Classical Physics

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Lecture No. # 02

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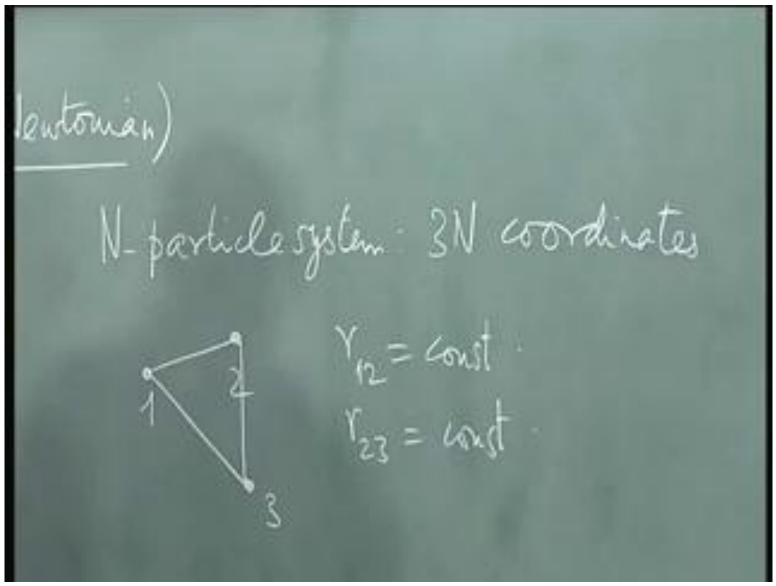

So, let us start now, the formal study of a first topic that we would like to look at in this course, which is classical dynamics, by which I mean Newtonian mechanics. And then, after we finish the study of classical dynamics, in some sense in the conventional way, we will go on to see how it merges with other areas of physics. The first topic we would like to take up is the idea that the mechanics of a particle, which you study in, say, physics one can be generalized to systems of particles, many, many particles taken together, and the first concept that we would like to get clear is that of degrees of freedom. So let me illustrate that. What I mean by the degrees of freedom of a physical system, such as a collection of particles or an object or rigid bodies obeying Newtonian mechanics in some sense.

So I start with a single particle, which can move in three-dimensional space. I need three coordinates - three independent coordinates - to specify its position at any given instant of time and it has 3 degrees of freedom to start with it. What happens if I have two particles moving in space? What is the number of degrees of freedom? I need 6 degrees of freedom, and if I have N particles, I need 3N degrees of freedom, provided these particles move in space and there are no constraints among them at all; they are, of course, subject to interaction; they interact with each other, they move under each other's influence and so on, but they move in three-dimensional space; each of them requires three coordinates. It does not matter whether these coordinates are Cartesian coordinates or spherical polar coordinates or cylindrical polar coordinates, does not matter; what matters is the number of coordinates. The number of coordinates for N particles; so N particle system, has 3N coordinates.

I am going to call these generalized coordinates. The reason is they do not have to be Cartesian coordinates; they do not necessarily have to be angular coordinates. There could be some of them Cartesian, some of them angular, some of them in other coordinate system and so on; it does not matter. What matters is the number here and we are going to use a certain symbol for these coordinates. I am going to use little q for these coordinates. But before I do that, I would like to introduce the idea of constraints.

If I tell you, that a particle is constrained to move on a plane such as this, then of course, it has only 2 degrees of freedom, and if you consign it, if you confine this particle to a line, then it has a single degree of freedom. What is important to recognize is that you must count the number of degrees a freedom before you start solving the equations of motion.

So first, you have to specify what the independent degrees of freedom are. Now we have seen that for N particles you have 3N coordinates, 3N degrees of freedom. But suppose I start putting constraints, suppose I tell you that particle one and particle two are connected up in such a way that the distance between them is fixed it cannot be changed. Then how many independent degrees of freedom are there? There are 5 independent of degrees of freedom, because you have 6 coordinates between these two particles, but then you also have a constraint that r 1 2 equal to a constant. The distance between 1 and 2 is constant, is fixed; it is given to you as part of the problem. In which case, you subtract one independent coordinate out and you have 5 independent coordinates left. Therefore, the lesson is to find the number of independent degrees of freedom, you first must take the total number of degrees of freedom and subtract out all those degrees, which you can eliminate due to the constraints on the problem.

Suppose I have three particles, the third one here, and I tell you that the distance between 2 and 3 is also constant. How many degrees of freedom do we have now? Independent; we have 7 independent degrees of freedom, and if I join all three and I tell you that all these three distances are constant - fixed - then of course, these three particles could be imagined to be at the vertices of a rigid triangle and you have 6 independent degrees of freedom left. So, now what is the general story? I have N particles, 3N coordinates, and now I tell you that all the distances between particles are constant.

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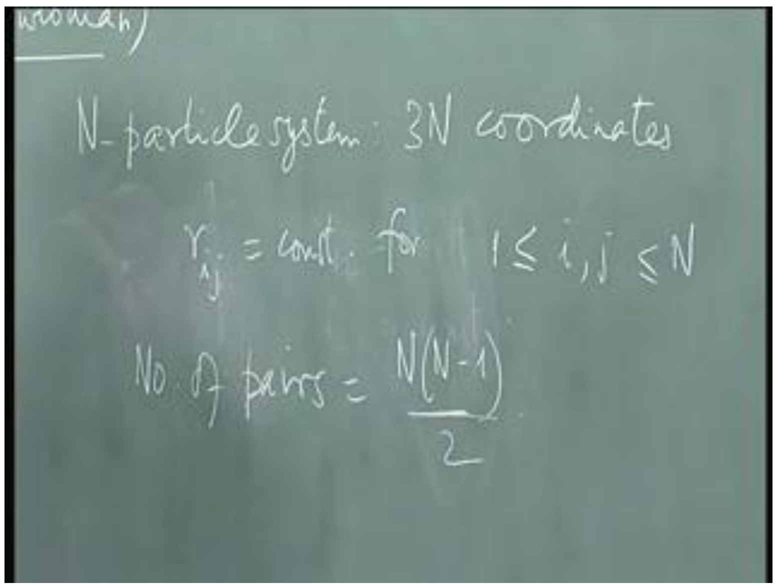

r i j equal to constant for 1 less than equal to i j less than equal to N. How many independent degrees of freedom do we have now? How many constraints are there? Well, I have N particles and there are 3N coordinates and how many constraints are there? I tell you that all the distances, all the pair-wise distances, are constant. Nc 2 is the number of constraints number of pairs equal to N times N minus 1 over 2, and therefore, how many independent degrees of freedom are there? Well let us try that out.

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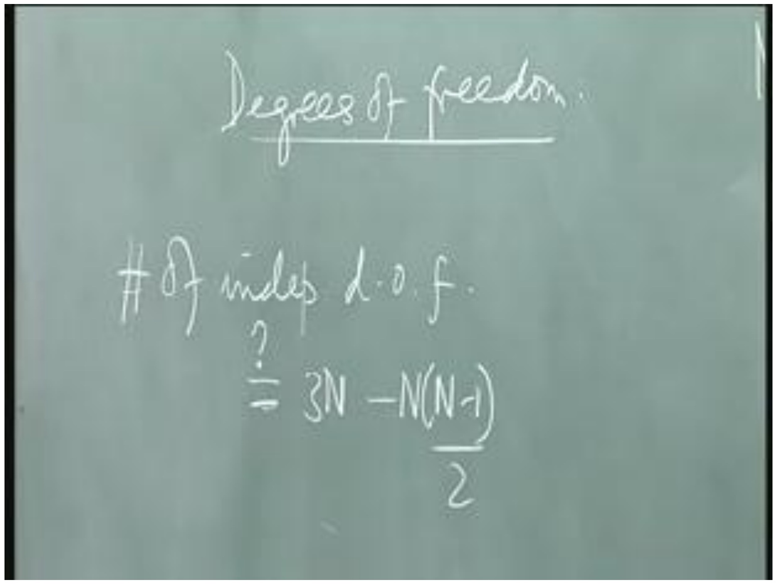

So number of independent degrees of freedom equal to… I put a question mark here. I am not altogether convinced 3N minus N minus 1, N times N minus 1. What happens if we put N equal to 4? What happens if we put N equal to 4 here? What happens if we put N equal to 5. What happens if we put N equal to 8? Becomes negative, but that is not physical; it is obvious that you cannot have a negative number of degrees of freedom; this cannot go on increasing; it is very clear from here; because this increases linearly in N, and that increases like the square of N; so pretty soon that is going to overtake it. So what would you say is wrong? What should we do?

