Classical Physics

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Lecture No. # 19

Let us take a problem two, and go through these questions once again, this was just the bunch of objective type questions. The first one says, the Lagrangian of a particle moving in a central potential V of r has two cyclic or ignorable coordinates; and remember, the definition of a cyclic coordinate something that does not appear, a generalize coordinate that does not appear in the Lagrangian at all. Is this true or false, its false in the reason is, only one, only the asymptote angle see does not appear, the theta appears because, it is appears is part of the matrix in the kinetic energy, there are terms which involves sin square theta and so on.

And therefore, there is only one cycle coordinate in this problem, yeah this is a very good question, is it possible to construct a coordinate system such that, there is a constant of motion corresponding to every cyclic coordinate or vice versa, for every cyclic coordinates there is a constant of the motion. In principle you see, the moment you have the cyclic coordinates there is automatically constant of the motion, because the derivative of the Lagrangian with respect to the corresponding generalize velocity give you the momentum and that is conserved, this quantity is conserved.

The question is can we construct the coordinate system such that, there is a constant of the motion corresponding to every cyclic coordinate; and the answer is, for a Hamiltonian system this is precisely what happens, when you have an integrable Hamiltonian systems, because you go to action angle variables, and which all the angle variables are in fact, cyclic coordinates automatically. So, in that sense you constructed a coordinate system, but it is not a global coordinate system, because when you go to an angle variable an action angle pair, it is some function of the original q's and p's.

And of course, this is like saying that at every point, I have a different change of variable numerically in the sense that, you have some function and it is not a once and for all like going from Cartesian into polar coordinates are anything like that. But, it is a change of variable what called the diffeomorphism at each point, so this is certainly possible but then, it is not guaranteed, not true for all Hamiltonian systems; for integrable Hamiltonian systems, for which the necessary and sufficient condition is that you must find n constant of the motion for n freedom Hamiltonian in evaluation with the each other.

There are deeper reasons as to why they should be in evaluation, let me just mention that, the fact is we know that if you have bounded motion, and it is an integrable Hamiltonian system then you can change to action angle variables, and then the motion effectively takes place on an N dimensional torus. Now, on an N dimensional torus, it is a very interesting topological object you can always find, you need to find what is called a vector field which parallelizes the torus, in other words you must have a basis set of vectors, unit vectors which as you move along this torus, would come back to the original orientation after any possible circuit.

This means, this torus is what is called parallelizable, in the sense of differential geometry, and the differentials of these n constants of the motion in evaluation form of a basis, the linearly independent of each other; and the only comeback N dimensional manifold is n-torus which is parallelizable is an n-torus. So, this is the reason why the torus appears so naturally, in problems of integrable Hamiltonians, so that is little bit of technicality.

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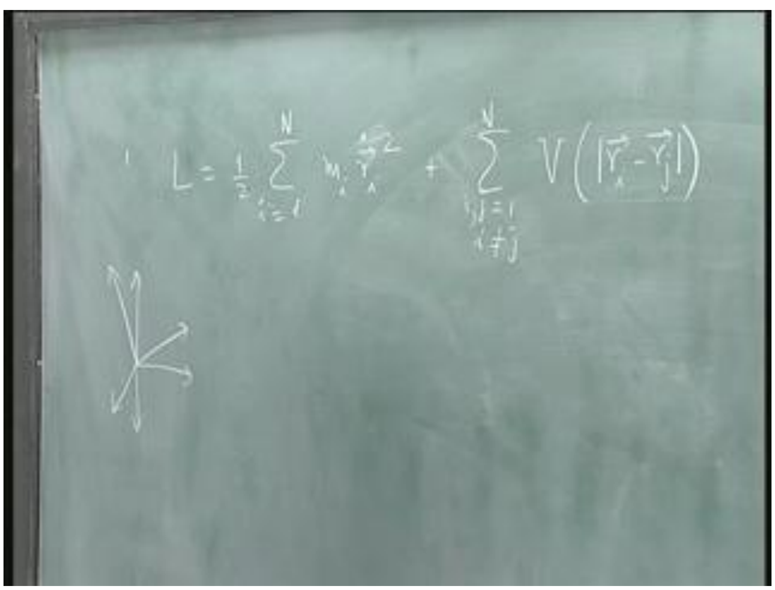

In the next four or five parts, they we consider a Lagrangian, which is just standard Lagrangian which is 1 half summation i equal to 1 to N m i r i dot square sorry r i vector dot square, the kinetic energy plus summation i j equal to 1one to N and i naught equal to j, so that no self interaction. The potential is a function only of the distances between pairs of particles; this is the kind of a as you can see general problem, in you have N particles they interact with each other, there non relative is take and the interaction proceeds by pair wise potentials.

Such that, the force between any pair of particles is directed along the line joining these two particles, that all that it says that is what this treatment says here. Then the total angular momentum of the system is a constant of the motion, is it true or false

It is true.

It is true, because the entire system is invariant and rotations of the coordinate system, if I rotate the coordinate axis and move from any possible, any set of orientations to any other set of orientations in this fashion. Then this Hamiltonian, this Lagrangian does not change its invariant, and by north-west theorem one can then find, constant of the motion corresponding to this set of transformations.

This transformation is generated by three rotation generators, because you are in three dimensions, correspondingly you have three constants of the motion, and they are the components of the total angular momentum of the system about in arbitrary origin of coordinates, so it is a constant of the motion.

The next statement is it is possible to make a Lagrangian transformation to the Hamiltonian in this case, is that true or false.

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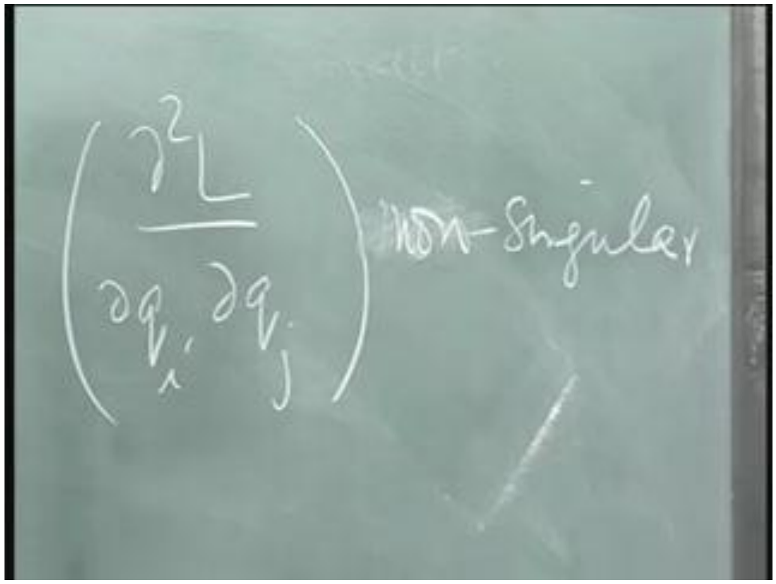

Well it is true, because what you required in general is that this Hessian matrix should be non singular, and indeed that is true because, in the kinetic energy here, all the different velocities are decoupled from each other; so imagine writing this, as a some of the Cartesian velocities for each of the particles. There are n particles and there are 3 n such analyze velocities, and that matrix there is a trivial one, it is actually a diagonal matrix because, you can see there are no cross terms at all.

Yeah, q i dot thank you yeah q i dot g j dot and those are these Cartesian components here, therefore this is certainly possible this case is completely trivial, we can find, by the by in general if you started with the set of particles, and you impose constraints between them, various kinds of constraints. Then in general what will happen is that this kinetic energy here, as you know from experience is going to have coefficients, it is going to be still quadratic in the general velocity, but it is going to have coefficients which would depend on the generalize coordinates.

