Classical Physics
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Lecture No. # 17

The question has been raised as to, whether the Lyapunov exponent is independent of the initial condition or not. Now, you see the situation is as follows.

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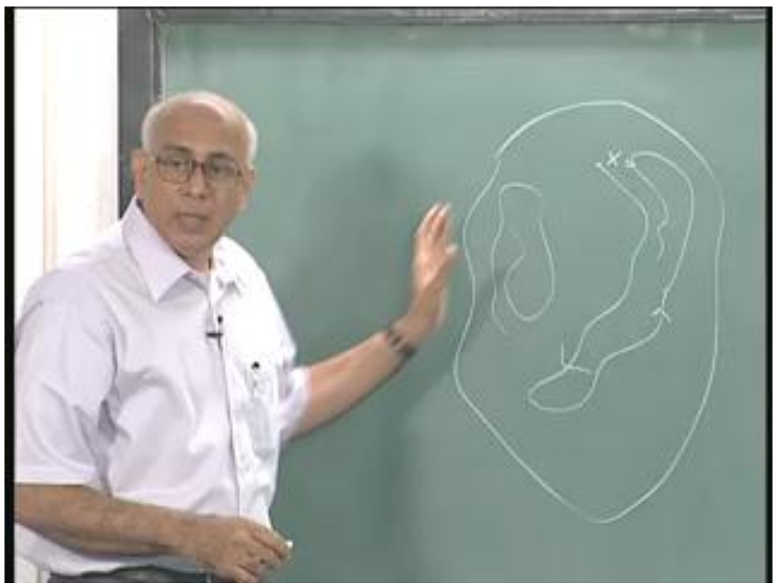

If this is your big phase space and you start with some point here x naught and it undergoes a trajectory of this kind, this point lies on a trajectory of that kind. Then if this initial condition is such that this trajectory comes arbitrarily close to almost all points in phase space.

Then of course, as time goes a long it is quite clear if I take the infinite time limit and compute the infinite time Lyapunov exponent, you have explore all of phase space in some sense. Therefore in that sense it should become independent of x naught itself. However we have to remember, that it is entirely conceivable that this part of this initial condition is in some stays in some region of phase space some attractor. And there could be another attracter in another portion initial conditions starting here, could stay here in which case the Lyapunov exponent could depend on where you are.

So, in general of course, it depends on where you are in phase space but, our assumption is in the simple models we are looking at any arbitrary initial condition. If it is sufficiently generate is going to eventually iterates are going to fill up the phase space and come arbitrary close to all points in phase space. We assume ergodicity of some kind, you might have multiple attracters, this could very well happen, there are situations many situations where it does but otherwise this statement is suddenly true.

The other point is the Lyapunov exponent is now a property of the trajectory rather than of the initial point itself that much is clear. If you try to find out the average exponential separation along this trajectory as you go long. We had looked at the case of the Bernoulli shift and a pointed out there it is Lyapunov exponent was log 2 and that it was positive and we had an invariant measure, which was constant and invariant density, which was constant.

Now, let us look at a situation, where you have measure which is preserve namely something like a Hamiltonian flow but, which is still chaotic. So, it is a very, very simple model of a conservative system this speed time system. Where you would have something mimic in the Hamiltonian in the sense that volume element are preserved or area elements are preserve. But, at the same time the system is chaotic and it is done by very simple extension of the Bernoulli map it is called the bakers map.

And it is modeled on the way a baker needs dough to make bread. So, what the baker does as you should take this piece of dough and it is stretches it and folds it back and stretches it folds it back and so, on. And in the process the dough gets mixed up and a very typical example is if you puts for instance human seeds or something like that in this dough.

Then gradually its gets mixed up, so that the seeds are uniformly distributed in the entire dough. And this transformation I am going to talk about mimics just that, what it what happen in the Bernoulli shift was exactly that in some sense.

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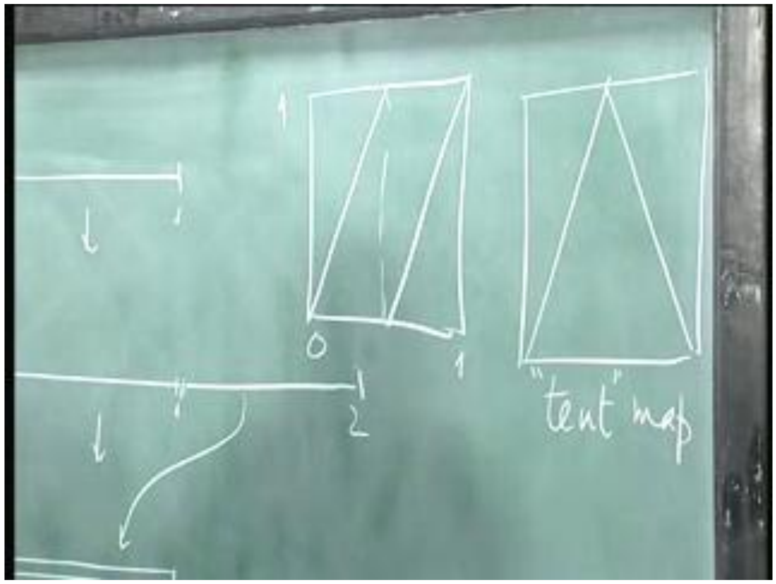

What you did was, let me draw the map here, in the Bernoulli shift the map function itself look like this 0 to 1. We add something, which was stretch to twice it is length by 2 x and then you cut off that piece and put it back here. So, that amount is you saying I start with the unit interval 0 to 1 and I stretch it to double it is length, the goes up to 2. Then, I cut that piece and put it back on top of this, so this is essentially what happen and then you had 1 piece here and another piece length immediately on top of it, that is sense to this piece was put down.

That is how you got the non-linearity here in this map, so what has happen is that you stretch this row and then cut it and put it back here. You could have done the other thing you could have stretch it and fold it back. And what that my correspond to, if I took this piece and sub cutting it and putting it back here, I just fold it back in this passion. Then of course, the map would look like this, this piece and then its fold it back on this side.

That is also non-linear this map is advantage that it is continuous unlike the Bernoulli shift, which was discontinue at half, this is continuous And it is got exactly the same properties in many senses as the Bernoulli shift Lyapunov exponent is still log 2, because the slope is still 2 in magnitude and the function is of course, 2 x here and then on this side it is twice 1 minus x to 2 minus 2 x. So, this map is called the tent map looks like a tent.

And it is also fully chaotic it is iterates also fill up the even even unit interval density for arbitrary typical conditions. And again once again all the rational points would lie on periodic orbit which are unstable. Now, what the baker map does you should take this idea of the Bernoulli shift and increase the number of dimensions of the map by one. In such a way that the measure is preserve, so let me write this down, and then we draw picture to see, what the transformation does.

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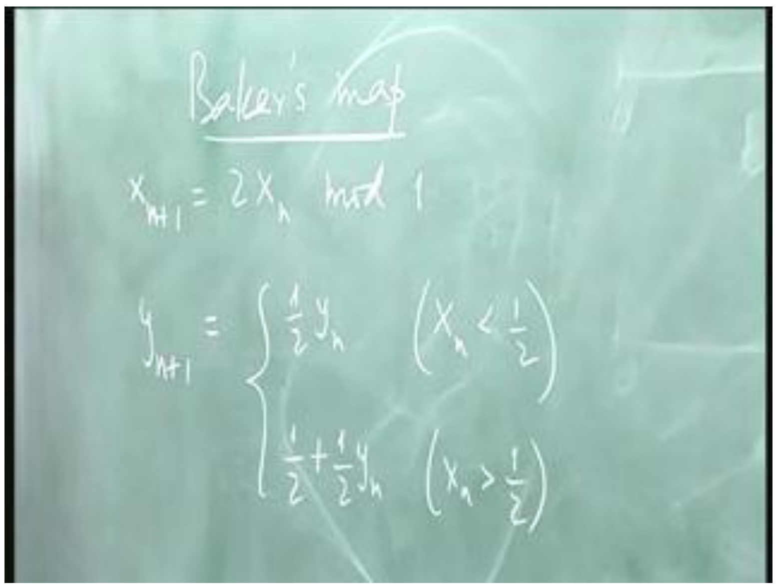

So, it is got 2 variables, now and it says x n plus 1 is twice x n modulo 1, that is the same as the Bernoulli shift fashion. And then y n plus 1 this guy here, is equal to 1 half y n provided x n is less than half and it is equal to 1 half plus 1 half by n, if x n is greater than half. So, you have 2 variables, now x and y and what you are doing is should take the unit square in the x y plane and mapping it onto itself in a non-linear fashion.

But a very strange now linear fashion by these rules and what does this look like well now it is hard to draw, because it is I cannot draw x n plus 1 versus x n.

