alignment. And would be a long range interaction, because it goes like one over r cube distance, but it is a very week interaction. There are dipole or magnets, there are magnets, where this becomes an important affect and so on. But in general the naive picture this would not be to Ferro magnetism to anti Ferro magnetism. So, therefore we will keep that also in the back of our minds that underlying some kind of interaction. We must include which favors parallel alignment and that happens we will see how that happens?

It turns out that this picture is not quite right or rather you do not have to wait to absolute zero. For this to happen at a finite temperature called the Curie temperature, it happens and below that the picture would in fact be this jump this continuously and you have a phase transition. But for the moment I want to call attention to the fact that we have not switched on interactions. And therefore you are still in the Para magnetic phase, which means that if you switch off the field the magnetization goes to zero. Now let us relax this assumption that we have made that the magnetic moment can be either just be parallel to the field or anti parallel, assume that it can be in any direction.

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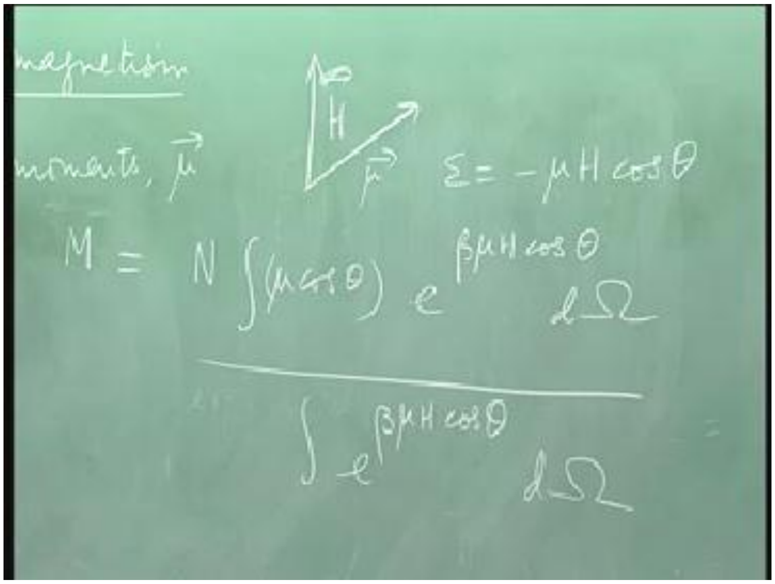

Then what does this magnetization look like? (( )) to compute. What does M look like in this case. This is equal to… Now remember that if I have a field in this direction and I have a magnetic moment dipole moment. In this direction, the energy epsilon e to the minus cos theta minus mu dot H; and now, if I choose the direction of the magnetic field to be along the polar axis, then spherical polar coordinates the field can the magnetic moment can be in any direction. What so, ever and where ever it is the angle between the two is theta, that is the polar angle the interaction the energy is minus mu H cos theta. Now let us compute the magnetization what is this equal to and I am interested in the magnetization in the direction of the field.

So, let me just put a scalar here take it to the z, what does this become? It is equal to again there is N, because there is no interaction mu times. This is the contribution that this guy would give along the direction of the field. The contribution will be mu cos theta times the probability that the angle is there, that is equal to e to the power beta mu H cos theta. Now, you have to integrate over all theta, you have to integrate over all directions and normalize. So, this is equal to an integral over all solid angles all directions divided by an integral e to the power beta mu H cos theta. That is the average magnetization. I have not put these angular brackets, because I have already used a capital letter M, it is a thermodynamic variable so; there is N factor here. This is the contribution from a dipole pointing like this to the magnetization. In this direction that is mu cos theta multiplied by the Boltzmann factor e to the minus beta times the energy, which is equal to minus mu H cos theta integrated over all possibilities and normalized.

You recognize this quantity is in fact, if I raise it to the power N, it is the partition. Now what is d omega, I have included the magnetic interaction, I am not looking at the kinetic energy or translation energy of these dipole moments. Only the magnetic portion that is all I am interested in. Now what is this equal to? What is this kind? What can we do this with you? We should do this integral what is d omega? This is equal to…

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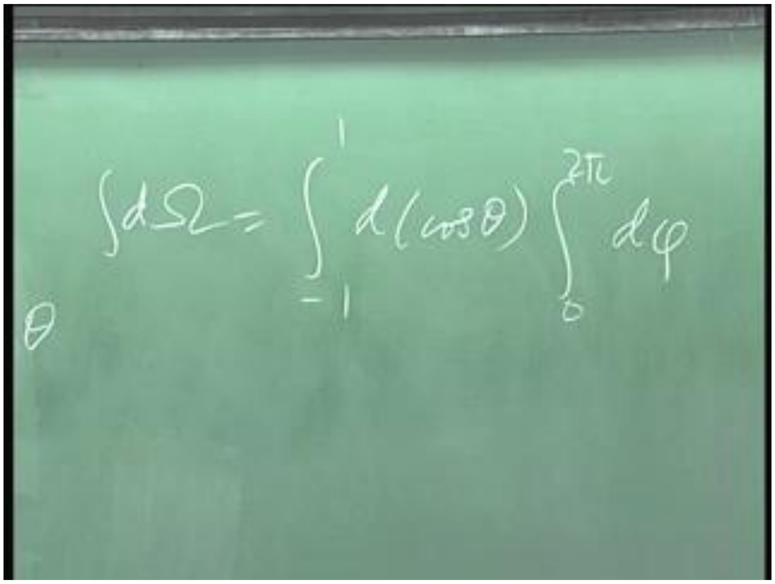

$$\int d\Omega = \int_{-1}^{1} d(\cos\theta) \int_{0}^{2\pi} d\varphi$$

So, integral d omega really stands for integral d cos theta minus 1 to 1 0 to 2 pi. It is sine theta d theta from zero to pi, but I like to write it as d of cos theta from minus one to one, that is what it is makes it easier? So, let us do that.