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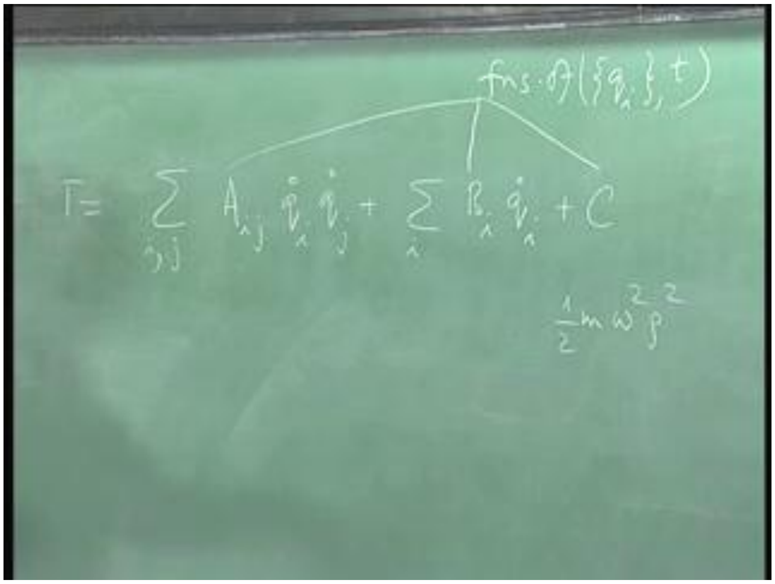

So, the general form of kinetic energy T in such cases, would be of the form i j some A i j q i dot q j dot plus summation over i some B i q i dot plus some function C, where A, B and C are functions of the set q i and possibly time; if you have time depended constraints you put a particle on a plane like this and you till the plane as a function of time. Then of course, you can see that the independent coordinates, would finally the kinetic energy would look like that, but A, B and C would have functions of both coordinates as well as time. But, the important thing is it would still remain quadratic in the generalize velocities, in this function but, it could have linear terms, could have a constant term as well.

If they are connected to each other, if there is constraints relating these 2, these 2 particles, then in general this would happen always, there would be a cross term; it is still independent, this coordinates are still independent, but they are related to each other through some constraint equations possible. Then in general the most general form you have is a matrix of this kind matrix A i j, the other properties you can write down about this matrix, we do not go to that here but, then you must remember they could also be linear term, there could be a constant term. You recall the problem of the bead on a hoop, there what happen was you ended up with just one independent coordinate, and that was the radial coordinate rho. But then you had a term which had, form like half m omega square rho square, there is no this was in the kinetic energy and there was no velocity here at all. And that was example of this point this C here, and of course the rho dot square this term here had a function of rho, a fairly complicated function on rho depending on the shape of this hoop, and that is an example of what A was.

What is curve?

Yeah

Yes, yes yes

Of course, if you have purely Cartesian coordinates it is not going to happen, but because you when in that case to cylindrical polo coordinates, and then the matrix is not just constants, if you write the kinetic energy down. These are just constants here in Cartesian coordinates, but the moment you go to spherical polo coordinates or anything like that, remember you are going to get r square theta dot square r square sin square theta phi dot square and so on, so automatically functions of the coordinates are rise.

The next one was, we can find capital N constant of the motion that are in involution with each other, true or false?

False

In general false, unless you are given much more information about this potential; trivial cases we switched off the potential of course this nothing there, it is immediately integrable, if this potential can you give me the example of a potential, where this would really be true. If you can find capital N constant of the motion, in fact there are 3 n degrees of freedom here, another system is integrable, you can find free and constant of the motion. Can you give me an example of when this, even with the potential this problem would be completely solvable and integrable?

What kind of potential would lead to this, suppose I had a collection of harmonic oscillators, suppose each particle was coupled to something else with a spring, this was quadratic, completely quadratic what would happen then, imagine do this in one dimensional first, I have a set of particles all are the line.

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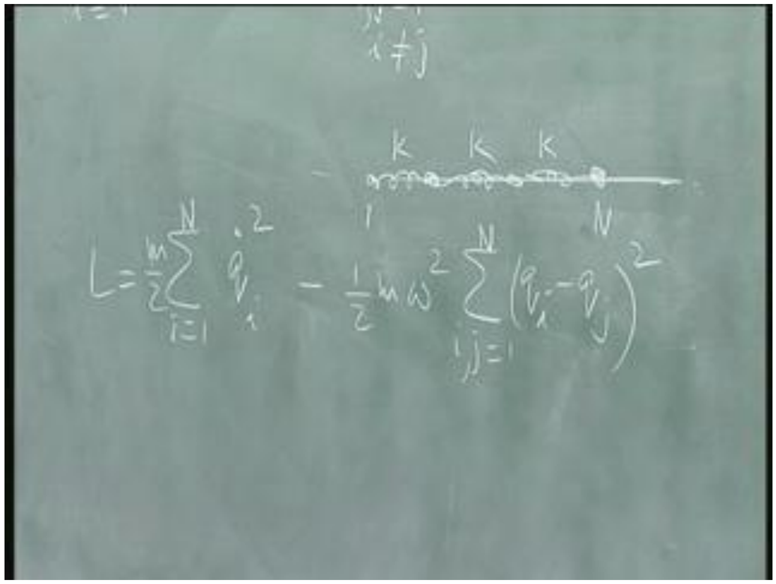

And each particle is couple by a spring to the next particle, couple by a spring to the next particle and so on, end of these particles.

Only to the next one, that is says it only to the next one, would this be doable, would this particle problem we doable, yes it would be doable it is a quadratic Hamiltonian, so you can actually go to a set of coordinates where the thing would look, there will be cyclic coordinate.

There is one cyclic coordinate here immediately, which you can identify I have N of these particles 1 to N couple by springs all of its, let us say have the same spring constant k each, and let us do everything in the x direction, and what is the what is the Hamiltonian of the system, what is the Lagrangian of the system? This is equal to summation 1 half, let us take the masses to be equal some simply to make like easy, i equal to 1 to N q i dot square, q i is the x coordinate of each of these particles minus 1 half m omega square summation q i minus q j whole square i j equal to 1 to N.

Because, I do not care even if i equal to j that gives your 0, so I do not have to put the sign r equal to j constraint this system here, do you think this is integrable, this is doable completely quadratic Hamiltonian, take two particles, take just two of that so let us put capital N equal to 2. In the connected by a spring, would you integrate this problem.

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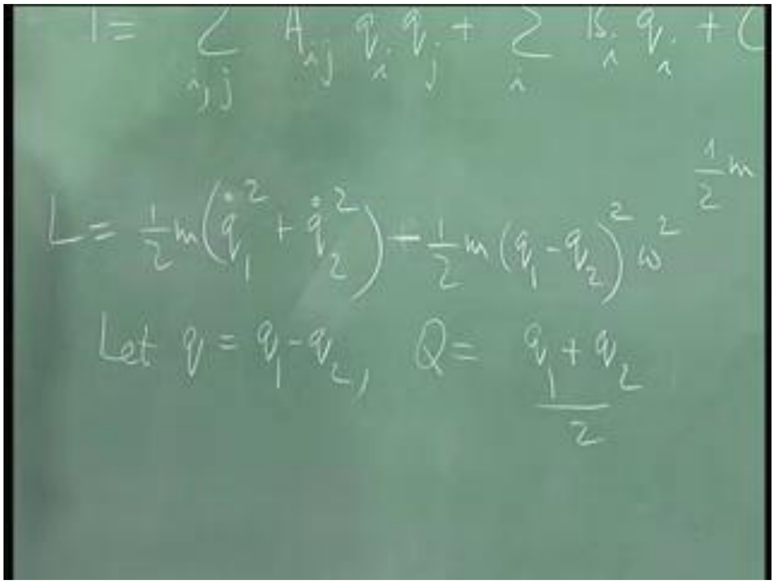

What could you do, 1 half n q 1 dot square plus q 2 dot square minus 1 half m q 1 minus q 2 whole square omega square.

What would you do now, it is got two independent degrees of freedom, but they couple to each other by this spring, now what to you do there?