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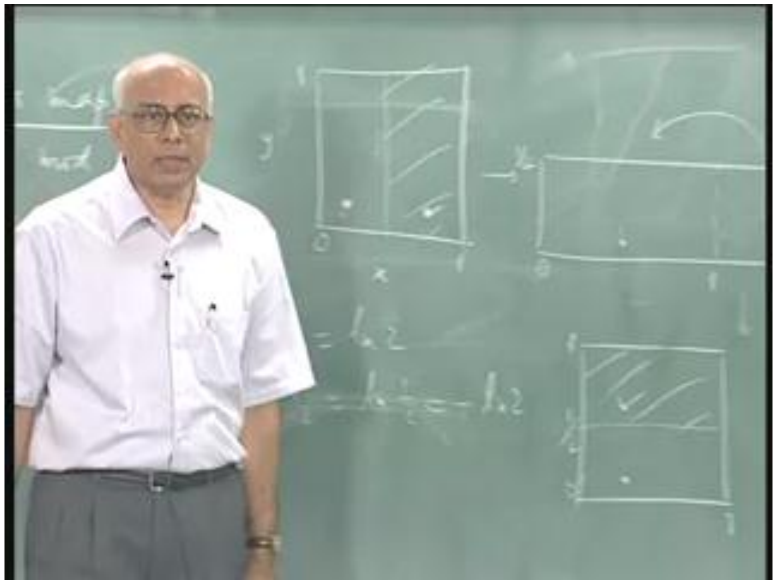

So, what I do is I draw picture of x n and y n and in the next picture, I draw x n plus 1 and y n plus 1. So, here was x n y n 0 to 1, 0 to 1, this is the y variable and this is the x variable and what is this map do. It takes this square and it is says expand in the x direction by twice a factor of 2 and simultaneously contract in the y direction by factor of half. So, in the first step this map would go into something which like this, it is zero this is 1, this is 2 this is the y direction it is comes to the half. And then it is says the y variable; however, first of all you have to do modulo 1, so it is clear that you have to cut this and put it back. But, the way you put it back is you take it on put it on top, then in the second stage does this.

So, this piece is cut and put on top of it, so as you can see all points x which are less than half are still here, they now between 0 and 1. All points x which are greater than half here on this side but, the movement you have x n bigger than half you add a half to y n. So, you add this piece, this much is added and then it is put on top, so it is clear that what has happen in this picture is if you started with the points say between 0 and half and here.

And y also less than half, some point here that point would have first gone to twice it is distance and come down to half it is height and then it would remain here. On the other hand if the point were here, this would have again them same thing except gone here and now it is come up and therefore, it is move there the point has ended up there. So, all points in this side are actually ended up there and all points on this lower half left half of this have ended up at the bottom.

So, everything in this shaded region has move to this shaded region but, the area is being preserved, you started with unit area unit square and you still have the unit square. So, therefore, the (()) of this transformation is plus 1 is 1. So, in that sense this map mimics a Hamiltonian system, because the measure is preserve. But, is there Lyapunov exponent is there loss of information definitely there is; because as you can see. In the x direction it is like Bernoulli shift the y direction is compensating in the other direction to make sure that the volume is area has kept constant.

This system has 2 Lyapunov exponents, because there is an x and y, what are the Lyapunov exponents log 2 is certainly one. One of them what is the other one yeah log of half which is minus log 2. So, the some of the Lyapunov exponents actually becomes 0 and that characteristic of a Hamiltonian system always as soon as measure is preserve the some of the Lyapunov exponents has to be 0. Because any volume element would some other mode expand some other mode contract.

So, volume element V b would have e to the power lambda, whichever the expanding directions and then e to the minus lambda different lambdas times and packing directions. What is the Lyapunov exponent or Lyapunov spectrum for an intergrable Hamiltonian system. All of them have to be 0 and the reason is if you had one of them positive you have Kaoyas and then have to be a contractive direction immediately to preserve the volume.

So, you can afford to have even one of them positive, because it is intergrable; therefore all of them have to be 0's and some has to adopt to 0. So, integrable system do not have any Lyapunov exponents they the motion is on chloride in action angle variables. And on torus the angle variables increase linearly with time therefore, to initial conditions would separate linearly in time and the Lyapunov exponent for power separation in time is 0 automatically.

So, Hamiltonian systems which are integrable have 0, Lyapunov all the Lyapunov exponents are 0. Those which are not integrable for every positive Lyapunov exponent you have corresponding negative one they would occur in pairs such that the sum would have up to 0.

And here is a instance of such a thing, the Lyapunov spectrum now Lyapunov exponents are log 2 and lambda 2 equal to log half minus log 2. Unlike the Bernoulli shift where I told you that it is not invertible you cannot tell one you can go in the forward direction but, given a final point you cannot tell there is no unique pre image, any number of pre images.

After n iterations there are 2 to the n pre images is this map invertible this map is (( )) invertible you can predict the feature of every point. You can go back again tell you given any point, here I can tell you exactly where it came from of there. It is also invertible and yet it displays kaoyas yeah, again because in the x direction you are losing the errors amplify.

So, it is such a point but, the fact is that in kaotic dynamics unlike the unless unlike differentiable dynamics in discrete time dynamics you may be able to write down the explicit solution to a map explicitly, but it could still be kaotic. That was in true in differential dynamics you do not write analytical solutions you do not write then explicit excess a function of t or y is a function of t for arbitrary log t.

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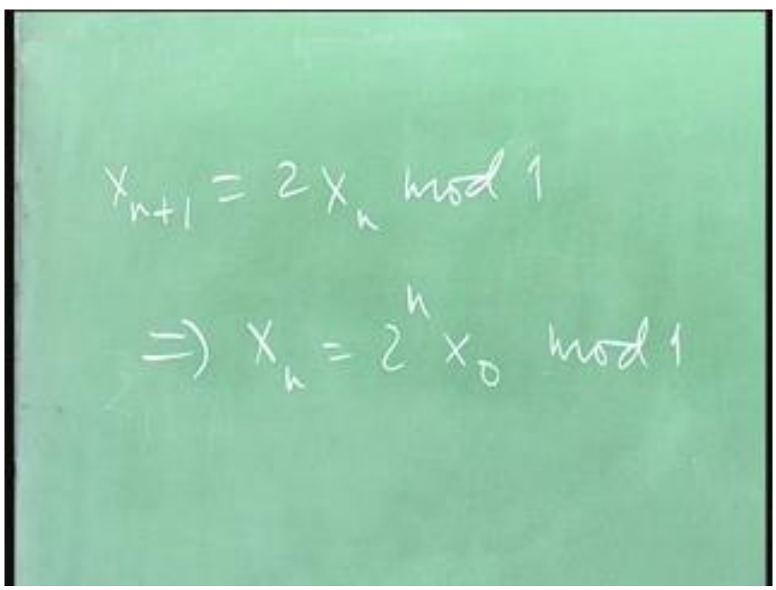

In discrete dynamics does not true because in the Bernoulli shift I know that x n plus 1 is 2 x n modulo 1, this would imply that x n is 2 to the n x naught modulo 1. So, I have written the solution down explicitly but, all the same it is kaotic completely. This an interesting way of seeing, why there is no loss of any information at all, in this baker transformation and that is the following.

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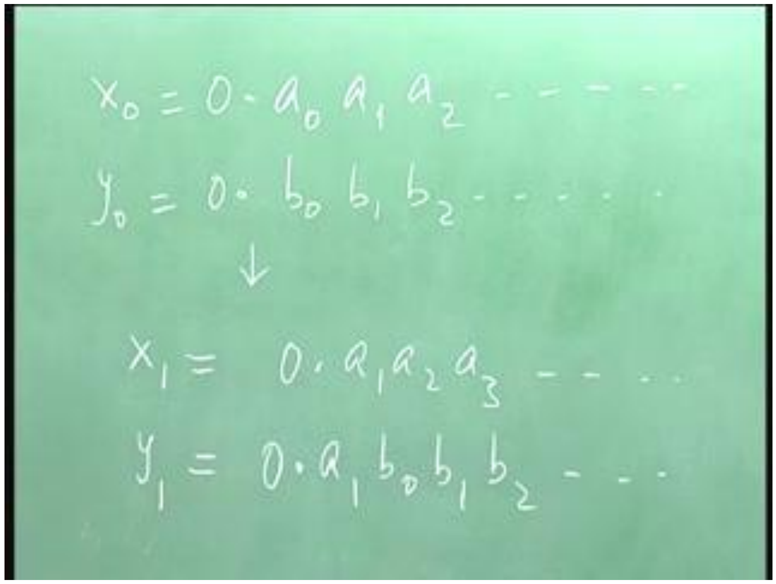

Let me start by writing x naught equal to 0.a naught, a 1, a 2 dot dot dot and y naught is 0. b naught, b 1, b 2. That is my initial binary decimal representation for x naught and y naught each a naught a i is 0 or 1 and each b i is 0 or 1 and after one iteration. What happens to x 1, this is 0. a 1 a 2 a 3 dot dot dot but what is y 1 equal to, y 1 depends on whether x naught was equal to x naught was bigger than half or less than half. If it is less than half a naught is 0, if it is bigger than half a naught is 1, so what is that (()). So, what is this 0.a naught b naught b 1 b 2, so this a 1 gets transfer for to the other place sorry a naught. So, you have that information here another way of saying this is to pretend that I am going to write the number x naught by y naught in a strange way.