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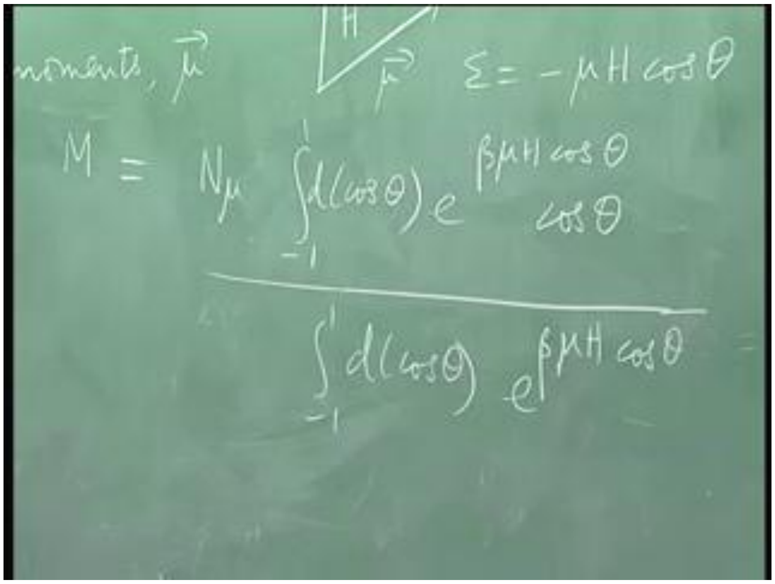

$\varepsilon = -\mu H \cos\theta$

$$M = N\mu \frac{\int_{-1}^{1} d(\cos\theta)\, e^{\beta\mu H\cos\theta} \cos\theta}{\int_{-1}^{1} d(\cos\theta)\, e^{\beta\mu H\cos\theta}}$$

This gives a 2 pi factor from the free integration integral minus 1 to 1 d of cos theta. There is N mu and then there is a cos theta divided by the 2 pi, the 2 pi cancels out and then you have minus 1 d cos theta e to the beta mu H cos theta. This is the integral and what does that work out? What does this guy work out?

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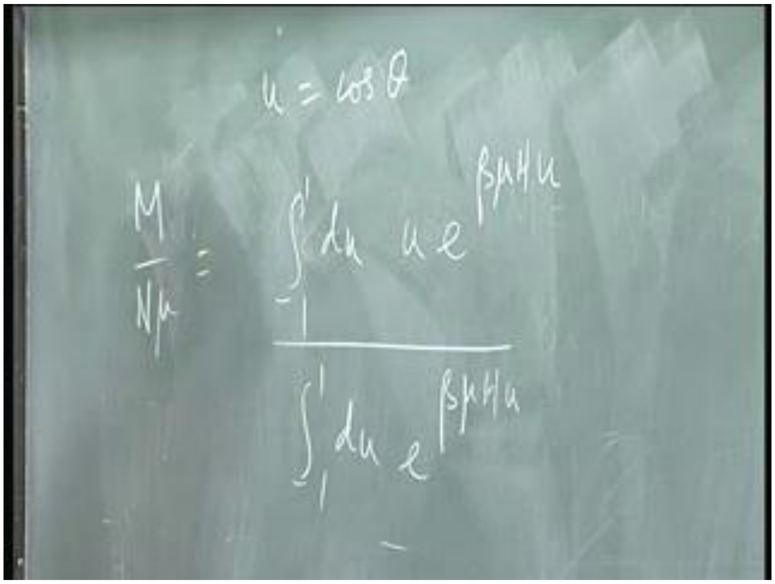

$u = \cos\theta$

$$\frac{M}{N\mu} = \frac{\int_{-1}^{1} du\, u\, e^{\beta\mu H u}}{\int_{-1}^{1} du\, e^{\beta\mu H u}}$$

So, it is M over N mu equal to this thing minus 1 to 1 d cos theta, by the way can I equate this to zero on the top? Because it is minus 1 to 1 d cos theta and there is a cos theta sitting here. Yes. It is an odd function. Can I equate it to 0? No, because this guy is sitting here. That is a mix function, I cannot do that right. So, I have an integral let me call that let U equal to cos theta so minus one to one d u e to the beta mu H mu divided by integral minus one to one d u e to the beta mu H mu, because once you do this integral and then say this signal is the derivative of this denominator with respect to beta mu H.

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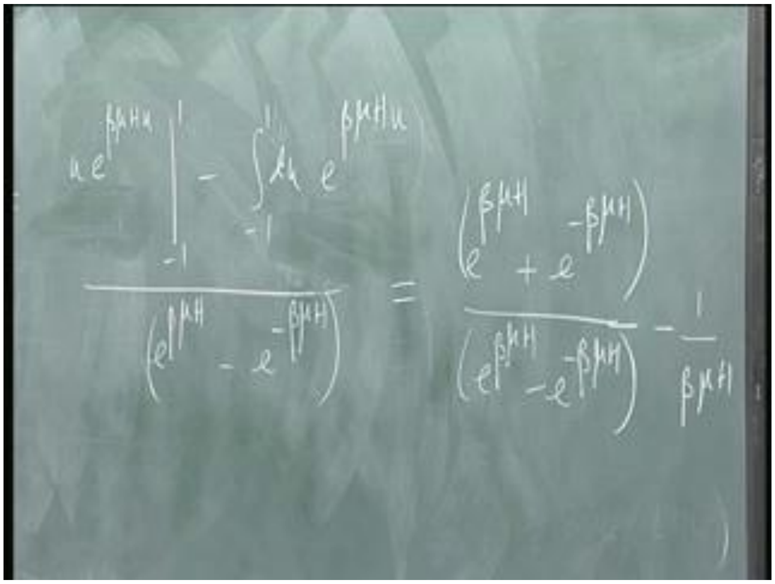

$$\frac{u e^{\beta\mu H u}\Big|_{-1}^{1} - \int_{-1}^{1} du\, e^{\beta\mu H u}}{\left(e^{\beta\mu H} - e^{-\beta\mu H}\right)} = \frac{\left(e^{\beta\mu H} + e^{-\beta\mu H}\right)}{\left(e^{\beta\mu H} - e^{-\beta\mu H}\right)} - \frac{1}{\beta\mu H}$$

Let us just do the integral so this is equal to integrate by parts. Therefore, you have u e to the beta mu H mu over beta mu H and slow at this minus 1 to 1 minus integral minus 1 to 1. By I can do this integral directly e to the beta mu H minus divided by beta mu H, which cancels. This is equal to plus minus, if I integrate this. I get exactly the same null set, but this factor and that is equal to cot hyperbolic. That is this is cos hyperbolic over sine hyperbolic so, it is cot. So, this is the function we get its different from the function we got earlier, which was a tan hyperbolic whatever it is.

