Classical Physics

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We have been looking at properties of dynamics dynamical systems, in which you have a prescribe rule of evolution say Hamilton in dynamics or Lagrange's dynamics or maps, discrete time dynamics and so on. Completely deterministic you give a rule and the future is there before you. In principle, the question is whether you can solve these equations of motion explicitly or not numerically you could always solve them for short time and then extrapolate.

But we talk the little bit about incredibility whether you can solve these problems analytically and so on. And the general conclusion is that for instance for Hamiltonians systems you generally do not have a sufficient number of analytic constant of the motion in order to be able to write the solutions down explicitly. The systems are generally non-integrable. The question then arises as to how one should tackle such systems, when we saw that in the case of chaotic dynamics for instance. The idea of an in variant density writing down of equation; and then finding time averages written in terms of statistical averages was a very powerful way of doing this.

The whole idea of statistical mechanics is precisely that, so what we should like to do now is to turn our attention to systems with a very large number of mutually interacting degrees of freedom to understand how these statistical techniques can help us to analyze the properties of successive.

I should say immediately that the subject is not a closed one, we really do not know all the possibilities that can occur when you have a very large number of degrees of freedom interacting with each other non-linearly many many possibilities happen, time evolution can be exceeding the complex very, very complicated. But in like most of these problems, one would like to ask or they are stationary state or they are steady states. Well systems asymptotically go in to some steady state of the other and if so what are the states etcetera. Very important class of such steady state concerns, what is called thermal equilibrium.

So, I am going to define what is mean by thermo dynamic equilibrium or thermal equilibrium for short; and we will see how the subject of equilibrium statistical mechanics, which is now over a 100 years old in this in this modern form as helped us to understand systems in specific situations, specifically in thermal equilibrium and close to thermal equilibrium.

So, I should start the mentioning that non equilibrium statistical mechanics is an open subject still are the great dealers known. But, equilibrium statistical mechanics is a very well developed subject great dealers known and we think we understand the basic ideas behind this kind of statistical mechanics. There have been lots of efforts over the last 150 years to try to derive the postulates of equilibrium statistical mechanics from those of dynamics from those of mechanics.

This is what to do with the foundational aspects of equilibrium statistical mechanics. But, the fact remains that there is no complete asymmetric derivation. And we have we know the reasons why this is generally not going to be possible, but the fact is one keeps trying to push the boundaries further and further and trying to understand and trying to understand exactly where this fundamentals postulate of equilibrium statistical mechanics comes from.

But, right the away I caution you that we need additional inputs, statistical mechanics is not a special case of classical mechanics. We need an additional input and this is what we will discuss and try to make classable with some level. Now, if we took the gas in this room, an ideal gas for instance in the classical case completely non relativistic. Even this problem which is extremely simple perhaps the simplest of all statistical problems is quiet complicated because, time depending questions about the system are rather hard to answer unless you have specific models.

On the other hand the equilibrium itself is rather easy to understand from the postulates of statistical mechanics. And one of our targets would be to try and understand this to try and see where the loss of thermo dynamics comes from. And to cut a long story short I should mentioned that you could regard thermo dynamics as the times of averages.

Thermo dynamics deals with average quantities, but we already have some experience in statistics. So, we know that the average never tells a whole story. There is always the standard deviation there is always the dispersion about the average there are the higher moments and so on. And the question is what about those do they play a role etcetera and these are precise questions which will get some answers in what we are going to do.

While thermo dynamics is the science of averages. It is also a science of time scales and line scales. In fact, one could go ahead and define thermo dynamics as the study of systems on the longest lengths scales and time scales. So, that all fluctuate that happens at shorter lengths scales and time scales are neglected.

And you look at things over very long time scales to very long lengths scales, this is another way of looking at thermo dynamics. And therefore, you can see that what I called thermo dynamics for one object, one system may be different from what I called thermo dynamics for another system. Just as what I called low temperature for one system could be very different from what is called low temperature for another physical system and will see where this comes about.

So, let us start with our old friend Hamiltonian dynamics, I have in mind classical picture to start with. We have a collection of particles for instance interacting with each other and let us in the beginning assume that this system is isolated from the rest of the universe completely and you have a closed isolated system.

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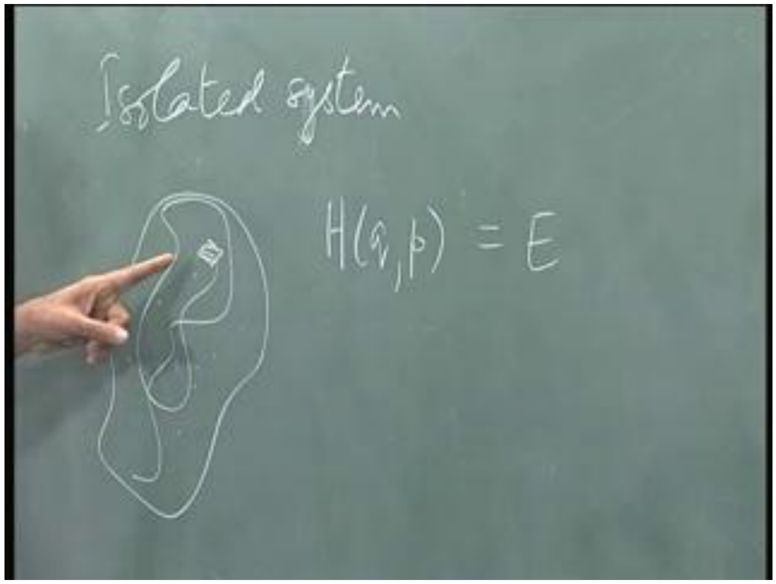

It exchanges no energy with the surroundings, it exchanges no matter with the surroundings, no particles go in or out no energy goes in or out and the system is sitting there in isolation, now what is it we can say about it. So, this is the first simplest for systems, here is this root system it contains a lots of object inside, this kind of a completely self contain system sitting in isolation.

We have to make some assumption, so we assume that there is a Hamiltonian for the system which describes what the decrease of freedom inside are doing. So, there are some kind of Hamiltonians specify with all the generalized co-ordinates in the momentum of the system. Now, of course, when we leave it alone and assume that these rules are all time translation in variant then, we know that this Hamiltonians is the constant of the motion. And therefore, this thing here has some prescribe value E, numerical value E which I called the total energy of the system.

Other than that, I know very little about the system which is sitting there is got this. Lots of interactions going on inside, but the total energy is some E. Of course this measurement of this e requires the degree of precession. So, I assume that there is the precession, some range delta E about which you know that energy is E then the question is what is going on inside very very complicated stuff is going on inside the phase space of the system is some very very high dimensional phase space.

If there are n particles the phase space is 6 n dimensional among this 6 n dimensional phase once you prescribe the value of this constant of the motion, you are on a hyper surface in a energy surface. And we saw that the only other things that could be constant for a very general system of this kind would be perhaps the total linear momentum of the system, the total angular momentum of the system the center of mass co ordinates there is a center of mass position of center of mass at t equal to 0, these are the constant of the motion.

For a prescribe state of the motion.

We going to assume we going to assume that this is an autonomous system it is everything is given inside that is it this is the assumption. You start by saying that. It is a collection of particles interacting with each other no external forces are put on it, it is just kept an isolation from the rest of the universe.