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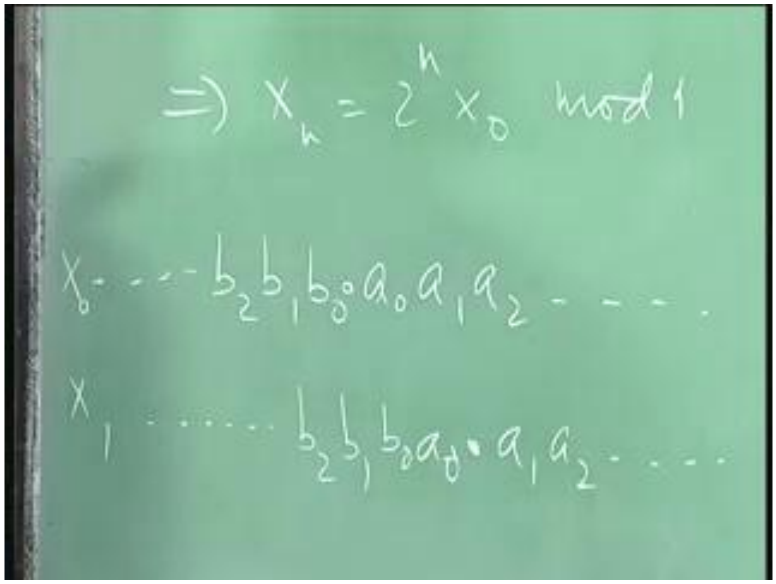

So, I put a decimal point and write a naught a 1, a 2 etcetera and I write this in b, so I write b naught b 1 b 2 on this side. It is a by infinite sequence let me pretend, that I represent the y variable by going the left hand side and the other one going from left to right or right to left. So, if this is a representation of x naught, then the representation of x 1 dot dot dot b 2 b 1 b naught a naught a 1 a 2.

So, you see you have an loss anything, what you loss in case you dot into the other and therefore, all that you do is still a shift. So, you keep simply shifting the decimal point one place to the right and that is what the action of this entire maps is and it has is very weared property. That it is in fact, kaotic map but, it is the same time its measure preserving and it is inverting.

General rule is that to produce kaoyas in the one dimensional map it has to been non inverted but, in two and higher dimensions even invertible maps can produce chaotic. So, it is a little different from what happens in the case of differential differential dynamics. Let me look at one more map of this kind is show you how complicated the system can get.

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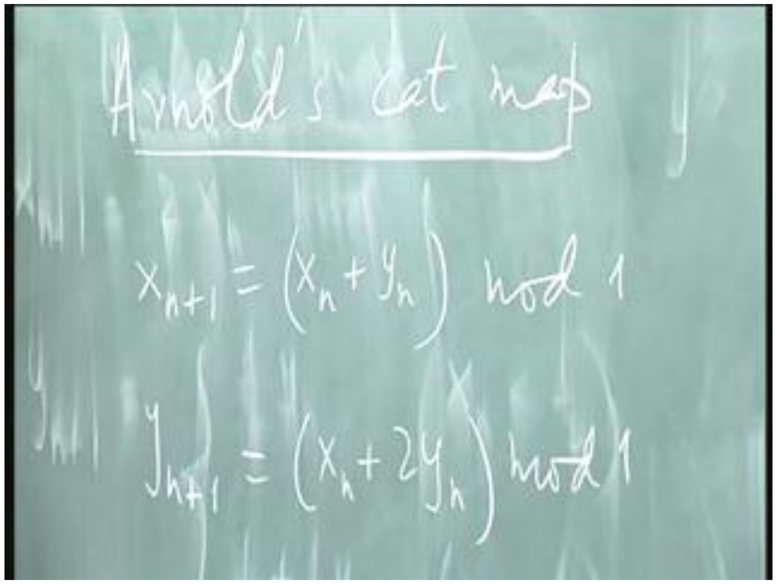

The idea is to mimic, what happens in Hamiltonian system and this is the famous Arnold's cat map. And again it is takes the units square and x and y and it does the following, x n plus 1 equal to x n plus y n again modulo 1 and y n plus 1 equal to x n plus twice y n modulo 1 and that is all it is needed.

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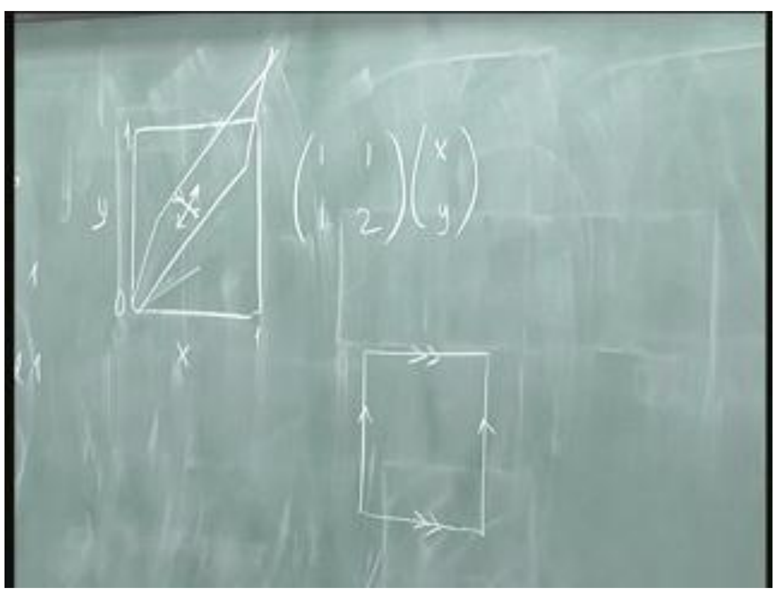

So, the map looks like this it is the unit square is the y direction and the x direction and this map says distract is in this fashion. So, as you can see what you are doing is to take this variable x y and you are applying to a certain transformation which is 1 1 and 2. So, that is the reason why x n plus 1 is found from x n and y n by applying this matrix on.

On the determinant of this matrix is 1, so it is plus 1. So, it is orientation preserving its measure preserving the unit square is map to the unit square but, what happens because of modulo 1 is that it is non-linear very badly non-linear. So, if you took this original square and you apply the transformation you of course, go out of the square in general and it would be somewhere thing like this it would look like this, it would perhaps look like this. This will become parallel and then maybe look like that.

And now you are suppose take that piece, because of modulo 1, you cut it put it back here inside and you keep doing this over and over again and the square gets completely scrambled up. The reason it is called a cat map is because Arnold, when you first gave this example do the picture of a cat here, the face of a cat and then of course, after few iteration you cannot recognize a cat at all it is a scramble cat, it is completely scrambled and mixed inside.

Now you can see in this map you can find what the Eigen values of this matrix are this login values are related to the golden mean 1 plus root 5 over 2. And it will turn out that this direction there is a stretching direction, there is an Eigen direction of stretching and there is a contraction direction inverse. And every point is an intersection of a stable and an unstable manifold every point on this is part of is actually hydrobolic point.

It is a stretching direction and this a concating direction and because x on it is modulo 1 its equivalent to saying that I put it on a torus. After all saying modulo 1 on a unit square a torus is form from a square by identifying opposite ends right. I start with the unit square and I put an orientation on it and say bend this and sold this on to this ticket on to this and then you get a cylinder. And then if I put in orientation on this and after I form a cylinder I take the cylinder and identify ends what do I get a torus.

So, torus is like a unit the unit square (()) is to like it completely periodic and this is on a torus and it is the map of the torus to the torus it is called an auto morphism and it is hyperbolic. So, this cat map is very fancy name it is called it is an example of what is called hyperbolic toral auto morphism wings to back to itself. It is just a big name but, lot of these maps have been studied as examples of simple model which mimic Hamiltonian kaoyas.

And once again you can show that the variant density in this problem is also constant, the entire the entire units square is completely uniformly filled up by the iterates of a generic point. By the by this particular map you can have other integers here, these have other applications this particular matrix is very strange one. It is also generates what is called a pento style was a periodic tyling two dimension and so, on; you would not get into that right now.