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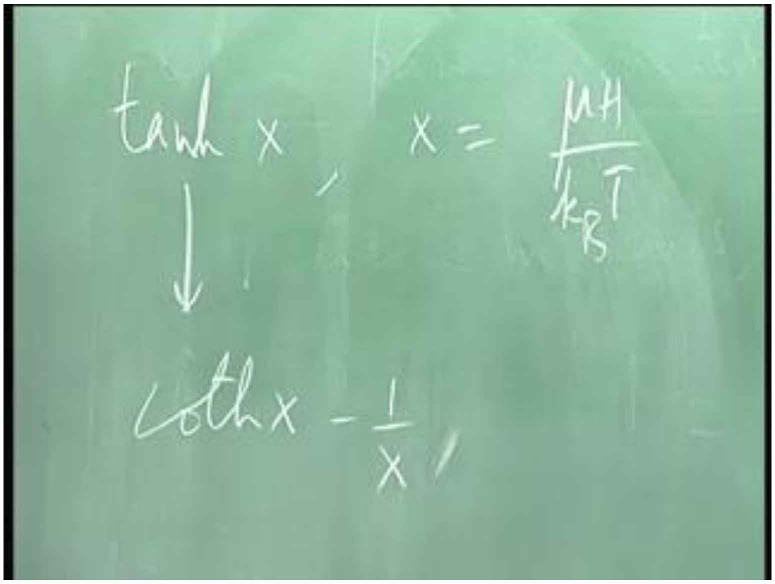

So, let us compare these functions earlier, we got tan hyperbolic some x where x equal to mu H over k Boltzmann T. That has simply been replaced by cot hyperbolic x minus one over x, where x is the same value mu H over k Boltzmann T. Now the time hyperbolic function we plotted we found out from minus one to one and pass through the origin linear, but this is a different story. We have to plot this function as a function of x.

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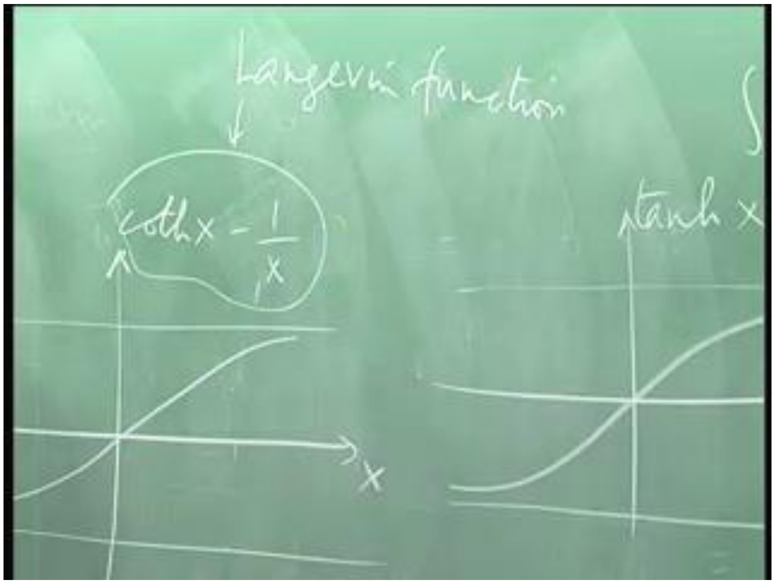

So, what does this look like? Here is x and here is cot hyperbolic x minus one over x. What does it look like? What does it look like Does it diverge is it finite at the origin. What does it look like? If tan hyperbolic x goes like x near the origin cot hyperbolic x must go like one over x and that neatly cancels the singular part cancels and you are left. Now, with rest of it and that is a regular function and what would it do well tan hyperbolic x saturates to plus minus one at infinity? We call the way this function looked as x saturated term like this and therefore we got x versus cot hyperbolic x.

This stayed out all the time it went like this so, now both are odd functions and now you have to subtract the one over x part. And when you do that you discover that once again you have a function, which behaves pretty much like this. And this function has a name it is called the Langevin function occurs again and again Para magnetism. It is called the Langevin function. Langevin was the first person to give the theory of Para magnetism he is a friend of great physicist, friend of Madam Curie's, and one of the founders of the subject non-equilibrium statistical mechanics. So, this is what the function looks like? Pretty much same as before except we got to decide, what the slope of this function at the origin is?

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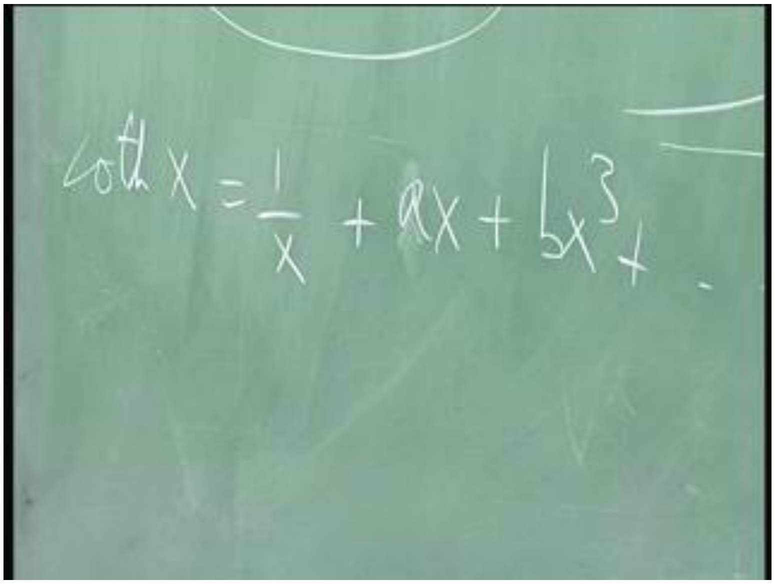

$$\coth x = \frac{1}{x} + ax + bx^3 + \ldots$$

Now, remember that cot hyperbolic x blows up at x equal to 0 and it is an odd function in singularities of one over x. So, it must go like one over x plus, what is the first correction going to look like. Near x equal to 0 and then you have to tell me what is going to be the next term near the origin. This is certainly one over x it will cancel the minus 1 over x and what is the next term? What will it be proportional to what power of x it will be proportional? It is not more singular so, it cannot be one over x square one over x cube or anything like that x. It cannot be proportional to x to the 0 why odd function.