Then of course, it is an assumption that it is an Hamiltonians system right do that because, as I need a starting point. Now, I have the 10 galleon invariance ten variance and for a given state of motion in some given frame of reference. All these quantities like capital P capital L and so on have some fixed values they are all given. And the phase trajectory of the system moves is a point in 6 n dimension phase space and it moves on the energy surface.

And that is about all you know about this system, to specify the state of the system accurately I have to tell you the point in space I have to tell you exactly where the point is in phase space. If this is the phase space of the system I have to say this, this point here.

So, that is specifies all the q's and p's at some instant of time and then there is a very complicated trajectory going on does not intersect itself it is a very high dimensional object and its wandering around here. In the absence of any further information this very little one can do except our experience with normal system tells us that this system is likely to be chaotic. It is likely to have liapunov exponents some of them which are positive and it is slightly to wander around completely in a very strange passion such that the energy surface is filled up the system is more or less ergodic on the energy surface this is the very mild assumption to make.

It does not have to make strictly ergodic, there is a great deal of mixing there is a exponential separation and so on. But, as you if you remember exponential separation stability implies mixing which implies ergodicity. So, the idea is that any small volume element and phase space given sufficient time points from this initial volume element would come arbitrarily close to all the points of the accessible phase space.

The whole phase space is not accessible to the system, because I have specified its total energy. So, it is only that particular energy shell on the surface if I say the energy is E, it lies between E and E plus delta E, then it is really an energy shell between E and E plus delta E. And the system lies on that point representative point lies on that energy shell for all time.

And is ergodic on it then I ask the following what mode can I say? It is very little one can say in a sense. Because, it is moving around in an random passion on this energy shell and it samples all portions of this energy shell. And very ignorant about what happens next and the assumption is again based on what happens in lower dimension which you have points occupied in this around this energy shell is uniform completely uniform.

This says any one portion of this energy surface is as likely to be occupied or frequented as often as any other portion. This is if you like the principle of maximal ignorance you know nothing about written you say therefore, all are comes are equally likely.

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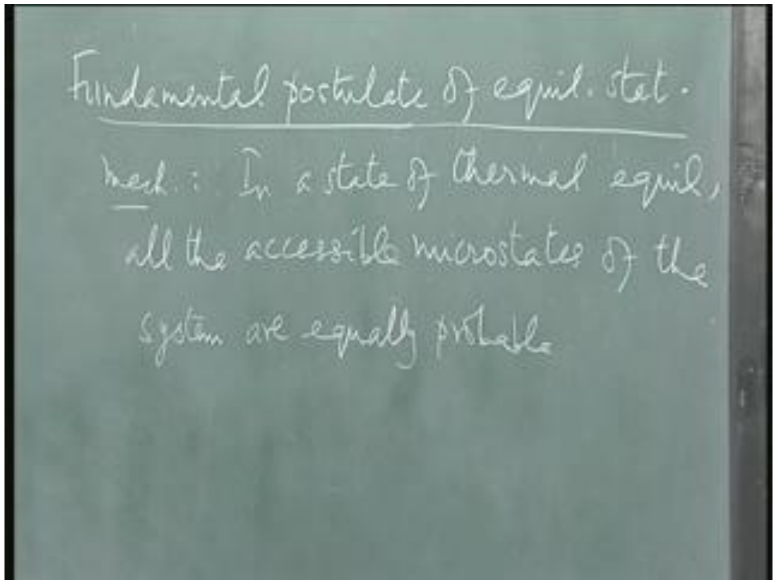

And this is the fundamental postulate of equilibrium statistical mechanics which says of equilibrium, equilibrium statistical mechanics this is important this is I have to define equilibrium yet and I will do so carefully and it says in a state of thermal equilibrium, which instantly I have defined as yet in a state of thermal equilibrium.

All the accessible micro states of an isolated system at equally probable, in a state of thermal equilibrium, there are many terms used here. Thermal equilibrium starts with microstate is another accessible microstates is yet another. And equally probable I have to define all these terms one by one, but this is the fundamental postulates. And it turns out remarkably this is the five steps to derive all of equilibrium statistical mechanics not only that thermo dynamics comes out as the special case. So, the remarkable statement is that this single statement is enough and the rest follows. Now, let us define all this quantities one by one what do I mean by thermal equilibrium.

Well I mean a system which is in thermo dynamic equilibrium by which I mean the time averages of all macroscopic quantities are independent of time the average values long time averages re some steady state and all physical quantities which I can measure macroscopic quantities such as total pressure total internal energy and so on. These quantities are all time independent they are steady. So, that is the good way to define, what I mean by an equilibrium state all physical measure of the quantities, macroscopic quantities are time independent average values are time independent.

Let us take that there is an operational definition to start. Of course you could say if I looked at the velocity of the single particle and think always of an ideal gas or a gas does not has to be ideal some kind of fluid inside which is interacting all the molecules are interacting.

It is certainly not true that if I look at the velocity of single molecule that is not time independent that is an undergoing collusion its getting forces acted on a or acting on this particle from other particles. Therefore, that changes where time that I would call that an microscopic quantity. It refers to an individual degree of freedom as suppose to collection the full average. So, if I look at the average velocity of all the particles that is going to be 0, because this container is not going anywhere, so that certainly 0, remains 0 in thermal equilibrium, total average momentum in average over all the particles that is 0 once again. The average energy of each particle average over all the particles that does not change with time either.

And in this case if it is an ideal gas with no interactions except collusions elastic collusion is the kinetic energy. But even if there is an interaction, even if the particles extract forces on each other you can still has said that some physical quantities associated with each particle is independent of time. But you have to average over all the particles. If you put in the little piece of paper in that and ask what is the average force per unit area on this piece of paper that will be the pressure of this gas that would be independent of time. So, such quantities would be independent of time and I call that state of thermal equilibrium make the 0 more precise.

Now, the average values are independent of time and the averages are taken with respect to some probability distribution, the only way this can happen is this, the probability distribution is itself independent of time, because that way you are sure all the moments of all the variables you take are all independent of time.

So, this is equivalent to saying that the probability distribution itself set as it is of whatever we have to specify is independent of time completely. The next thing that we have to ask is what is mean by the microstate of the system, I am going to give some simple examples of microstate and macro state. By a microstate I mean specifying the state of every constituent of the system. If it is a collection of molecule then it is equivalent telling you what is this molecule doing what is that doing what is this doing and so on. Since, we are talking about classical particles the variables that you used to describe each particle would be its generalized co ordinates and generalized momentum.

Now, if I tell you the generalized co ordinates and generalized momentum of each and every particle I would call that the micro state of the system. On the other hand if I will tell you the total momentum at any instant of time of all the particle I would call that a macro states a descriptor of a macro state of the particle. We will come across examples of microstates I am going to give a simple probabilistic example of micro states what do you mean by accessible micro states. Well suppose the range of energy that each particle can takes run from 0 to infinity kinetic energy comes from 0 to infinity.

It is clear that if you specified the total energy of the system as capital E some finite number a given particle cannot have an energy larger than that kinetic energy larger than that. If you do not have any potential in the problem, but just the sum of kinetic energy is the total energy its evident that if you give me total energy E no individual particle can have an energy larger than that. Therefore, some micro states are not accessible any more to the system. So, what I mean the accessible micro states is all those micro states which are compatible with whatever conditions you put on the entire system, rest are not accessible to you. If you increase the capital E, obviously, most states become accessible. If you decrease the capital E towards 0, it is clear almost all the particles must now get to rest, because any one of them zipping very fast will exceed the total energy.