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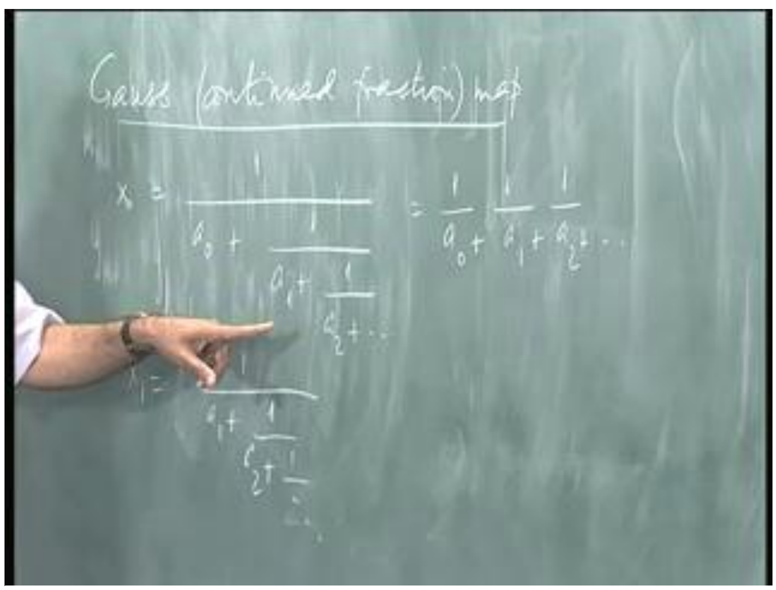

Let me go back and show you another map, which is very similar to the Bernoulli map, but uses the slightly different representation of real numbers. And this is the map which gauss first down and it is called the Gauss continued fraction method. And very similar to the Bernoulli shift, maybe said write the number in binary and moved the move the first digit and put it decimal point and so on. Here, what does is you start with a number x naught and you can write this number as a continue fraction every number between 0 and 1 in between as a continued fraction.

If the fraction terminates then it is a rational number, if it does not terminate and goes on forever then of course, it is and irrational number. So, every number like this can be written us a naught plus in standard form you can be written us 1 over a naught plus 1 over a 1 plus 1 over a 2 plus etcetera, where a 1 a naught a 1 a 2 and so, on; are positive integers every number can be written in this form.

Is this thing continued fraction terminates at some point, then I would say the number is rational, because you can rationalize it whole thing and get it as p over q. But, is it is goes on forever it is irrational and this is represented for ease of writing they also written us 1over a naught the plus sign is put here.

The gauss idea was very simple he said from this you form an x 1, which is equal to 1 over a 1 plus 1 over a 2 plus 1 over etcetera. So, through this out, so find the map such that this first a not is removed completely, now how do we do that now let us clear that what is should do to take the reciprocal of this number. Throw away the integer part right and keep this portion the fraction fraction of portion.

So, what is the map now says x n plus 1 equal to 1 over x n minus the integer part of 1 over x this is the integer part, that notation stands for the integer part of 1 over x. And then you are guarantee to get a fraction between 0 and 1 once again right. Now, what is the map function look like for draw it as a picture.

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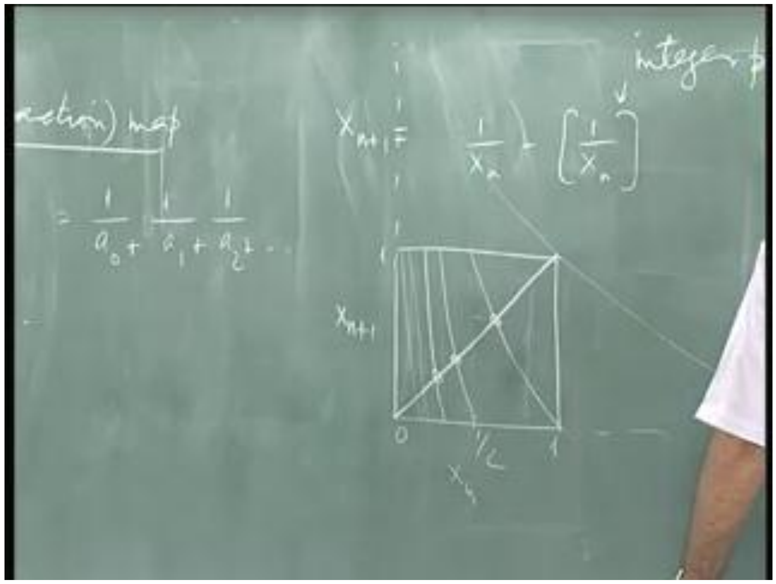

So, here 0 to 1, this is a half this is 1 y equal to 1 over x, it is just a hyperbola right simply hyperbola. And that hyperbola would cut the point one and it look like this if I could expand this it could be a rectangle hyperbola, which goes long like this. And if x is between x naught is between half and 1, 1 over x naught between 1 and 2. Therefore, you have to subtract 1 from it in other words you take this piece cut it and put it back here.

So, this is what the piece would look like and then if x naught is between 1 3rd and half. Then of course, the reciprocal is between 1 and 3 and you have to remove 2 from it. So, when you subtract that and put it back here is 1 3rd, so it is look like this. And then between 1 3rd and 1 4 th, it would even steeper, it would look like that and you have an infinite number of branches of this bank.

So, this keeps going its gets more and more steep, then infinite number of branches. So, you can see it is a very non-linear map, the number of pre images of any point is actually infinitive. So, if you give me any x n, if you give me and x n plus 1, there are infinite number of x n, in which it could have come. So, takes this Bernoulli shift makes it much versions some sense. Now the slope is; obviously, bigger than 1 everywhere and because of that this map is suddenly hyperbolic, you can see all it is fixed points are unstable all the iterate fixed points are unstable and so, on; and the map is actually fully chaotic.

And now how is came across this in a very interesting way, you want it to solve a problem in number theory and it is question was a following. I start with the number between 0 and 1 in a arbitrary number I take it reciprocal throw the integer out, I takes it is reciprocal throw the integer out. If I keep doing this a very long time for long, long time, what is the probability that the final number I get is less than some value alpha between 0 and 1 we found the answer.

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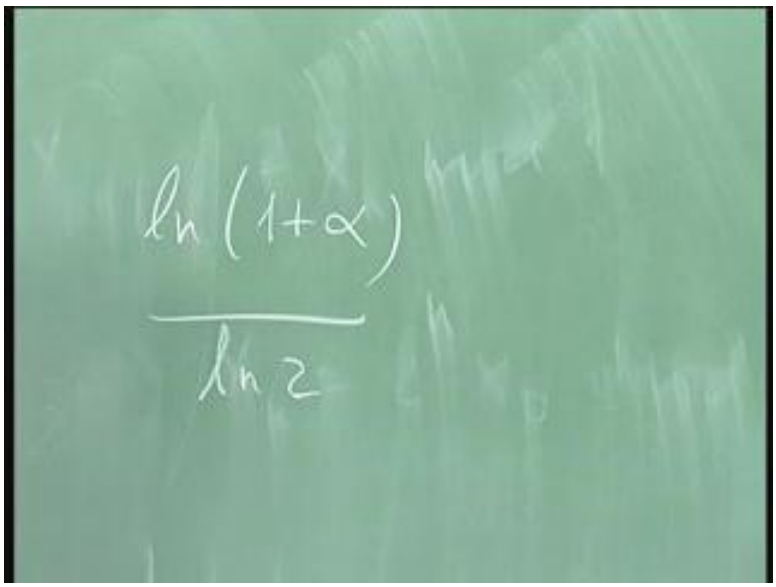

And the answer you got was the probability that the final answer is between is less than some number alpha which is between 0 and 1 is, log 1 plus alpha over log 2. You got, so excited about this, that you wrote to a plus or you saying this found the solution. And now we can understand this in the language of dynamical system, all we have to do is the fine the invariant measure of this map.

Once we do that we can integrate this invariant measure up to alpha and that should give you this answer. So, even though we are dough's, we can at least guess the answer no invert dough there. Let us see we can do this, what would be the Frobrenius Perron equation in this case rho of x, before I do that lets look a little bit at the fixed point of this map, which are unstable of course. So, here it is this is the fixed point, this is the fixed point, this is the fixed point and, so on, what is the first fixed point of this map, how would you find it.

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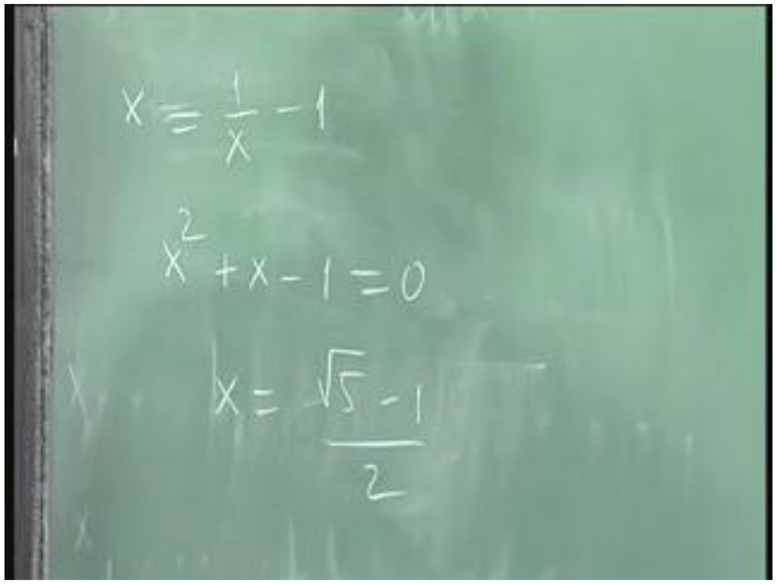

Well this fixed point must satisfy x equal to 1 over x minus 1 in that is also the x squared plus x minus 1 equal to 0. So, x is equal to minus 1 plus or minus square root of 1 plus 4 is five over 4 and of course, you want the positive root between 0 and 1. So, it is really root 5 minus 1 over 2, that is the fixed point does that ring a bell, this is the golden mean, this number is a golden mean.