Yes, I would change variables to a center of mass coordinates, because there is no external force on the system and therefore, the center of mass motion must be like that of a free particle, with a mass equivalent to the total mass of the system. So, what would I do I choose new variables and immediately say let little q, the q 1 minus q 2 the relative coordinate between these two particles, and capital Q equal to q 1 plus q 2 divided by 2, in this case inside equal masses, if they unequal what would I do, m 1 q 1 plus, can I only do m 1 q 1 plus m 2 q 2 over 2, cannot I do something else, cannot I do something else, cannot I do a linear combination, this is the good question.

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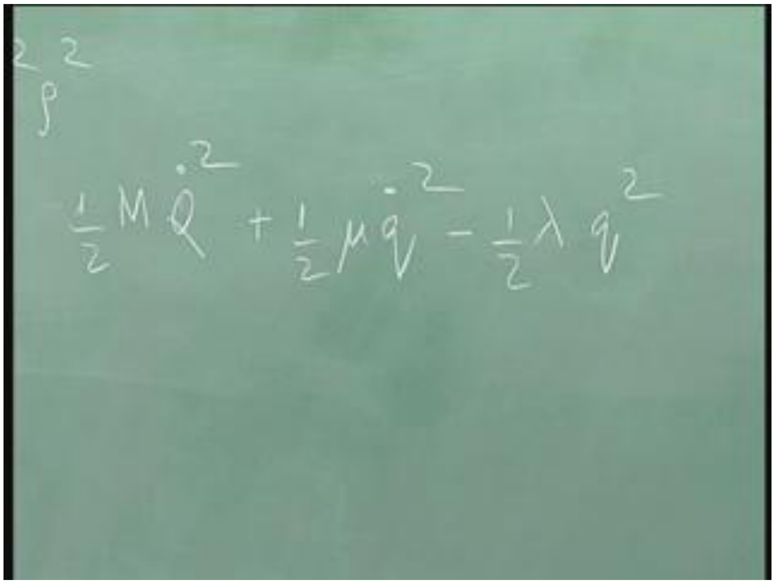

Why should I do q 1 plus q 2 over 2, in this problem of course once I do this since the masses are equal this Lagrangian would simplify, and what would it become, this one become 1 half M q dot square total mass M q dot square, and then what well, it would become 1 half mu q dot square where mu is the reduce mass in general. If I have 2 masses M 1 and M 2, and then it would have some spring constant times, some constant minus 1 half some lambda times q square it would look like this.

In the general case, if I have two different masses M 1 and M 2, this is what it would look like, capital M would be M 1 plus M 2, and little and mu would be what?

If there unequal masses, it would be M 1 M 2 divided by M 1 plus M 2, it would be reduce masses, and then what happens is there cyclic coordinate, capital Q is a cyclic coordinate, and what is that imply?

Which momentum

Total momentum is conserved, capital P which is conjugate to capital Q that turns out to be a constant of the motion, little p of course is not a constant of the motion, there is a force, there is a relative force between these 2 guys. So, you can change variables, and get read of one of these variables, what happens?

i less than j, then of course this do not happen, you some over i and j, take i less than 0, apart from some factor this is certainly true, so that problem is integrable when you have 2 particles. What happens, when you have 3, still do this we can still do this, you can go to the center of mass they get set of one coordinate, there is one cyclic coordinate and then two relative coordinates; say q 2 minus q 1 and q 3 minus q 2 and still work it is.

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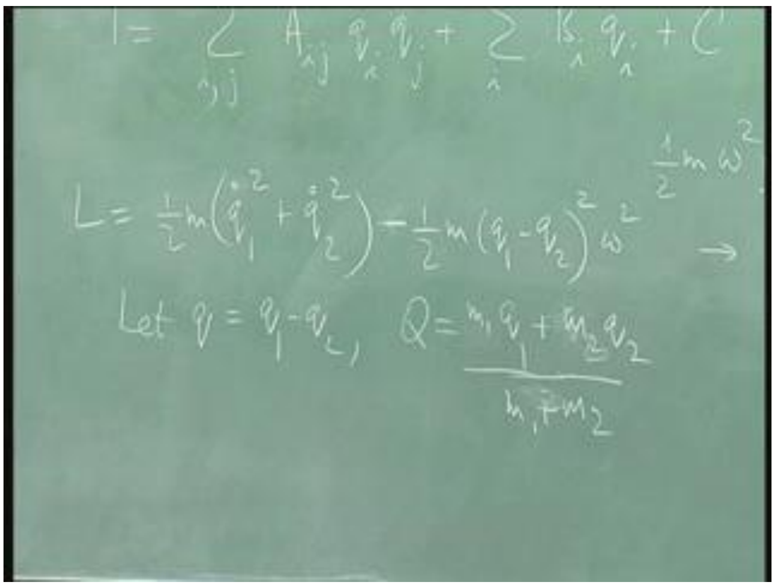

But coming back to my earlier question, assume the masses are all different for instance, why should I chose this, why should I chose in general m 1 q 1 plus m 2 q 2 over m 1 plus m 2, what happens if I chose some other linear combination, after all I need to liner combinations one of them is sort of dictated by the form of the potential its minus this. But, the other guy could be anything else, which is linearly independent some alpha q 1 plus beta q 2 I could choose that, what could not be a constant of the motion?

The center of masses not a constant of the motion

The conjugate momentum, the total momentum is a constant of the motion yes, so if I chose capital Q is equal to some alpha q 1 plus beta q 2, what happens?

Yes yes, so it will turn out that the conjugate momentum is no longer a constant of the motion, in order to fixed that one chooses the center of mass. What happens, if I chose 6 times m 1 k 1 plus m 2 q 2 over m 1 plus m 2, some constant times that why should I chose the one times that, y not some constant times, what would happen then?

Your constant of the motion would turn out to be not the total momentum, but is the multiple of the total momentum right turns out its ridicules do that, for convention you want it to be the absolute actual momentum, and that is the reason you choose a center of masses. So, you have to ask yourself these questions all the time, what if I do this, what if I do something else and so on; it is easy to go by the beat and path where people say do this, do this, and then do this that is the algorithmic approach, but you should ask suppose I do not do it, what happens then?

So, this problem with n masses and pair wise potentials of this kind completely solvable gets much harder and three dimensions, we do not have nearest neighbor and so on, but the practice once you have quadratic Hamiltonians, then things are much easier to handle. The case N equal to 2 alone is integrable, but not N greater than equal to 3 for a general V, true. Because we know for N equal to 2 you can straight away go to the center of mass and relative coordinate in the problem is solved, the moment you have 3 or more for an arbitrary potential, this is no longer true for general potential.

Is it true, if you had 3 particles interacting with each other pair wise, but by the capillary potential 1 over r potential, say gravitational potential between three part recursive; is that integrable?

It is not integrable, famously not integrable two yes, the two body problem is solve, but three no in general no, there are special cases which could be integrated, but the fact is in general per arbitrary masses, this is not true, this is the famous three body problem which people spend a lot of time, and eventually Poincare I think establish that its, essentially not integrable, its not solvable problem. There are three cyclic or ignorable coordinates in this system true or false?