So, it is going to give us the concept of the density of states, because it is .

That could be an accessible microstate absolutely.

Yes indeed.

Indeed it saying that you see what is happening, you will see why that is not going to happen in practice. That is exactly right. In fact, his point is suppose you have a collection of million particles in the total energy is 1 joule say or 10 joules. There is a possibility that instantaneously, you might look at this system and discover that one particular particle has almost all the energy close to 10 joules and the rest of it is almost 0 energy.

This two is an accessible microstate definitely. And that postulate of equal probability says that this is perhaps as equally probable as anything else, that is true that is true that is necessary as we will see right. This is certainly going to happen. But I hope you the idea of an accessible micro state is clear it will become clear as we go around, so to repeat the micro state of a system, which I have to precisely define. More carefully for the specific system is telling you what is the state of each microscopic constituent of the system is and macro state is giving you some blows information some over all information about the system. And the statement here is that in the state of thermal equilibrium which, we have define by for a operationally as the system the state in which all macroscopic quantities have time independent averages.

In that state all the accessible microstate of the system is equally probable this is the postulate. You would ask can I derive this can I derive this mechanics. Well its turns out that what you can derive is the converse of it. You can show that if at any instant of time all the microstate of an system accessible microstate of the system are equally probable if you can prove that then you can show it remains show for all time. It remains show for all time it means the system is in equilibrium all the average is become time independent. Unfortunately from mechanics or dynamics you can only prove the converse of what you want, whereas what we have is the situation where you have the thermal equilibrium and you are trying to ask what is the probability distribution?

You are trying to work backwards you all have the average quantities and you discover they are all time dependent. And you say oh I am thermally equilibrium, but what is the probability distribution that could lead to this. There is no way of deriving to make a postulate, but you can go the other way if you give me information about the probability distribution, then I can go back and say oh all micro states equally probable implies that they remains show for all time.

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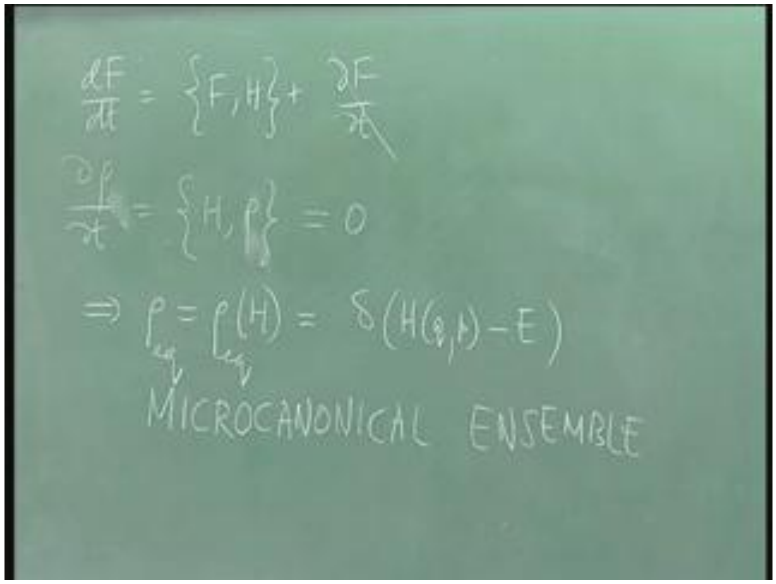

And what is that imply we know already what this mean we will talk little bit about density and so on probability density is etcetera. Remember that in phase space the probability density obeys in equations of the form H rho. This is the labial equation which I mentioned in earlier occasion in Hamiltonians dynamics classical Hamiltonian dynamics. It says that the phase space density of the system obeys the partial differential equation delta over delta t is the Poisson bracket of the Hamiltonians of the system with rho.

If this distribution probability distribution is independent of time it implies this must be equal to 0 in equilibrium. So, for the equilibrium distribution this is 0, so you have a equations like this. So, now what is this imply what is it imply for rho. It is a constant of the motion. So, what can I dependent on it is not an observable it is not a physical observable probability distributions are not physical observable. Average values means square values moments of physical observables physical variables or observables measurable.

But, a probability distribution is not a measurable by itself what is it implies remember that we also found for any physical quantity dF over dt was F H plus delta F over delta t. If there is no explicit time dependent that goes away and then this was the equation of motion for any quantity F which is the function of the phase space variables the p's and q's.

And if it is a constant of the motion then it Poisson to the Hamiltonian otherwise this the equation of motion. But, that is different from the equation of motion for the density because the phase space density itself the probability density itself obeys in equation, which has the minus sign compare to that equation.

This is very important this minus sign, this immediately tells you that the probability distribution phase space density is not a physical observables. It is what you need in order to calculate average values of physical observables. So, this says if it is 0, if delta over delta t is 0, it says row equilibrium must be a function of H and possibly the other constants of the motion possibly the other constant of the motion.

And what is it we are saying now they are saying that for a given system for a given state of motion in which all the P capital P capital L and so on are all fixed in some frame of reference. It is only the Hamiltonians now that you are worried about and that Hamiltonian has a value E numerical value E. Therefore, you can immediately conclude that under this condition under these conditions for the closed isolated systems in thermal equilibrium.

This equation tells you immediately that rho is rho of H equal to, what can be on the right hand side your all the energy shell. So, the Hamiltonian has the numerical value E and that is all you know about it it should be. So, H must be equal to 0, so what function would you have the probability density what possible function would you have yes, yes what function of h can it possibly be?

You are on the energy shell, so remember H is constraint to be equal to E numerically a direct delta function nothing consist possible. That is it, so in fact, we found the equilibrium density. Then, now if you take the energy shell and partition it into little cells this thing does not distinguish between one shell and another.

And therefore, it is clear immediately that, this is true if this is your energy shell probability of being here or here or here any equal sized element is exactly the same. A statistical and sample with that probability distribution in phase space which corresponds to an isolate system in thermal equilibrium has the special name this is called what is the name given to this example, because it says something about micro states it is saying something about each microstate now, they are equally probable. What would this corresponds what do we called this, this is called micro canonical ensemble.

I told you that the micro canonical ensemble has this very singular probability density, just delta function on the energy shell on a given energy shell. You still have to come to terms with what is this mean how do I classify states so on and so forth. That is an next task is what we are going to do about it, how you are going to use that information. But the fact is we run into a problem right away.

And the problem is if this is phase space and I have to specify particular point on this phase space, in order to tell you a state of the system that is in six n dimensional phase space. The probability of a single point is a set up measures 0, the probability in a continue you find any particular specific value sharp value is 0. And the number of micro states is infinity because number of point is unaccountably infinity in the phase space.

So, you write away no difficulty, you do not know how to translate that into specific prescriptions of finding probabilities of micro states, because number of points is infinity right away. And therefore, you need to now make a further assumption that is, that you have some finite resolution. That we cannot specify a point in the phase space, we do not have infinite precision available to us.

If I do that write away, if I say this is the smallest area or volume element in phase space that I can specify then you are in good shape, because as long as the relative element is finite momentum of how big this phase space is you can always count the number of cells in phase space that you have. So, it is a total phase space volume divided by the volume of a cell that is equal to the total number of states or micro states.