Speak in a mike; yes.

If we have a case structure with a triangle, some constraints become redundant.

Some of the constraints become redundant. It is quite clear that you do not need all these constraints; just some of them are enough to hold the object rigidly; how many would those be?

Well in the case of a triangle, you have… No, it is not 2, because if you have these two guys, you can still move this around, you can still do this, distances are still fixed, but if it is a third one, then its rigid. So how many independent degrees of freedom do we really have?

Now this is mimicking a rigid body; a rigid body is one in classical mechanics - in Newtonian mechanics - you define a rigid body as one where the distance between any pair of points is fixed once and for all.

So, it is quite clear, that this cannot be a formula. This cannot be the number of independent degrees of freedom, simply because this increases like the quadratic function of N, whereas this is linear; where does this stop? 6. 6 degrees of freedom; a rigid body has 6 independent degrees of freedom. So no matter how large N is, it is quite clear that you have just 6 independent degrees of freedom; the remaining constraints are redundant completely. And what are the 6 degrees of freedom? Now, we can start counting them, and when you compute degrees of freedom, the way to check if something is right or wrong is to do it in two different ways. Compute the number of degrees of freedom in two different ways, and if the answers match, then you know you are on the right track.

How many degrees? We have said that a rigid body has 6 degrees of freedom. I should like to count these degrees of freedom and tell you what they correspond to. You can choose coordinates in different ways, but I would physically like to understand what these coordinates correspond to.

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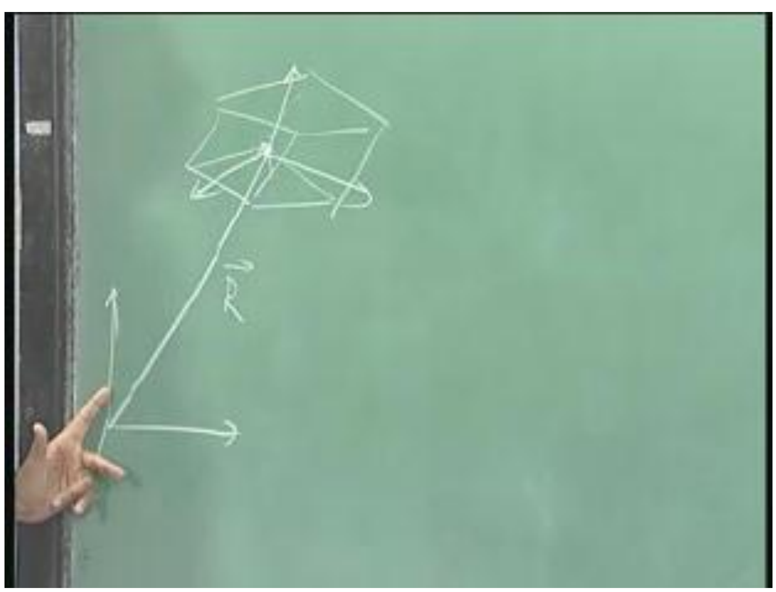

So here is a rigid body. Let us take something like a cube and this has 6 degrees of freedom - independent degrees of freedom - we should like to count them; here is one way of counting them. I need 3 degrees of freedom to tell you where the center of mass of this object is in space. So with respect to some fixed coordinate system, the center of mass of this object is at some point - position vector R, and it has of course, three coordinates associated with it and that takes care of 3 degrees of freedom. There are 3 more degrees of freedom, which tell you what the orientation of this object is in space and how do you tell what that is.

Well there are several ways of doing this, but a simple way of doing it is the following: associate - here is my fixed coordinate axis in space - with the body associate a coordinate system; a body fixed coordinate system. So associate with it a coordinate system, which looks like this - body fixed coordinate system - and then the orientation of body in space relative to this fixed coordinate system is dependent on how this coordinate system is twisted or turned with respect to this coordinate system. Now, what this means is, you cannot translate this R, put it here and compare and see how the new coordinate system, how the body fixed coordinate system, is oriented with respect to the space fixed coordinate system. And what this means is you start with a space fixed coordinate system and you want to go to a body fixed coordinate system, to go to this set of coordinates from this, you have to do 3 Euler rotations, you rotate about three different axis and you need three angles for this purpose.

Therefore, you have three coordinates for the center of mass, and three angles, which will specify the manner in which this coordinate system is turned to reach this coordinate system from here. What sort of rotations do you need to reach this coordinate system from here? That is 3 more degrees of freedom, and therefore, you have 6 degrees of freedom.

Another way of looking at it is to say here is a rigid body, and now I start here, and here is the center of mass of this rigid body. So three coordinates are gone there, and then to reach the orientation of this coordinate system from this coordinate system, I take this coordinate system and turn it about some axis through a certain angle. So to specify the axis of rotation, I need two angles, because the axis of rotation in three-dimensional space is a unit vector, and unit vector requires two numbers to specify, because the sum of the squares of the three numbers is unity. So I need to specify a latitude and a longitude if you like, on the surface of a unit sphere to reach to specify an axis, and about that axis I can rotate through any angle from 0 to 2 pi. Now that is one more degree of freedom. So we have computed this in two different ways, and each time, we discover that for a rigid body, you have 6 degrees of freedom;

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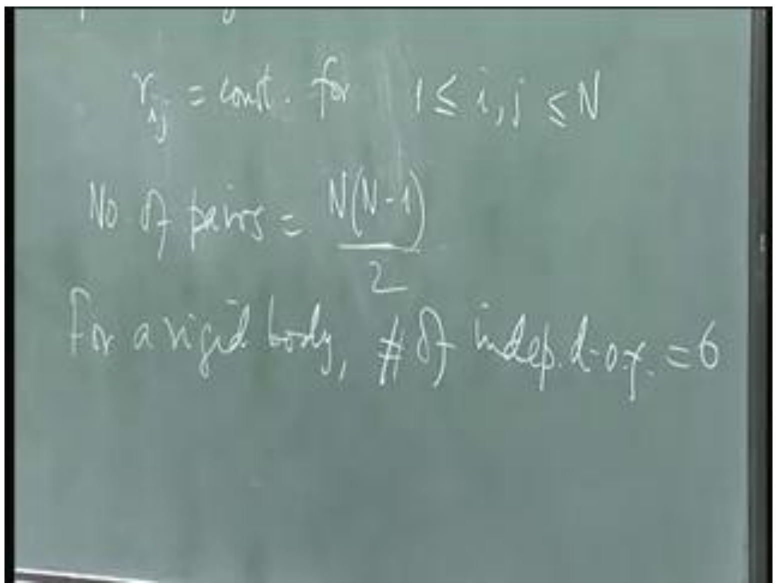

now that is a general rule; for a rigid body, number of independent degrees of freedom equal to 6 for any N, is this clear that you have just 6 degrees of freedom.

And of course, you are used to this from spectroscopy, when you study the spectra of diatomic molecules. For example, you are told there are 3 translation degrees of freedom, the 2 rotational degrees of freedom, and the reason, of course, is that in the diatomic molecule, you have two molecules connected by a invisible bar, and there is no moment of inertia for rotations along that axis; and therefore, you have 2 degrees of rotational freedom and then 3 degrees of translational freedom.