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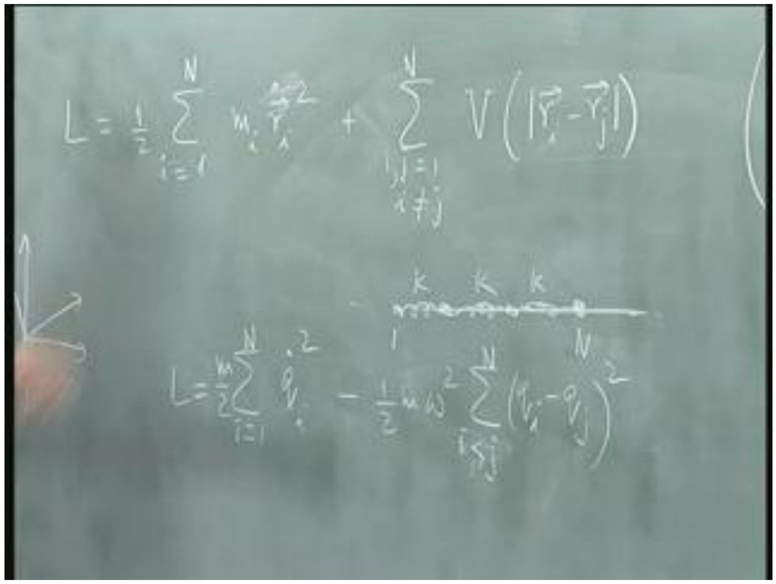

This system, why do we say that why do we say that, it is seen to me, that if I go to the center of masses system, the center masses coordinate, which is the vector is actually cyclic coordinate. So, there are indeed cyclic coordinates in this system, center of mass coordinates you do not appear in the Lagrangian, because the Lagrangian depends, a potential depends only on write a plus here, so minus. The potential depends only on the relative distances between particles and therefore, the total center of masses coordinate does not appear anywhere in the Lagrangian.

By the way, if you have more than 3 or more particles, then it is not always the best choice to chose the relative coordinates, always we have a potential of this form is naturally dictated, but in general if you got a 3 body problem or a 4 body problem and so on. The other choices of coordinates there coordinates and so on, which could be more efficient in certain instances, but getting read of the center of the mass is always useful thing to do, because that is gives you three cyclic coordinates right away.

And N dimensional dynamical system is given by x dot is gradient of 5 the gradient system, if del square phi is less than 0 everywhere, then the system is conservative, a conservative dynamical system, true or false?

Its false, in this case in practice like to have does quite 5 equal to 0, then in the standard definition of a conservative system the vector field as vanishing divergence, and then its conservative. Like h of q p V the Hamiltonian of a an autonomous system with 1 degree of freedom, if a of q p is any function of the dynamical variable such that, the Poisson bracket A with h is 0, then A is either a constant or a function of h itself, true or false, 1 degree of field just 1 degree of field; it is an autonomous Hamiltonian system therefore, how many independent constants are motion can you have 1, the Hamiltonian that is it, time independent constant of the motion.

So, it says if A with h is 0, then A is either a constant or a function of h itself and that is only true.

Not necessary, but definitely there related to each other.

It is a function of h, and may not be able to invert it you at all points, but it is a function certainly true, then not be a single valued function everywhere, but it is certainly true. Is this true by the way if you had more than 1 degree of freedom

Is this true, you have more than 1 degree of freedom, not true, not true, definitely, so you cannot make a statement of this kind, it could be another constant of the motion which is an involution this, incidentally a related question if Poisson bracket of A with B is 0 and A with C is 0, is a Poisson bracket of B with C is 0 necessarily, not necessarily true. Again if you solve 1 degree of freedom, then this this whole thing reduces we know that B and C would be functions of A any of the Hamiltonian; so not true in general. The critical points of an autonomous Hamiltonian system can only be saddle points and centers.

That is true, because it is a conservative system for a Hamiltonian system there are no attractors in phase space, no limits cycle, no change attractors and no critical points, which are asymptotically stable and so on, no spiral points and so on. You can have saddle points you can have centers, and then you can have more complicated kinds of behavior, you could have a chaotic behavior, completely chaotic behavior; but you would not have the so called strange attractors, which exist in dissipative systems, there do not exist and Hamiltonian systems.

The particle moves on the x axis in a potential k mod x to the alpha, were k and alpha are positive constants, all the phase trajectories are close trajectories in the x V plane, true or false?

This is true, this is like a bounded potential which is going up in some fashion, and all physical trajectories are close trajectories. And continuing the time period of oscillator the motion of the particle is independent of its total energy, in the cases alpha equal to 2 and alpha equal to minus 1, another words you told if the potential is k x squared, then the time period is independent of the. And the case alpha equal to minus 1, there is k over x and that is not true necessary, this is not true.

x x squared is true

x minus 1 is not true x minus 1 is not true, what about x minus 2

What about, what about x square plus 1 over x square?

What about x square plus 1 over x square, that is an interesting problem, it actually takes on field something totally different.

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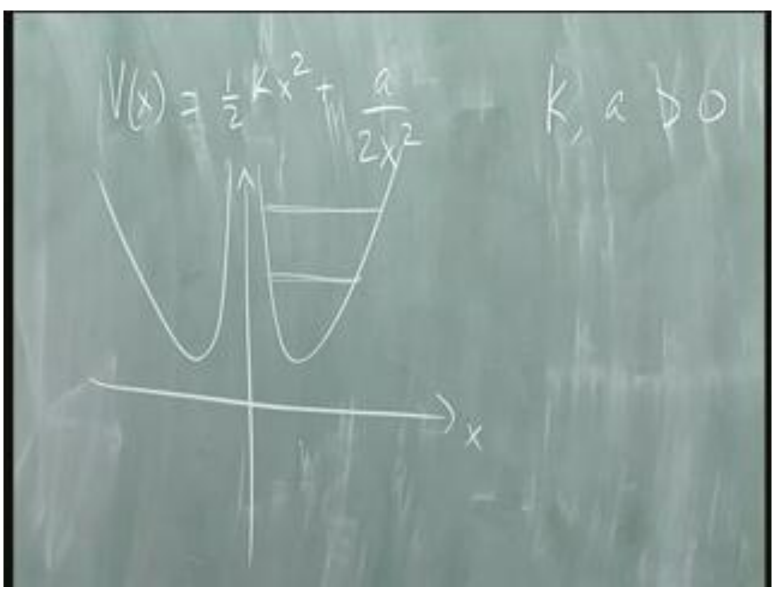

Look at that potential a is the positive constant, k, a positive what is the shape of v of x.

It will be two, it is going to be infinitive here, so it goes up like that parabolically and then it is a infinitive here its goes up like that parabolically on this side, the symmetric completely, and you can have periodic motion in either of these two potential wells. So, this question asked is is a time period for this motion, the same as a time period for that motion of course, you have by symmetry exactly the same sort of motion possible here. Remarkably enough, the answer is yes, the time period for this motion is independent of the amplitude, even though it is not a harmonic oscillator potential, this is called an isospectral, isochronous potential, it is called exactly the same behavior as the harmonic oscillator.

And if that is the case, then what should the time period B, it would be exactly the same as this was not present at all, so the time period in this problem is independent of a, I want it to work this out, it can actually be done write down the time period it is not hard to do because, you know the Hamiltonian, you know the phase trajectories, and you know what it is completely symmetry motion, so you can actually find out what the time period is. What you need to do is, to take some turning point here, and turning point here for a given value of the energy, and integrate to find the time period which takes, twice the time it is takes to go from here to there, and back to there.

And you will discover that this integral is actually independent, finally of the position of these turning points, its depends only on this constant k. So, it is a remarkable fact, it have whole family, because now I can change a it is a parameter and as I tune a the time period remains independent; this is the starting point of a whole series of investigations, because you can add make the potential more complicated, this is part of what is called the Sutherland model.

And then from here, you go on to the quantum version, then you can put all these potentials on a line pair wise potential and integrate, and from there you get other integrable models and so on, so this a huge field of study the worded to integrable models, starts with this very simple example. And when you quantize this problem, then you have again in a situation, where you have harmonic oscillator like behavior, which means the energy levels are equally spaced, you have non quadratic potentials, for which the energy levels are equally spaced, and they called isospectrul oscillators.

So, there is a huge field of study associated with this apparently simple problem here once again, what would you guess is the reason, why this thing becomes independent of a take a guess, well there is no deep reason in that sense that I except of you, but what would you thing is what is the mechanism by which is this happening? It would imply that you can go to some new set of coordinates, you can make some change of variables such that, it looks like a harmonic oscillator right, so that eventually what happens.