So, it is clear that is coming out in a very, very natural way here, what is it is continued fraction representation.

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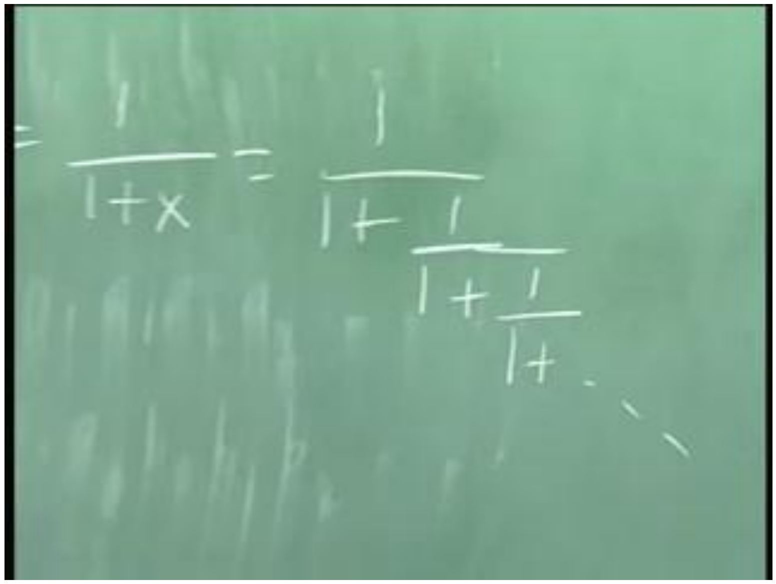

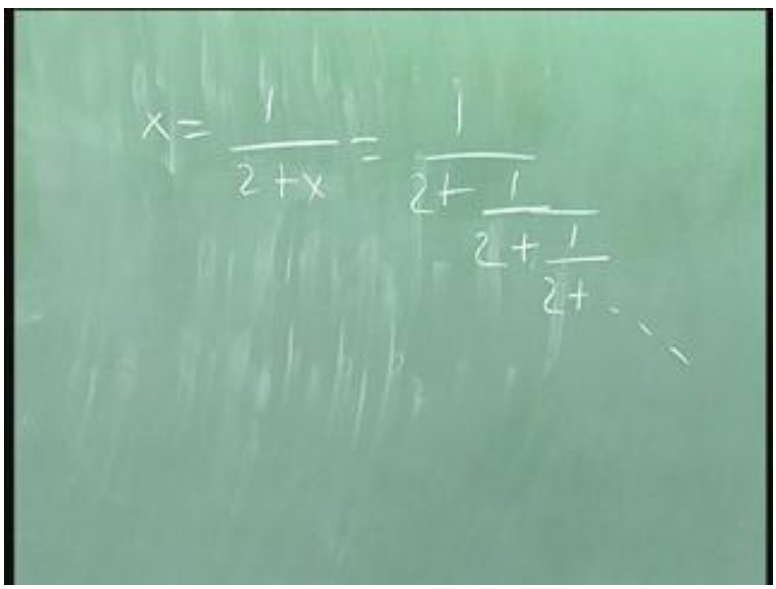

Because, that is very easy to find from this representation it says 1 plus x equal to 1 over x this implies x equal to 1 over 1 plus x equal to 1 over 1 plus 1 over 1 plus x and so on; keeps going. So, this is in some sense the most irrational number of all, in the reason is the most irrational number in the very precise sense is because, you can see that. If in a continued fraction expansion of this kind at some stage 1 of the a is becomes very large compare to 1.

Say 293 like Ramanujan found for phi after three or four research places, then you can neglect the rest of it. And you have a very goof approximation to the irrational number very good rational approximation. The Ramanujan actually found rational approximation to phi with about three steps but, one of these numbers ended up being 293 or something like that therefore, it is very number of large number of decimal places you had a rational approximation.

But that fails completely even all these are equal to each other and what is the smallest value that a can have 1. So, the moment you have once everywhere this is an some sense the most irrational number of all. The one that least acceptable to a rational, where you would end this at what says you would end it.

So, root 5 minus 1, what is the value of root 5 minus 1 .618 we are good the worth remembering this number, that number of 0.618 dot dot dot etcetera is. In fact, golden mean the most irrational one of all, what is this number, what is how are you going to find this?

When I put a 2 there is in it is subtracted 2, so you do this minus 2, so this is 2 plus x and therefore, that number becomes 2 plus 2 plus 2 plus and so, on. Now what you will work out to, we have to solve this quadratic equation right, what does that work out square root of 2 minus 1 or something like that.

Let us called the silver mean another one is called golden mean this is silver mean, next to this the original number this is, in fact, of very, very poorly assertable to a rational approximation. So, you can see that as you go down here, this is a continued fraction expansion with 3's everywhere 4's everywhere and so on.

What sort of number is this, what sort of number could be x equal to 1 over 1 plus 1 over 2 plus 1 over 1 plus 1 over 2 plus 1 over etcetera, what would this be. It what if be a fixed point of this map, now what you would be then? Yeah it would be a fixed point of the first iterate of the map or a period two cycle. So, this number could alternate the two part of a period two cycle.

One of which would start with the 1 here and the other 1 would start with the 2 here, the first place and. So, so you can actually generate all the periodic points of this map by this procedure but, they are all guarantee to be unstable and the question is what is the invariant density, for that we need to find we need to solve the frobenius Perron equation.

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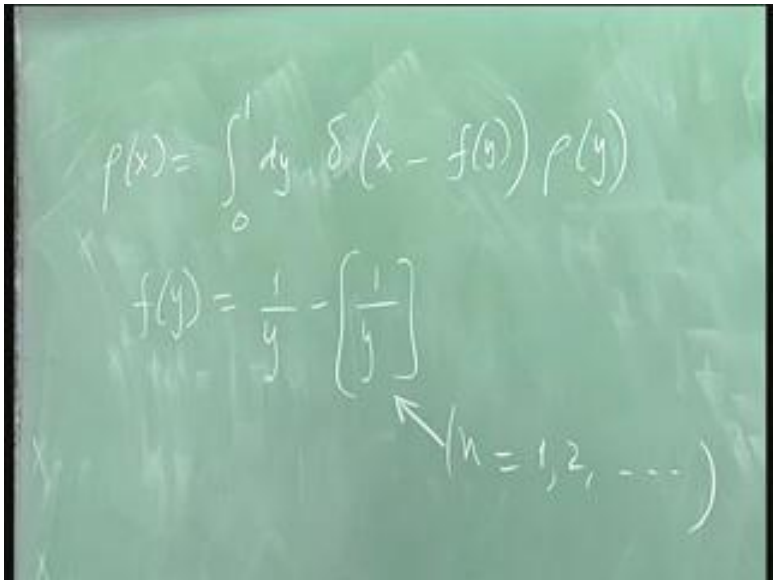

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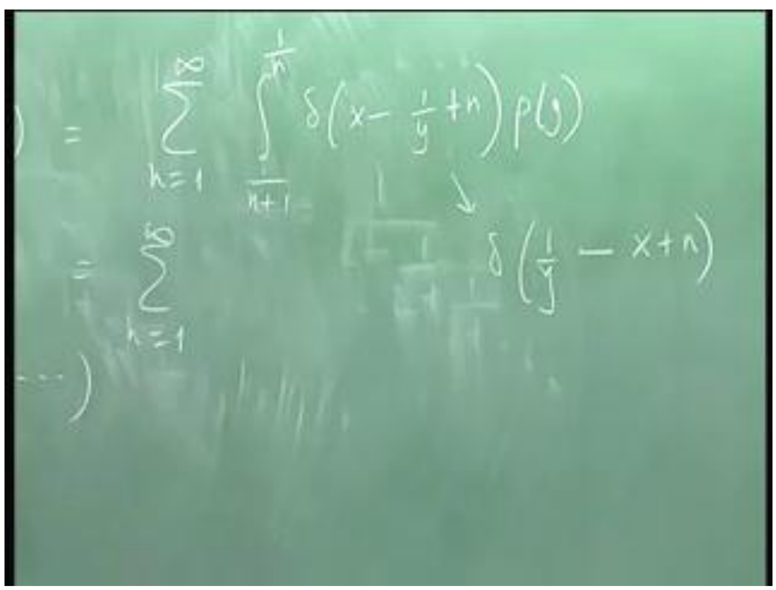

So, you have this equation rho of x equal to integral 0 to 1 d y delta of x minus f of y rho of y. But what is the f of y the map function it is equal to 1 over y minus n, some integer n when inform 1 to infinitive right. It is actually 1 over y minus 1 over this guy here and this is n. You keep subtracting that for each of the branches.