In the moment, you have a tri-atomic molecule you have 6 and so on, provided the molecule is not deformable. If, of course, it is deformable the distances between the atoms changes, then you have 9 degrees of freedom and depending on the energy you can excite any number of these modes. But for a rigid body, as far as we are concerned, we have 6 degrees of freedom, 3 of which I will call translational, because they tell you where you should move to take it center of mass from a fixed set of coordinates, and then 3 orientational degrees of freedom - tell you how the system is oriented. So, henceforth, what we are going to do is to pretend that we always have eliminated the constraints and computed the number of independent degrees of freedom.

Now, you have to be a little careful here, because the constraint does not always eliminate a degree of freedom. For instance, if I told you that you have a particle moving along on this table, and the table has boundaries, and you cannot get out of those boundaries, then what you doing, is to confine the particle to a certain region. It has 2 degrees of freedom because it is on a plane, but it has got inequalities, it says its x coordinate not exceed this value and the y coordinate cannot exceed that value. Those are inequalities, but they do not reduce the number of degrees of freedom. If I told you that on the xy plane, a particle moves on the xy plane in the first quadrant only, it still has 2 degrees of freedom nothing is eliminated.

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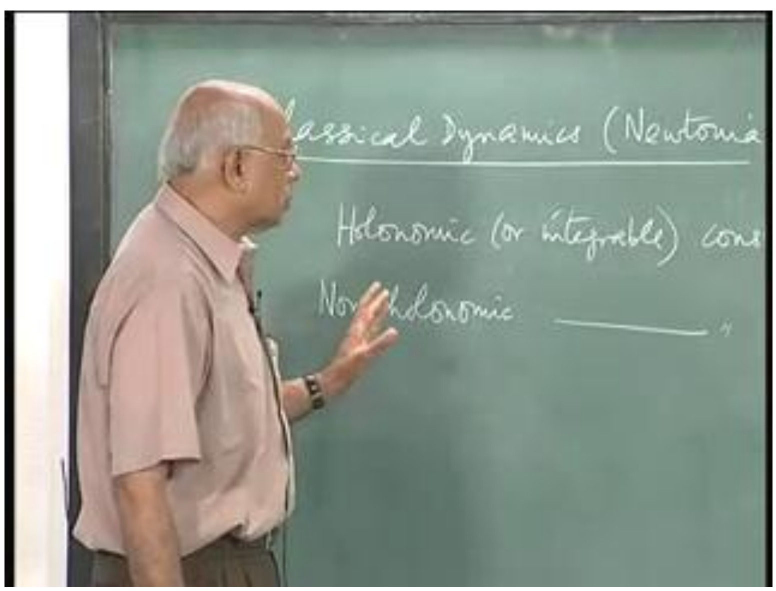

So, whenever you can eliminate degrees of freedom, these are called holonomic constraints or integrable. Never mind why I call it integrable for the moment, and then you have non holonomic. This cannot be used a non holonomic constraint cannot be used in general to eliminate a degree of freedom. In general, non holonomic constraints would be inequalities of the kind I just illustrated or they could involve velocities as opposed to positions alone or coordinates alone, in which case it is not all clear that you can integrate matters out to eliminate a degrees of freedom, we will come across some examples of this as we go on, but by and large, in this course, we will restrict ourselves to holonomic constraints.

Now, I made the point that you must first identify the number of independent degrees of freedom, and then start writing down the equations of motion, and solving them; not the other way about. A very simple example is the following - I imagine this piece of chalk to be a point particle moving in space under the action of the earth's gravity, I hold this particle here - at this point. So I know its initial position, and I know the force on it due to gravity. Can you predict its future from that? Can you predict its future motion from that?

Well, I take this chalk, and I drop it from rest, and of course, if it drops straight down, on the other hand, I give it a little horizontal velocity, and then it moves in a parabolic path. Strictly speaking, an elliptic path - part of an elliptic part - I give it a slightly higher velocity; it goes into orbit round the earth; I give it even higher horizontal velocity and it escapes in a hyperbolic orbit. So, how could you say, that if I tell you the initial position and the initial force, the future is predictable? It is not; it looks like it is not. Is there sufficient data? Does it suffice to tell you what the initial positions are and the initial forces are? You need initial velocities also. Newton's equations are second order in time. When you need to grade it you have two constants of the motion in the one-dimensional case, and then of course, you need to know both the initial position as well as the initial velocity to predict what is going to happen next.

So, it is immediately clear that dynamics occurs in a space, which is not just the coordinates, but also the velocities. This space is called phase space. We are going to do a lot about phase space, but it is good to get this idea right in the beginning, straight right in the beginning, that the equations of motion require you to specify both the initial coordinates as well as the initial velocities or the slopes of the trajectories, not just the point on the trajectory, for you to be able to solve this equations uniquely, and this was a simple example of it.

That immediately tells you that dynamics is not happening in real space; it is happening in something called phase space, which we will study in greater detail, but it is a lesson. And secondly, let me ask you - how many degrees of freedom does this particle have? It has 3. You must not make the mistake of saying, if I drop this from rest, it moves in a straight line; therefore, it has 1 degree of freedom; you should not say that; that is not true, because you cannot count degrees of freedom after you solve the equations of motion. You have to count the degrees of freedom - independent degrees of freedom -before you solve the equations of motion. You must then identify the corresponding velocities as well, and then, the equations of motion are written down in terms of these coordinates and velocities, and solve with some specific initial conditions. The idea is once you specify the initial conditions the motion in principle is solved.

So, this is the idea behind dynamics. We are going to do this in great generality. You going to do it in much greater generality than even mechanics itself, as we go along, but you have to get this idea right in the beginning, that the number of independent degrees of freedom has to be determined first, and the corresponding generalized velocities have also to be added on to the set of dynamical variables.

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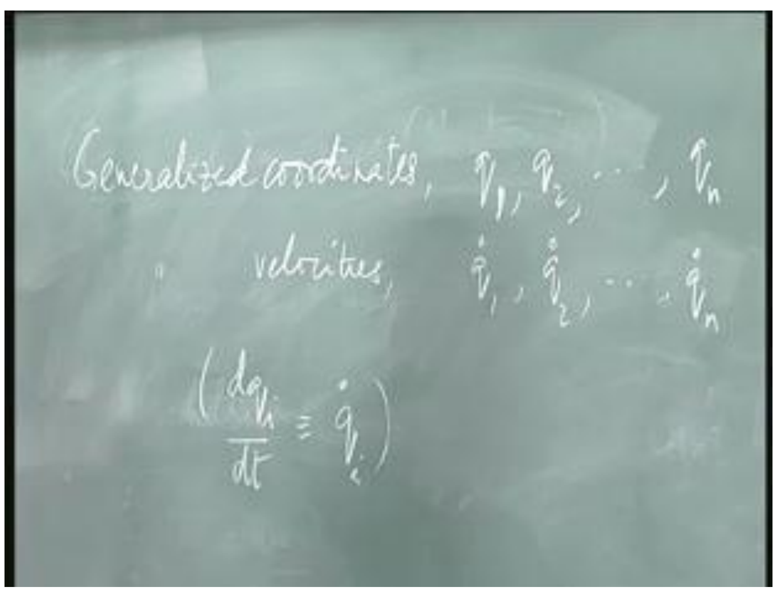

So, let me write that down here - generalized coordinates… q n and generalized velocities. I am going to use overhead dot for the time derivative of any dynamical variable, and these are the so-called generalized velocities, and this is a notation I am going to use for a system, and I do not care what kind of system it is, which has N independent degrees of freedom, and I label them q 1 to q n. Some of them could be Cartesian coordinates, some of t hem could be angular coordinates, they could be mixtures of the two, we do not care, but this is the general framework.

And then, to solve the equations of motion, you need to know a certain amount of initial data. You need to know the values of these generalized coordinates and the generalized velocities at an initial instant of time, at some initial instant of time. And then, the task is to write the equations of motion down, and to solve these equations. And you guaranteed then suitable conditions, the solution is unique. It is in this sense the Newtonian dynamics is deterministic. You give me the initial data and the equations of motion, and the future is uniquely determined. We going to write these equations once again, look at them. The simplest instance, of course, you have Newton's equations. Let us do this for the very, very simple case of a single particle moving along the x-axis, just in one-dimension; there is just 1 degree of freedom. A particle of mass m, moving along the x-axis has a single degree of freedom.