So, the possibility of the the uncountable number of points that you have in the continuous space as be termed by saying I have discretisized it, I give you resolution. And I will now define the phase space the micro state something that inside the elementary volume element in phase space. Now, of course, if this is where it ended you would still have operational difficulties, you would have to ask where does this resolution comes from, where does this finite resolution come from? In principle, if I make my measurement better and better and better then I should be able to get over this. And then what would happen in the limit it will still lead to this problem that you have a uncountable infinity of states. Fortunately for us provident very kind to us and given as a finite resolution by planks constant. So, we know that there is plank's constant, the nature is quantum mechanical and going to be playing out games classically, we know that the principle at the ultimate level there is quantum and certain things.

And definitely you cannot specify p and a q of a particle conjugate mode where of variables to better than planks constant position. So, the uncertainly in principle comes to help. You could give the whole theory of classical statistical mechanics pretending that there is a finite resolution, but the cost of the finite resolution lies in quantum mechanics.

It is exactly the same as saying that effectively I could do rigid body dynamics. But classically you cannot explain the rigidity of matter. You finally, have to take refers to the fact there is a Pauli exclusion principle that may solids impenetrable. But of course, when you are trying to find out, what the stress or what the hills stress of the young's modulus of piece of metal is you do not need to go to the Pauli exclusion principle. We use the fact that have made matter region and then you use an operational effective theory, in exactly the same way all of classical statistical mechanics is going to be predicated on the assumption that there is a exit resolution ultimate resolution. And the interesting thing will be you will see than appears in such a form then as for as thermo dynamics quantity are concerned the actual precise size of this volume element is irrelevant, because all physical things become independent of that to a certain extent.

There are other pleases where it would appear, and if you go to the region where you have for instance lower and lower temperatures; then quantum effect would starts playing a role and you cannot make these assumptions. And that of course, classical statistics would fail in that case. But we will see carefully in what region it plays a role and where it applies.

Is like a.

We have no doubt at all that this constant exist and we have no doubt at all now, of course its finally, based on experiment over the last 100 years. So, we know that there exist the quantum of action we know that planks constant exist we know the certain principle is valid we know the quantum mechanics is valid in its present form, so that is it given that is an assumption.

So, we are for away from planks postulates of energy quantum be you know, energy come radiation coming in quanta and so on, we are far away from that we know now based on the 100 years of experience and direct physical observation; that is planks constant exist and provides a fundamental limitations on the specification of p's and q's conjugate pairs of p's and q's, so we take that as given.

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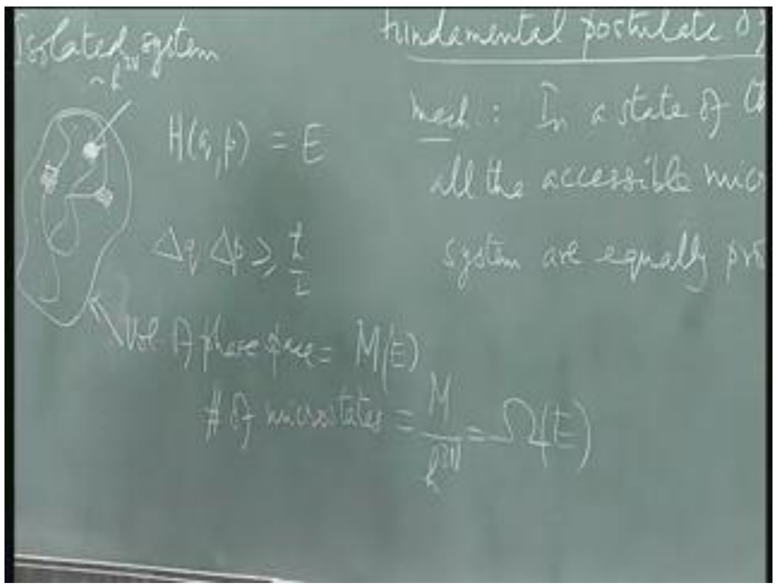

That immediately implies this, this little relation delta q delta p is greater than equal to h cross over 2. This immediately imply that in this phase space for every conjugate pair of q's and p's, you have a little resolution problem there and you can only try the values of q's and p's to within the little area, whose value, whose magnitude is some constant time planks constant; h over 2 by or square root of 3 times it does not matter something proportional to planks constant.

And therefore, if this full phase space is six n dimensional phase space the small volume element that you are talking about is ultimately proportional to h to the power of 3. There is an h for each q and p and three of them in the present particle in h q and there are n of them. So, this volume element is what dimension volume element is six n dimensional, and therefore, little thing here is of size h to the power 2 n whatever it be.

So, we keep that at a back of . When if this full phase space has the volume omega remain not use omega here let me call in full phase space volume. Volume of phase space equal to some I do not want to use capital V because I am going to use that for the physical volume and configuration space for systems.

So, let me call this volume of phase space want a better word, so let me call it M capital M is very bad notations where with me for a minute we do not need this. When the number of microstates is equal to M divided by over h 3 volt and then you call that omega and that is a countable number. And for a finite M this is finite because h is now a non-zero quantity and therefore, this number here is some finite number very large number of micro states. Then, the fact that your density is simply saying that H is equal to E and nothing more implies that each of these microstates on this energy shell has as much weight as anything else. And therefore, what is the probability, once I have set the microstates are all countable then what is the probability each of them is equally probable.

It just one over capital , so…

M is the volume of of the energy shell that is right, and the number of microstates is omega and as you points out just a minute as you points out this m is therefore, a function of E. Because you have the different E might have a different total volume of this energy shell and therefore, this is omega of E. And that is one over omega of E that will become relevant.

Yes

Well it could be three times h for h to the 3 n, it could be square root of 7 times you do not know numerical value here, so all the equations are write down or modulus some numerical factor here some factor. Obviously, if I want to specify this state more and more accurately, I try to make it as best as I can I could not choose something which is 10 to the 100 times planks constant, then of course, you lose information.

So, you would like to re finite as much as you can, but the point of making is there is an intrinsic fundamental limitation to how accurately can be, and incidentally we will talk a little bit about it perhaps quantum mechanics course.

This uncertainty, this limitation has absolutely nothing to do with measurement, it has nothing do with your inability to measure or your the fact that your appraise is not good enough for anything. It is just that for quantum mechanical objects, you cannot define momentum and position conjugate momentum and position with infinite precision in it at the same time. Simultaneously, you cannot define a state of such a system quantum system in which both the position and the conjugate momentum have arbitrarily precise values, not possible.

You might even say the property does not exist that is semantic now, but it might even say that you might say that an electron does not have the property of a precise momentum and position in any given state. So, it is fundamental limitation and we are using it in a very crude way here, we simply saying finally, amounts to saying this some kind of uncertain and let me just say that gives you resolution in phase space and enables me to start counting the number of microstates.

The first thing we realize is that the probability is one over the total number and that is normalize to unit, because you some over all these guys and you get one. Now, having set this much let us now take a break from this and look at a system in very trivial system for a few minutes, whether the idea of microstates and macro states become very clear it is actually easy to write down.

And once it is easy to write down in that system then we can come back here. This is the little more complicated because the microstate of the single particle of the particular state in a single particle itself it is a very large number of possibilities, where as we would like to look at a toy model where each constituent can have a countable number of states like two or three or something like that. Then it gives a focuses of idea is better, what I mean by microstates and microstates.