So, what would this give you, if I write this out this is equal to 0 to 1, it should be a sum from n equal to 1 infinitive, because n infinitive number of branches. And then an integral running from where to where? 1 over n 1 over n plus 1 to 1 over n this fraction times at delta function of x minus 1 over y minus n to rho of y. At is equal to summation n equal to 1 infinity, now I have to convert this guy to a delta function in y.

So, let us write this as should have been minus n, so this is plus sorry this is minus n here, so it becomes a plus. So, this is equal to delta function of 1 over y minus x plus n and I need to converted to a delta function in y.

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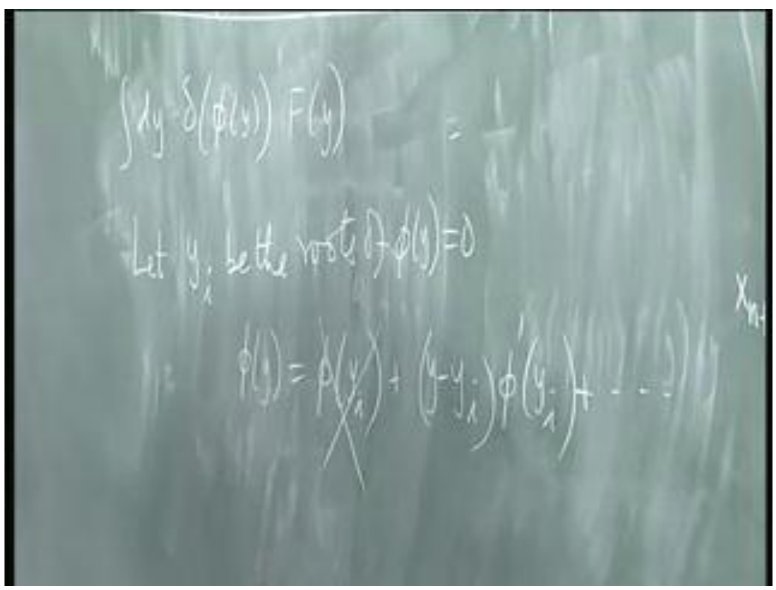

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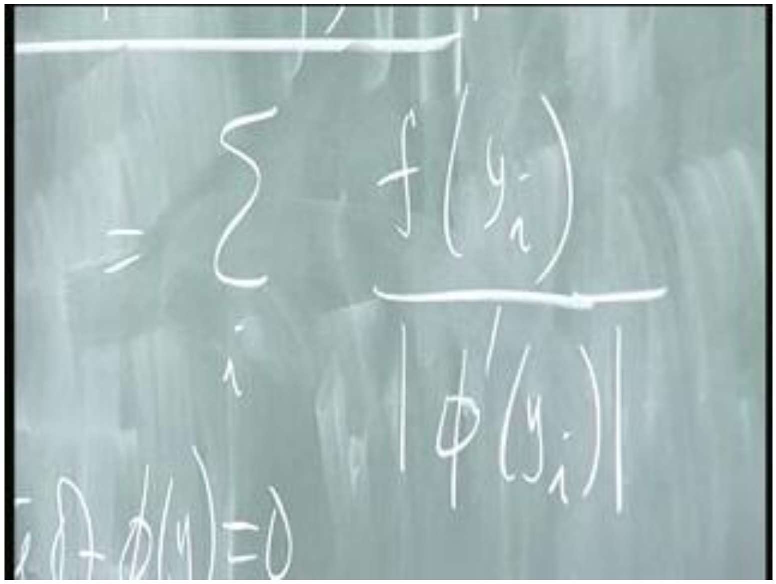

So, what should I do, how do I do that, how do I handle at delta function of a function of the variable y, function like this, what is that stands for. It is obviously, contribute when you integrate and y wherever phi of y is zero, so it all the roots of phi of y it is going to contribute right

If I do an integral like this d y f of y then this is going to have all those values corresponds into the roots of phi of y. It is going to pickup contributions from all the roots of phi of phi and assuming the roots are simple roots, which in that case it is because y is equal to 1 over x plus 1, it is exactly 1 root in each of those intervals of integration.

So, what is the general formulas like this what is the general formula integral d y delta of phi of y f of y assuming that the interval covers, this 0 here is equal to what. Well let y i be the roots of phi of y equal to 0 that is be the roots. Near one of those roots I can do a Tailor expansion. So, phi of y can be written as phi of y sub i plus y minus y i phi prime of y sub i plus dot dot dot near one of those roots.

But this is 0 by definition and then this is the term that is picked up and this is the constant; that is a constant. So, you have a situation delta y minus y i times a constant and what you do with the constant you pull it out. So, this quantity here is equal to summation over the roots of function of y i divided by the ecopian modulus modulus. Delta of a x is 1 over mod a times delta of x equal the that is why I need a simple roots otherwise it is called the function is not meaningful.

So, I need to use that here, I need to put that in here if I do that what is the what is the ecobian of this transformation. Derivative 1 over y with respective 1 over u squared, we take it is modulus it is just 1 over y squared and you put y equal to 1 over x plus n.

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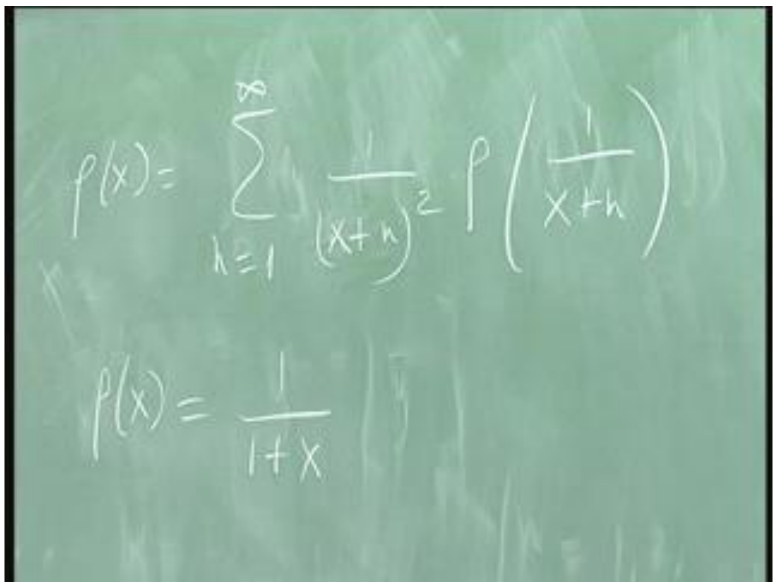

So, what you get finally, rho of 1 over x plus n divided by x plus n whole square right that is the function equation to which the Frobrenius Perron equation has got reduced. But, this is not an easy equation to solve, I mean you have a formidable task, because you have to find rho of x such that rho of x is summation n equal to 1 to infinitive 1over x plus n the whole square rho of 1 over x plus n. This is the equation to which Gauss found the solution.

Since we not Gauss we can we can try to find what the solution is by putting it in and seeing if it is works or not. We put rho of x is 1 over 1 plus x it is got to the non negative it is got to be normalizable its got to be integrable from 0 to 1, which case and you have to normalize it. So, what is the integral of this from 0 to 1 it is log one plus x and then you have to put x equal to 0 x equal to 1 and then divide by rho, so log 2. So, I leave it you to check it out that this is true, put that in an see that really works.

And you have to use this most complicated of mathematical tricks, which you learn in the third standard called resolving into partial fractions. You need one stage you have to resolve in the partial fraction everything will cancel out and give you that.