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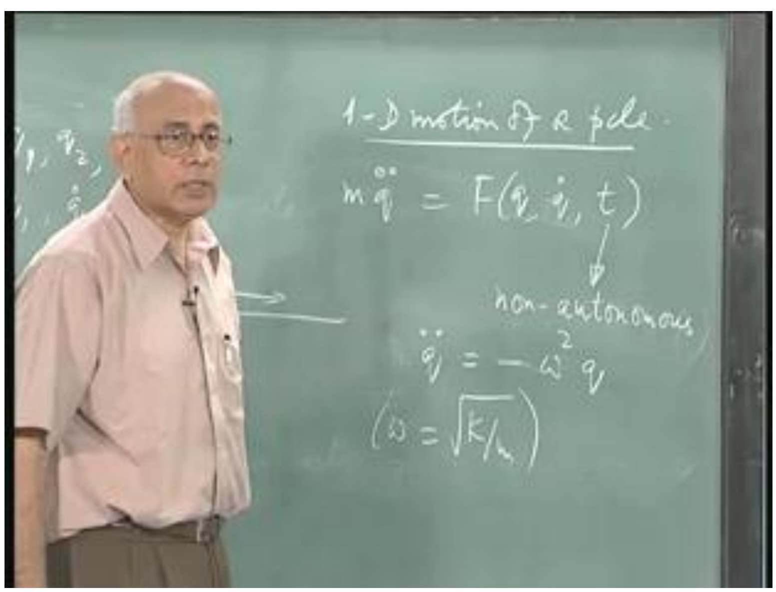

So one-d motion of a particle and let me call this just q; the x-coordinate of this particle just q; just to use to get this notation. What is the equation of motion of this particle? If it is subject to some force f of q, what is the equation of motion? What is Newton's equation? It is equal to mass times m q double dot equal to the force on this particle, and this force could, in fact, be quite a general force; it could depend on where the particle is; certainly, it could do that. So it is general a function of q. Could it depend on q dot as well? Can you give me the example of a force, which actually depends on the velocity of the particle?

Viscous force. Anything else? Magnetic force. Force due to a magnetic field; it is velocity dependent. So certainly, in general, this would have also a q dot. Could it depend on time explicitly? Could it depend on time explicitly? Yes or no? Yes. What is to stop me from pushing this particle up and down by some prescribed external force, which changes with time? So, in general, it could depend on time, could depend on time as well. Certainly could. Yes?

Pardon me.

If the force depends on q and t, it depend on q dot?

He says if the force depends on q and t it accounts for q dot; is that true? It is a good question. Is that true? You have to appreciate the fact that q and q dot are independent dynamical variables, because I could specify them independently, initially. After you have solved an equation of motion under certain conditions, then of course, it turns out that q is a function of t, and q dot is just found by differentiating it, whether it is after you solve the equation of motion, for a specific motion, not true in general.

So, again you have to get used to this idea, that coordinates and velocities are actually independent dynamical variables, because the equation of motion depends on the acceleration; that is Newton's equation of motion; that is a way it is. We will see why as we go along. But accepting Newton's equations for the moment, this is an equation for the rate of change of the velocity and it is a second order equation in the coordinates; it is a second order differential equation. So positions and velocities are independent dynamical variables, because initially you could specify them independently of each other.

So you must not make the mistake of saying q dot is determined by giving q; it is determined only if you give me q already as a function of t, which happens only after you solve the equation of motion.

That is a very good question - why cannot f depend on higher forces? q double dot, for example, q triple dot and so on. Well our experience has been, first of all Newton's equations do not say anything about what kind of force you have here at all. You could have in principle very complicated forces, could depend on the history of the particle as well. Now, we are making an assumption that it does not do so, that it depends only on q, q dot, and t and that is an assumption; based on experience to start with. There are situations where the force depends on q double dot.

For example, I give you a simple example we will not consider much of this at the moment. If you have a charged particle that is subject to external electric and magnetic fields, and it moves in space, if it accelerates, then principle it radiates, and once it radiates - electromagnetic radiation - that radiation could act back on the particle and produce what is called radiation reaction, leading to radiation damping, and that force indeed turns out in classical physics to depend on the acceleration of the particle. So, you could have situations where you have q double dot here as well; this is possible, but these are the simplest instances most of the time.

Now, already this is not an easy problem as it stands, because the forces in general are coordinate dependent, velocity dependent, and it is time dependent, and when it is time dependent, this implies that they are naturally changing the rules of the game as time goes on. I am applying a force, which is explicitly time dependent, and such a dynamical system is said to be non-autonomous. If it did not have this time dependence, I would say it is autonomous. Of course, non autonomous systems are bound all round us. If, for instance, I took a simple, single particle, and you applied a time dependent force explicitly, it was charged, you applied an electric field, and you change the electric field as a function of time, then it will be a non-autonomous system. We will come to non- autonomous systems, just tell you the possibility this exists.

But in the simplest instances, when you do not have explicit time dependents, and you do not have magnetic fields, you do not have viscous forces and so on, you have just f of q; nothing more than that; this would be the simplest instance. Now let us look at a simple harmonic oscillator and ask - what does it do? Just a single simple harmonic oscillator.

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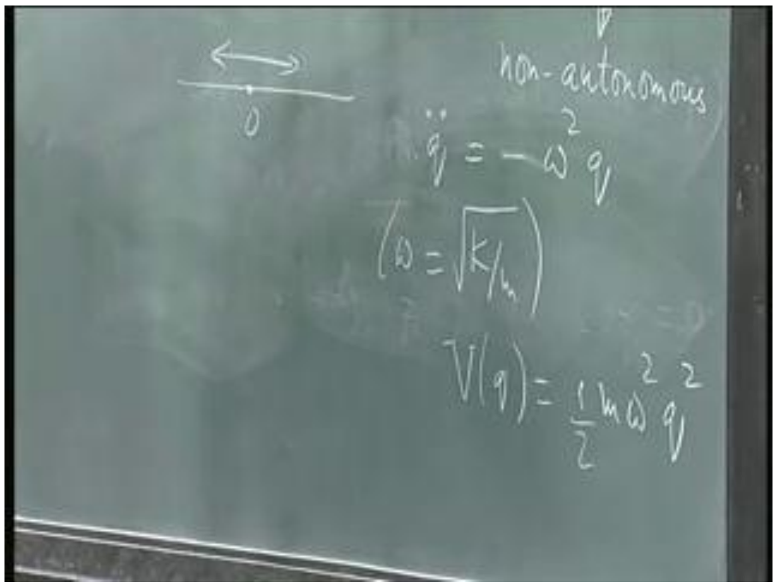

So you have the x-axis; this is 0 and you have a particle oscillating back and forth. What is the equation of motion, if the frequency with which it is oscillated - the natural frequency is omega - what is the equation of motion? It is mq double dot equal to… Is this is correct? q double dot is minus omega squared q. So move this m, this is minus kq, where k was a spring constant and you could write this as minus omega square; just to recall. What is special about this force? It is directed towards center of force, center of attraction, but what is special about it? It is a conservative force; this force is conservative in the sense that it is derivable from a potential.