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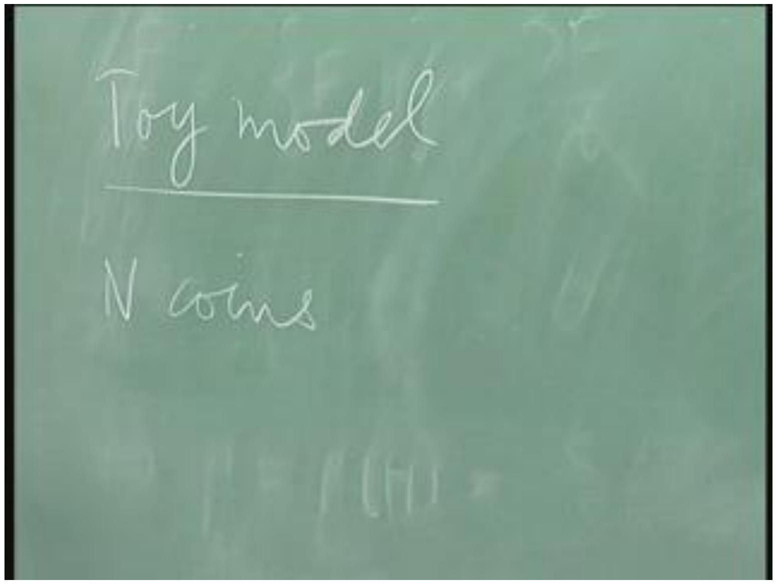

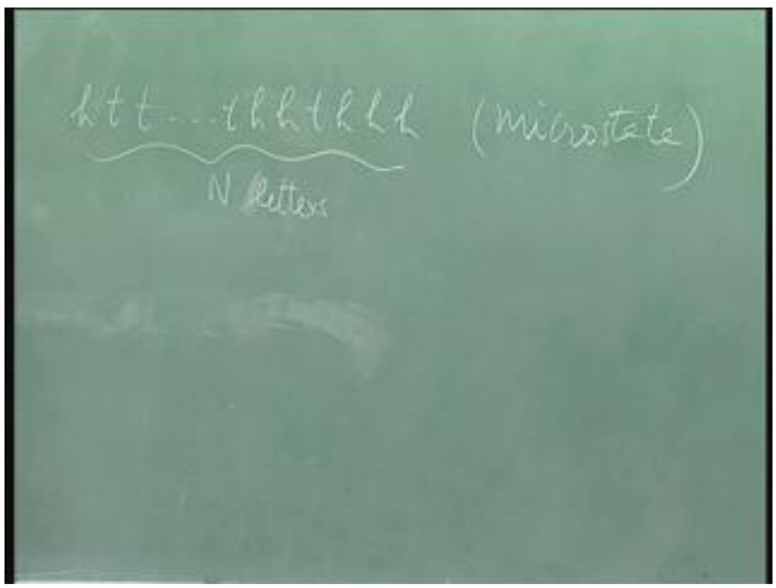

So, let us do that in the system I am going to look at this is a it is a toy model that is going to be my system and this is just a set of coins I just take a set of coins N coins capital N coins. And each coin has two possible states I assume that it is either a heads or a tails you can use this paradigm for various other things.

So, it is N identical coins, but identifiable coins that is important here and like to say that I have coin one here coin two here and coin three there and so on. And each coin has two possible states either a head or a tail and I will denote them by h and t. And I put these guys together after coin toss experiment and I discover I have something like this. So, I have a string of t's and h's N letters that is a possible string I called this a microstate of the system, because it sets something about each constituent. Like jumble of these h and t's I have another microstate of the system. So, this is my set of microstate, this is a microstate. Just a matter of caution you must not imagine that a microstate is a state of a given constituent. It is still a state of the entire system, but it is a state with specifies at microscopic detail, what is happening to every constituent and that is why it is called a microstate. But it is still a state of the system as a whole because, I put I told you the state of all the coins in the system.

How microstate the system have it has 2 to the power n microstate you see we have right the way been able to count, 2 to the n microstates. Let us make life a little simple, and assume that there is no bias many of these coins, so that in a given coin toss the h and t appear with equal probability; if there is a bias we can allow for that in some ways and so on, but to start with you have two to the n microstates of the system.

Now, would you call a macro state of the system. A given number of heads, so after I do this experiment I lay them all out I count all the h's and I count all the t's and of course, it is a constraint on the system.

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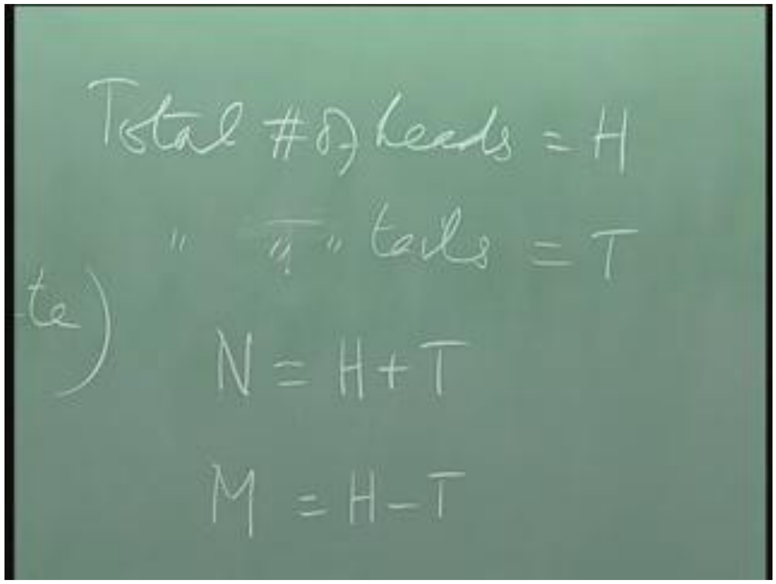

So, shells that total number of heads equal to h and total number of tails equal to T.

So, let us is called N is equal to H plus T of course, total number of heads plus number of tails. And for a given capital N and capital H, I have specified a macro state of a system; because capital T is known immediately. If capital N itself is variable then I would tell you what is capital N and capital H or capital T, it does not matter.

It is sometimes convenient to do the following. Let us also define I am sorry I used N there I do not want to use M there let me call this something else. Let me call this some curly m this is why we must all practice writing script. So, that we can add more symbols to it for some other script.

So, let me call capital M for reasons which will become clean later on I want use this from magnetization, that is why I put it m here. Let we call this the number of head minus the number of tails difference between the number of tails and number of tails.