It has to be proportional to x could be proportional to x cube, but that will be an accident. If it started so, the next term is in fact proportional to x. Some a x plus may be b x cube plus dot dot dot A and B and so on are real constants could be positive negative. We do not know. Would you expect that to be positive or negative? A positive what is a can we find out and of course do this painfully?

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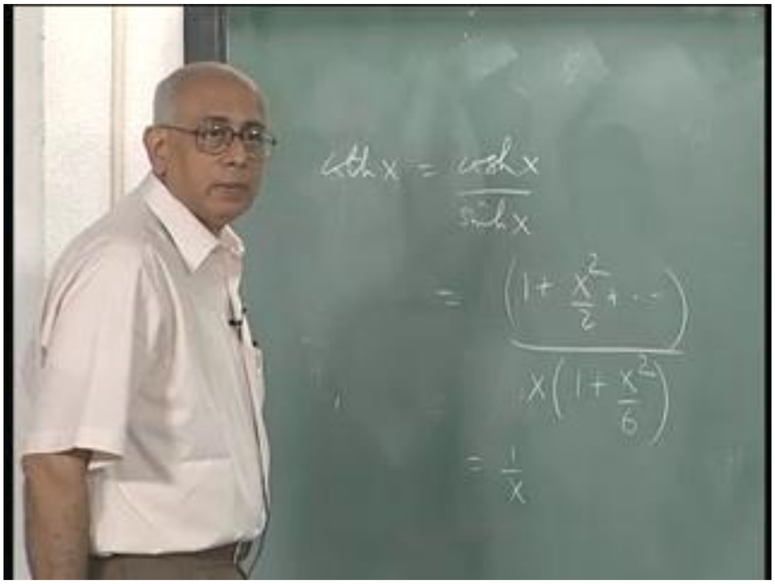

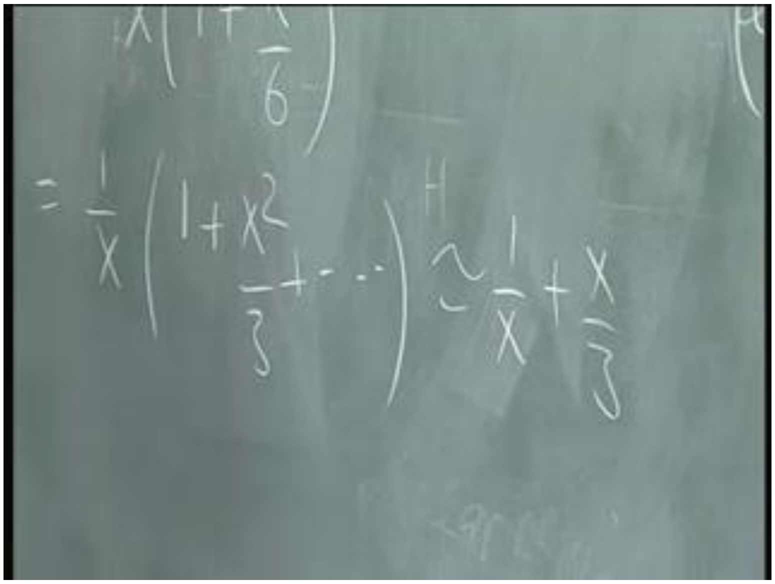

So, let us write cos hyperbolic h equal to cos hyperbolic x over sine hyperbolic x. Now what is cos hyperbolic x power series in x 1 plus what is the definition of cos? Cos is one minus x square by two factorial plus x over 4. So, what is cot x one plus everything is plus one plus x square by two and so on. So, this is one plus x square by 2 plus order x 4 and what does sine hyperbolic x do x plus x cube over three factorial etcetera. So, there is x, which I take out then this becomes one plus x square by six plus higher terms. So, this whole thing goes like 1 over x times this is 1 plus x square by 2. This is 1 plus x square minus 1 so that is one minus x square over six to start with therefore, you get one plus half x square minus one sixth x square and what is half minus one-sixth one-third? So, this goes like 1 plus x square over three plus higher order terms and therefore this is 1 over x plus x over three plus higher order terms.

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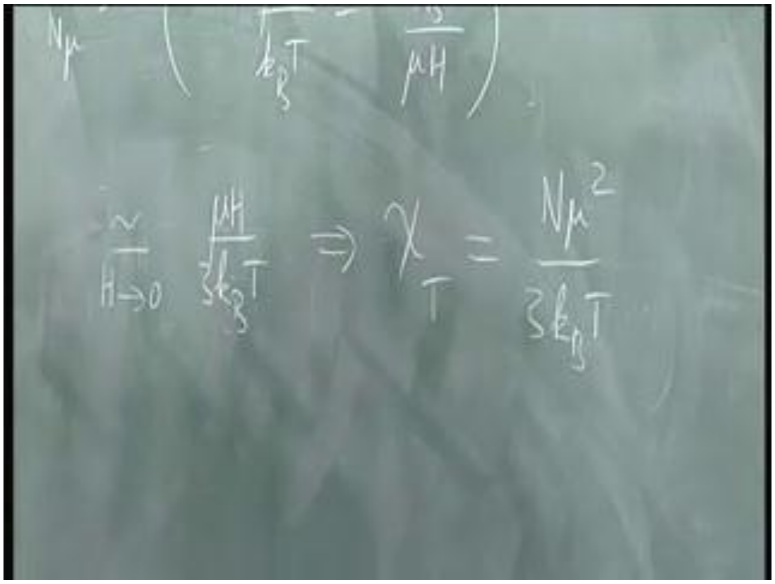

Therefore, this quantity here, now that we found magnetization lets write it down M over N mu equal to cot hyperbolic mu H over k Boltzmann T minus k Boltzmann T over mu H. This quantity is (()) function goes as H goes to 0 to what term? What is the leading function leading quantity? It goes like mu H over three k Boltzmann three, which came from half minus one sixth that came one over two factorial minus one over three factorial. It came from the exponential function and turned out to be this. So, this would imply that the isothermal susceptibility, in this case is equal to one third of the earlier N mu square over three k Boltzmann T.