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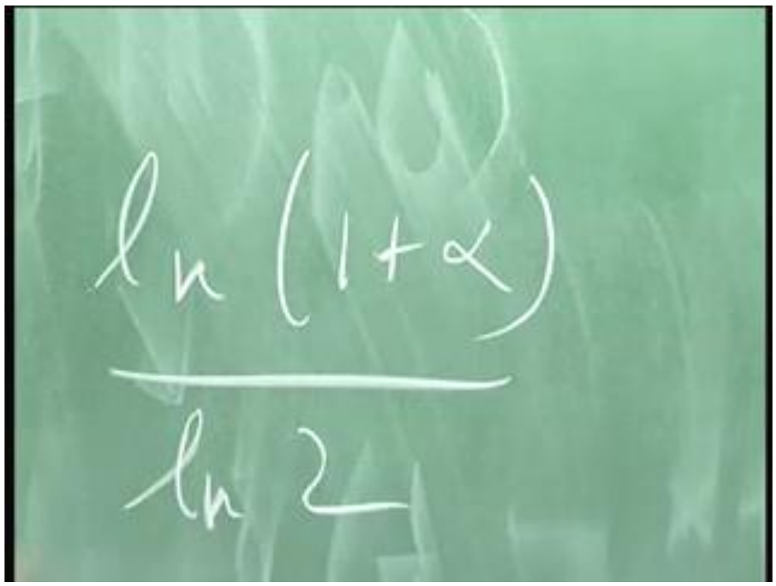

So, this is the reason why Gauss was able to assert that the probability, that the iterates x less than equal to some alpha equal to log 1 plus alpha over log 2. Because, you have to integrate this density from 0 to alpha then you got this log answer. So, this was Gauss discovery and this map here is called the gauss continued fraction map, what would be it is Lyapunov exponent.

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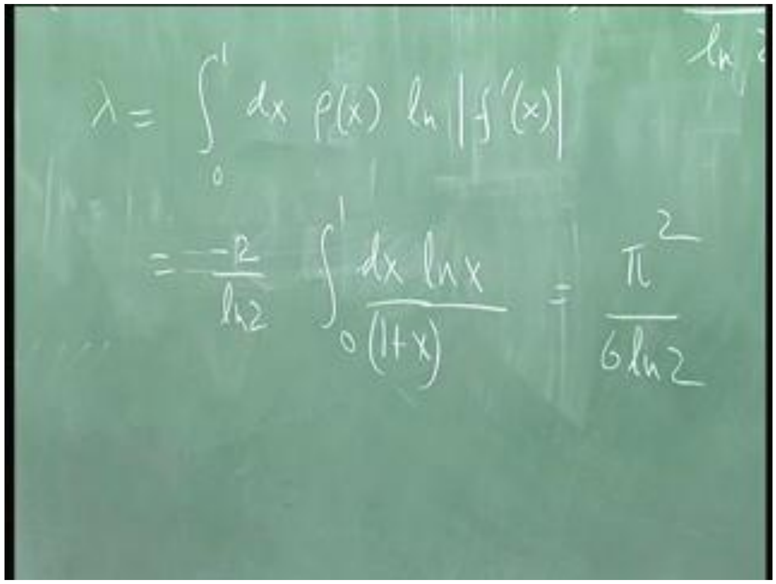

The Lyapunov exponent mind you look at the formula we have for the Lyapunov exponent, we have lambda equal to integral and this case 0 to 1 d x rho of x log modulus f prime of x. So, you could regard this Lyapunov exponent as the average value of the log of the stretch factor modulus f prime of x, we could regard it as that, because after all its an integral over this probability density.

So, you could also take it to be average value of log f prime of x over the interval. Now, what is the slope look like to you. It is clear that the slope is getting steeper and steeper and steeper, so it is not even clear whether this is finite. Some of these branches are becoming almost infinitely speed and therefore, you can expansion rate is enormous out there. But, the region in which its happening is getting smaller and smaller, so there is a possibility that the whole thing will actually be finite, if you integrate it out if you put that n then what you get here.

This is equal to, well f of x is 1 over x, so f prime of x is one over x squared the modulus of that is equal to. So, what should I write here 1 over log 2 that this portion that is come 0 to 1 d x rho of x is 1 over 1 plus x and then log modulus 1 over x square, that is equal to minus 2, whatever it is minus 2 log x is it positive or negative.

It is positive log x itself negative between 0 and 1, so the minus n cancels out and it is in positive. You have to do this integral, this is not a trivial integral to do by elementary from means, because you cannot integrate by part it is just get verse as you go long. It is one of those wired integrals which involves much more complicated function, this one involves zeta of two the (( )) zeta function of argument 2. And the final answer terms out to be equal to phi squared over 6 log 2.

Those who you interested I leave it to you to work this integral out, look up tables or whatever, but you can actually reduce this do a binomial expansion of 1 over 1 plus x try to integrate term by term and see. And then you end up with the series, if you worried about convergence and so, on; but still the final answer is this. This combination pi squared over 6 is 1 over 1 squared plus 1 over 2 squared plus 1 over 3 squared all the way to infinite that is the zeta function at argument 2 and this is the answer.

So, it is bigger than log 2 this number how big this thing it is pi square is the order of 10 and log 2 is like 0.7. So, that is like 10 over 4 it is like 2. something 2.3 something bigger than log 2 much bigger than log 2. So, it is, in fact, a very, very hyperbolic map as you can see. And unlike the earlier cases, the map is not uniform in the sense that it does not constant slow, the slope does not changing. And find the average value of this slope over the invariant measure turns out to be that, the log of that turns out to be that number.

You can do a lot more with this map, but I do not want to get into for the details here. Let me return to somewhat more general considerations. One of the things, that we going to be interested in statistical physics is given this kind of complex dynamics it is clear. Now, we have to we can do much more with this, the subject by itself you take off from here and one can study the various kinds of mixing properties ergodicity properties and so, on of various dynamical systems. Look at intermittent cases look at more complicated situations and so, on.

But what we would like to do is to turn off focus slowly towards statistical physics for this we need to have some inclines. Some general theorem, general idea of what the behavior of the systems is going to be like in phase space; a very relevant question is the following.

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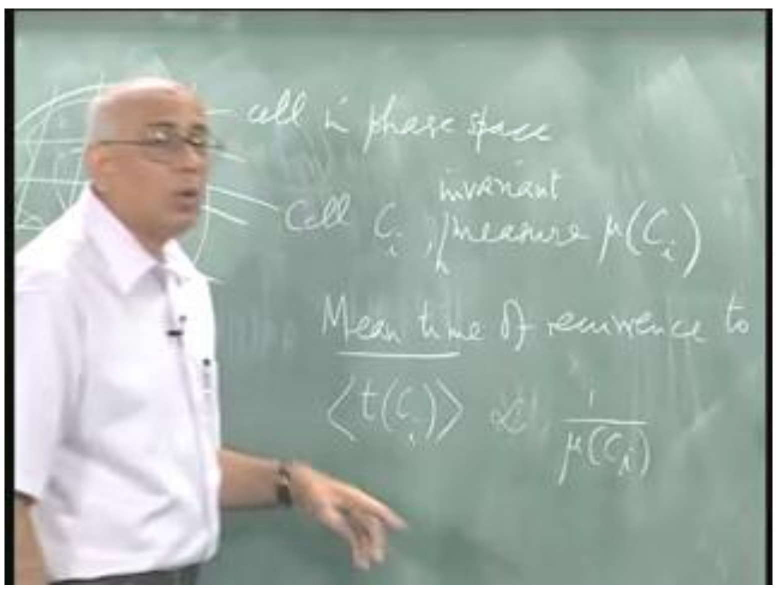

Here is my phase space and I am going to assume that this system is ergodic, even chaotic in general. Then an initial volume element here is going to get scrambled up get mixed up badly twisted around and so on, so fourth in a very, very complex fashion. I could ask if the system is egodic, then I know that portions of this initial volume element would return after certain amount of time to this original place.

Wherever to original location was, so, I ask the following question what is the time of return on the average and this is called the recurrence time. What is the mean time of recurrence and guaranteed that is going to be recurrence. So, question is, what is the average time it could take for things to come back; for that, question can be answered precisely and what one does for that is to partition the phase space into cells.

So, this is a typical cell in phase space, assume that this dynamics as in invariant measure in the sense that as the dynamics keeps going, you end up with some invariant density or invariant measure of a cell. Let me call this cell C label at i C sub i, I runs over all the cells and I ask and it is measure is mu of C i. If you like if the system has invariant density rho of x integrate rho of x over the volume of the cell and that is the measure of the cell it is the invariant measure.

And now I ask suppose I start with my representative point in the cells C i as time goes on it moves out of this cell it wander around and you put return to see. If it get back into C i somewhere I call that a recurrence, I start here may be does this comes back here, I call that a recurrence. In fact, this is the first recurrence and because system is ergodic and ongoing it would again around and then come back what intersect itself this is a higher dimension.

I call that the next recurrence and so, on; I could ask then what is the mean time of recurrence you come back here and there is a very general formalism which helps you to calculate this number without any ambiguity at all.

So, you can ask for mean mean time I do not want to go into this algebra because it requires a little bit of manipulation of integrals but, instead doing that let me just scope the answer. And it is not very hard to derive I will give you in a problem seat I will give you sufficient input in order to be able to derive this result yourself. One correct derive this in the early part in the early part of it 19 century maybe beginning of the 20th century.