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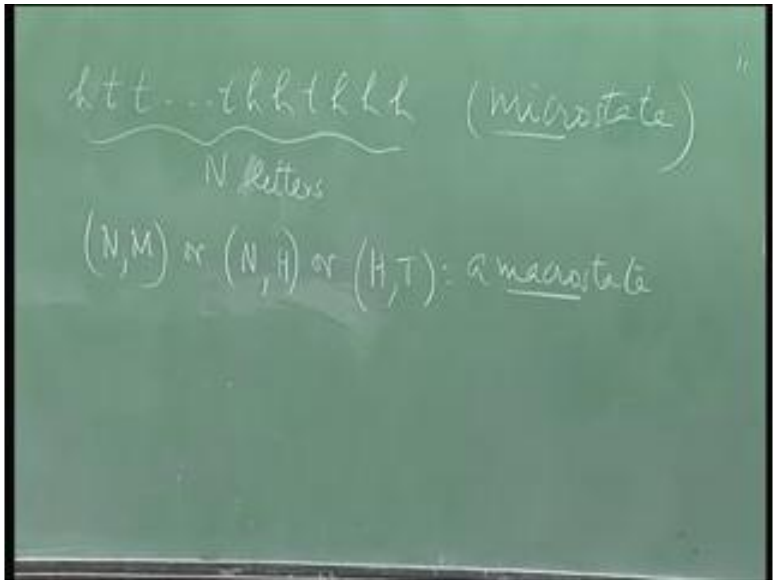

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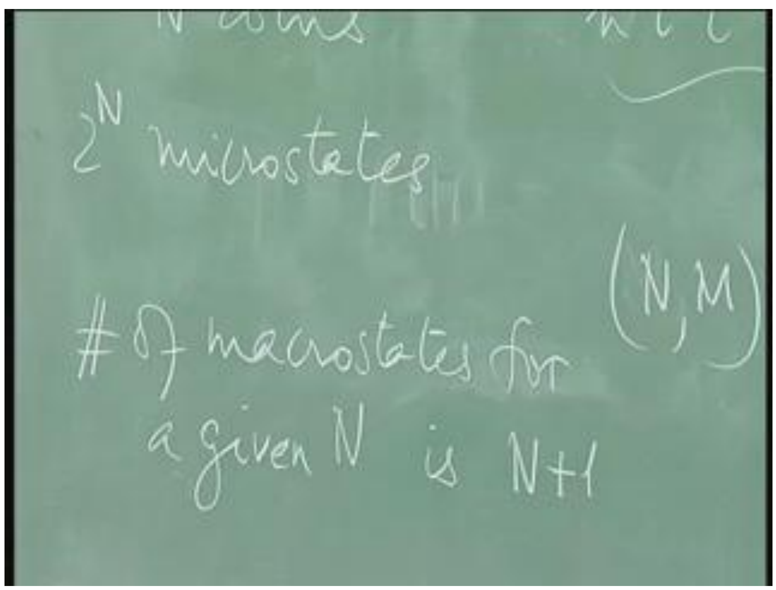

When a specification of the pair of numbers capital N capital M is equal to telling me a macro state of the system, N M or N H or H T a macro state of the system as suppose to microstate. How many possible macro states are there for a given capital N for a given capital N how many possible.

Well one let us let us look at this way for a given capital N, the number of macro states, what is the least value of capital H, what is the largest value. So, how many macro states are there N plus 1. So, the number of macro states is N plus 1. The number of micro states is much much larger, exponentially slow its E to the N log 2 and that guy is a linear energy.

So, the number of microstates linear is exploding as you can see the number of macro states is going much more , just going like N its slight the number of degrees of freedom this system. But this is going exponentially in the number of degrees of freedom and that is going to give as an important lesson.

Now, the coin is unbiased what is the probability that I get any particular microstate 1 over 2 to the N. So, it is a small probability it is a vanishingly small probability, probability of a given microstate is going to be vanishingly small, because it is 1 over 2 the power N.

But what is the probability of a given macro state, it depends on the micro it depends on the macro state.

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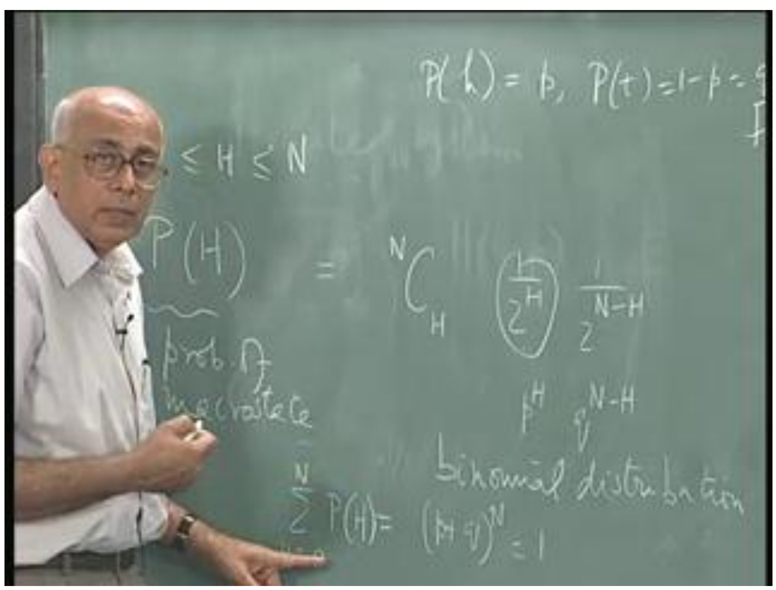

So, what is the probability now let us ask this question here. So, I have 0 less than equal to H less than equal to N and I ask what is the probability of H, what is this equal to. Well, it is easy to see this because I must have a capital H number of little h s in that string in the probability of any one of the is the half. So, this is 1 over 2 to the power H and must have here. That is very clear and then the rest of them must be tails and the probability of that is 1 over 2 to the N minus H.

Because these are independent these coins are independent of each other and therefore, the probability of two of them being H is 1 over 2 square in cube and so on. And you require capital H of them, so therefore, this is the probability. But, now comes the important fact that I do not care in which order the heads appear I do not care at all I am just counting a macro state I am looking at a macro state I am not interested in the particular sequence, therefore this is now multiply by N C H.

This is the number of alternative ways in which you can achieve that this by the way is this many heads and that many tails, so it is multiplied because the real meaning of the word and is multiplication of probabilities. And the meaning of r is addition of probabilities. So, you can have this way or that way or that way and you add up all the probabilities and then they answers is this is the number the combinatorial number factor which tells you in how many ways you can have capital H little h s in a string of length N of h's and t's. So, that is the probability distribution what would you call this probability distribution? It is called the binomial distribution.

And incidentally this particular problem this factors cancels again that by this where not the case and you had a bias coin, you had a bias coin such that the probability of a little h probability of h is equal to p probability of k equal to 1 of p equal to q. Then again I am sorry to use p's and q's these are not positions of momentum, these are probabilities.

What would this become? This would become instead of this we would have the p to the power H and this would be the power q to the n minus h. And since this guy looks like the term in a binomial expansion of p plus q raise to the power of n sum over h, this is called binomial distribution. Incidentally it is a, obviously that summation H equal to 0 up to n p of h in the most general case here this guy is equal to p plus q to the power n is equal to 1. It immediately follows that the total probability has to be 1 because it says the total number of h's must heads must be either 0 1 or 2 etcetera up to N and that and that of course, is normalized automatically.

And a given P of H is the coefficient of little p to the h in this exponential. So in fact if I put a generating function here, if I put x to the power h, then this would become p times x plus q to the power N and that is the generating function for the binomial distribution.

As you know for any probability this distribution the quick way to find all the moments of this distribution is define what is called the generating function for this distribution and then the operation of finding averages higher moments reduce to differentiating this generating function and then setting x equal to 1. So, generating function for this guy since I am going to use generating functions let us.