Now, the next one was consider an autonomous four dimensional dynamical system, where x is in and f for elements of r 4, the system can have at most free functionally independent constant of the motion, that do not have any explicit time dependents, true this is certainly true, it is just general statement. Continuing the system can have at most four functionally independent constants of the motion, of each at least one must be explicitly time dependent, also true. The product of all the Eigen values of any rotation matrix in N dimensional Euclidean space must be equal to minus 1 to the n, false. What should they product of all Eigen values be, plus 1 the proper rotation should be plus 1, because these are problem matrix is and they are connected to the identity and therefore, the determinant is plus 1.

The system consist of n particles moving a space and the k integrable constraints, some other constraints maybe time dependent, in the question is the number of independent generalize coordinates of the system is 3 times n minus k.

What is it equal to…

x 3 minus k naught 3 times n minus k, the kinetic energy of the system in general is of the form that I wrote down there, is that true or false, where n is the number of independent degrees of freedom functions of q'a and t's I already said that is true, so this is. A particle moves is space under the potential k times x 4 plus y to the 4 plus z to the 4, where case of positive constant, this look like a harmonic oscillator potential, but its 4 powers the angular momentum of the particle about the origin of coordinates is a constant of the motion.

Why is it false?

It is not rotationally symmetrical, it is not invariant and rotations, it is invariant under the rest of transformations which would exchange the x, y, z axis, and otherwise 90 degree rotations about the 3 axis, but that is a discrete set of transformations; it is got Cartesian symmetry you can exchange y for z, z for x and so on, and so forth. But, this is not the same as a continuous set of rotations, therefore this problem the angular momentum of the origin is not constant in this case. The next one is a particle moving in space, under the potential V of r and this is not a central potential, it is a function of the position itself.

I V of r is invariant under the parity transformation r to minus r this implies the existence of a constant of the motion according to north west theorem, it is also a false, because its again a discreet transformation; and you do not have and you have variant current associated with it, and therefore there is no such requirement. I should mention here, and we are going to talk about symmetry is later on in the course, that there are other sources of symmetries; there are continuous such of transformations, then there are discrete symmetries like parity, may be time reversal invariants and so on. And then there are others symmetries which are not induce through changes of coordinates systems, rotations, translations and so on, would be things you due to the coordinates, but you could have other symmetries which are not, which are nothing do with the coordinate system, like a gauss transformation on a electromagnetic potentials.

Those transformations could also lead to symmetries, they could also lead to conserve quantities, equations of continuity and so on, but they have nothing to do with coordinate transformations. And finally, there is another source of symmetry in nature, and that is there are topological symmetries, there are some there is topological conserve quantities there are some conserve quantities, which would completely topological in nature they have do with the nature of the space that you are end, and nothing to do or so ever with space time transformations are anything like that; and I mention that when we come to it that stage.

Once again applying north west theorem in those cases, not immediately obvious (()), the simple example would be, you can just given example, so I do not mystify you, you take a long string, very long string and you put a not on it. Now, obviously you can move the not around, but it is there somewhere, in some sense the fact that you have one not on it, continuous no matter what do you do, no matter where you move; unless you do something drastically, if I go to finite n point and move it or you go to infinitive or if you cut the string.

If you do not do those things, but you take a long circle of species of rho with single not on it, no matter what you do this always one not on it, and that is a topological fact which you get it off. Next one was the canonical momentum of a charge particle moving in an electromagnetic field is dependent on the gate chosen for the electromagnetic potentials yes, this is in fact true. The dynamical symmetry group of the N-dimensional isotropic harmonic oscillator is s o 2 n

Its s u n, it is actually s u n and not s o 2 n, how many generators thus s o 2 n have?

2 n times 2 n minus 1 over 2 that is n times 2 n minus 1, and how many generators does this does the simplected group of this problem have, there are n degrees of freedom, so its its n times 2 n plus 1, that is the number of degrees of freedom. So, that is a larger group than the group of rotations, and how many generators does s u s u n have?

n square minus 1, so much smaller group, so the symmetry group is the intersection of the simplected group s p 2 n with the group of invariants is the Hamiltonian, which is s o 2 n and that turns out to be s u n, much smaller group. As I said we will talk about these groups towards the end of the course; the system has 2 degrees of freedom the Lagrangian is q 1 minus q 2 q 1 dot minus q 2 dot square plus an arbitrary potential, its not possible to make a Lagrangian transformation do the Hamiltonian in this case, true or false?

It is true, because this hessian matrix become singular in this problem, so here is the case where you cannot do it, there is no way of going to a Hamiltonian system. Other remaining question were groups on the on the simplistic matrix is, the group of 2 n by 2 n simplistic matrix is with real elements, is a sub group of the group of 2 n by 2 n orthogonal matrix is with real elements; that is not true, we just found out that s o 2 n is actually smaller than s p 2 and so.

M denotes the Ecobian matrix of a canonical transformation, the transformation whose Ecobian matrix is given by M transpose is also canonical transformation.

True, true because remember, m transpose is related to m inverse and a canonical transformations invertible, so it is certainly true.

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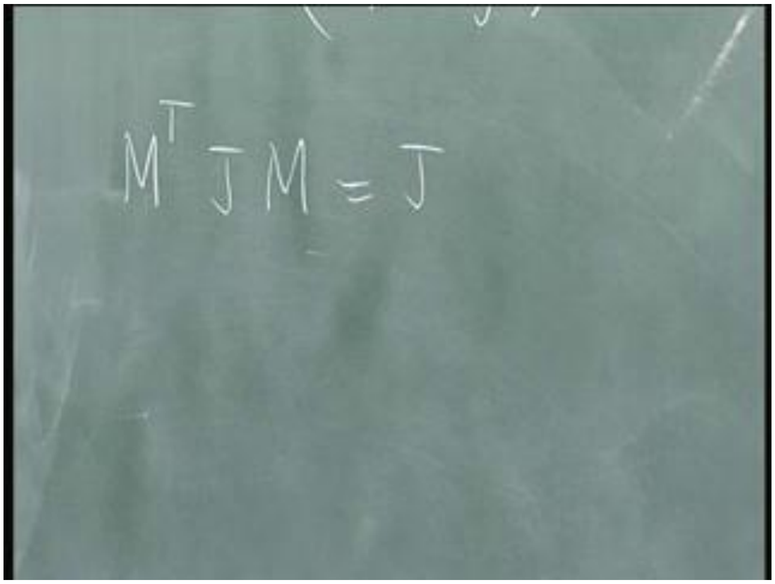

And what is the definition of a simplistic matrix, you must have M transpose J M equal to J on the side, and it is easy to show from here, that M inverse and M transpose or both also some (()) transformations. Now, this theorem this this is an important statement, north west theorem enables as to find a conserve, quantity associated with a group of transformations of the dynamical variables of a Lagrangian system; continuous transformations.

The statement is in order for the theorem to be applicable; the Lagrangian must be unchanged under the transformations belonging into the group

Not in general, not in general of course, if it unchanged, you end up with the conserve quantity; but in general, what can happen?

The total derivative of a function of a coordinates and time could be the change, now what is the implication, what quantity is unchanged, the Euler Lagrangian equations are unchanged, but as a consequence of what remaining unchanged, what is that remain unchanged, the action does not change, the action does not variable right therefore, this is suddenly true. In fact, this is important there are instances such as, the gauss transformation on the electromagnetic potentials where we saw, that the Lagrangian changes to precisely by the total time derivative of a function of the coordinates and time; and you still have a conservation law, you have a conservation of charge.

There is an analog of the (()) vector, and then be vector constant of the motion for the motion of the particle in any central potential of the form, V of r is k r to the n

Not true.

Not true, not true in general at all, there is no such thing, what is true is that if you have an r square potential, the harmonic oscillate a potential then there are other constant of the motion that is an integrable system, that is super integrable, in the sense that if you have a symmetric group, which is a large symmetric group s u n, in the case of the oscillator, on three dimensions if you as you three.