In the mean time of recurrence to see I, let me call it t C i average mean over all realization. Sometimes it could comeback for sometimes it could comeback much slower depending on where you started etcetera. But, over the long period long time average over the invariant density, what is the mean time of recurrence. This terms out to be equal to is proportional to 1 over the measure of this cell, it is a very simple formula.

The numerator would be your time step dough, whatever that be; however, small it is, it depends the units of time of course, you depend on your time step. But, the mean time is proportional to the reciprocal of the measure of the cell. This of course, tells you this reciprocal tell you the larger this measure is, the shorter the mean time and as you can see its equal to 1 if you look at the full phase space the system is where somewhere.

So, in one time steps it is still there and that is the end of it, but as you get smaller and smaller as a cell gets smaller and smaller or at least it is invariant measure get smaller and smaller. This number become becomes larger and larger and if it is a point, if it is a single point, then this becomes infinite, which is as it should be, because it is tells you it is not periodic motion it is kaotic motion. So, really there is no period here at all, so what is doing understood here is that on the average as the measure of itself gets smaller as you refine your course training here, the recurrence time will naturally go up more and more.

But, it is not just the physical volume in phase space of the cells that is important it is actually the invariant measure depends on how much time is append at various points. When we wrote down all those invariant measures the rho of x you can actually ask what fraction of time does it an iterate of an arbitrary points spend between some regions, some other region and that is proportional to the measure of that x.

So, this thing here is called the Poincare recurrence formula or theorem or whatever you call it such a recurrence is called Poincare recurrence. I have to average over what this is already an invariant measure. So, it is average or the question is what are these what is this average over, this average is over all realizations of this process.

So, you have to start at this point, at this point, at this point, at this point you have to start everywhere and each time you have to find all the recurrence times and then average over it yeah it is not clear. So, it depends on how it depends on how close, this is it is depends on how refined this course training is. If it is extremely refined there is no reason why one portion and another portion should have exactly the same recurrence time not at all.

So, even within this cell the invariant density maybe weighted more towards this point than this point definitely. So, now we are getting into the fine integrity of how it is goes training done and so on. So, in general the most general case this thing here could be typically cases if it sufficiently hyperbolic sufficiently kaotic then it is not matter, where you are in this side itself. This point is that wherever you start, if the equivalent to starting anywhere else in this cells, but this could really depend on how what you call is cell.

The most convenient thing to do could be to see how random this could get. It could be very nice if you could partition this cell into such partition in the phase space into such cells that the entire process the dynamics is mimic by a random process. Like in the earlier case we set this random process could be for example, coin toss if its mimics by what is called the mark (( )) process.

Then we can write the answer down completely, we can write down the measure, we can write down the mean we can write down explicitly what the formulas are because then for a mark of process what you do is the following. You assume that each of these cells represents a state of the system. And then you say that the transition probability per unit time from any state to any other state is prescribe to you and if it is a mark of process. The probability that you go from any given state to any other state in n times steps could be related to the nth power of this transition matrix.

Some suitable element the nth power of the matrix of transition probability, that is the property of mark of process. And if you have practitioner of that kind it is called a mark of partition. And you would like to see if in general cases you can produce work of partition, because it is also like to use the power of probability and its takes in order to solve dynamical problems, the other thing you can ask is I do course gaining in phase space.

So, clearly I am losing, I am blurring information but once I have done the course training can I reconstruct the trajectories completely from the course training information. And that is an interesting question let us go back and look at the Bernoulli shift.

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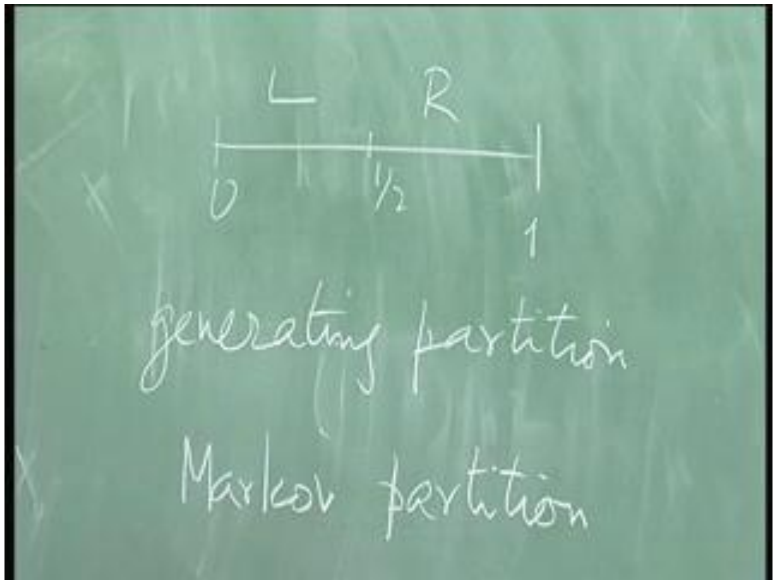

In the Bernoulli shift the course training that I suggested was you took 0 to 1 and this was the half I said call this L and call this R call the bin from 0 to half L and half to 1 R. And then as you iterated you just keep track whether the random the point the iterate was in L or in R and you had a strings of L and R, but you see the L's and R's corresponded to whether the a i is 0's or 1's.

So, if you give me the L's and R's I can give you the A's completely I can regenerate the original dynamics from the course training dynamics. I think it is does that is called the generating partition, because the dynamics of the course grained course grained dynamics it is called symbolic dynamics because you have to let an alphabet with two letters L and R. So, keeping track of just the L's and R's if you are able to reconstruct what the original trajectory was the original point was, when you got the generating partition.

And of course, it is should be the dream of every in every system to discover a generating partition wonderful or in the absence of that if you look at Markov partition. Where the entire process of transitioning from one state to one cell to another is simply replace by a Markov process with some prescribe set of transition probabilities. Of course it would be a wonderful thing if you could have a generating Markov of partition then of course, a problem is solved completely.

But, these are approximations in general it is not easy to find in higher dimensional dynamical systems things of this kind. But, they provide the starting point for this whole subject of symbolic dynamics there is a large literature on this in the subject by itself. You could ask what happens in Hamiltonian dynamics, what sort of, what sort of behavior you have you have Kaoyas, Hamiltonian dynamics.

It turns out the Hamiltonian dynamics when it is becomes kaotic is, so complicated that in the vicinity of the kaotic points you can do symbolic dynamics and transform everything things which look like Bernoulli shift. But, the alphabet you need itself infinite, you cannot map Hamiltonian dynamics Gauss Hamiltonian dynamics recurresly, to symbolic dynamics with the finite alphabet; that is in infinite alphabets.

So, you can imagine how complex the system is how complex Hamiltonian dynamics, how delicately balanced integrability is you see that as soon as you loose intergrability you have a enormous complications this kind. This takes as into many, many other things and I will slowly move into statistical physics from next time. In particular when you look at molecules of gas in this room, you ask what is the behavior, like the answer is it is kaotic.

Even in your (( )) dynamics it is completely chaotic, what is the Lyapunov exponent like remarkably enough. The Lyapunov exponent is like the number of decrease of freedom and that is of the order of 10 to by 23. We are talk about Lyapunov exponent is log 2 and pi square over 6 log 2 that is of order 10 to the by 23 in principle. What is the power carrier recurrence time like well now this will depend on how close you are watching the system how much course grained you are going to do.

The exact recurrence time it will depend on the size of your phase space, how many degrees of freedom you are willing in put in. But, you can also show that if you are going to look at the microstate of the system, namely I am going to look at the state of every particle in this room and called that state of the system.

Then the Poincare recurrence time become of the order this becomes of the order of e to the power the number of degrees of freedom. So, that is like e to the power 10 to by 23 this is the stacked in large number inconceivably large and that is why things look irreversible to us; so one of the reasons, why you have irreversibility appearing even if nothing else happened, and that because things are never going to look reversible, because a Poincare recurrence time is actually so large.

So, the observation time is much smaller than the recurrence time things good look irreversible, even with reversible dynamics going on e to the 10 to the 23 is, so large that I do not bother to write down the units. Whether I measure it in femto seconds or in edges of the universe it is only a matter of 10 to the 42 or 10 to the 32 that is nothing even the units do not have to be written out.

So, this is just completely outside this scope of reality it is really, really in irreversible system because of the very large number of degrees of freedom. Now we will see how some of these concepts play a deep role in our understanding of statistical physics. I am going to assume a certain knowledge of femodynamics in the part of all of you we will; however, revise thermodynamics in 15 minutes and after that go on to a elementary statistical physics of this is what I start with next week.

Meanwhile I give you another problem see it put down some decides and in particular how to derive this relationship here, which is very easy to do but, in a little bit of algebra and also you can do more things. In the formula that you derive you can also ask what is the meantime to escape from some cell? What is the mean exist time what is the mean return time what is the mean time to go from one place to another and so, on. All of these can be answered fairly straight forwardly, once you have some information about the invariant measure the system. So, let me stop here today.