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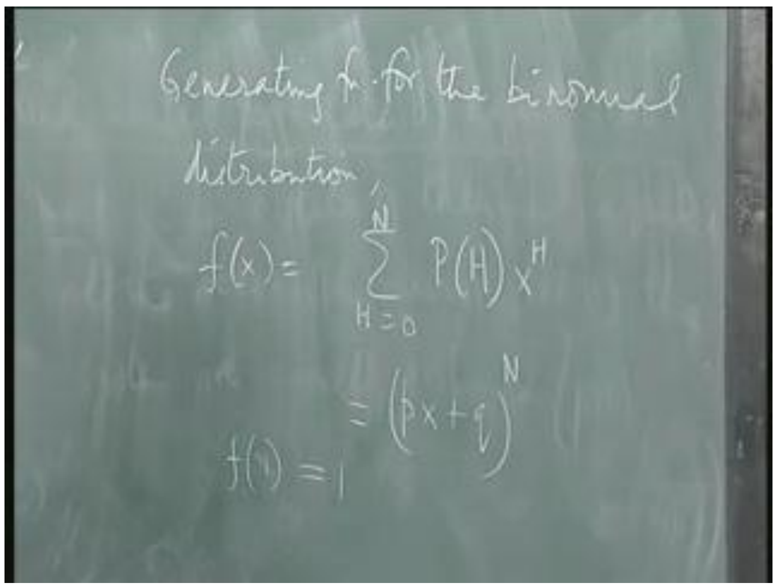

Let us for the moment take bias coins and say that the probability of a head for each coin is little p and tail is q; otherwise of course, p and q become equivalent things come out things become little easier, but it is good to keep the general case in mind. So, this is what the generator function is P of H is therefore, the coefficient of x to the power h in the binomial expansion of this generating function.

What is the first property this generating function of this f of 1 is equal to 1, you put p x equal to 1, then of course, its simply says the total probability conserved is 1. What is the average number of heads what is the average number of heads? How do you define the average number of heads? Now, do you define the average numbers?

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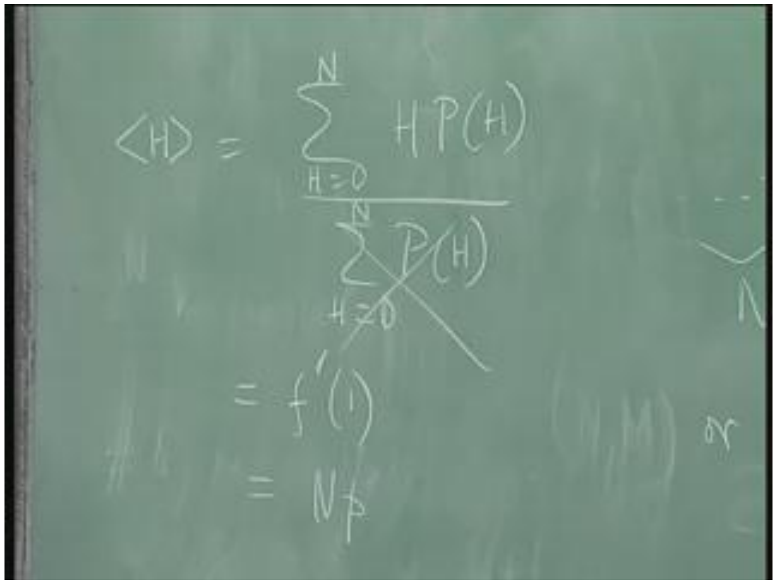

So, let us define the average here numbers the average equal to summation H P of H divided by summation P of H 0, 0 to N that is the definition that is the very definition of the average number of heads.

But what is that equal to we know that the denominator is already one, so we do not need this then how you find the numerator you go back here to this expression here and you differentiate with respect to x. If we do that you get H coming out here and then x to the power H minus 1. And then you put x equal to 1 and you get f p of h, so this is also equal to f prime of 1. That is the great advantage of using generating functions because finding averages becomes trivial now, what is this equal to? I differentiate this you see the left hand side is given by this that is equal to this guy here differentiated and set x that equal to 1, but I can also differentiate the right hand side and set x equal to 1.

If I do that I get an N coming here, px plus q to the n minus 1 multiplied by p and I put x equal to 1 px plus q become p plus q which is 1 right. And therefore, this is equal to NP right you must always check that if little p is 0 the average number of x must be 0. If little p is one then the average number is number of coins tosses itself capital N and indeed we know that if the coin is fail, then the average half of them are going to be H. So, you get N over 2.

Therefore, this is completely checks out this is what the average number exists, what is the mean square deviation, what is the mean square value what is the standard deviation? We need to find H square, so let us do that H square is easily found.

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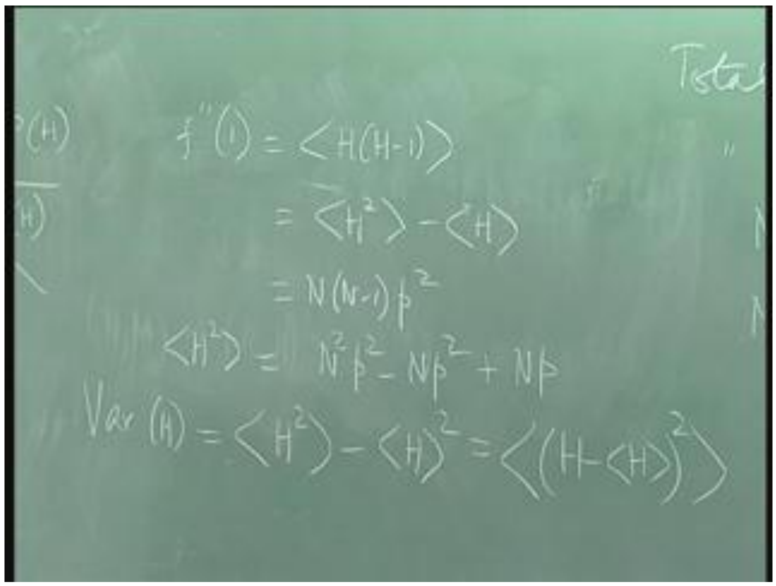

Well, if I differentiate this twice what do I get f double prime of 1 is equal to what I differentiate twice I get an H times and H minus one then I put x equal to 1, so the average value of H times H minus 1, but that is the same as average of H square minus the average value of H.

But, if I differentiate this twice I get an n times p and second if I differentiae I get an n minus 1 time p. So, this is also equal to N times N minus 1 P square and that is equal to this quantity here. Therefore, H square equal to N square P square minus NP square plus the average value of H which is NP we already know that.

So, therefore, what is the standard deviation or the variance the variance of H equal to H square minus H average square that is equal to the average of H minus average H whole square. As you know the variance of a random variable is the mean square minus square of the mean.

But, as the same as saying it the mean value of the square of the deviation of the variable from its mean value and that is the non zero quantity it is a positive quantity unless it a sure variable. So, this thing here turns out to be equal to N square T square minus NP square plus NP minus N square T square because the average is NP and if I square it I get this and this terms [doubt] cancels out and this is n times NP times 1 minus P.