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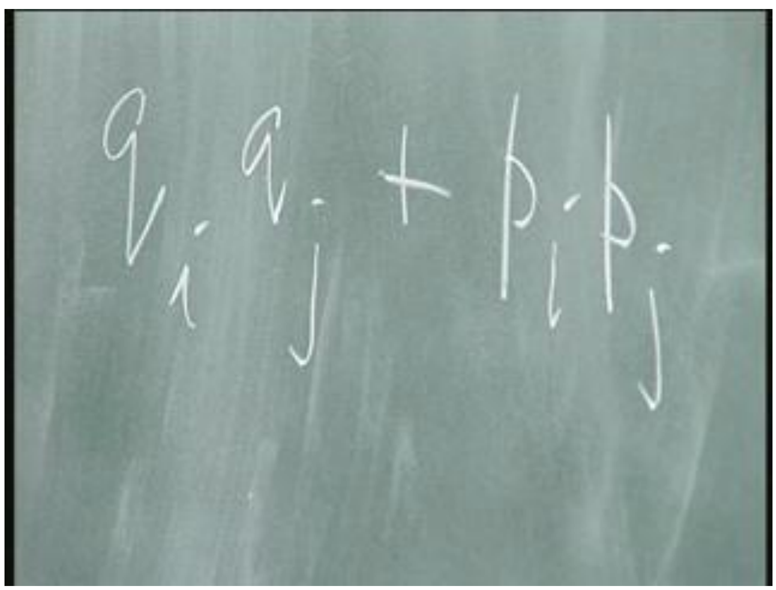

Now, that quantity is not a vector, the constant of the motion for a three dimensional isotherophic oscillator, is in fact this quantity q i q j plus p i p j, and that is a tensor of rank 2, and you put i equal to 1, 2, 3 and you have all these quantities in their constants, all the components they all not independent of each other. Some later for example the trace of this guy would be the Hamiltonian itself and so on, but you have this tensor constant of the motion, because of the symmetric group that is involved, so this statement is not true.

But, what is interestingly true is that, and this is a theorem that is will prove, many years ago by one of our best mathematical physics and Mukuntha from the Indian institute of science, he prove that four of its central potential of this kind, locally at each point in the phase space, you can find the change of coordinates; such that the problem has either the symmetry of the Capillary problem or the symmetry of the oscillator locally. Of course, you could do it globally everywhere with the same change of variables, then the problem is integrable, which is not in general.

But, what is interesting is that you can show in three dimensional, when you have a, when you have a central potential, locally you can map the problem on either to the isotropic oscillator or to the Capillary problem, which is interesting result. Euler equations of motion for the post pre motion of the rigid body this discribe a completely integrable system, true or false, this is true this is true. The equations are motion they come back, we saw the constant of the motion also, we would write down the trajectories which . The next one had to do with fill in the blanks and most of these are standard problems here, so let me not go through this, there is a problem here which we did not talk about earlier, let us do that, let us look at that, because there is a problem, which we did not mention and I did not talk about this at all.

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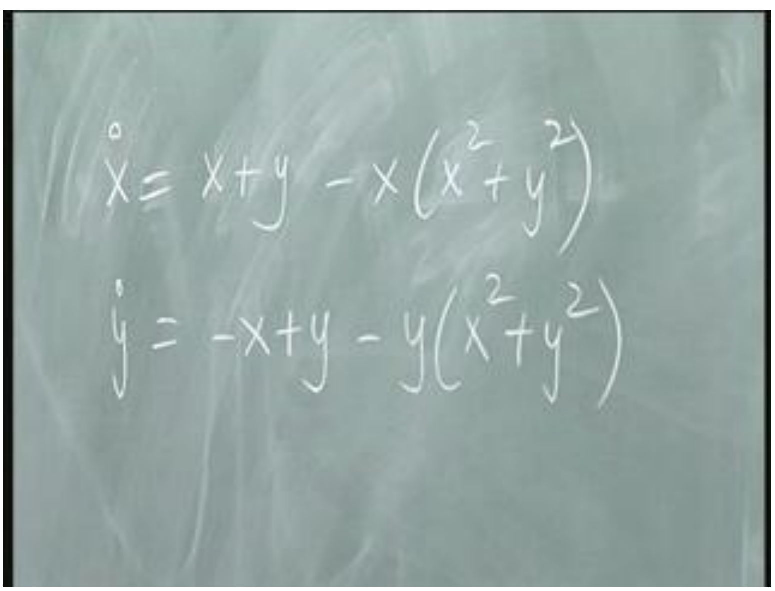

A switch causes a bit, its x plus y minus x times x square plus y square, and y dot equal to minus x plus y minus y into x square plus y square, I talks about the and criterion, which I did not mention in class, which I did not talk about, but let us look at this two dimensional dynamical system. The first question is where the critical points of the system, but the origin is clearly critical point of the system, let us see what kind of critical point it is at the origin; and what you do of course is to neglect this, and just look at this portion that is the linear portion of it.

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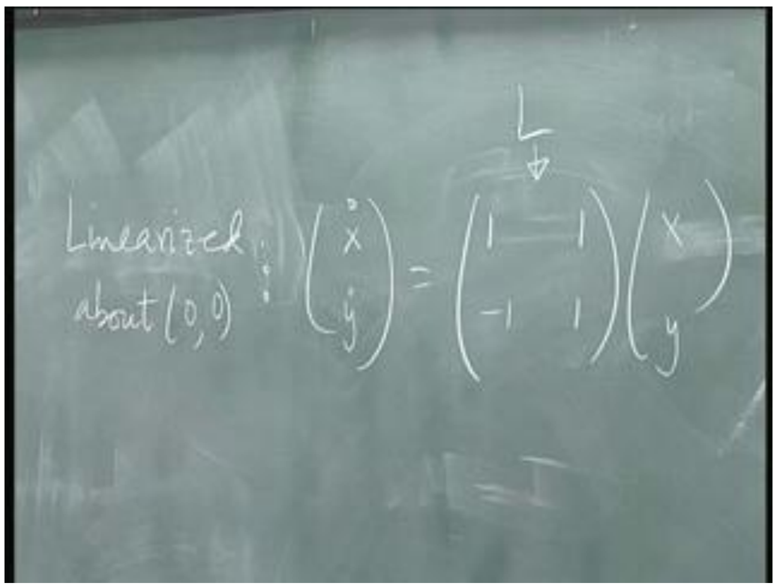

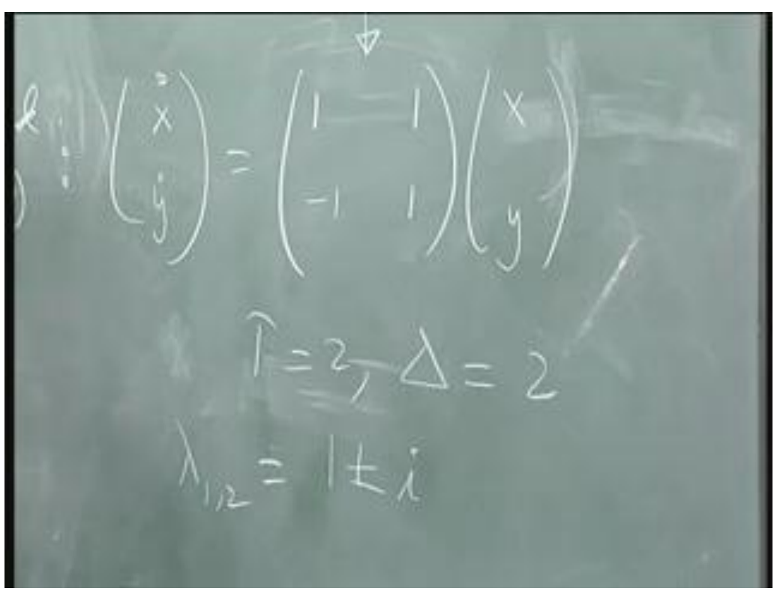

So, the linearised version, the linearise one is x dot y dot equal to this matrix L, which in this cases 1 1 minus 1 1 on x y, this is the matrix one, linearised gives you this matrix here, what kind of critical point is it, well we need to it Eigen values, so let us find what is its trace and what is its determinant and that should tell us immediately, what happens, so the trace T equal to 2 and the determinant delta is equal to, therefore what sort of Eigen values to you have?