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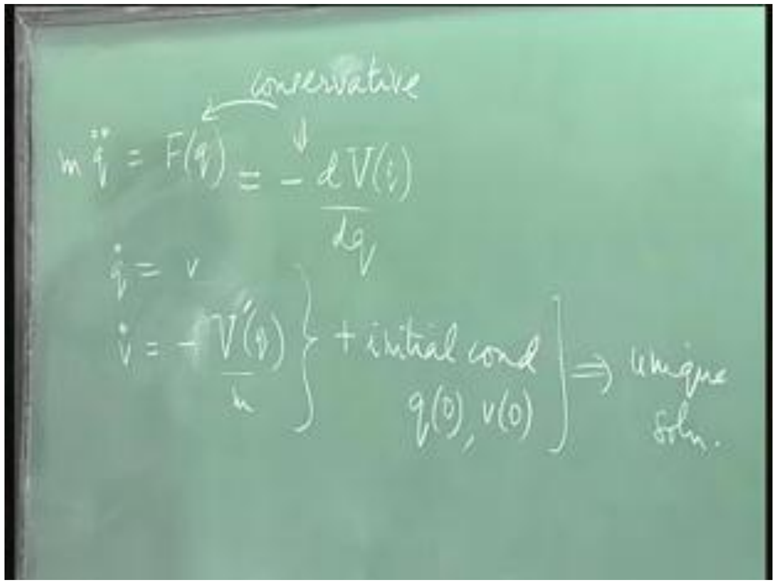

So as soon as you have a force that is conservative, we know that the q double dot equal to mq double dot is F of q, and if this is a conservative force, this implies this force is minus dV; it is the derivative of a potential. Now what is the potential here? What is V of q? It is one half m omega squared q squared. In a general sense, if I took an arbitrary potential V of q, and I said I have this particle moving on the x-axis under the influence of a conservative force derived from a potential V of q, then this is the equation of motion: N q double dot minus dV q over dq. But now you would say, look this is a second order equation, and to solve this set of equations, this single equation, you need two initial conditions, what should those be? Positional velocity. For instance, I could choose q of 0 and q dot of 0, I could specify them independently and I can solve this in principle.

Therefore, it suggests that what we should really do is the following - we should write q dot equal to V; you should write V dot equal to… this side minus v prime q over m; I use the prime for a derivative with respect to the argument. This is the way I should write these two equations. That is a set of two first order equations and this explicitly tells you that the independent dynamical variables are q and V or q and q dot. And, then of course, this plus initial conditions q of 0, v of 0, together implies a unique solution, together.

It turns out this is the most fruitful approach in mechanics, we write the whole thing down in terms of first order differential equations, always. And what is the advantage of writing this set of first order equations? They are coupled to each other, because the q equation involves V, the V equation involves q once again, but they are first order differential equations, and therefore, specifying the complete set of initial conditions, in principle, leads to a unique solution.

Now let us pursue this in our case of our harmonic oscillator. We know the solutions; we can write the solution down completely; it is quite trivial to solve this you get cosines and sine, in general, as functions of t, and then, you fit in the initial conditions and you got the solution uniquely. But this is not as interesting as finding out what the general kind of solution looks like, in this problem, it is periodic, you know that; what is the most general kind of solution, one can write down? You do not even need to solve this problem, the reason is - I could take the attitude, and I plot q versus V.

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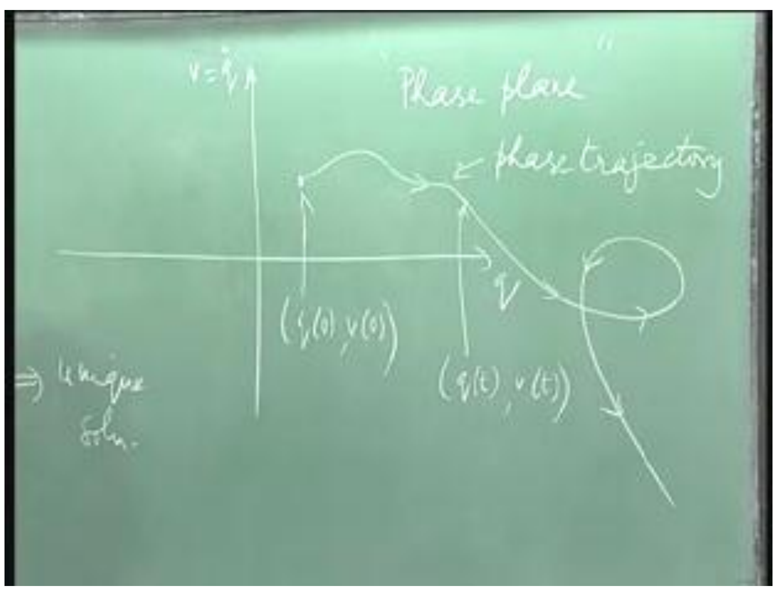

That is q dot on the vertical axis, and then, since, the initial conditions imply specifying a point on this plane - some initial q of 0 and v of 0 - as time goes along q changes to q of t and v changes to v of t, this point moves on the plane and this plane is called the phase plane. A point on this plane specifies this system completely, and this point changes as a function of time, and it traces a trajectory called the phase trajectory. So what to do is to start at this point - this is the initial point - and as a function of time to move in this way.

Now, tell me, can this trajectory cut itself? This is the phase trajectory. This point here q of 0, v of 0 and at any arbitrary instant of time this is q of t. I am interested in looking into the future. So I start with a initial condition, and I say what happens as a function of time, as things go along. But surely the initial condition itself - had the motion been going along forever - would have been reached from some earlier time, and therefore, it is really part of a half trajectory. If I had started the whole thing at t is equal to minus infinity, there would have been a trajectory, which comes along q, at 0 it would take this point and then it keeps going further down.

Now, can this trajectory do the following? Can it do that? Could this trajectory do that? Why not?

If the point of intersection is an initial condition the future is supposed to be determined uniquely.

If the point of, very good, if the point of intersection is an initial condition the future is supposed to be determined uniquely from this initial condition, and the force equations, and the equation of motion. So, you start with this point, and you have two outward arrows, and therefore, the future is not unique, because you could have started with that point of intersection as the initial state.

And then, you are told the future is uniquely determined, cannot branch of two different trajectories, and therefore, the phase trajectories of such a system cannot intersect themselves. And this is true in any number of dimensions; it is true in general for dynamical systems, provided you do not change the rules in between. In other words, provided you do not have a non-autonomous system. If you have a non-autonomous system, then this is possible, because then it distinguishes in time, it is not time translation invariant; the equations of motion are changing as a function of time. The rules are changing; therefore, there is nothing to stop you from intersecting the phase trajectories from intersecting themselves.

For autonomous systems, phase trajectories cannot intersect themselves; they cannot intersect each other either. Two different phase trajectories corresponding to two different initial conditions cannot intersect. We will write that down; it is so important that we should write this down; not intersect each other.

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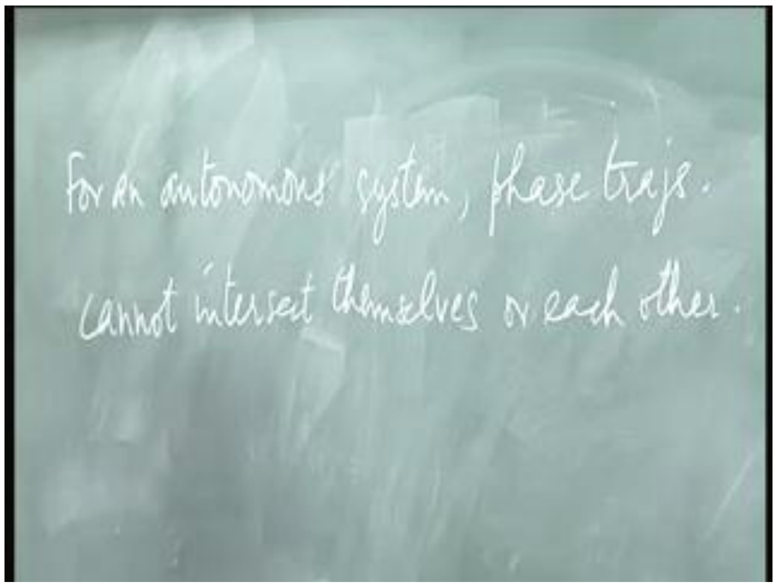

Is this possible? It is possible that this system does the following?