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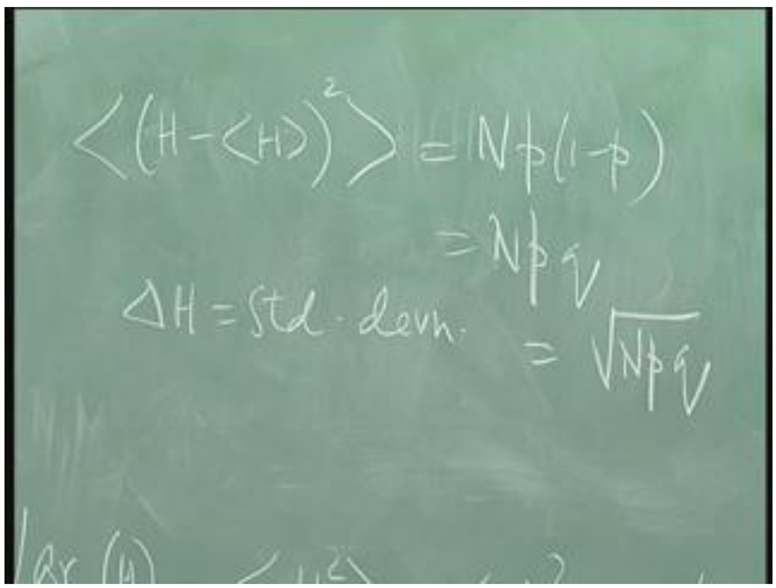

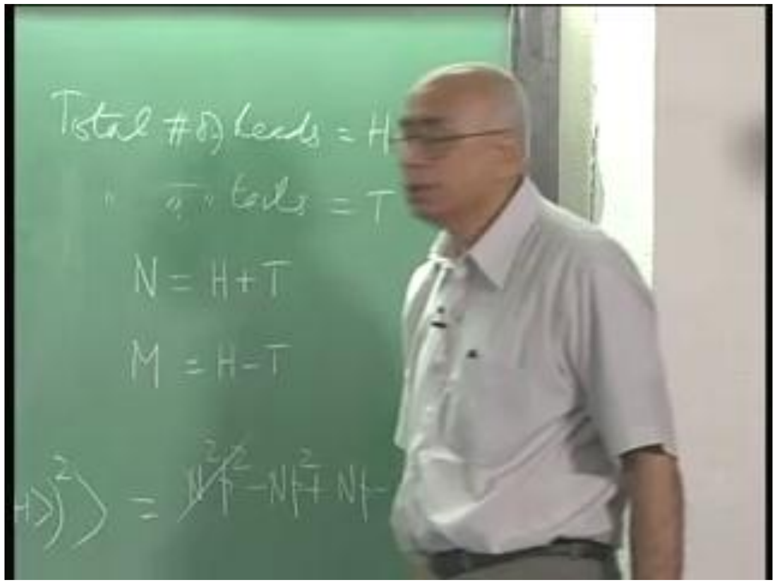

So, it tells us that while the average was this H minus H whole square equal to NP 1 minus p equal to NPq, q is 1 minus p. Therefore what is the standard deviation square root of this n. So, it says delta H equal to standard deviation equal to square root of NPq.

If you like that is the uncertain, if you define the uncertainly, I just the standard deviation its equal to square root of NPq. For a fair coin p and q are half. So, there is no problem there NPq is square root of n over 2.

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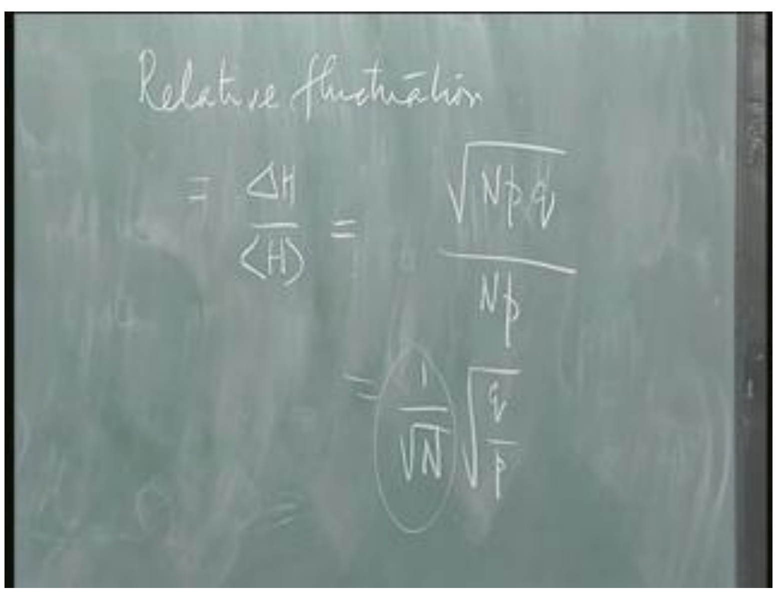

On the other hand what we like to find out is what is the dispersion about the mean, therefore the relative fluctuation equal to delta H over the average value and this is equal to square root of NPq divided by NP.

This is equal to one over square root of N square root of q over p. So, the fluctuation about the mean value divided by the mean value tells you the relative scatter, and that quantity is proportional to one over this square root of N times square root of q over P and for the un bias found that factor cancels out.

So, the important thing is the this factor it says the relative fluctuation is becoming smaller and smaller, when the number of coins access become larger and larger. If the number of coin tosses is 1000, this guy here which is a already of 10000 its already 1 percent 1 over 100.

If it is 10 to the 23, 10 to the 24, it is one part trillion. So, it says the fluctuation about the mean relative part in a trillion and that is what is going to happen in the thermal dynamics and equilibrium conditions it will indeed turn out to be this. It will indeed turn out go like one over square root of N it is very characteristics of all these distribution and this is what is going to eventually, when the day, this is why thermo dynamics works we will see that here.

But going back there you can see that you can find all the moments once have the generating function which is really the crucial point, how do we find the generating function? The word for the generating function in statistical mechanics is called is called the partition function. It is really finally, just a generating function you can call it has Laplace transform green function with proper you can call it by many names, but it really just a generation function finally it is going to tell you how to calculate average values its not has trivially define as in this case. Little less trivially define they should say and the measure need not be uniform and so on and so forth

But the rest is detail matter of detail what must do and that going to be done next time immediately is to ask why it is that is more probable to have 50 heads and 50 tails in your 100 coin tosses. Then 1 head 99 tails the reason is per head in the combinatorial factor . Every sequence is equally probable, but that sequence or that macro state will dominate, which has a largest number of contributing micro states and that is all the world works that is it. So, although intensically it is possible that there is a micro state in which one particle carries all the energy and all the other carries no energy in a large system; that micro state is no more probable than any other micro state. But, if you ask what is the most probable macro state this has contribution from maybe one or two micro states where as other states should have contribution very large number of .

So, that is what is going to happen over well means, even in this coin example you says it is going to you will see that nch, if n is 100, 100 50 is going to be exponentially larger than 100 C 1 and that is why statistics works; for this is how large is large. And of course, looks like a question which cannot be set, but you would ask how large is large compared to one if it is integers.

How large is large, what would you say is large? Now, you got be a little more precise unfortunately for us there is a definite theorem this is called the center limit theorem. And its very powerful formula called sterling formula which will tell us how large is large. And you see indeed that fairly small numbers are quite large, but these purposes 100 is already very large 10 to the 24 is un inconvincible show. And that is why this whole business works, you are looks like this is a large numbers have a life of your own really is show we make it presence.