How come we got a one k Boltzmann T initially and now we get a three what is going on? It is a different model. We assume first it goes up or down and now we are saying that it can be in all directions and so on. So, what has changed the number of dimensions has changed? We have said that this magnetic moment lives in one line go either up or down. Now, we are saying it can live in three dimensions so, really this one, this three has to do with the fact. That you are in three dimensional spaces, although it came from someone over 2 factorial minus 1 over three factorial really is the reflection of the fact that you are in three dimensions.

You trace it back carefully you discover that this guy here, the Curie law comes from the number of dimensions in which you have allowed magnetic moment to move a test would be to ask. Now, we did it for one, we did it for three what should we do it for 3? Do it for two. We will say what if the magnetic moment moves only in the vertical plane or something like that a plane containing the magnetic field. Let us do that just to see there is a reason for doing all this, because phase transitions turn out to depend these exponents on the properties. Critical properties turn out to depend on two universal two basic objects one is the number of dimensionality of the space.

So, if your system is one dimensional two-dimensional three dimensional things change drastically. The other has to do with whether the quantity you are looking at is a vector or scalar or tenser. What is the number of components it has these seem to be the universal quantities upon, which critical depend independent of interactions. We will see why that is so? So, let us see if our guess is right. Let us assume heck of it see whether we can do this for a planar magnet.

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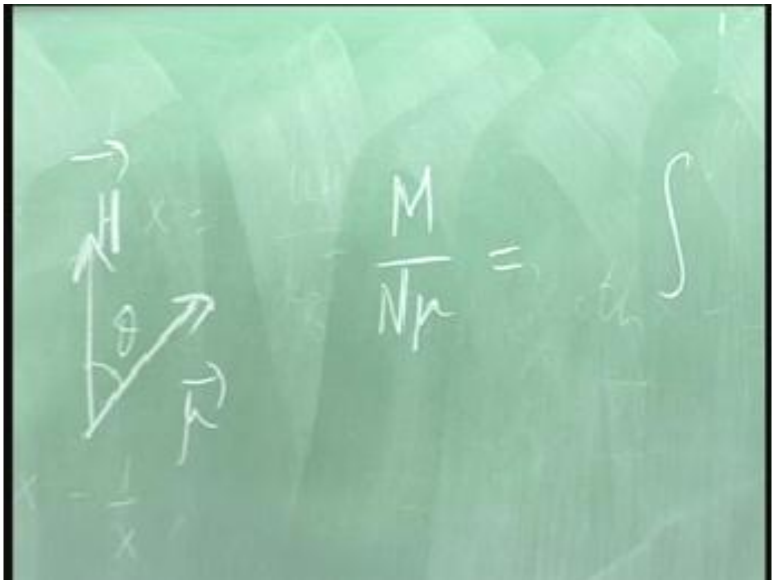

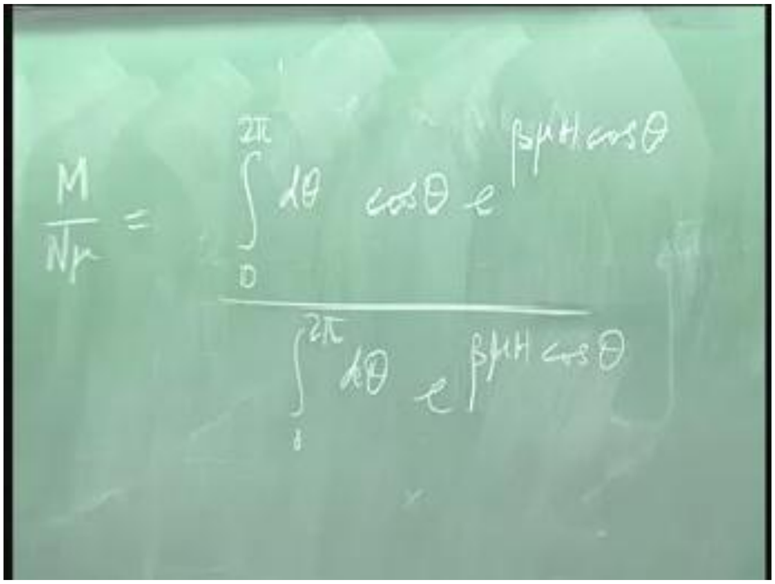

$$\frac{M}{N\mu} = \frac{\int_0^{2\pi} d\theta \, \cos\theta \, e^{\beta\mu H \cos\theta}}{\int_0^{2\pi} d\theta \, e^{\beta\mu H \cos\theta}}$$

So, what would you now say here is H, and here is the magnetization mu and I am going to assume that it moves in this plane and nothing more right. And I would like to find the magnetic susceptibility. I would like to find the magnetization, let us see if we can do this would you expect you can do this?

Yes. We expect we should be able to do this. We did it one; we did in three so, why not in two. So, what should I write M divided by the N mu the average is equal to and now I have to integrate over all possibilities and I integrate. I have to integrate over this angle theta, I continue to call it theta, but what is the range of theta 0 to 2 pi or minus pi to plus pi does not matter, because you are in this plane it is like a plane polar angle term, it is not a polar angle term in three dimensional. So, it is 0 to 2 pi 0 to 2 pi and what is the measure? It is just theta. There is no sine theta d theta, it is just d theta and then cos theta e to the beta mu H cos theta divided by integral 0 to 2 pi d theta e to the beta mu H cos theta. This is the integral we have to do.

That we think we can do this well the denominator looks easier. Can we do it? How to do this integral? We can always do it numerically, if you give me values of beta mu H, I put that in and I can do it, but now I am trying to find out what is the functional dependence. Unfortunately this is not a easy integral to do and the reason is that there is no d cos theta. If there had been a sine theta d theta I would have done it in short, but you cannot do this otherwise. So, what is this integral equal to? What is this? It is a Bessel function this guy here is related to j 0. So, you see already strange things can happen to change the dimensionality. The fact that this measure has gone the sine theta has gone in given dimensional space it made it much worse.