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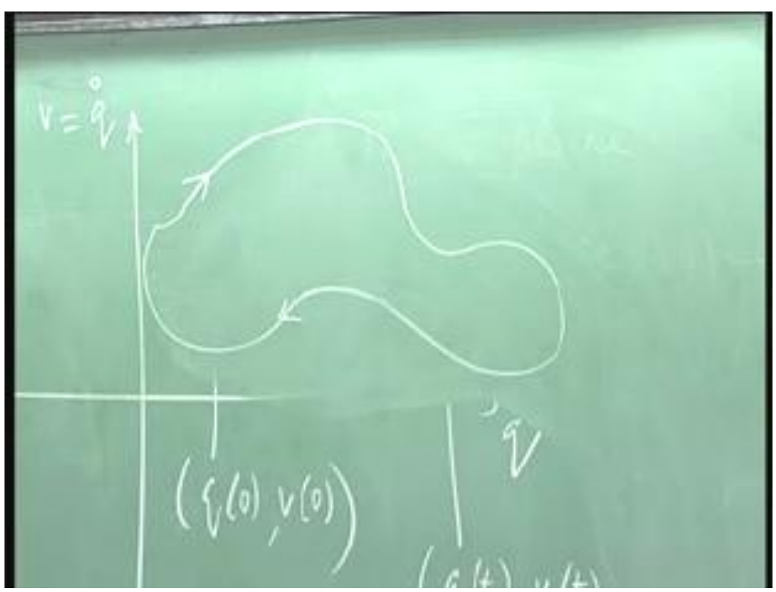

It starts at some point it goes along and comes back to its initial point; is this possible? This is eminently possible, because once it reaches this point it has no choice, but to retrace what it did earlier and comes right back. And what do you call such motion where the system comes back to itself after some time? Periodic motion; it is periodic motion. The only exception is periodic motion. So, we say closed - simple closed - implies and this is implied by periodic motion.

Let us look at the case of the harmonic oscillator once again, and ask what happens. We know in this case every initial condition corresponds to periodic motion of some kind. So let us look at that case.

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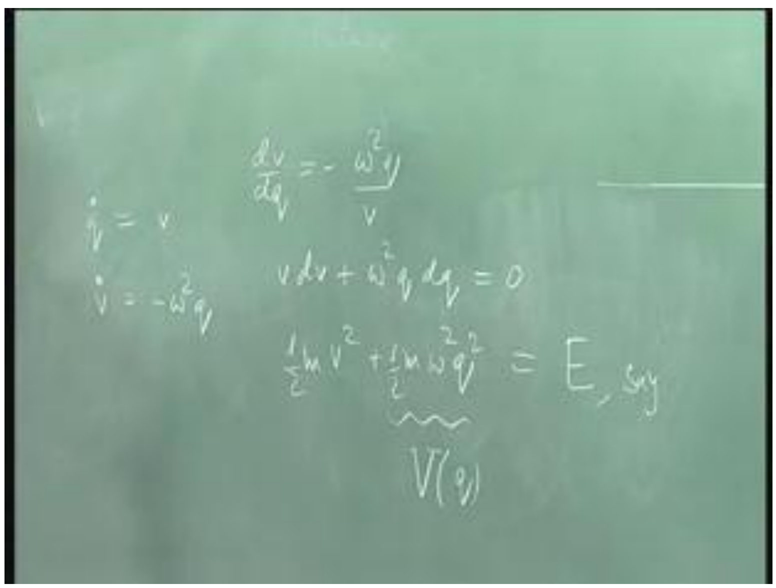

Now, I specialize to the harmonic oscillators. So q dot is v and v dot is minus omega squared q; so the q double dot is equal to minus omega squared q. What will the phase trajectories look like? You are going to integrate this; you have to take this equation of motion and integrate it. So let us do that. Let us integrate this equation. We have to take these two equations, find out what the slope of the phase trajectory is like, and then integrate it.

So I divide one equation by the other, and I get dv over dq equal to minus omega squared q over v. I need to integrate; this is the slope of the phase trajectory at any point. What should I do to integrate this? Well the variables are separable, as you can see it is very simple; so it says vdv plus omega square qdq equal to 0. And if I integrate this, what happens? What is the integral of this? Half v squared; so it says half v squared plus half omega squared q squared equal to constant; a constant of integration. What is the significance of this constant of integration?

Energy…

Almost; if I multiply this by m, then of course, you immediately see that this is nothing but writing down the fact that the total energy of the system, and this remember is v of q, potential energy; this is equal to e; I choose a symbol e for this constant of the motion; very evocative, because I know it is the energy in the system.

Now what is this equation? What does it look like? What kind of equation is this? In general it is an ellipse; unless, of course, these numbers are such that the coefficients of these two are equal; in general it is an ellipse. Now, what kind of ellipse? Where are the principal axis of the ellipse? The coordinate axis themselves; this is of the form ax squared over a squared plus y squared over b squared is 1; so it is some kind of ellipse in this fashion.

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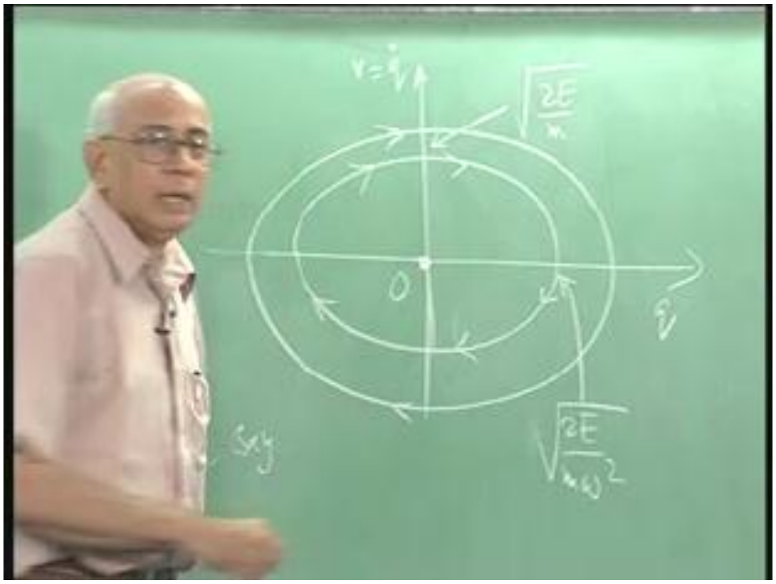

Wherever you start on this ellipse, whatever be the initial conditions, you are going to remain on this ellipse and come back forever. But you should never draw a phase trajectory without drawing an arrow on it to show the orientation; namely, how does the system move on the phase trajectory as time increases. So what should I do? There are just two possibilities - either it goes clockwise or counter clockwise. What does it do here? This physical problem corresponds to a harmonic oscillator, which is oscillating with center of oscillation at the origin of coordinates; therefore, you can determine what the direction of motion is in this phase trajectory; in which direction does it move? this way; clockwise counter; how do you say that?

q dot is one axis; q dot is positive.

Well, if I start, if this is the center of oscillation, I am going back and forth on this axis, this is 0, if I stretch this oscillator to the end of its amplitude and let go from rest, this corresponds to being here at this point. q is at its maximum. the velocity is 0. And then as you can see, when I let go from rest, the velocity directs it back towards the center of motion, and therefore, it is directed to the left; in other words, the velocity goes negative. So inside it does this; goes negative, and then, hits at this point, it passes through the origin again going rapidly leftwards, goes to the left most extreme - the amplitude here, and starts moving to the right again so the velocity becomes positive until it comes back here. Therefore, this is the phase trajectory for a simple harmonic oscillator.

We know, of course, that the amplitude of the oscillator determines the energy and vice versa. So it is quite clear that this point - the maximum value of q - is given by what? In terms of e, what is it equal to? It is square root of 2 e over m omega squared. That is another way of remembering the fact that the energy of an oscillator is one half ka squared, where k is a spring constant, and a is the amplitude; it is exactly the same statement.