Complex, so lambda 1, 2 equal to

1 plus or minus over 2 or 1 plus or minus i that is it, therefore what kind of critical point

It is an unstable spiral, things are going to flow away from it, and the question is where are they going to go to, because my argument is following.

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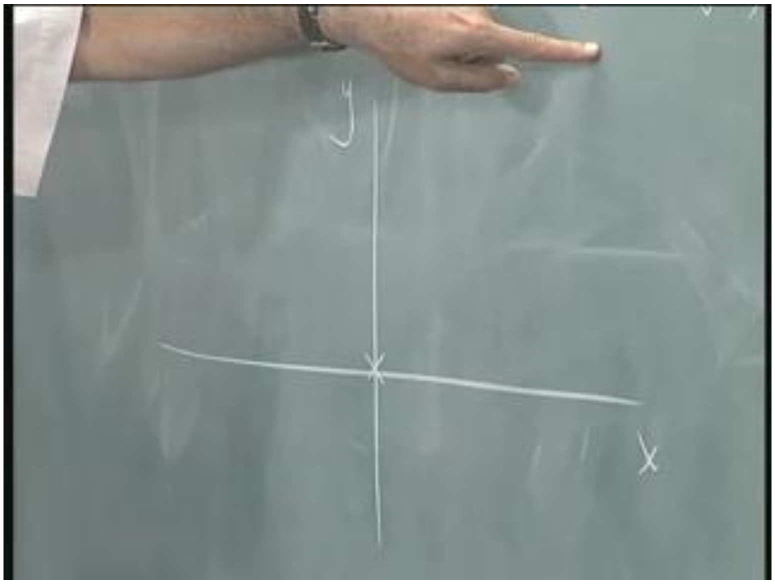

I look at the phase plane, I know this is an unstable spiral. But if I look for example at this point here, very far away large x and large positive y, then this is the negative term and that is a negative term, so both x and y are decreasing, I should be leaving coming in if I am way out here. And yet this is unstable its slowing out, how would you recognize these two statements, from the origin things are going to flow out and if I am sufficiently far away I am going to flow in, unless other critical points, are there critical points in this problem?

1 0 0 1, let us look at the system and polo coordinates is suggest that, we should look at in polo coordinates, because you have this here and this presently this nonlinearity, may be able to handle it exactly.

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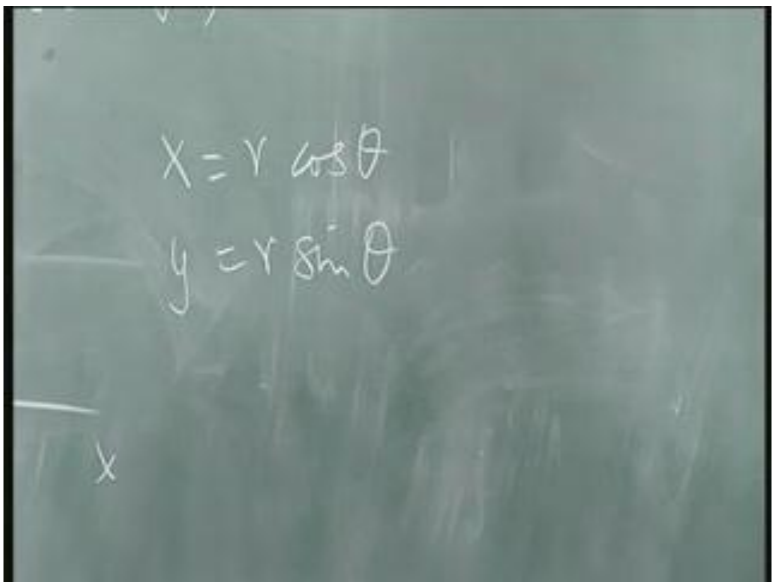

So, let us go to polo coordinates, I put x equal to r cos theta and y equal to r sin theta, then how do I find r dot, I like to find r dot in theta dot, what do I do.

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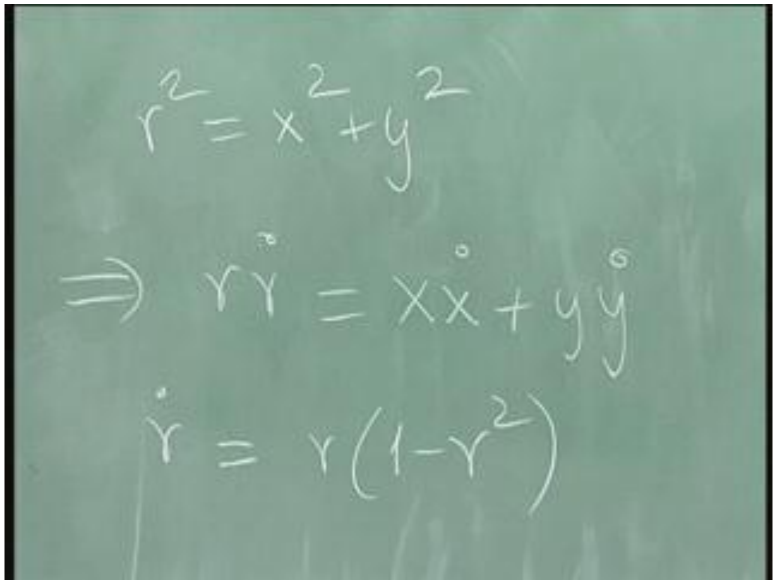

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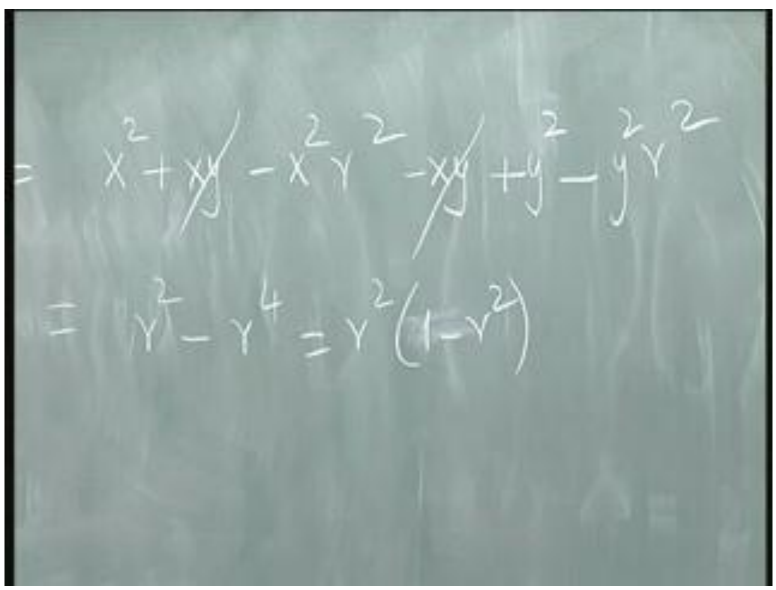

So, I have r square is x square plus y square, this immediately implies that if I differentiate r r dot is x x dot plus y y dot this differentiate both sides with respective time.

So, what is r dot become in this case, I multiplied by x dot here, so let us do that here, its x x dot, so its x square plus x y minus x square r square plus y y dot so there is a x y plus y square minus y square r square. So, this term is cancels out, and you get r square minus r 4 1 minus r square.

So, we have r dot in this problem is equal to r times 1 minus r square, so this actually suggest immediately that, if r is sufficiently large bigger than 1.