So, it is not a trivial thing one can evaluate it in terms of Bessel functions and so on, but our purpose is not that. I want to find out what the magnetic susceptibility is? And I want to find out if you get N mu square over two k T. That is my target, I would like to find out, but what is the susceptibility? It is the slope at the origin, which means I need to find this quantity to what order N the magnetic field linear. I need the linear behavior. So, let me expand it and do everything to first order. Let us expand so, we put everything and try to expand everything to first order in H. I am only interested in that.

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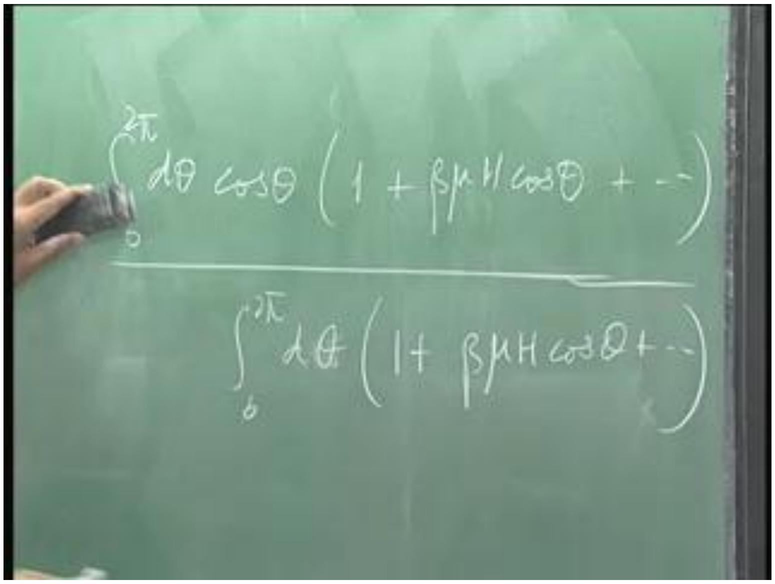

$$\frac{\int_0^{2\pi} d\theta \cos\theta \left(1 + \beta\mu H\cos\theta + \cdots\right)}{\int_0^{2\pi} d\theta \left(1 + \beta\mu H\cos\theta + \cdots\right)}$$

So, this is approximately equal to 0 to 2 pi d theta cos theta and the first term is one plus beta mu H cos theta plus dot dot dot divided by integral 0 to 2 pi d theta and one plus beta mu H cos theta plus etcetera.

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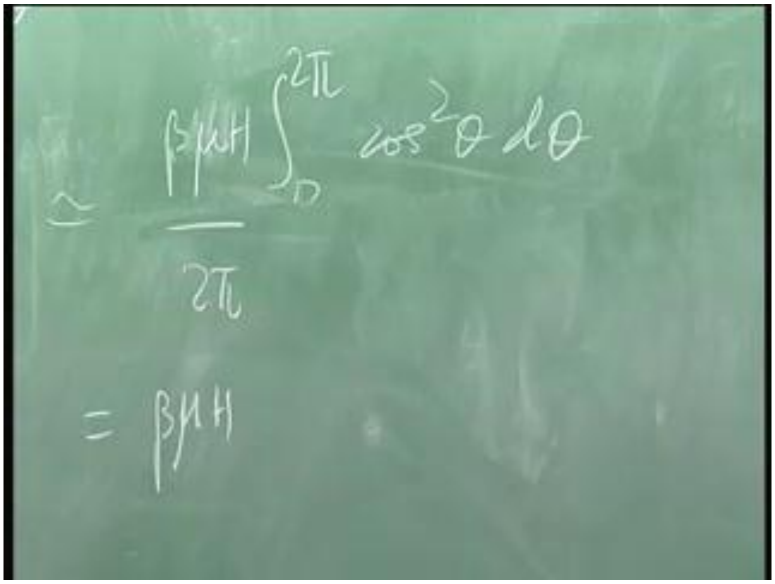

$$\simeq \frac{\beta\mu H \int_0^{2\pi} \cos^2\theta \, d\theta}{2\pi}$$

$$= \beta\mu H$$

What is d theta cos theta 0 to 2 pi that 0? You have to get rid of that immediately and I get a beta mu H and then a cos square theta plus higher orders divided by what is in the denominator? This term goes away d theta the first order term in H goes away and you just get a 2 pi in the denominator, what is this integral equal to? This is equal to beta mu H, if I write this as cos square theta as one plus cos 2 theta divided by 2 and the cos 2 theta parts will just vanish. You just get a two in the denominator, but you also get a 2 pi.

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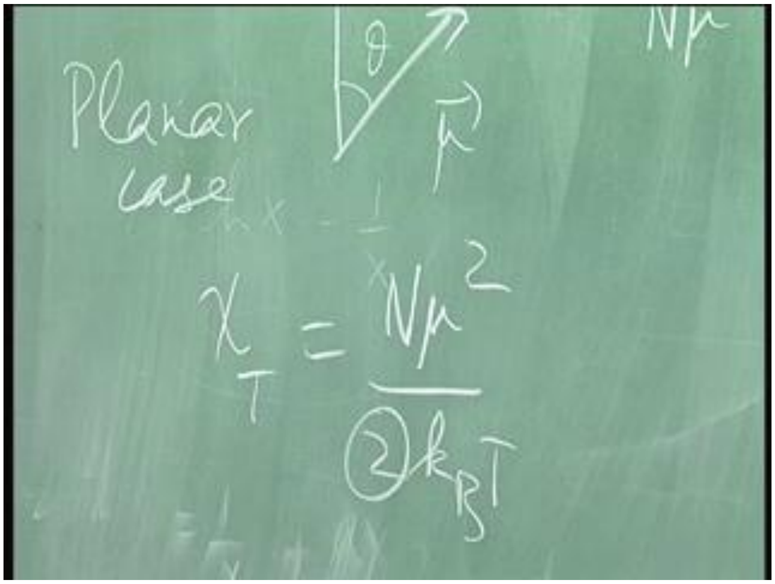

So, indeed in this planar model indeed susceptibility this T is equal to N mu square over 2 k Boltzmann T planar. So, sometimes you have to notice these facts this two this three looks very innocuous and so on. It comes from various places here, it came from cos square theta; fact that it is one plus cos 2 theta over 2 and so on, but underlying this whole business is the fact. That it is really telling you this guy is really telling you how many dimensions the magnetic moment is free to move?