What does this point correspond to? What is the semi minor axis? 2 e over m. What happens if I start with an oscillator, which has a slightly higher energy? What would the ellipse look like? It would be a concentric ellipse; exactly the same way; so that would be another ellipse, which will go like this. And therefore, as you can see, every time you specify a positive number e, the entire phase trajectory is determined. The system is constraint to then, confined to this ellipse depending on what the value of e is, the plane is therefore laminated by these ellipses.

There is just one exceptional point; there is just one initial condition, which does not fall into this picture. And what is that? If it starts at the origin; in other words, you do not stretch it; it is at the origin and it has zero velocity, then it remains there forever. The origin in the phase plane is a trajectory all by itself; user does not move, just stays, the ellipse there; it is the equilibrium point. We will come back to the significance of this equilibrium point, because it is a special trajectory all by itself. It corresponds to putting e equal to 0, and then of course, both v and q are compelled to be 0.

What is special about the harmonic oscillator? What is really special about harmonic oscillations, which are there all around us? The time period, is well… the time period is constant, in what sense? Independent of the amplitude; harmonic motion is the only motion where the time period is independent of the amplitude. There are other potentials for which also you have time period which is independent of the amplitude, we will come to that. They are related to the harmonic oscillator, but there is a simple way of showing that harmonic motion is the only one that is independent of the amplitude or the energy equivalent. For the moment, I want to point out that our statements here are borne out, independent phase trajectories, different phase trajectories, do not intersect themselves, and each phase trajectory that is a closed orbit corresponds to periodic motion; that is it.

We would like to show that the time period is independent of the amplitude; we would like to see if we can do that without doing any hard work. Is there a simple way of doing this? Of course, you can solve the harmonic oscillator problem, it is very trivial moment the moment you solve it, but is there is a simple way of doing this? There is simple way of understanding that there is no time dependence, no amplitude dependence on the time period. By the way, could you write down a formula for the time period from this phase trajectory?

Notice, that I have not solved the equation of motion. I have not started writing cosines sines or anything like that; I have just looked at the phase trajectory and that is sufficient actually, because what it is telling me is that an oscillator, which starts here or here or here or here, anywhere on this trajectory, they are just minor changes in initial conditions, the motion is exactly the same. So one of the primary advantages of looking at phase trajectory is that you do not have to worry about specific initial conditions; we are not so worried about that.

And in the case of one degree of freedom, we were able to write down the phase trajectory immediately. We did not solve the equations of motion. You are able to integrate this directly. If you have a conservative system - mechanical system - we know the total energy is constant. If therefore, you write the total energy equal to constant and I write this equation.

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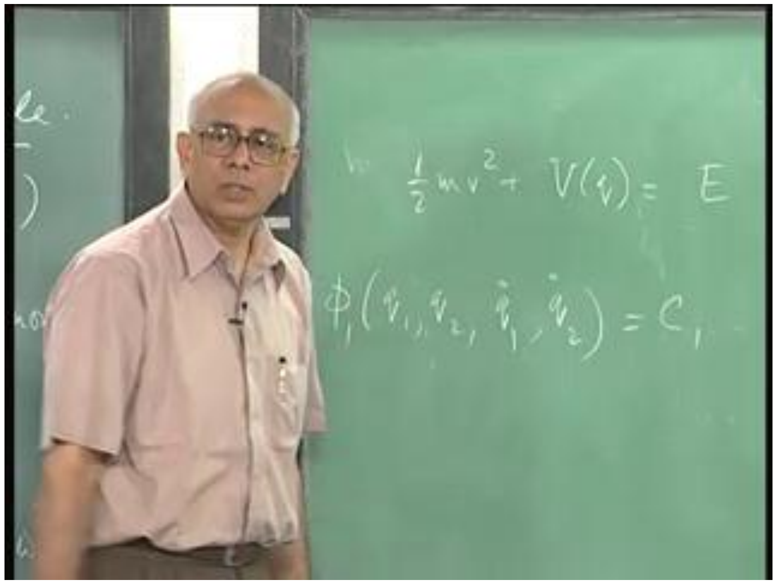

I write one half mv square plus v of q equal to constant e, this specifies the phase trajectory already. Because to specify a curve on a plane, I need one equation between two variables and this provides it; there is nothing more to be done. What happens if I have 2 degrees of freedom? I have a problem with q 1 and q 2. Then, of course, as you can see the independent degrees of freedom q 1 and q 2 and associated with it there will be a q 1 and a q 2 dot, what is a dimensionality of phase space in this problem? It is four; so I cannot draw picture of this kind. But the phase trajectory after I solve the equations of motion would still be a one-dimensional object, in a four dimensional space, and the same thing goes through the phase trajectories cannot intersect themselves, any closed phase trajectories periodic motion, and so on. But is it enough to find one constant of the motion in that problem? Would that suffice to tell you what the trajectory is? No. Between four variables, you have one equation, you do not specify a curve; you specify a three-dimensional surface.

How many independent equations do you need before you can specify a curve? I have some functions, suppose I say phi 1 equal to c 1; I discovered a constant of the motion; some function of q 1, q 1 dot, q 2, q 2 dot is constant, may be the total energy discovered, let us say, and if get an equation of this kind, this is going to specify in q 1 q 2 q 1 dot q 2 dot space a four-dimensional object, a three-dimensional hyper surface.

But I need a phase trajectory. I, therefore, need, yes, I therefore need more constants of the motion. I need a phi 2 of this c 2 and I need a phi 3 of this; I need all these three and the mutual intersection of these constant surfaces could give me the line, could give me the phase trajectory. So you begin to see already that is not a trivial matter to solve problems with more degrees of freedom than one or two; already with two it becomes complicated. If it is of the order of Avogadro's number, as in the gas in this room, it is a hopeless task. So our trick would be not to attempt, could not be to solve equations of motion in general, except for simple system.

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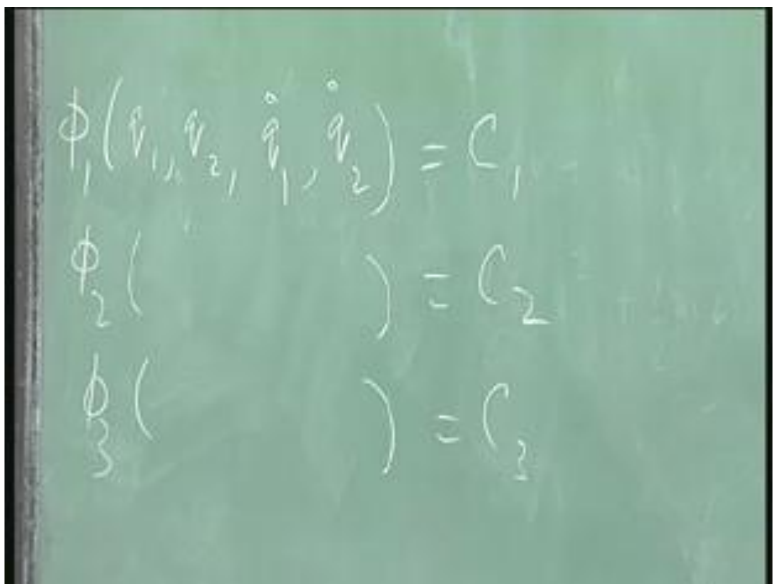

But we need to go little bit further with this see how we can find the constants of the motion what we need to do do so. But this should be clear already that you need more constants of the motion; the more you find the closer you are to solving the problem. But with one degree of freedom; if it is a conservative system the fact that the total energy of the constant of the motion is enough; if you write this energy as the function of the coordinate and the velocity, the job is in principle done, complete. We will write down phase trajectories and see what they look like. What if I change the sign of this potential? What if for shear perversity, I wrote this as minus...What would this motion look like minus half? What could this look like? We have to be a little cautious.