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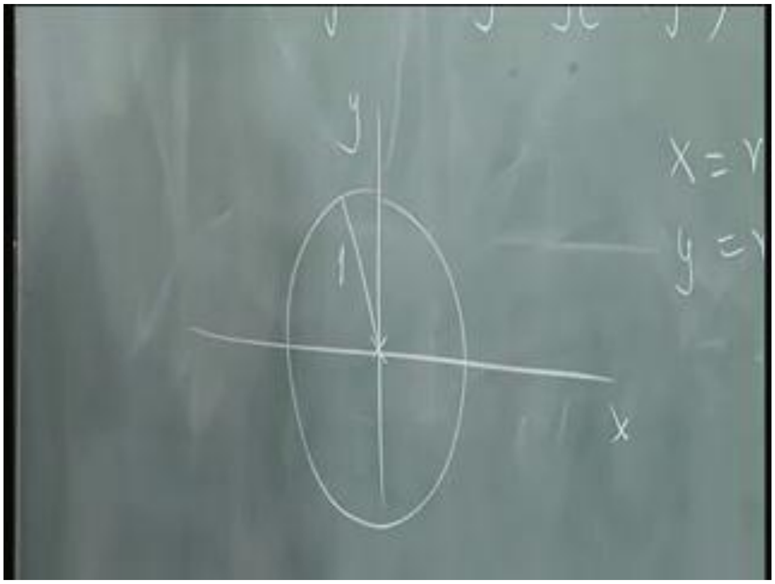

Then you know immediately this is a circle here at radius 1. And if you are outside the circle, then r decreases, because r dot is negative therefore things must flow in, but if you are inside the circle it must flow out because, 1 minus r square is positive. In which direction, this is going to happen clockwise or counter clockwise this depends on what is theta dot does.

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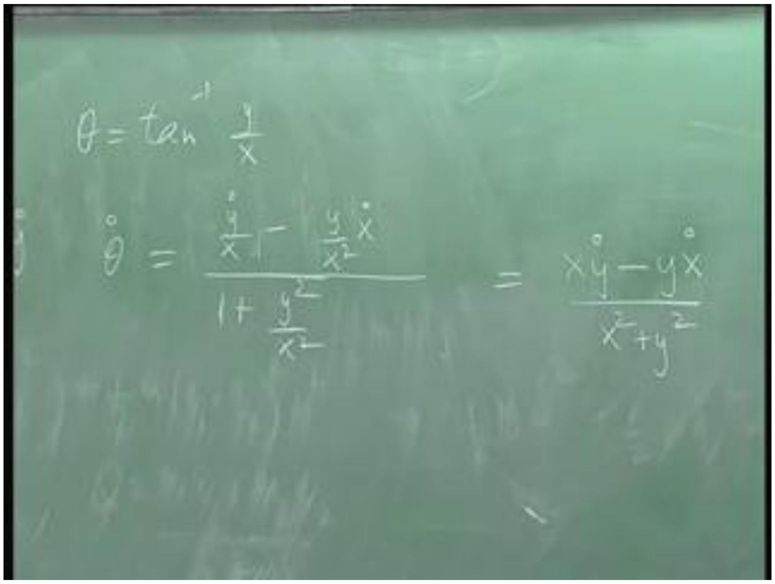

But what is theta, theta is equal to tan inverse y over x and therefore, theta dot equal to for differentiate this 1 plus y squared over x squared, and then the derivative of this guy which is y dot over x minus y over x square x dot, I differentiate y over x with respective time, and negative two term of this kind. So, this could be written as equal to x square cancels out, and you get x y dot minus y x dot divided by x square plus y square and that is the useful formula, that is the way theta change is, and let us find out what theta dot is in our problem.

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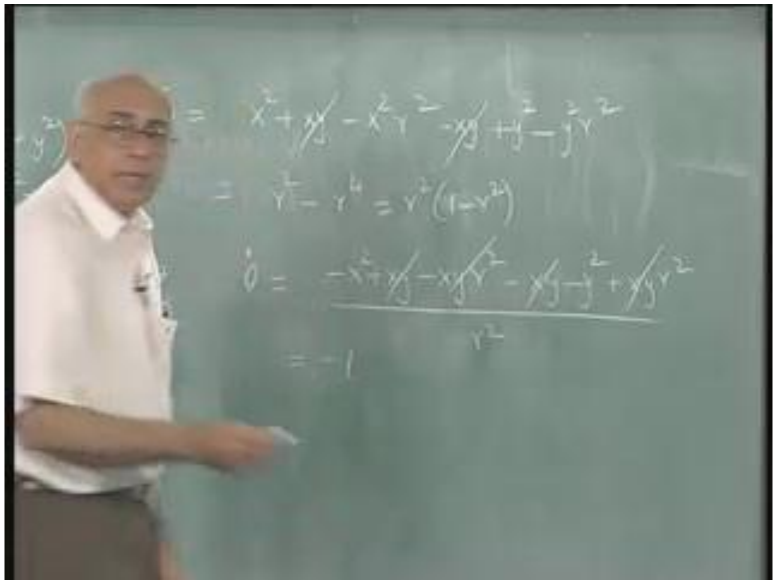

So, theta dot is equal to x y dot, so I multiply this by x minus x square plus x y minus x y r square, and a subtract from it y x dot, so minus x y minus y square is it correct, this is correct plus x y r square plus x y r square and that is got to be divided by r square, this is correct?

Yeah, so that is minus 1, the term cancels out and that is equal to minus 1, that is says that you are going to move in the clockwise direction.

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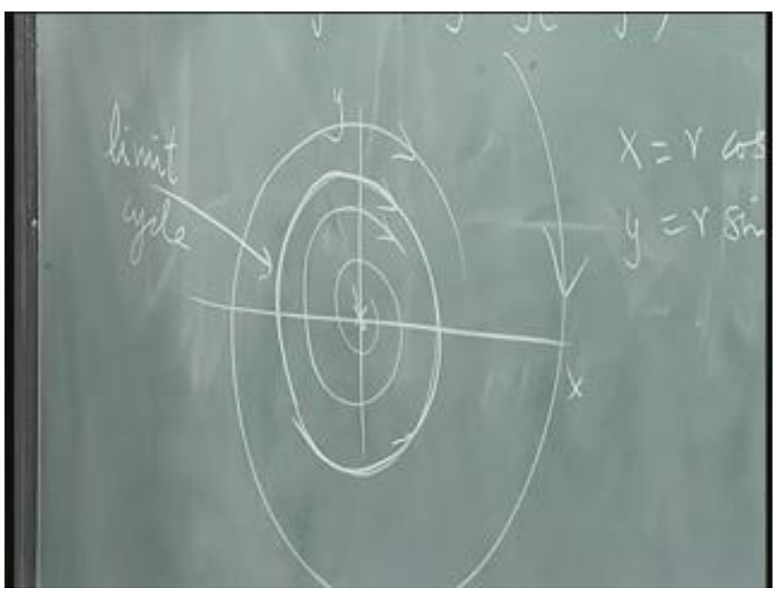

So, this immediately says that, if you start here you going to spiral out in this fashion, and approach this trajectory. The continuity this trajectory also moves in this fashion, because if I said r equal to 1, I start with r equal to 1 any point on the unit circle, I am just keep going to keep going round it this fashion.

And if I am anywhere outside there, I am going to spiral end, and fall into this circle here, and this trajectory therefore, is a very special trajectory and this is a limits cycle, so this problem actually has very different kind of attractor, there is an unstable spiral point at the origin. But, then it ends up the stable thing is, the stable attractor is not a fixed point, its not a critical point, but rather a whole trajectory one dimensional object and its limits cycle, its an isolated periodic trajectory; it appears limits cycle occur only in dissipative systems, not in Hamiltonian and conservative systems.

And its clear in this problem, the phase space volume element here all shrinking and the finally going to end up on this line and volume element here, and also going out and going to end up here. Every point other the origin inside the unit circle is going to find the end up asymptotically approach in this limits cycle, and from the outside too. I think we run out a time, summary at a class today, so we will stop here.

And then, we will take it up the next time, I would like to move now to slowly towards statistical considerations, which is what we will start with next week. Mean while, if there other problems here which are, if you have any questions on them, let me know and we will work those problems out explicitly. I did think in terms of writing out the solutions and sending it to you by previous files or I thought that it will be a little too much to do; so not going to do that, if that questions I will answer that, its fine.