So, that is the reason I did the one dimensional cases the easy model, but it has implications a numerical purely numerical, but this is worth bearing in mind. In all cases it is proportional to one over T. That is the Curie law and that is the law for Para magnetism, but we know it cannot be valid till T goes to 0. So, now we got to put in the fix, we got to put in the fact that you have Ferro magnetism, you have an interaction. How should we do this? You have to write the model for Ferro magnetism that is much harder to do than this elementary models. Incidentally you can ask the following question.

If I really took into account magnetic interactions that quantum mechanical interaction between dipoles, then what does it look like? What would be the interaction? How do I find the Eigen values energy values and so on? You get essentially for what is called spin half; you get the maxima function for higher spins. You get what is called the function, which would have similar behavior very similar. Eventually you would end up with a Curie law some constant multiplying that is not so, important. What is crucial is to ask? How do you change this to Ferro magnetism? And just as in the ideal gas we go from there to a real gas. We have to put in interactions in a realistic way, but we did this by a short cut of saying that this is the phenomenology called the Vanderwaal’s equation.

Similarly, here there is a phenomenon called the Weiss molecular field. It is the analog of the Vanderwaal’s equation and the fix is very simple. You could take any one of these models; let us take the simple one dimensional model.

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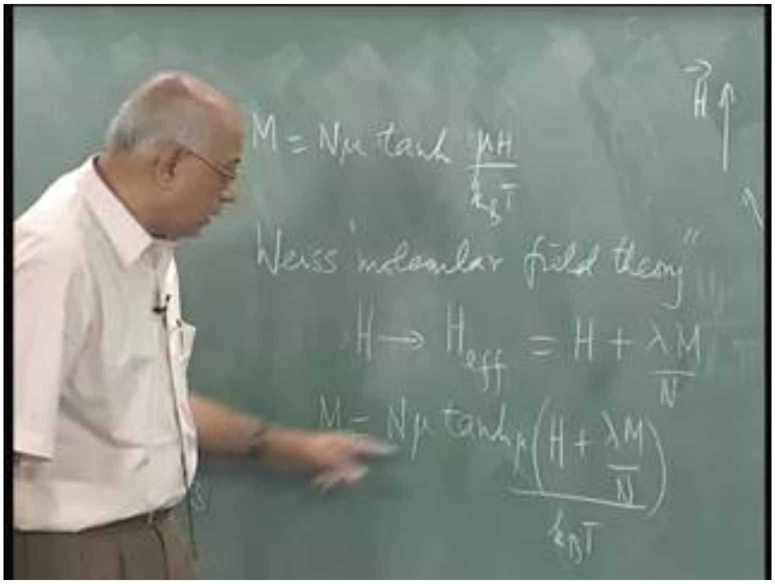

So, we have M equal to N mu tan hyperbolic mu H over k Boltzmann T and the fix consists in the following this is called the Weiss. Remember, all this was done in pre history before we understood quantum mechanics properly. Before Iceberg came out with the idea, that this is a exchange interaction so on. The idea was that in a real medium the Ferro magnetic medium. Each dipole moment sees not only the external field, but also the internal magnetic field. Due to the other magnetic dipole moments and this would act like an effective field.

It gets added to this stage so, it says replace H by H effective, which is equal to H plus. The part which says if this is the external field H and you have a dipole moment here and it is surrounded by other fellows of this kind. These are the fellows produce on your given dipole moment this guy and effective field, which tends to align. In this direction it adds to the external and what is it going to be proportional to it will be proportional to the magnetization itself?

The idea is proportional to how much of it is already magnetized? So, there is a portion which goes like M multiply by some constant lambda, but you have to be little careful in doing. This I would like this lambda to be some constant, which is for a unit can be evaluated, but this H is an intensive quantity supplied from outside. The M is an extensive quantity, if I increase the size of the system M increases. So, I must normalize this M to the magnetization per unit volume or better still per unit magnetic dipole moment. So, let me call it by N mere make it a little intensive quantity like H itself, you should add an intensive and extensive quantity.

If you do then the constants multiplying, it will become of order N. So, I put this in and ask what happens? Now then my magnetic equation of state is M equal to N mu tan hyperbolic plus lambda M by N times mu over k Boltzmann T. That is much harder equation, because it is not an explicit equation for M as a function of H and T, because M appears on this side also. And you cannot solve this equation in closed form for M, because it is a transcendental equation an algebraic function M on the left. There is an exponential function and hyperbolic on the right s. So, no closed form solutions can be given, but you can certainly do this numerically or graphically. And we in fact we are interested in asking what happens to this equation for small values of H near the origin. We would like to see what is the magnetization at all? What is near the origin? What does it look like?

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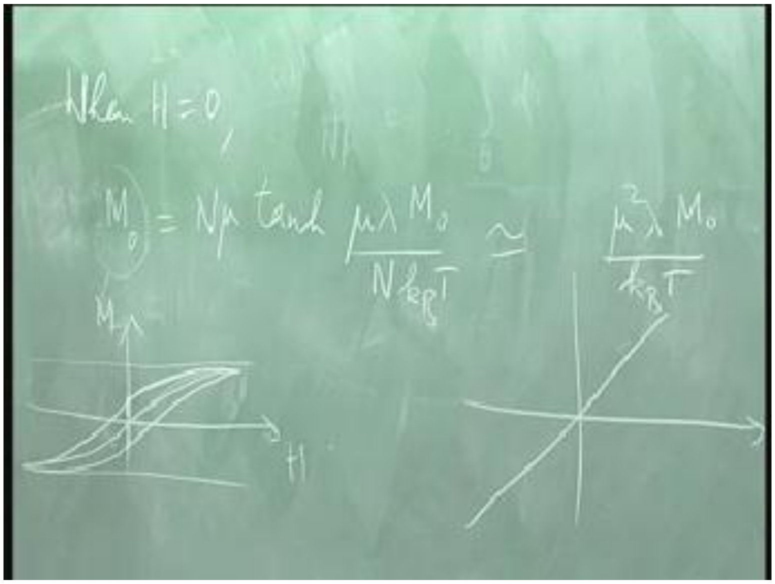