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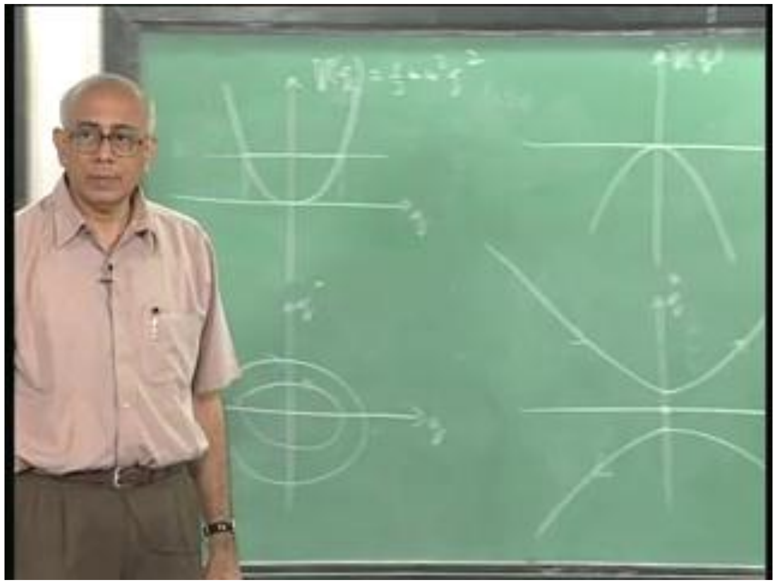

In the case of the harmonic oscillator, if I plot q versus v of q this is one half m omega squared q squared, this was parabolic, and then the motion took place as if you had a particle moving in this well, in this potential well, back and forth. So for any given specified total energy, this was the amplitude of the particle and the particle moves back and forth here; when it is here, it is all potential energy; when it is here, it is all kinetic energy, because a potential is 0; and when it is here, its again all potential energy; it moves back and forth, and this was simple harmonic motion. And then, we found the corresponding phase trajectories; so here is q and here is q dot; we found these phase trajectories was simply ellipses, concentric ellipses.

What happens if I took this potential and inverted the sign? So this is an inverted parabola. What would this motion look like? Let us try to guess. So here is q dot, here is q, if the particle starts with q equal to 0 and q dot equal to 0, it stays there; it stays there, because there is no change any more.

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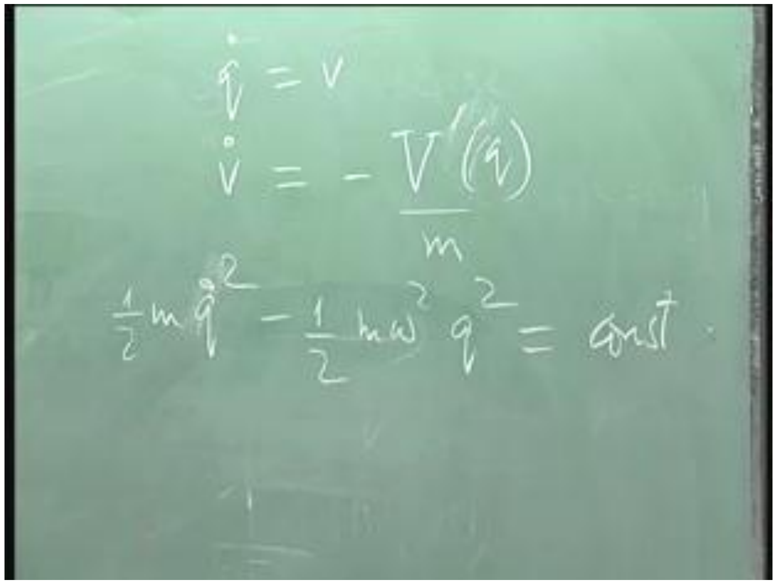

Remember the equations of motion, always. So the equations of motion are q dot is v and v dot is equal to minus v prime of q over m. So if it starts at the origin with zero velocity, if this is 0, then and that 0; this is an extremum of the potential; the maximum of the potential at the origin. Therefore, this quantity is 0 and that is 0 to start with here. Therefore, neither q nor v can change with time, because both the derivatives are 0 to start with and remains 0. So the particle would remain here at this point.

On the other hand, if you start it with a slight positive velocity at this point, what would happen? It would just roll down this hill and escape to infinity. If you start it here with a slight negative velocity, it will roll down this hill and escape to minus infinity. Does this correspond to periodic motion? The motion, there is no periodic motion here at all. Therefore, the trajectories cannot be closed phase trajectories.

So what would they look like? What would they look like? They would look like hyperbolas, and you can do that very easily, because in this case we know that the equation of motion says that one-half… let me just write q dot q dot squared minus 1 q squared equal to constant; the total energy. Now, that thing suggests immediately that this is a hyperbola.

You know x squared minus y squared equal to constant hyperbolas, family of hyperbolas and they are not closed curves, they are open curves. What do they look like? What would they look like here? So think physically. I start at this point at q equal to 0, but I give it a slight initial positive velocity. So I am really here, at this point, and then q increases as a function of time. What happens to q dot? It also increases in the forward direction. So where does this guy go? Goes off in this fashion. Now what happens if I start here and push it to the left? q decreases. Then what happens? So what should I draw? So we are here and then where does it go? It goes from here in this fashion.

What happens if I start at infinity and give just enough energy to crawl up this well and reach this point? What would then happen? Well it would crawl up here and end here. Then, of course, if it stop exactly at this point. If I gave it a little more energy, it would go with the barrier to the other side. So what is the complete set of phase trajectory look like? They are only half trajectories. What would complete set look like? You must look at all possible initial conditions. By the way, the motion on this, goes in this direction, goes in this direction. These would getting completed, so you would have this is part of a half trajectory, gradiates like this, in this fashion; it is a families of trajectories.

I would like you to tell me if this is correct - If this picture is correct - and to complete this phase portrait. The full set of phase trajectories, all different kinds of phase trajectories is called the phase portrait. And the phase portrait for the simple harmonic oscillator was very simple; it was just family of concentric ellipses. The phase portrait for this inverted parabolic potential does not have any close trajectories. No periodic motion, it consists of hyperbolas, but you have to tell me what these hyperbolas look like.

The other thing you have to tell me, is whether you could have negative energies in this problem. It has no motion corresponding to a negative energy in this problem. Remember if I took any total energy to be negative, and this is the potential energy, this would imply the kinetic energy is negative, which is not possible, because that is a square of the velocity. So, no physical motion happens for less than for e less than 0; equal to 0 you are at the equilibrium point and e greater than 0, you have physical periodic motion.

In this problem, in contrast, you could have e less than 0; you could have a total energy which is this and this implies that it could be anywhere in space, in coordinate space, except between these two points, because between these two points if this is a total energy, and that is the potential energy, the kinetic energy is negative, which is not allowed. So this would be the allowed region for you, either on this side or on that side, but for e greater than 0, you can be anywhere.

Pardon me. So this is what I would like you to complete. You need to complete in this problem a typical phase trajectory for e less than 0, the phase trajectory or trajectories for e greater than or equal to 0, and the phase trajectory for e greater than 0. So you need to complete that and this not altogether trivial. There are three possible kinds of e negative, positive, as well as 0, and when e is 0, remember, you can have a trajectory which corresponds to just this point.

But you could also have other trajectories, because if you put e equal to 0 on the right hand side it says q dot squared, apart from constants, is equal to q squared. Therefore, q dot is plus or minus q with a certain slope; these are lines which go through the origin asymptotically and we will see this, what happens. So already you begin to see, that the phase plane analysis is much more powerful than trying to solve the equations of motion, but at the same time it tells you the difference between qualitatively different kinds of dynamical behavior, some of which would be stable, some would be unstable, and so on.

Now, we take it from this point next time, where we will complete this phase trajectory, this phase portrait, and then see what happens in higher numbers of degrees of freedom. And the other thing I would like to do is to show you that in this particular problem, in the case of the single simple harmonic oscillator, there is a very simple dimensional argument, which will tell you that the time period is independent of the amplitude. Of course, as a caution, there are the potentials which would do this, but they are much more complicated and I will introduce a few of them as we go along.