If you do that you immediately realize when H is equal to zero switch it off. Then in Para magnet the magnetization goes to 0, but in a Ferro magnet with this equation of state we have M naught equal to N mu. I will denote the magnetization by M 0, when the magnetization in the absence of the field. I will call it M naught this is equal to tan hyperbolic mu lambda M naught over N, that is it. This is again a transcendental equation and you have to ask is there a solution at all. M naught equal to 0 is always a solution, because the right hand side vanishes the left hand side vanishes and that is the Para magnetic solution.

Because it says in the absence of an external field then magnetization of the sample is also zero. This is how a Para magnet behaves, but a Ferro magnet does not do that. A Ferro magnetic curve has a hysteresis curve in general. So, what it does is to behave in this fashion to plot H versus M from this side instead of the Para magnetic behavior like this. What are Ferro magnetic nulls after you stabilize it is behavior is that you start with a sample and you switch on the field. It will saturate, but then when you switch the field back to do this go there and then it would do this. So, that the hysteresis loops therefore, when you switch off the field from positive side, there is a remnant magnetization. That is what I called M naught and if you do it from the negative side there is a remnant magnetization from the opposite direction.

So, you have to allow for the possibility that there is this solution rather than that solution and that should come out of this equation. Now how do you solve this equation? You would solve it graphically; we will do this next time. What would happen is that? If I plot it M naught the left hand side is a 90 degree, 45 degree line M naught equal to N naught. The right hand side is N mu lambda M naught over N k Boltzmann T, that quantity for M naught approximately 0 is N mu square lambda M naught over N k B T.

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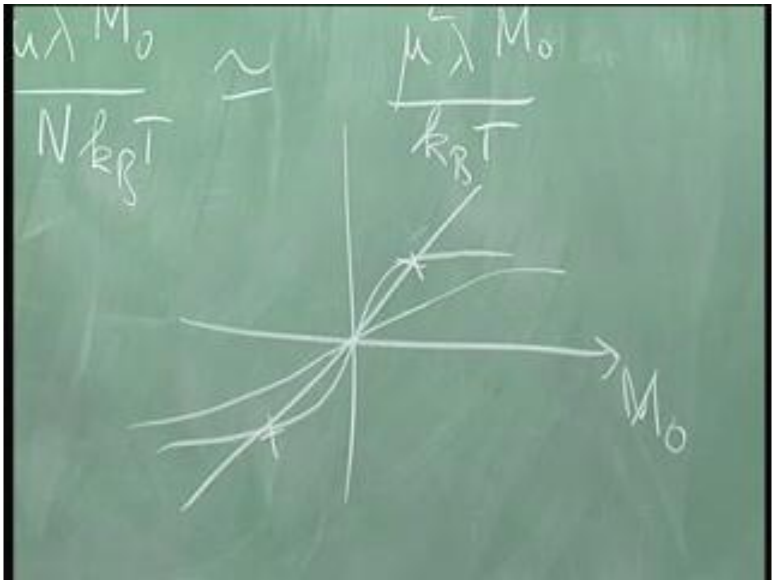

So, it goes like here M naught is equal to 0 the N cancels out and we have mu square lambda M naught by k B T. Now, that is a straight line of slope mu square lambda M naught over k B T. So, if T is very high this slope would look like this and the curve would then saturate, but if T is sufficiently low the curve would look like this. And then saturate and then you have a root here and a root here other than the root at the origin. That is how you get permanent magnetization and when would this happening? It would start happening as soon as the slope exceeds 1.

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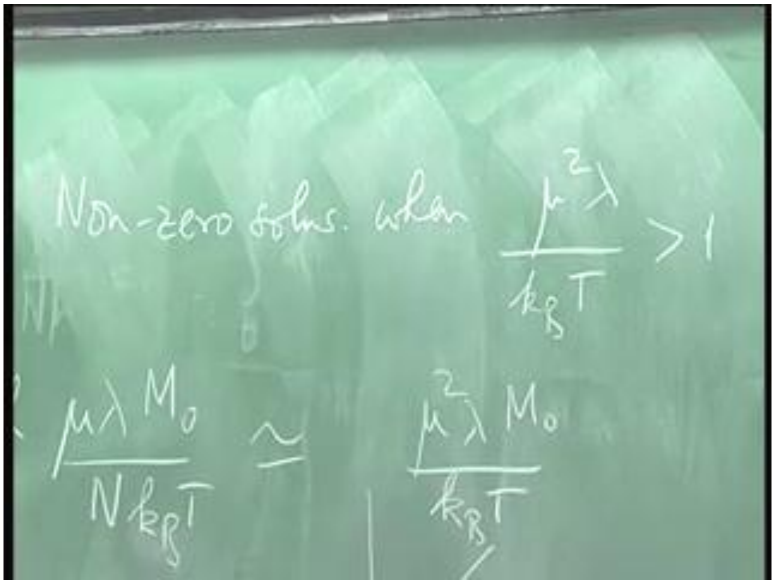

So, you have a non trivial non-zero solution, when the slope mu square lambda over k B T is greater than unity, which implies that T is less than mu square lambda over T. That is your critical point below that you have these new solutions. Then of course you have this new solution, but now we have to look at the free energy and show that solution is unstable and this is how Ferro magnetism appears in a simplest phenomenon. So, the whole idea is below a certain physical below a certain physical temperature. The cooperative tendencies of all these magnetic dipole moments to line up will dominate over the disrupting tendency of entropy or fluctuations and you get order. But above the critical point the disorder wins and the order is the energy interaction is the strong enough to maintain. All this is how the phase condition occurs between the Para magnet and the Ferro magnet? Now, we got to take all this and put it back and see where the connection is with. We will do this in the next class.